Final Examination (Take Home)

MATH 2043: Honors Analysis I

Spring 2021, HKUST

May 27th, 2021, 16:30 - 20:00

READ THE INSTRUCTIONS CAREFULLY:

- This is a **Take Home Exam**. However discussion with any person (online or offline) is **strictly prohibited**, and is a **serious violation of the Honor Code**. Posting related questions in any online forum, whether it is answered or not, is also a serious violation of the Honor Code.
- The is an **Open-Book Exam**. You are **allowed** to consult **any** online materials or computer software (e.g. to simplify expressions). However you must write up your own solution, and you **must only** quote results from the Lecture Notes, Lecture Slides, Worksheet, Homework and Tutorial.
- **You will automatically score zero for the whole part of a Problem if you use any results considered outside the syllabus of this course. (Please ask me if you are not sure.)**
- The Examination is **Proctored.** You should join the Zoom Meeting at all time.
  If you have any questions, please send me a **private message** via the Zoom Meeting.
- Please show your work clearly and **justify your answers**.
  Points will be deducted if we find the logic incomplete.
- **Write neatly** – answers which are illegible for the graders cannot be given credit.
- Scores are indicated in the square brackets.

ABOUT HINT SYSTEM:

- You can **spend points to obtain hints** to the problems on Canvas Quizzes.
- The hints may or may not be useful, and may or may not worth the points you spend.
- You are still required to write down the full solution and it will be graded as usual.
- The points you spent will be deducted **separately** from the Exam score.

ABOUT SUBMISSION:

- Answer **ALL** 4 Problems on blank white paper, single-lined paper, or export from a whiteboard app on a tablet. If you use a tablet, you should only use a **single color** for answering.
  DO NOT use "Soc Paper" or other paper with patterns.
  DO NOT LaTeX type solution. They will not be graded.
- **Scan and upload 4 PDF files (one for each Problem), or a single PDF file with all Problems.**
  Please use a scanner / scanner app. Large file size may cause Canvas to timeout and log you out.
- (Optional) In order to help us grade (yeah∼), **you will score 1 bonus point for the course grade** if you label your 4 filenames as `Q1.pdf`, `Q2.pdf`,... and so on in a **SINGLE** submission.
- Submit your solution by **20:00, May 27th, 2021** at the Final Examination Assignment Page.
  Double-check that your PDF files are scanned and uploaded correctly!
  2-point penalty per minute late for late submission, according to the **last timestamp** on Canvas.

Time allowed: 3.5 Hours (including submission time.)

In this Exam,

- **The field is the real numbers $\mathbb{R}$. All functions are real-valued.**
- **All sequences with $n \in \mathbb{N}$ start with $n = 1$.**
- **$[a, b]$ denotes a closed interval in $\mathbb{R}$ with $a < b$.**
- **"Integrable" means Riemann Integrable.**
- **"Measure zero" means Lebesgue measure zero.**

Hint System:

- **No hints for Q1(a).**
- **Hint costs 2 points for 5-point questions.**
- **Hint costs 3 points for 10-point questions.**

PROBLEM 1. Short Questions (35 points)

[10] (a) Determine the pointwise limit of the following functions on the given domain, and whether the convergence is uniform or not. (**No justifications are needed.**)

(i) $f_n(x) = x^{\frac{1}{n}}, \quad x \in [0, 1].$

(ii) $f_n(x) = \dfrac{nx}{1 + n^2 x}, \quad x \in [0, \infty).$

(iii) $f_n(x) = x \arctan nx, \quad x \in (-\infty, \infty).$

(iv) $f_n(x) = \displaystyle\sum_{k=1}^{n} x^k (\log x)^2, \quad x \in (0, 1].$

(v) $f_n(x) = \displaystyle\sum_{k=1}^{n} \frac{kx}{(1 + x)(1 + 2x) \cdots (1 + kx)}, \quad x \in [0, \infty).$

In (b)–(f), **prove or disprove** the following statements:

[5] (b) There exists a closed subset $E \subset \mathbb{R}$ that is not measure zero, but it does not contain any rational numbers.

[5] (c) There exists a continuous function $f(x)$ on $\mathbb{R}$ such that it is differentiable only at $x = 0$ and $x = 1$.

[5] (d) There exists a function $f \in \mathcal{C}^1[-1, 1]$ such that $f(-1) = f(1) = 1$, $|f'(x)| \leq 1$ and $\left| \displaystyle\int_{-1}^{1} f(x)dx \right| \leq 1$.

[5] (e) If $f(x)$ is integrable on $[0, 1]$, then $\|f\|_p$ is a bounded increasing function in $p \in (0, \infty)$.

[5] (f) There exists a sequence of *even* polynomials $\{p_n\}$ on $[-1, 1]$ such that

$$\lim_{n \to \infty} \int_{-1}^{1} |p_n(x) - f(x)| dx = 0$$

where f is an *odd* function defined by $f(x) = \begin{cases} 1 & x \in K \\ -1 & -x \in K \\ 0 & \text{otherwise} \end{cases}$ and $K \subset [0, 1]$ is the Cantor set.

PROBLEM 2. (25 points)

Let $f(x) \in \mathcal{C}^1(\mathbb{R})$. Define the subset of $\mathbb{R}$ by

$$X := \{x \in \mathbb{R} : f'(x) = 0\}.$$

Assume X has measure zero.

[10] (a) Show that if $f'(x) \neq 0$ on an interval I and $Y \subset I$ has measure zero, then $f(Y)$ has measure zero.

[10] (b) Using (a) and the inverse function theorem, show that if Y has measure zero, the inverse image

$$f^{-1}(Y) := \{x \in \mathbb{R} : f(x) \in Y\}$$

also has measure zero.
(Free Hint: Every open subset in $\mathbb{R}$ is a countable union of disjoint open intervals (HW#2 Q5).)

[5] (c) Let g be an integrable function on $f([a,b])$. Using (b), show that $g \circ f$ is integrable on $[a,b]$.

PROBLEM 3. (20 points)

Let $a, b > 2$ be two constants. Recall the Gamma function defined by the improper integral

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt := \lim_{R\to+\infty}\int_0^R t^{x-1}e^{-t}dt$$

which is finite for all $x > 1$.

[5] (a) Prove that $\displaystyle\sum_{n=1}^\infty x^{b-1}a^{-nx}$ uniformly converges on $[0, \infty)$.

[5] (b) Let $g_n(t) = \displaystyle\int_0^{1/t} x^{b-1}a^{-nx}dx$. Prove that $\displaystyle\sum_{n=1}^\infty g_n(t)$ uniformly converges on $[0, \infty)$.
(Here $g_n(0)$ is defined to be $\displaystyle\lim_{t\to 0^+} g_n(t)$.)

[10] (c) Extend $\dfrac{x^{b-1}}{a^x - 1}$ to a continuous function on $x \in [0, \infty)$. Using (a) and (b), prove that

$$\lim_{R\to+\infty}\int_0^R \frac{x^{b-1}}{a^x-1}dx = \frac{\Gamma(b)\zeta(b)}{(\log a)^b}.$$

where $\zeta(s)$ is the Riemann zeta function.

PROBLEM 4. (20 points)

Let $\{f_n\}$ be a sequence of differentiable functions $\mathbb{R}$ such that $f_n(0) = 20$ and $|f_n'(x)| \leq 43$ for all $x \in \mathbb{R}$.

[5] (a) If $K \subset \mathbb{R}$ is compact, prove that there exists a subsequence of $\{f_n\}$ converging uniformly on K.

[10] (b) Prove that there exists a subsequence of $\{f_n\}$ converging pointwise to a continuous function on $\mathbb{R}$.

[5] (c) Give an explicit example of $\{f_n\}$ that is uniformly bounded on $\mathbb{R}$ and satisfying the assumptions above, but the convergence in (b) is not uniform on $\mathbb{R}$.

–End of Paper–