Math 2043 Final Exam

May 20, 2023

Your Name ___________________________

Student Number ______________________

Number	Score
1	
2	
3	
4	
Total	

(1) (25 points) Suppose $f(x)$ is a continuous function on $[a, b]$. Suppose $\alpha(x), \beta(x) \geq 0$, and $\alpha(x)$ is strictly increasing, $\beta(x)$ is strictly decreasing. If

$$\int_a^b f(x)\alpha(x)dx = \int_a^b f(x)\beta(x)dx = 0,$$

prove $f(x)$ vanishes at at least two places in (a, b).

(2) (25 points) Prove that the product of two bounded variation functions has bounded variation.

(3) (25 points) Suppose $f(x) = x + ax^2 + o(x^2)$ at 0. Suppose $\lim x_n = 0$.

1. Prove that $\sum x_n$ absolutely converges if and only if $\sum f(x_n)$ absolutely converges.
2. Prove that, if $\sum x_n$ and $\sum x_n^2$ converge, then $\sum f(x_n)$ converges.
3. Prove that, if $\sum x_n$ and $\sum f(x_n)$ converge, and $a \neq 0$, then $\sum x_n^2$ converges.
4. Explain that the condition $a \neq 0$ is necessary in the third part.

(4) (25 points) For $p, q > 0$, study the uniform convergence on $(0, +\infty)$

$$f_n(x) = \frac{nx^p}{1 + n^q x}.$$

Math 2043 Final, Spring 2023

answer not absolutely guaranteed to be correct

(1) If $f(x)$ does not vanish on (a,b), then by the intermediate value theorem, we have $f(x)>0$ on the whole (a,b), or $f(x)<0$ on the whole (a,b). Then by $\alpha(x)\geq 0$ and strictly monotone, we get $\int_a^b f(x)\alpha(x)dx<0$ or $\int_a^b f(x)\alpha(x)dx>0$, contradicting the assumption.

If $f(x)$ vanishes at exactly one point $c\in(a,b)$, then by the intermediate value theorem, we have $f(x)>0$ on (a,c) and $f(x)<0$ on (c,b), or $f(x)<0$ on (a,c) and $f(x)>0$ on (c,b). Then by $\alpha(x),\beta(x)\geq 0$ and strictly monotone, we get $\alpha(c),\beta(c)>0$. Moreover, by $\alpha(x)$ strictly increasing and $\beta(x)$ strictly decreasing, the function $\gamma(x)=\beta(c)\alpha(x)-\alpha(c)\beta(x)$ satisfies $\gamma(x)>0$ on (a,c) and $\gamma(x)<0$ on (c,b). Then $f(x)\gamma(x)<0$ on $(a,c)\cup(c,b)$. This implies

$$0>\int_a^b f(x)\gamma(x)=\beta(c)\int_a^b f(x)\alpha(x)dx-\alpha(c)\int_a^b f(x)\beta(x)dx=0,$$

a contradiction.

(2) Suppose $f(x)$ and $g(x)$ have bounded variations on $[a,b]$. Then

$$|f(x)-f(a)|\leq V_{[a,b]}(f).$$

Therefore $f(x)$ is bounded, and $g(x)$ is also bounded. Let $|f(x)|\leq B$ and $|g(x)|\leq B$.

For any partition $P\colon a=x_0<x_1<\cdots<x_n=b$, we have

$$\begin{aligned}V_P(fg)&=\sum|f(x_i)g(x_i)-f(x_{i-1})g(x_{i-1})|\\&\leq\sum|f(x_i)-f(x_{i-1})||g(x_i)|+|f(x_{i-1})||g(x_i)-g(x_{i-1})|\\&\leq B\sum|f(x_i)-f(x_{i-1})|+B\sum|g(x_i)-g(x_{i-1})|\\&\leq BV_P(f)+BV_P(g)\leq BV_{[a,b]}(f)+BV_{[a,b]}(g).\end{aligned}$$

This proves fg has bounded variation.

(3.1) For $\epsilon=1>0$, there is $\delta>0$, such that

$$\begin{aligned}|x|<\delta&\implies|f(x)-x-ax^2|\leq|x^2|\\&\implies|f(x)-x|\leq(|a|+1)|x|^2\\&\implies(1-(|a|+1)|x|)|x|\leq|f(x)|\leq(1+(|a|+1)|x|)|x|.\end{aligned}$$

Then $|x|<\delta'=\min\{\delta,\frac{1}{2(|a|+1)}\}$ implies $\frac{1}{2}|x|\leq|f(x)|\leq\frac{3}{2}|x|$.

By $\lim x_n=0$, we have $|x_n|<\delta'$ for sufficiently large n. Then we have $\frac{1}{2}|x_n|\leq|f(x_n)|\leq\frac{3}{2}|x_n|$ for sufficiently large n. Then by the comparison test, this implies $\sum|x_n|$ converges if and only if $\sum|f(x_n)|$ converges.

(3.2) Let $f(x)=x+ax^2+R(x)$. For $\epsilon=1>0$, there is $\delta>0$, such that

$$|x|<\delta\implies|R(x)|\leq\epsilon x^2=x^2\implies|ax^2+R(x)|\leq(|a|+1)x^2.$$

By $\lim x_n = 0$, we have $|x_n| < \delta$ for sufficiently large n. Then we have $|ax_n^2 + R(x_n)| \le (|a|+1)x_n^2$ for sufficiently large n. Then by the the convergence of $\sum x_n^2$ and the comparison test, this implies $\sum(ax_n^2 + R(x_n))$ converges. Combined with the convergence of $\sum x_n$, we know $\sum f(x_n) = \sum x_n + \sum(ax_n^2 + R(x_n))$ converges.

(3.3) For $\epsilon = \frac{1}{2}|a| > 0$, there is $\delta > 0$, such that

$$|x| < \delta \implies |R(x)| \le \epsilon x^2 \implies |ax^2 + R(x)| \ge (|a| - \epsilon)x^2 = \frac{1}{2}|a|x^2.$$

By $\lim x_n = 0$, we have $|x_n| < \delta$ for sufficiently large n. Then we have $|ax_n^2 + R(x_n)| \ge \frac{1}{2}|a|x_n^2 \ge 0$ for sufficiently large n. On the other hand, by the convergence of $\sum x_n$ and $\sum f(x_n)$, we know $\sum(ax_n^2 + R(x_n)) = \sum f(x_n) - \sum x_n$ converges. Then by the comparison test, we find $\sum x_n^2$ converges.

(3.4) Let $f(x) = x + x^3$, and $x_n = \frac{(-1)^n}{\sqrt{n}}$. Then by the Leibniz test, both $\sum x_n = \sum \frac{(-1)^n}{\sqrt[3]{n}}$ and $\sum x_n^3 = \sum \frac{(-1)^n}{n^{\frac{3}{2}}}$ converge. Therefore $\sum f(x_n) = \sum x_n + \sum x_n^3$ converges. However, $\sum x_n^2 = \sum \frac{1}{n}$ diverges.

(4) (25 points) For $p, q > 0$, study the uniform convergence on $(0, +\infty)$

$$f_n(x) = \frac{nx^p}{1 + n^q x}.$$

We have

$$\lim_{n\to\infty} f_n(x) = \begin{cases} \infty, & \text{if } q < 1 \\ x^{p-1}, & \text{if } q = 1 \\ 0, & \text{if } q > 1 \end{cases}.$$

Suppose $q > 1$. We have $|f_n(x) - 0| = f_n(x)$. For $p > 1$, we have $\lim_{x\to+\infty} f_n(x) = \infty$. Therefore the convergence is not uniform. For $p = 1$, we have $|f_n(x) - 0| = f_n(x) < \frac{nx}{n^q x} \le \frac{1}{n^{q-1}}$, and the convergence is uniform.

For $0 < p < 1$, to get upper bound of $f_n(x)$, we find its maximum. We have

$$f_n'(x) = \frac{nx^{p-1}(p + (p-1)n^q x)}{(1 + n^q x)^2}.$$

Then $f_n'(c_n) = 0$ for $c_n = \frac{p}{(1-p)n^q}$, and $f_n(x)$ is increasing on $(0, c_n]$ and decreasing on $[c_n, +\infty)$. Therefore

$$0 \le f_n(x) \le f(c_n) = \frac{nc_n^p}{1 + nc_n} = p^p(1-p)^{1-p}\frac{1}{n^{pq-1}}.$$

The uniform convergence is equivalent to $\lim_{n\to\infty} \frac{1}{n^{pq-1}} = 0$, which is the same as $pq > 1$.

Suppose $q = 1$. Then

$$h_n(x) = |f_n(x) - x^{p-1}| = \left|\frac{nx^p}{1 + nx} - x^{p-1}\right| = \frac{x^{p-1}}{1 + nx}.$$

If $p > 2$, then $\lim_{x\to+\infty} h_n(x) = \infty$. If $p < 1$, then $\lim_{x\to 0^+} h_n(x) = \infty$. If $p = 1$, then $\lim_{x\to 0^+} h_n(x) = 1$. Therefore the convergence is not uniform. If $p = 2$, then $h_n(x) = \frac{x}{1+nx} < \frac{1}{n}$, and the convergence is uniform.

For $1 < p < 2$, to get upper bound of $h_n(x)$, we find its maximum. We have

$$h_n'(x) = \frac{x^{p-2}((p-1)+(p-2)nx)}{(1+nx)^2}.$$

Then $h_n'(c_n) = 0$ for $c_n = \frac{p-1}{(2-p)n}$, and $h_n(x)$ is increasing on $(0, c_n]$ and decreasing on $[c_n, +\infty)$. Therefore

$$0 \le h_n(x) \le h(c_n) = \frac{nc_n^p}{1+nc_n} = \frac{(p-1)^p}{(2-p)^{p-1}} \frac{1}{n^{p-1}}.$$

This implies the convergence is uniform.

In conclusion, $f_n(x)$ converges uniformly on $(0, +\infty)$ if and only if $q > 1$, $p \le 1$, and $pq > 1$, or $q = 1$ and $1 < p \le 2$.