MATH2043 HW1

Agastya Singh
asinghao@connect.ust.hk
21103283

§1.8(5) Let $\varepsilon_1 = 2\varepsilon$. For any $\varepsilon > 0, \exists N : |x_n - l| \leq 2\varepsilon \equiv |x_n - l| \leq \varepsilon_1$. Since $\varepsilon_1 = 2\varepsilon$, The second statement is true for all $\varepsilon_1 > 0$.

§1.15 For all $\varepsilon > 0$, There is N such that for all $n > N$, $|x_n - l| < \varepsilon$. For even n, $\exists k : 2k = n$. Hence, above implies $x_{2k} - l| < \varepsilon$. For odd n, $\exists k : 2k + 1 = n$. Hence, $|x_{2k+1} - l| < \varepsilon$.
For the converse, there exists K_1 such that $k \geq K_1 \Rightarrow |x_{2k} - l| < \varepsilon$, and K_2 such that $k \geq K_2 \Rightarrow |x_{2k+1} - l| < \varepsilon$. Set $N = \max\{2K_1, 2K_2 + 1\}$. FOr any $n > N$,
if n is even, write $n = 2k$. Since $n > 2K_1$, $k > K_1$ so $|x_{2k} - l| < \varepsilon$.
if n is odd, write $n = 2k + 1$. Since $n > 2K_2 + 1$, $k > K_2$ so $|x_{2k+1} - l| < \varepsilon$.

§1.19 The error in this proof is assuming $\lim_{n \to \infty} x_n$ exists and performing mathematical operations on it.

§1.22 No, the converse is not true. Consider

$$x_n = \begin{cases} n \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

The sequence is obviously unbounded, but the sequence does not tend to infinity.

§ 1.31(1) Suprema is $2 + \sqrt{3}$.

§ 1.31(4) No suprema, set n to 1, m is has no upper bound, making $\frac{m}{n}$ grow to infinity.

§1.33 TODO

§1.34(8) $|x - y| \leq c \Rightarrow x - y \leq c$. For any $\varepsilon > 0$, choose $x_\varepsilon : x_\varepsilon > \sup X - \varepsilon$. We have

$$x_\varepsilon - y \leq c \Rightarrow \sup X - c - \varepsilon \leq y \text{ for all } y \in Y.$$

$\sup X - c - \varepsilon$ is a lower bound of Y. Therefore,

$$\sup X - \varepsilon - c \leq \inf Y$$

and it follows that

$$\sup X - \varepsilon - c \leq \sup Y$$

it can be concluded that

$$\sup X - \inf Y \leq c$$
$$\sup X - \sup Y \leq c$$

Symmetrically, $\sup Y - \inf X \leq c$ and $\sup Y - \sup$. Hence, $|\sup X - \inf Y| \leq c$ and $|\sup X - \sup Y| \leq c$.

Conversely,

$$\inf X \le x \le \sup X \ , \text{ and } \inf Y \le y \le \sup Y$$
$$x - y \le \sup X - \sup Y \le c$$

: Symmetrically, $y - x \le c$. $\square$

§1.40   TODO

§1.45(5)   The function is periodic for every 12 indices. Let

$$x_n = \frac{(12k + r) \sin\left(\frac{(12k+r)\pi}{3}\right)}{(12k + r) \cos\left(\frac{(12k+r)\pi}{2}\right) + 2}$$

Let $n = 12k + r$ for some $k, r \in \mathbb{N}$ (including 0). We get

$$x_n = x_{12k+r} = \frac{(12k + r) \sin\left(\frac{(12k+r)\pi}{3}\right)}{(12k + r) \cos\left(\frac{(12k+r)\pi}{2}\right) + 2}$$

$$= \frac{(12k + r) \sin\left(\frac{r\pi}{3}\right)}{(12k + r) \cos\left(\frac{r\pi}{2}\right) + 2}$$

As $k \to \infty$, this approaches $\frac{\sin\left(\frac{r\pi}{3}\right)}{\cos\left(\frac{r\pi}{2}\right)}$, for $r \in \{0, 2, ..., 11\}$. Computing the value of the function for all 12 of these, we see that $\mathrm{LIM}\{x_n\} = \left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$.

§1.45(8)   Let

$$x_n = \sqrt[n]{2^n + 3^{(-1)^n n}}$$

For even $n = 2k$,

$$x_n = x_{2k} = \sqrt[2k]{2^{2k} + 3^{2k}}$$

$$= 3 \sqrt[2k]{\left(\frac{2}{3}\right)^{2k} + 1}$$

As $k \to \infty$, x_{2k} approaches 3.
For odd $n = 2k + 1$,

$$x_n = x_{2k+1} = \sqrt[2k+1]{2^{2k+1} + 3^{-(2k+1)}}$$

$$= 2 \sqrt[2k+1]{\left(\frac{1}{6}\right)^{2k+1} + 1}$$

As $k \to \infty$, x_{2k+1} approaches 2.
Therefore, $\mathrm{LIM}\{x_n\} = \{2, 3\}$.

§1.46

1. $\mathrm{LIM}\{x_n\} \cup \mathrm{LIM}\{y_n\} \subseteq \mathrm{LIM}\{z_n\}$
  Assume $l \in \{x_n\}$, then there is a subsequence $\{x_{n_k}\}$ converging to l. Since $\{x_n\} \in \{z_n\}$, $\{x_{n_k}\} \in \{z_n\}$. Symmetrical argument for y_n.

2. $\mathrm{LIM}\{z_n\} \subseteq \mathrm{LIM}\{x_n\} \cup \mathrm{LIM}\{y_n\}$

Assume $l \in \{z_n\}$, then there is a subsequence $\{z_{n_k}\}$ converging to l.

Case 1: $\{z_{n_k}\}$ **contains infinte terms from** $\{x_n\}$

Then, for some K, there is a sequence $\{x_{m_k}\}$ that has the exact same terms in the same order as $\{z_{n_k}\}$.

Let $J = \{k : z_{n_k} \in \{x_n\}\}$. We can write for each $k \in J$, $z_{n_k} = x_{m_k}$. Note that the indices $\{m_k\}$ are increasing, since order of $\{x_n\}$ is preserved in $\{z_n\}$. It is clear that $\{x_{m_k}\}$ is a subsequence of $\{x_n\}$. Since $\{x_{m_k}\}$ converges to l, this means that $l \in \mathrm{LIM}\{x_n\}$.

Case 2: $\{z_{n_k}\}$ **contains infinte terms from** $\{y_n\}$

The proof is symmetrical.

$\square$

§1.48(5) $\overline{\lim} \, x_n = \frac{\sqrt{3}}{2}$, $\underline{\lim} \, x_n = -\frac{\sqrt{3}}{2}$.

§1.48(8) $\overline{\lim} \, x_n = 3$, $\underline{\lim} \, x_n = 2$.

§1.49(7) Assime $\{x_n\}$ and $\{y_n\}$ are bounded. F, or all $m_k, n_k \geq k \geq N$, we have $x_{m_k} \geq y_{n_k}$. By Bolzano Weierstrass, we know that there are convergent subsequnces $x_{m_{k_i}}$ and $y_{n_{l_i}}$, such that $\{x_{m_{k_i}}\} \geq \{y_{n_{l_i}}\}$. By the order property, we have

$$\lim_{i \to \infty} x_{m_{k_i}} \geq \lim_{i \to \infty} y_{n_{l_i}} \tag{1}$$

Since both $\{x_{m_{k_i}}\}$ and $\{y_{n_{l_i}}\}$ are subsequences of $\{x_n\}$ and $\{y_n\}$ respectively, we also have

$$\overline{\lim} \, x_n \geq \lim_{i \to \infty} x_{m_{k_i}}$$
$$\lim_{i \to \infty} y_{n_{l_i}} \geq \underline{\lim} \, y_n \tag{2}$$

Combining (1) and (2), we get

$$\overline{\lim} \, x_n \geq \underline{\lim} \, y_n$$

$\square$

§1.49(8) TODO **§1.51 TODO**

§1.55

Note that

$$\left| \frac{x_{n+1}}{x_n} \right| \leq \left| \frac{y_{n+1}}{y_n} \right| \Rightarrow \frac{|x_{n+1}|}{|x_n|} \leq \frac{|y_{n+1}|}{|y_n|}$$

1 By writing the inequality for the first few n, we see that

$$\frac{|x_2|}{|x_1|} \leq \frac{|y_2|}{|y_1|}$$

$$\frac{|x_3|}{|x_2|} \leq \frac{|y_3|}{|y_2|}$$

$$\vdots$$

$$\frac{|x_n|}{|x_{n-1}|} \leq \frac{|y_n|}{|y_{n-1}|}$$

Multiplying the inequalities (since $x_n \neq 0$), we get

$$\frac{|x_n|}{|x_1|} \leq \frac{|y_n|}{|y_1|}$$

$$|x_n| \leq |y_n|\frac{|x_1|}{|y_1|}$$

c is $\frac{|x_1|}{|y_1|}$.

2 if $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty} |y_n| = 0$. By squeeze theorem, we have $\lim_{n\to\infty} |x_n| = 0$.

3 We have $x_n \leq |x_n| \leq c|y_n|$. For any M, choose N such that for any $n > N \Rightarrow \frac{x_n}{c} > M$ (such n exists, since $x_m \to \infty$). We have

$$M < \frac{x_n}{c} \leq |y_n|$$

Thus, $|y_n| \to \infty$. $\square$