

MATH2043 HW1

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§1.8(5) Let $\varepsilon_1 = 2\varepsilon$. For any $\varepsilon > 0$, $\exists N : |x_n - l| \leq 2\varepsilon \equiv |x_n - l| \leq \varepsilon_1$. Since $\varepsilon_1 = 2\varepsilon$, The second statement is true for all $\varepsilon_1 > 0$.

§1.15 For all $\varepsilon > 0$, There is N such that for all $n > N$, $|x_n - l| < \varepsilon$. For even n , $\exists k : 2k = n$.

Hence, above implies $|x_{2k} - l| < \varepsilon$. For odd n , $\exists k : 2k + 1 = n$. Hence, $|x_{2k+1} - l| < \varepsilon$.

For the converse, there exists K_1 such that $k \geq K_1 \Rightarrow |x_{2k} - l| < \varepsilon$, and K_2 such that $k \geq K_2 \Rightarrow |x_{2k+1} - l| < \varepsilon$. Set $N = \max\{2K_1, 2K_2 + 1\}$. For any $n > N$,

if n is even, write $n = 2k$. Since $n > 2K_1$, $k > K_1$ so $|x_{2k} - l| < \varepsilon$.

if n is odd, write $n = 2k + 1$. Since $n > 2K_2 + 1$, $k > K_2$ so $|x_{2k+1} - l| < \varepsilon$.

§1.19 The error in this proof is assuming $\lim_{n \rightarrow \infty} x_n$ exists and performing mathematical operations on it.

§1.22 No, the converse is not true. Consider

$$x_n = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

The sequence is obviously unbounded, but the sequence does not tend to infinity.

§ 1.31(1) Suprema is $2 + \sqrt{3}$.

§ 1.31(4) No suprema, set n to 1, m is has no upper bound, making $\frac{m}{n}$ grow to infinity.

§1.33 TODO

§1.34(8) $|x - y| \leq c \Rightarrow x - y \leq c$. For any $\varepsilon > 0$, choose $x_\varepsilon : x_\varepsilon > \sup X - \varepsilon$. We have

$$x_\varepsilon - y \leq c \Rightarrow \sup X - c - \varepsilon \leq y \text{ for all } y \in Y.$$

$\sup X - c - \varepsilon$ is a lower bound of Y . Therefore,

$$\sup X - \varepsilon - c \leq \inf Y$$

and it follows that

$$\sup X - \varepsilon - c \leq \sup Y$$

it can be concluded that

$$\sup X - \inf Y \leq c$$

$$\sup X - \sup Y \leq c$$

Symmetrically, $\sup Y - \inf X \leq c$ and $\sup Y - \sup X \leq c$. Hence, $|\sup X - \inf Y| \leq c$ and $|\sup X - \sup Y| \leq c$.

Conversely,

$$\inf X \leq x \leq \sup X, \text{ and } \inf Y \leq y \leq \sup Y \\ x - y \leq \sup X - \sup Y \leq c$$

: Symmetrically, $y - x \leq c$. \square

§1.40 TODO

§1.45(5) The function is periodic for every 12 indices. Let

$$x_n = \frac{(12k + r) \sin\left(\frac{(12k+r)\pi}{3}\right)}{(12k + r) \cos\left(\frac{(12k+r)\pi}{2}\right) + 2}$$

Let $n = 12k + r$ for some $k, r \in \mathbb{N}$ (including 0). We get

$$\begin{aligned} x_n = x_{12k+r} &= \frac{(12k + r) \sin\left(\frac{(12k+r)\pi}{3}\right)}{(12k + r) \cos\left(\frac{(12k+r)\pi}{2}\right) + 2} \\ &= \frac{(12k + r) \sin\left(\frac{r\pi}{3}\right)}{(12k + r) \cos\left(\frac{r\pi}{2}\right) + 2} \end{aligned}$$

As $k \rightarrow \infty$, this approaches $\frac{\sin(\frac{r\pi}{3})}{\cos(\frac{r\pi}{2})}$, for $r \in \{0, 2, \dots, 11\}$. Computing the value of the function for all 12 of these, we see that $\text{LIM}\{x_n\} = \left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$.

§1.45(8) Let

$$x_n = \sqrt[n]{2^n + 3^{(-1)^{n_n}}}$$

For even $n = 2k$,

$$\begin{aligned} x_n = x_{2k} &= \sqrt[2k]{2^{2k} + 3^{2k}} \\ &= 3 \sqrt[2k]{\left(\frac{2}{3}\right)^{2k} + 1} \end{aligned}$$

As $k \rightarrow \infty$, x_{2k} approaches 3.

For odd $n = 2k + 1$,

$$\begin{aligned} x_n = x_{2k+1} &= \sqrt[2k+1]{2^{2k+1} + 3^{-(2k+1)}} \\ &= 2 \sqrt[2k+1]{\left(\frac{1}{6}\right)^{2k+1} + 1} \end{aligned}$$

As $k \rightarrow \infty$, x_{2k+1} approaches 2.

Therefore, $\text{LIM}\{x_n\} = \{2, 3\}$.

§1.46

1. $\text{LIM}\{x_n\} \cup \text{LIM}\{y_n\} \subseteq \text{LIM}\{z_n\}$

Assume $l \in \{x_n\}$, then there is a subsequence $\{x_{n_k}\}$ converging to l . Since $\{x_n\} \in \{z_n\}$, $\{x_{n_k}\} \in \{z_n\}$. Symmetrical argument for y_n .

2. $\text{LIM}\{z_n\} \subseteq \text{LIM}\{x_n\} \cup \text{LIM}\{y_n\}$

Assume $l \in \{z_n\}$, then there is a subsequence $\{z_{n_k}\}$ converging to l .

Case 1: $\{z_{n_k}\}$ contains infinite terms from $\{x_n\}$

Then, for some K , there is a sequence $\{x_{m_k}\}$ that has the exact same terms in the same order as $\{z_{n_k}\}$.

Let $J = \{k : z_{n_k} \in \{x_n\}\}$. We can write for each $k \in J$, $z_{n_k} = x_{m_k}$. Note that the indices $\{m_k\}$ are increasing, since order of $\{x_n\}$ is preserved in $\{z_n\}$. It is clear that $\{x_{m_k}\}$ is a subsequence of $\{x_n\}$. Since $\{x_{m_k}\}$ converges to l , this means that $l \in \text{LIM}\{x_n\}$.

Case 2: $\{z_{n_k}\}$ contains infinite terms from $\{y_n\}$

The proof is symmetrical.

□

$$\S 1.48(5) \quad \overline{\lim} x_n = \frac{\sqrt{3}}{2}, \underline{\lim} x_n = -\frac{\sqrt{3}}{2}.$$

$$\S 1.48(8) \quad \overline{\lim} x_n = 3, \underline{\lim} x_n = 2.$$

\S 1.49(7) Assume $\{x_n\}$ and $\{y_n\}$ are bounded. For all $m_k, n_k \geq k \geq N$, we have $x_{m_k} \geq y_{n_k}$. By Bolzano Weierstrass, we know that there are convergent subsequences $x_{m_{k_i}}$ and $y_{n_{l_i}}$, such that $\{x_{m_{k_i}}\} \geq \{y_{n_{l_i}}\}$. By the order property, we have

$$\lim_{i \rightarrow \infty} x_{m_{k_i}} \geq \lim_{i \rightarrow \infty} y_{n_{l_i}} \quad (1)$$

Since both $\{x_{m_{k_i}}\}$ and $\{y_{n_{l_i}}\}$ are subsequences of $\{x_n\}$ and $\{y_n\}$ respectively, we also have

$$\begin{aligned} \overline{\lim} x_n &\geq \lim_{i \rightarrow \infty} x_{m_{k_i}} \\ \lim_{i \rightarrow \infty} y_{n_{l_i}} &\geq \underline{\lim} y_n \end{aligned} \quad (2)$$

Combining (1) and (2), we get

$$\overline{\lim} x_n \geq \underline{\lim} y_n$$

□

\S 1.49(8) TODO

\S 1.51

Let $\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = r < 1$. We know that there must be an N such that for all $n > N$, $\left| \frac{x_{n+1}}{x_n} \right| < r$. (Not sure if this needs to be proved, but in short, if there were infinitely many $\left| \frac{x_{n+1}}{x_n} \right| \geq r$, then there would be a converging subsequence with limit $\geq r$ which would belong to LIM, so contradiction).
for any $k > 0$, we have

$$\begin{aligned} |x_{N+k}| &< r |x_{N+k-1}| < r^2 |x_{N+k-2}| < \dots < r^k |x_N| \\ |x_{N+k}| &\leq r^k |x_N| \end{aligned}$$

Since $|r| < 1$, $r^k \rightarrow 0$. Thus, $|x_{N+k}| \rightarrow 0$, and therefore $x_n \rightarrow 0$.

\S 1.55

Note that

$$\left| \frac{x_{n+1}}{x_n} \right| \leq \left| \frac{y_{n+1}}{y_n} \right| \Rightarrow \frac{|x_{n+1}|}{|x_n|} \leq \frac{|y_{n+1}|}{|y_n|}$$

1 By writing the inequality for the first few n , we see that

$$\begin{aligned} \frac{|x_2|}{|x_1|} &\leq \frac{|y_2|}{|y_1|} \\ \frac{|x_3|}{|x_2|} &\leq \frac{|y_3|}{|y_2|} \\ &\vdots \\ \frac{|x_n|}{|x_{n-1}|} &\leq \frac{|y_n|}{|y_{n-1}|} \end{aligned}$$

Multiplying the inequalities (since $x_n \neq 0$), we get

$$\begin{aligned} \frac{|x_n|}{|x_1|} &\leq \frac{|y_n|}{|y_1|} \\ |x_n| &\leq |y_n| \frac{|x_1|}{|y_1|} \end{aligned}$$

c is $\frac{|x_1|}{|y_1|}$.

2 if $\lim_{n \rightarrow \infty} y_n = 0$, then $\lim_{n \rightarrow \infty} |y_n| = 0$. By squeeze theorem, we have $\lim_{n \rightarrow \infty} |x_n| = 0$.

3 We have $x_n \leq |x_n| \leq c|y_n|$. For any M , choose N such that for any $n > N \Rightarrow \frac{x_n}{c} > M$ (such n exists, since $x_m \rightarrow \infty$). We have

$$M < \frac{x_n}{c} \leq |y_n|$$

Thus, $|y_n| \rightarrow \infty$. \square