

# MATH2043 HW1

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**§1.8(5)** Let  $\varepsilon_1 = 2\varepsilon$ . For any  $\varepsilon > 0$ ,  $\exists N : |x_n - l| \leq 2\varepsilon \equiv |x_n - l| \leq \varepsilon_1$ . Since  $\varepsilon_1 = 2\varepsilon$ , The second statement is true for all  $\varepsilon_1 > 0$ .

**§1.15** For all  $\varepsilon > 0$ , There is  $N$  such that for all  $n > N$ ,  $|x_n - l| < \varepsilon$ . For even  $n$ ,  $\exists k : 2k = n$ . Hence, above implies  $|x_{2k} - l| < \varepsilon$ . For odd  $n$ ,  $\exists k : 2k + 1 = n$ . Hence,  $|x_{2k+1} - l| < \varepsilon$ . For the converse, there exists  $K_1$  such that  $k \geq K_1 \Rightarrow |x_{2k} - l| < \varepsilon$ , and  $K_2$  such that  $k \geq K_2 \Rightarrow |x_{2k+1} - l| < \varepsilon$ . Set  $N = \max\{2K_1, 2K_2 + 1\}$ . For any  $n > N$ ,  
if  $n$  is even, write  $n = 2k$ . Since  $n > 2K_1$ ,  $k > K_1$  so  $|x_{2k} - l| < \varepsilon$ .  
if  $n$  is odd, write  $n = 2k + 1$ . Since  $n > 2K_2 + 1$ ,  $k > K_2$  so  $|x_{2k+1} - l| < \varepsilon$ .

**§1.19** The error in this proof is assuming  $\lim_{n \rightarrow \infty} x_n$  exists and performing mathematical operations on it.

**§1.22** No, the converse is not true. Consider

$$x_n = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

The sequence is obviously unbounded, but the sequence does not tend to infinity.

**§ 1.31(1)** Suprema is  $2 + \sqrt{3}$ .

**§ 1.31(4)** No suprema, set  $n$  to 1,  $m$  has no upper bound, making  $\frac{m}{n}$  grow to infinity.

**§1.33** TODO

**§1.34(8)**  $|x - y| \leq c \Rightarrow x - y \leq c$ . For any  $\varepsilon > 0$ , choose  $x_\varepsilon : x_\varepsilon > \sup X - \varepsilon$ . We have

$$x_\varepsilon - y \leq c \Rightarrow \sup X - c - \varepsilon \leq y \text{ for all } y \in Y.$$

$\sup X - c - \varepsilon$  is a lower bound of  $Y$ . Therefore,

$$\sup X - \varepsilon - c \leq \inf Y$$

and it follows that

$$\sup X - \varepsilon - c \leq \sup Y$$

it can be concluded that

$$\begin{aligned} \sup X - \inf Y &\leq c \\ \sup X - \sup Y &\leq c \end{aligned}$$

Symmetrically,  $\sup Y - \inf X \leq c$  and  $\sup Y - \sup X \leq c$ . Hence,  $|\sup X - \inf Y| \leq c$  and  $|\sup X - \sup Y| \leq c$ .

Conversely,

$$\begin{aligned}\inf X &\leq x \leq \sup X, \text{ and } \inf Y \leq y \leq \sup Y \\ x - y &\leq \sup X - \sup Y \leq c\end{aligned}$$

: Symmetrically,  $y - x \leq c$ .  $\square$

#### §1.40 TODO

**§1.45(5)** The function is periodic for every 12 indices. Let

$$x_n = \frac{(12k+r) \sin\left(\frac{(12k+r)\pi}{3}\right)}{(12k+r) \cos\left(\frac{(12k+r)\pi}{2}\right) + 2}$$

Let  $n = 12k + r$  for some  $k, r \in \mathbb{N}$  (including 0). We get

$$\begin{aligned}x_n = x_{12k+r} &= \frac{(12k+r) \sin\left(\frac{(12k+r)\pi}{3}\right)}{(12k+r) \cos\left(\frac{(12k+r)\pi}{2}\right) + 2} \\ &= \frac{(12k+r) \sin\left(\frac{r\pi}{3}\right)}{(12k+r) \cos\left(\frac{r\pi}{2}\right) + 2}\end{aligned}$$

As  $k \rightarrow \infty$ , this approaches  $\frac{\sin\left(\frac{r\pi}{3}\right)}{\cos\left(\frac{r\pi}{2}\right)}$ , for  $r \in \{0, 2, \dots, 11\}$ . Computing the value of the function for all 12 of these, we see that  $\text{LIM}\{x_n\} = \left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$ .

**§1.45(8)** Let

$$x_n = \sqrt[n]{2^n + 3^{(-1)^n n}}$$

For even  $n = 2k$ ,

$$\begin{aligned}x_n = x_{2k} &= \sqrt[2k]{2^{2k} + 3^{2k}} \\ &= 3 \sqrt[2k]{\left(\frac{2}{3}\right)^{2k} + 1}\end{aligned}$$

As  $k \rightarrow \infty$ ,  $x_{2k}$  approaches 3.

For odd  $n = 2k + 1$ ,

$$\begin{aligned}x_n = x_{2k+1} &= \sqrt[2k+1]{2^{2k+1} + 3^{-(2k+1)}} \\ &= 2 \sqrt[2k+1]{\left(\frac{1}{6}\right)^{2k+1} + 1}\end{aligned}$$

As  $k \rightarrow \infty$ ,  $x_{2k+1}$  approaches 2.

Therefore,  $\text{LIM}\{x_n\} = \{2, 3\}$ .

#### §1.46

**1.**  $\text{LIM}\{x_n\} \cup \text{LIM}\{y_n\} \subseteq \text{LIM}\{z_n\}$

Assume  $l \in \{x_n\}$ , then there is a subsequence  $\{x_{n_k}\}$  converging to  $l$ . Since  $\{x_n\} \in \{z_n\}$ ,  $\{x_{n_k}\} \in \{z_n\}$ . Symmetrical argument for  $y_n$ .

**2.**  $\text{LIM}\{z_n\} \subseteq \text{LIM}\{x_n\} \cup \text{LIM}\{y_n\}$

Assume  $l \in \{z_n\}$ , then there is a subsequence  $\{z_{n_k}\}$  converging to  $l$ .

**Case 1:**  $\{z_{n_k}\}$  contains infinite terms from  $\{x_n\}$

Then, for some  $K$ , there is a sequence  $\{x_{m_k}\}$  that has the exact same terms in the same order as  $\{z_{n_k}\}$ .

Let  $J = \{k : z_{n_k} \in \{x_n\}\}$ . We can write for each  $k \in J$ ,  $z_{n_k} = x_{m_k}$ . Note that the indices  $\{m_k\}$  are increasing, since order of  $\{x_n\}$  is preserved in  $\{z_n\}$ . It is clear that  $\{x_{m_k}\}$  is a subsequence of  $\{x_n\}$ . Since  $\{x_{m_k}\}$  converges to  $l$ , this means that  $l \in \text{LIM}\{x_n\}$ .

**Case 2:**  $\{z_{n_k}\}$  contains infinite terms from  $\{y_n\}$

The proof is symmetrical.

□

**§1.48(5)**  $\overline{\lim} x_n = \frac{\sqrt{3}}{2}$ ,  $\underline{\lim} x_n = -\frac{\sqrt{3}}{2}$ .

**§1.48(8)**  $\overline{\lim} x_n = 3$ ,  $\underline{\lim} x_n = 2$ .

**§1.49(7)** Assume  $\{x_n\}$  and  $\{y_n\}$  are bounded. For all  $m_k, n_k \geq k \geq N$ , we have  $x_{m_k} \geq y_{n_k}$ . By Bolzano Weierstrass, we know that there are convergent subsequences  $x_{m_{k_i}}$  and  $y_{n_{l_i}}$ , such that  $\{x_{m_{k_i}}\} \geq \{y_{n_{l_i}}\}$ . By the order property, we have

$$\lim_{i \rightarrow \infty} x_{m_{k_i}} \geq \lim_{i \rightarrow \infty} y_{n_{l_i}} \quad (1)$$

Since both  $\{x_{m_{k_i}}\}$  and  $\{y_{n_{l_i}}\}$  are subsequences of  $\{x_n\}$  and  $\{y_n\}$  respectively, we also have

$$\begin{aligned} \overline{\lim} x_n &\geq \lim_{i \rightarrow \infty} x_{m_{k_i}} \\ \lim_{i \rightarrow \infty} y_{n_{l_i}} &\geq \underline{\lim} y_n \end{aligned} \quad (2)$$

Combining (1) and (2), we get

$$\overline{\lim} x_n \geq \underline{\lim} y_n$$

□

**§1.49(8) TODO**

**§1.51**

Let  $\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = r < 1$ . We know that there must be an  $N$  such that for all  $n > N$ ,  $\left| \frac{x_{n+1}}{x_n} \right| < r$ . (Not sure if this needs to be proved, but in short, if there were infinitely many  $\left| \frac{x_{n+1}}{x_n} \right| \geq r$ , then there would be a converging subsequence with limit  $\geq r$  which would belong to LIM, so contradiction). for any  $k > 0$ , we have

$$\begin{aligned} |x_{N+k}| &< r|x_{N+k-1}| < r^2|x_{N+k-2}| < \dots < r^k|x_N| \\ |x_{N+k}| &\leq r^k|x_N| \end{aligned}$$

Since  $|r| < 1$ ,  $r^k \rightarrow 0$ . Thus,  $|x_{N+k}| \rightarrow 0$ , and therefore  $x_n \rightarrow 0$ .

**§1.55**

Note that

$$\left| \frac{x_{n+1}}{x_n} \right| \leq \left| \frac{y_{n+1}}{y_n} \right| \Rightarrow \frac{|x_{n+1}|}{|x_n|} \leq \frac{|y_{n+1}|}{|y_n|}$$

**1** By writing the inequality for the first few  $n$ , we see that

$$\begin{aligned} \frac{|x_2|}{|x_1|} &\leq \frac{|y_2|}{|y_1|} \\ \frac{|x_3|}{|x_2|} &\leq \frac{|y_3|}{|y_2|} \\ &\vdots \\ \frac{|x_n|}{|x_{n-1}|} &\leq \frac{|y_n|}{|y_{n-1}|} \end{aligned}$$

Multiplying the inequalities (since  $x_n \neq 0$ ), we get

$$\begin{aligned} \frac{|x_n|}{|x_1|} &\leq \frac{|y_n|}{|y_1|} \\ |x_n| &\leq |y_n| \frac{|x_1|}{|y_1|} \end{aligned}$$

$c$  is  $\frac{|x_1|}{|y_1|}$ .

**2** if  $\lim_{n \rightarrow \infty} y_n = 0$ , then  $\lim_{n \rightarrow \infty} |y_n| = 0$ . By squeeze theorem, we have  $\lim_{n \rightarrow \infty} |x_n| = 0$ .

**3** We have  $x_n \leq |x_n| \leq c|y_n|$ . For any  $M$ , choose  $N$  such that for any  $n > N \Rightarrow \frac{x_n}{c} > M$  (such  $n$  exists, since  $x_m \rightarrow \infty$ ). We have

$$M < \frac{x_n}{c} \leq |y_n|$$

Thus,  $|y_n| \rightarrow \infty$ .  $\square$