MATH2043 HW1

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§1.8(5) Let $\varepsilon_1 = 2\varepsilon$. For any $\varepsilon > 0, \exists N : |x_n - l| \leq 2\varepsilon \equiv |x_n - l| \leq \varepsilon_1$. Since $\varepsilon_1 = 2\varepsilon$, The second statement is true for all $\varepsilon_1 > 0$.

§1.15 For all $\varepsilon > 0$, There is N such that for all $n > N$, $|x_n - l| < \varepsilon$. For even n, $\exists k : 2k = n$. Hence, above implies $x_{2k} - l| < \varepsilon$. For odd n, $\exists k : 2k + 1 = n$. Hence, $|x_{2k+1} - l| < \varepsilon$.
For the converse, there exists K_1 such that $k \geq K_1 \Rightarrow |x_{2k} - l| < \varepsilon$, and K_2 such that $k \geq K_2 \Rightarrow |x_{2k+1} - l| < \varepsilon$. Set $N = \max\{2K_1, 2K_2 + 1\}$. FOr any $n > N$,
if n is even, write $n = 2k$. Since $n > 2K_1$, $k > K_1$ so $|x_{2k} - l| < \varepsilon$.
if n is odd, write $n = 2k + 1$. Since $n > 2K_2 + 1$, $k > K_2$ so $|x_{2k+1} - l| < \varepsilon$.

§1.19 The error in this proof is assuming $\lim_{n\to\infty} x_n$ exists and performing mathematical operations on it.

§1.22 No, the converse is not true. Consider

$$x_n = \begin{cases} n \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

The sequence is obviously unbounded, but the sequence does not tend to infinity.

§ 1.31(1) Suprema is $2 + \sqrt{3}$.

§ 1.31(4) No suprema, set n to 1, m is has no upper bound, making $\frac{m}{n}$ grow to infinity.

§1.33

=>

1 supremum is the least **upper bound**, therefore if $\lambda = \sup X, x \leq \lambda$ for all $x \in X$.

2 Let $x_n = \frac{1}{n}$. For every n, we have $r \in X$ such that $\lambda - x_n < r < \lambda$, because otherwise λ would not be the least upper bound. $\lambda - x_n \to \lambda$.

§1.34(8) $|x - y| \leq c \Rightarrow x - y \leq c$. For any $\varepsilon > 0$, choose $x_\varepsilon : x_\varepsilon > \sup X - \varepsilon$. We have

$$x_\varepsilon - y \leq c \Rightarrow \sup X - c - \varepsilon \leq y \text{ for all } y \in Y.$$

$\sup X - c - \varepsilon$ is a lower bound of Y. Therefore,

$$\sup X - \varepsilon - c \leq \inf Y$$

and it follows that

$$\sup X - \varepsilon - c \leq \sup Y$$

it can be concluded that

$$\sup X - \inf Y \leq c$$
$$\sup X - \sup Y \leq c$$

Symmetrically, $\sup Y - \inf X \leq c$ and $\sup Y - \sup$. Hence, $|\sup X - \inf Y| \leq c$ and $|\sup X - \sup Y| \leq c$.
Conversely,

$$\inf X \leq x \leq \sup X \text{ , and } \inf Y \leq y \leq \sup Y$$
$$x - y \leq \sup X - \sup Y \leq c$$

: Symmetrically, $y - x \leq c$. $\square$

§1.40 By Bolzano-Weierstrass, there is a convering subsequence x_{n_k}. Consider y_{n_k}, Since y_n is bounded, so is y_{n_k}, therefore, it has a converging subsequence $y_{n_{k_i}}$. $x_{n_{k_i}}$ also converges.For somplicity, let $m_i = n_{k_i}$, then m_i is the common index.
To extend this to an arbitrary number of sequences, we use induction. Let the sequences be $x_n^{\{1\}}, x_n^{\{2\}}, ..., x_n^{\{r\}}$. Base cause is $r = 1$, trivial by Bolzano-Weierstrass. Assume true for r, then there is a common index sequence n_k such that $x_{n_k}^{\{1\}}, x_{n_k}^{\{2\}}, ..., x_{n_k}^{\{r\}}$ all converge. Consider the sequence $x_{n_k}^{\{r+1\}}$. It has a converging subsequence $x_{n_{k_i}}^{\{r+1\}}$. Now let $m_i = n_{k_i}$. For $r' \leq r$, $x_{m_i}^{\{r'\}}$ converges. The new common index is m_i.

§1.45(5) The function is periodic for every 12 indices. Let

$$x_n = \frac{(12k + r)\sin\left(\frac{(12k+r)\pi}{3}\right)}{(12k + r)\cos\left(\frac{(12k+r)\pi}{2}\right) + 2}$$

Let $n = 12k + r$ for some $k, r \in \mathbb{N}$ (including 0). We get

$$x_n = x_{12k+r} = \frac{(12k + r)\sin\left(\frac{(12k+r)\pi}{3}\right)}{(12k + r)\cos\left(\frac{(12k+r)\pi}{2}\right) + 2}$$
$$= \frac{(12k + r)\sin\left(\frac{r\pi}{3}\right)}{(12k + r)\cos\left(\frac{r\pi}{2}\right) + 2}$$

As $k \to \infty$, this approaches $\frac{\sin\left(\frac{r\pi}{3}\right)}{\cos\left(\frac{r\pi}{2}\right)}$, for $r \in \{0, 2, ..., 11\}$. Computing the value of the function for all 12 of these, we see that $\text{LIM}\{x_n\} = \left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$.

§1.45(8) Let

$$x_n = \sqrt[n]{2^n + 3^{(-1)^n n}}$$

For even $n = 2k$,

$$x_n = x_{2k} = \sqrt[2k]{2^{2k} + 3^{2k}}$$
$$= 3\sqrt[2k]{\left(\frac{2}{3}\right)^{2k} + 1}$$

As $k \to \infty$, x_{2k} approaches 3.
For odd $n = 2k + 1$,

$$
\begin{aligned}
x_n = x_{2k+1} &= \sqrt[2k+1]{2^{2k+1} + 3^{-(2k+1)}} \\
&= 2\sqrt[2k+1]{\left(\frac{1}{6}\right)^{2k+1} + 1}
\end{aligned}
$$

As $k \to \infty$, x_{2k+1} approaches 2.
Therefore, $\mathrm{LIM}\{x_n\} = \{2, 3\}$.

§1.46

1. $\mathrm{LIM}\{x_n\} \cup \mathrm{LIM}\{y_n\} \subseteq \mathrm{LIM}\{z_n\}$
Assume $l \in \{x_n\}$, then there is a subsequence $\left\{x_{n_k}\right\}$ converging to l. Since $\{x_n\} \in \{z_n\}$, $\left\{x_{n_k}\right\} \in \{z_n\}$. Symmetrical argument for y_n.

2. $\mathrm{LIM}\{z_n\} \subseteq \mathrm{LIM}\{x_n\} \cup \mathrm{LIM}\{y_n\}$
Assume $l \in \{z_n\}$, then there is a subsequence $\left\{z_{n_k}\right\}$ converging to l.

Case 1: $\left\{z_{n_k}\right\}$ **contains infinte terms from** $\{x_n\}$
Then, for some K, there is a sequence $\left\{x_{m_k}\right\}$ that has the exact same terms in the same order as $\left\{z_{n_k}\right\}$.
Let $J = \left\{k : z_{n_k} \in \{x_n\}\right\}$. We can write for each $k \in J$, $z_{n_k} = x_{m_k}$. Note that the indices $\{m_k\}$ are increasing, since order of $\{x_n\}$ is preserved in $\{z_n\}$. It is clear that $\left\{x_{m_k}\right\}$ is a subsequence of $\{x_n\}$. Since $\left\{x_{m_k}\right\}$ converges to l, this means that $l \in \mathrm{LIM}\{x_n\}$.

Case 2: $\left\{z_{n_k}\right\}$ **contains infinte terms from** $\{y_n\}$
The proof is symmetrical.

□

§1.48(5) $\overline{\lim}\, x_n = \frac{\sqrt{3}}{2}$, $\underline{\lim}\, x_n = -\frac{\sqrt{3}}{2}$.

§1.48(8) $\overline{\lim}\, x_n = 3$, $\underline{\lim}\, x_n = 2$.

§1.49(7) Assime $\{x_n\}$ and $\{y_n\}$ are bounded. F, or all $m_k, n_k \geq k \geq N$, we have $x_{m_k} \geq y_{n_k}$. By Bolzano Weierstrass, we know that there are convergent subsequnces $x_{m_{k_i}}$ and $y_{n_{l_i}}$, such that $\left\{x_{m_{k_i}}\right\} \geq \left\{y_{n_{l_i}}\right\}$. By the order property, we have

$$
\lim_{i\to\infty} x_{m_{k_i}} \geq \lim_{i\to\infty} y_{n_{l_i}} \tag{1}
$$

Since both $\left\{x_{m_{k_i}}\right\}$ and $\left\{y_{n_{l_i}}\right\}$ are subsequences of $\{x_n\}$ and $\{y_n\}$ respectively, we also have

$$
\begin{aligned}
\overline{\lim}\, x_n &\geq \lim_{i\to\infty} x_{m_{k_i}} \\
\lim_{i\to\infty} y_{n_{l_i}} &\geq \underline{\lim}\, y_n
\end{aligned} \tag{2}
$$

Combining (1) and (2), we get

$$
\overline{\lim}\, x_n \geq \underline{\lim}\, y_n
$$

□

§1.49(8) This follow directly from 1.49(7), it is the contrapositive of (7).

$$\exists N \forall m, n x_m \geq y_n \Rightarrow \overline{\lim_{n\to\infty}} x_n \geq \underline{\lim_{n\to\infty}} y_n$$

$$\overline{\lim_{n\to\infty}} x_n < \underline{\lim_{n\to\infty}} y_n \Rightarrow \neg(\exists N \forall m, n\, x_m \geq y_n)$$
$$\Rightarrow \forall \exists m, n\, x_m < y_n$$

□

§1.51

1 Let $\overline{\lim}_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = r < 1$. We know that there must be an N such that for all $n > N$, $\left|\frac{x_{n+1}}{x_n}\right| < r$. (Not sure if this needs to be proved, but in short, if there were infinitely many $\left|\frac{x_{n+1}}{x_n}\right| \geq r$, then there would be a converging subsequence with limit $\geq r$ which would belong to LIM, so contradiction). for any $k > 0$, we have

$$|x_{N+k}| < r|x_{N+k-1}| < r^2|x_{N+k-2}| < ... < r^k|x_N|$$
$$|x_{N+k}| < r^k|x_N|$$

Since $|r| < 1$, $r^k \to 0$. Thus, $|x_{N+k}| \to 0$, and therefore $x_n \to 0$.

2 Let $\underline{\lim}_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = r > 1$, There must be N such that $n > N \Rightarrow \left|\frac{x_{n+1}}{x_n}\right| > r$.(Similar proof as the one above). Then

$$|x_{N+k}| > r|x_{N+k-1}| > r^2|x_{N+k-2}| > ... > r^k|x_N|$$
$$|x_{N+k}| > r^k|x_N|$$

Hence $x_n \to \infty$. □

§1.55

Note that

$$\left|\frac{x_{n+1}}{x_n}\right| \leq \left|\frac{y_{n+1}}{y_n}\right| \Rightarrow \frac{|x_{n+1}|}{|x_n|} \leq \frac{|y_{n+1}|}{|y_n|}$$

1 By writing the inequality for the first few n, we see that

$$\frac{|x_2|}{|x_1|} \leq \frac{|y_2|}{|y_1|}$$
$$\frac{|x_3|}{|x_2|} \leq \frac{|y_3|}{|y_2|}$$
$$\vdots$$
$$\frac{|x_n|}{|x_{n-1}|} \leq \frac{|y_n|}{|y_{n-1}|}$$

Multiplying the inequalities (since $x_n \neq 0$), we get

$$\frac{|x_n|}{|x_1|} \leq \frac{|y_n|}{|y_1|}$$
$$|x_n| \leq |y_n|\frac{|x_1|}{|y_1|}$$

c is $\frac{|x_1|}{|y_1|}$.

2 if $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty} |y_n| = 0$. By squeeze theorem, we have $\lim_{n\to\infty} |x_n| = 0$.

3 We have $x_n \le |x_n| \le c|y_n|$. For any M, choose N such that for any $n > N \Rightarrow \frac{x_n}{c} > M$ (such n exists, since $x_m \to \infty$). We have

$$M < \frac{x_n}{c} \le |y_n|$$

Thus, $|y_n| \to \infty$. $\square$