Math 2043 Honors Mathematical Analysis

Midterm Test, Spring 2015

19:00 - 21:30, March 21, 2015

You only need to solve 3 problems among the following 4.

1. (25 Marks) Suppose $\{a_n\}$ is a positive sequence such that $\lim_{n\to\infty} a_n = 1$. Show that $\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = 1$.

2. (25 Marks) For which positive numbers a does the curve $y = a^x$ intersect the line $y = x$?

3. (25 Marks) Suppose f is a continuous function on $\mathbb{R}$. Show that if at every point, f attains either local maximum or minimum, then f must be a constant function.

4. (25 Marks) Suppose f is continuous on $[0, 1]$, $f''(x) < 0$ in $(0, 1)$, and $f(0) = f(1) = 0$.

   (1) For each positive integer n, there is a unique $x_n \in (0, 1)$ such that $f'(x_n) = M/n$, where M is the absolute maximum of f on $[0, 1]$.

   (2) Show that $\{x_n\}$ is increasing. So the limit $\lim_{n\to\infty} x_n = x^*$ exists.

   (3) Find $f(x^*)$.

SOLUTION

1. (25 Marks) Suppose $\{a_n\}$ is a positive sequence such that $\lim_{n\to\infty} a_n = 1$. Show that $\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = 1$.

Solution We first show that if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} \dfrac{b_1 + b_2 + \cdots + b_n}{n} = 0$.
In fact, since $\lim_{n\to\infty} b_n = 0$, for each $\varepsilon > 0$, there is an N_1, such that when $n \geq N_1$, $|a_n| \leq \frac{1}{2}\varepsilon$. Since

$$\lim_{n\to\infty} \frac{b_1 + b_2 + \cdots + b_{N_1}}{n} = 0,$$

there is an N_2, such that when $n \geq N_2$,

$$\frac{|b_1 + b_2 + \cdots + b_{N_1}|}{n} < \tfrac{1}{2}\varepsilon.$$

Thus, when $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned}
\left|\frac{b_1 + b_2 + \cdots + b_n}{n}\right| &= \left|\frac{b_1 + b_2 + \cdots + b_{N_1} + b_{N_1+1} + \cdots b_n}{n}\right| \\
&\leq \frac{|b_1 + b_2 + \cdots + b_{N_1}|}{n} + \frac{|b_{N_1+1}|}{n} + \cdots + \frac{|b_n|}{n} \\
&\leq \tfrac{1}{2}\varepsilon + \frac{\frac{1}{2}\varepsilon}{n} + \cdots + \frac{\frac{1}{2}\varepsilon}{n} \\
&= \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \cdot \frac{n - N_1}{n} \\
&< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.
\end{aligned}$$

Hence $\lim_{n\to\infty} \dfrac{b_1 + b_2 + \cdots + b_n}{n} = 0$.
Now, we let $b_n = \ln a_n$. Since $\lim_{n\to\infty} a_n = 1$, we have $\lim_{n\to\infty} b_n = 0$. So,

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} &= \lim_{n\to\infty} \exp\left[\frac{1}{n} \ln (a_1 a_2 \cdots a_n)\right] \\
&= \lim_{n\to\infty} \exp\left[\frac{\ln a_1 + \ln a_2 + \cdots + \ln a_n}{n}\right] \\
&= \lim_{n\to\infty} \exp\left[\frac{b_1 + b_2 + \cdots + b_n}{n}\right] \\
&= \exp\left[\lim_{n\to\infty} \frac{b_1 + b_2 + \cdots + b_n}{n}\right] = e^0 = 1.
\end{aligned}$$

2. (25 Marks) For which positive numbers a does the curve $y = a^x$ intersect the line $y = x$?

Solution To determine whether $y = a^x$ and $y = x$ intersect, we consider the function $f(x) = a^x - x$ $(a > 0)$. We will determine the values of a such that f has real root(s).

Case 1: $0 < a < 1$. In this case, the function f has at least one real root. In fact, by l'Hospital's Rule

$$\lim_{x\to-\infty} \frac{x}{a^x} \overset{\text{H}}{=} \lim_{x\to-\infty} \frac{1}{a^x \ln a} = 0,$$

we have

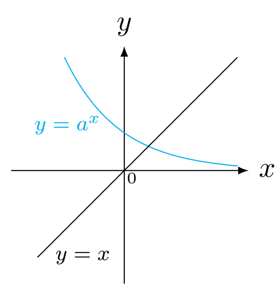
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (a^x - x) = \lim_{x\to-\infty} a^x \left(1 - \frac{x}{a^x}\right) = \infty.$$

It is easy to have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (a^x - x) = -\infty.$$

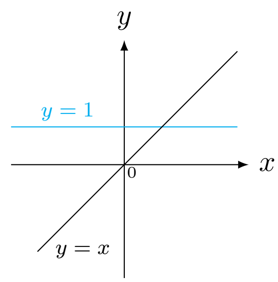
By the Intermediate Value Theorem, f has at least one real root.

The following sketch shows that when $0 < a < 1$, the curve $y = a^x$ intersects the line $y = x$.



Case 2: $a = 1$. This is a trivial case since $y = a^x = 1$ for all x.

The following sketch shows that when $a = 1$, the curve $y = a^x$ intersects the line $y = x$.



Case 3: $a > 1$. In this case, we will show that if $1 < a \le e^{1/e}$, f has real root(s); if $a > e^{1/e}$, f has no real root.

In fact, we will first show that f has one local minimum at some finite x_0. Indeed, by solving $f'(x) = a^x \ln a - 1 = 0$:

$$\begin{aligned} a^x \ln a - 1 = 0 \iff a^x = \frac{1}{\ln a} &\iff x \ln a = \ln\left[\frac{1}{\ln a}\right] \\ \iff x \ln a = -\ln(\ln a) &\iff x = -\frac{\ln(\ln a)}{\ln a}, \end{aligned}$$

we see that f has one critical number at $x_0 = -\frac{\ln(\ln a)}{\ln a}$. Since

$$\begin{aligned} f'(x) > 0, &\quad \text{if } x_0 < x < \infty; \\ f'(x) < 0, &\quad \text{if } -\infty < x < x_0, \end{aligned}$$

f has a local minimum value

$$f(x_0) = a^{x_0} - x_0 = \frac{1}{\ln a} - \left[-\frac{\ln(\ln a)}{\ln a}\right] = \frac{1 + \ln(\ln a)}{\ln a}.$$

Since f is increasing on (x_0, ∞), and decreasing on $(-\infty, x_0)$, f also attains its absolute minimum at $x = x_0$.

Now we are ready to determine the values of a such that f has real root(s). In fact, it is easy to have, for $a > 1$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (a^x - x) = \infty.$$

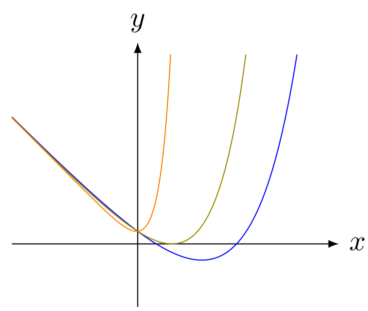
By l'Hospital's Rule, for $a > 1$,

$$\lim_{x \to \infty} \frac{x}{a^x} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{1}{a^x \ln a} = 0,$$

we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (a^x - x) = \lim_{x\to\infty} a^x \left(1 - \frac{x}{a^x}\right) = \infty.$$

Thus, the curve of $y = f(x)$ must intersect to the x-axis at either two points, one point, or no point, depending that the minimum value $\frac{1+\ln(\ln a)}{\ln a}$ is negative, zero, or positive, as illustrated in the following figure.



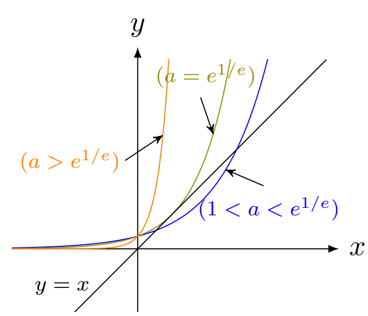
Since $a > 1$, we have that

$$\begin{aligned}
f(x_0) = \tfrac{1+\ln(\ln a)}{\ln a} < 0 \quad &\Longleftrightarrow \quad \ln(\ln a) < -1 \quad \Longleftrightarrow \quad \ln a < 1/e \quad \Longleftrightarrow \quad a < e^{1/e};\\
f(x_0) = \tfrac{1+\ln(\ln a)}{\ln a} = 0 \quad &\Longleftrightarrow \quad \ln(\ln a) = -1 \quad \Longleftrightarrow \quad \ln a = 1/e \quad \Longleftrightarrow \quad a = e^{1/e};\\
f(x_0) = \tfrac{1+\ln(\ln a)}{\ln a} > 0 \quad &\Longleftrightarrow \quad \ln(\ln a) > -1 \quad \Longleftrightarrow \quad \ln a > 1/e \quad \Longleftrightarrow \quad a > e^{1/e}.
\end{aligned}$$

Thus, if $1 < a < e^{1/e}$, f has two roots. If $a = e^{1/e}$, f has one root.

Therefore, if $1 < a \leq e^{1/e}$, the curve $y = a^x$ intersects the line $y = x$.

The following sketch shows that when $a > 0$, the curve $y = a^x$ and the line $y = x$ have either two, one, or no intersection.



3. (25 Marks) Suppose f is a continuous function on $\mathbb{R}$. Show that if at every point, f attains either local maximum or minimum, then f must be a constant function.

Solution Suppose f is not a constant function. Then there are $a_1, b_1 \in \mathbb{R}$ such that $f(a_1) \neq f(b_1)$. Without loss of generality, we assume $f(a_1) < f(b_1)$. By the Intermediate Value Theorem, there is $c_1 \in (a_1, b_1)$ such that

$$f(a_1) < f(c_1) = \frac{f(a_1) + f(b_1)}{2} < f(b_1).$$

If $b_1 - c_1 \leq \frac{1}{2}(b_1 - a_1)$, we denote $a_2 = c_1$. Since $f(a_2) = f(c_1) < f(b_1)$, by the Intermediate Value Theorem, we can take $b_2 \in (a_2, b_1) = (c_1, b_1)$ such that $f(a_2) < f(b_2) < f(b_1)$. Thus,

$$f(a_1) < f(a_2) < f(b_2) < f(b_1).$$

If $c_1 - a_1 \leq \frac{1}{2}(b_1 - a_1)$, we denote $b_2 = c_1$. Since $f(a_1) < f(b_2) = f(c_1)$, by the Intermediate Value Theorem, we can take $a_2 \in (a_1, b_2) = (a_1, c_1)$ such that $f(a_1) < f(a_2) < f(b_2)$. Thus,

$$f(a_1) < f(a_2) < f(b_2) < f(b_1).$$

In any case, we have $f(a_1) < f(a_2) < f(b_2) < f(b_1)$. Thus, there is $c_2 \in (a_2, b_2)$ such that

$$f(a_2) < f(c_2) = \frac{f(a_2) + f(b_2)}{2} < f(b_2).$$

We proceed in the same fashion as above to obtain the following:

(1) $[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$;

(2) $0 < b_n - a_n \leq \dfrac{b_1 - a_1}{2^n}$;

(3) $a_n < c_n < b_n$;

(4) $f(a_1) < f(a_2) < \cdots < f(a_n) < f(c_n) < f(b_n) < \cdots < f(b_2) < f(b_1)$.

It is clear that item (1) implies that $\{a_n\}$ is increasing and $\{b_n\}$ is decreasing. Both are bounded in $[a_1, b_1]$. So, $a_n \to a^*$ and $b_n \to b^*$. By item 2, we have $a^* = b^*$. Item (3) implies that $c_n \to c^* = a^* = b^*$. From item (4),

$$f(a_n) < f(a_{n+1}) < f(c_{n+i}) < f(b_{n+1}) < f(b_n), \quad i > 1.$$

Letting $i \to \infty$, we have

$$f(a_{n+1}) \le f(c^*) \le f(b_{n+1}),$$

so that

$$f(a_n) < f(c^*) < f(b_n).$$

Since $a_n \to c^*$ and $b_n \to c^*$, we know that f attains neither a maximum nor minimum at c^*. This contradicts to the hypothesis.

Consequently, f is a constant function.

4. (25 Marks) Suppose f is continuous on $[0, 1]$, $f''(x) < 0$ in $(0, 1)$, and $f(0) = f(1) = 0$.

   (1) For each positive integer n, there is a unique $x_n \in (0, 1)$ such that $f'(x_n) = M/n$, where M is the absolute maximum of f on $[0, 1]$.

   (2) Show that $\{x_n\}$ is increasing. So the limit $\lim_{n\to\infty} x_n = x^*$ exists.

   (3) Find $f(x^*)$.

   Solution

   (a) Consider the function $g(x) = f(x) - \dfrac{M}{n}x$. Since $f'' < 0$ on $(0, 1)$, f is not a constant function, so that the absolute maximum $M > f(0) = 0$. Because f is continuous on $[0, 1]$, there is $x_M \in (0, 1)$ such that $f(x_M) = M$. Clearly,

$$g(x_M) = f(x_M) - \frac{M}{n}x_M = M\left(1 - \frac{x_M}{n}\right) > 0,$$
$$g(1) = f(1) - \frac{M}{n} = -\frac{M}{n} < 0.$$

   By the Intermediate Value Theorem, there is $c \in (x_M, 1) \subset (0, 1)$ such that $g(c) = 0$. By the Mean Value Theorem, there is $x_n \in (0, c) \subset (0, 1)$ such that

$$0 = \frac{g(c) - g(0)}{c - 0} = g'(x_n).$$

   Hence, $f'(x_n) = \dfrac{M}{n}$. Because f' is strictly decreasing on $(0, 1)$, x_n is unique.

   (b) Since $f'' < 0$, f' is strictly decreasing on $(0, 1)$. Thus,

$$f'(x_n) = \frac{M}{n} > \frac{M}{n+1} = f'(x_{n+1}),$$

so that

$$x_n < x_{n+1} < 1.$$

Thus, $\{x_n\}$ is increasing and bounded. So the limit $\lim_{n\to\infty} x_n = x^*$ exists.

(c) By the continuity,

$$f'(x^*) = f'(\lim x_n) = \lim f'(x_n) = \lim \frac{M}{n} = 0.$$

Since $f''(x^*) < 0$, we know that $f(x^*)$ is a local maximum. Since f' is strictly decreasing, x^* is the only root of f' on $(0,1)$, and

$$f' > 0 \text{ on } (0, x^*) \text{ and } f' < 0 \text{ on } (x^*, 1).$$

Thus, f is increasing on $(0, x^*)$ and decreasing on $(x^*, 1)$. So, the local maximum is also the absolute maximum. Consequently, $f(x^*) = M$.