A Note on Semi-linear Wave Equations

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Abstract

Inspired by the work of Wang and Yu [21] on wave maps, we show that for all positive numbers $T_0 > 0$ and $E_0 > 0$, a large kind of semi-linear wave equation on $\mathbb{R} \times \mathbb{R}^3$ has a solution whose life-span is $[0, T_0]$, and the energy of the initial Cauchy data is at least E_0.

1 Introduction

We consider in $\mathbb{R} \times \mathbb{R}^3$ the equation:

$$\Box \phi = \pm |\phi|^{k-1} \phi \tag{1}$$

where

$$\Box = -\partial_{tt}^2 + \Delta_x,$$

and k is a odd number satisfying $k \geq 3^1$. The equation (1) has a conserved energy

$$E(\phi(t)) = \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t \phi(t,x)|^2 + |\nabla_x \phi(t,x)|^2 \pm \frac{2}{k+1} |\phi(t,x)|^{k+1} dx \tag{2}$$

The equation (1) is called defocusing, if there is a plus sign in front of the nonlinearity, otherwise it is called focusing. In view of the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, one refers to the range $k < 5$ as the energy-subcritical regime, to $k = 5$ as the energy-critical regime and to $k > 5$ as the energy-supercritical regime. So the super-critical wave equations ($k > 5$) (both defocusing and focusing case) are included in our present note.

The study of the Cauchy problem for (1) has a long history. For the defocusing case, Rauch [13] showed global existence for arbitrary smooth data for subcritical equations and for small energy smooth data in the critical case. Struwe [16] obtained global existence for large but radially symmetric data in the critical case, then Grillakis [5] removed the radially symmetric condition on the data. After that Shatah and Struwe [15] studied the energy-class solutions. See also [1], [2] and [18] for further results. The study of focusing case is initiated by Krieger and Schlag [12] as well as Kenig and Merle [8]. Up to now, only a few results are known about the energy-supercritical case. See [9], [10] and [19].

Inspired by the recent work [21], we study long time solutions of semi-linear wave equations by a different approach. Following the "short-pulse" method, which was first introduced by Christodoulou [3], and extended by Klainerman and Rodnianski [11], we establish the following long-time existence result for equation (1):

Main Theorem *For any $T_0 > 0$ and $E_0 > 0$, there exist $(\phi_0, \phi_1) \in C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3)$ such that the Cauchy problem for (1) with initial data $(\phi, \phi_t)|_{t=0} = (\phi_0, \phi_1)$ has a unique solution $\phi \in$*

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[1] In this case, small data generates global solution, see [7]. And we shall see that to establish the main estimates, we only need the nonlinearity to be a smooth function of ϕ which vanishes at the origin and whose growth is at least $|\phi|^2$ when $|\phi|$ is large.

$C^\infty([0, T_0] \times \mathbb{R}^3)$ *with energy of at least* E_0.

In [21], Wang and Yu constructed a solution for 2+1 wave maps with $\mathbb{S}^2$ as its target, by using a bootstrap argument. Since the characteristic initial data (so called *shortpulse*) is chosen to be highly-oscillating, they can close the bootstrap. Then the solution will automatically have a life-span $[0, T_0]$, where T_0 is an arbitrary positive number given priorly. If the initial data is chosen properly, then the initial energy will be at least E_0, where E_0 is also an arbitrary positive number given priorly. The crucial point in their work is that the nonlinearity of wave maps into $\mathbb{S}^2$ is a "null form", this means that the nonlinearity is "not too bad", so that they can absorb the term original from the nonlinearity in the a priori estimates if the characteristic initial data oscillates heavily enough. Wang and Yu also studied the 3+1 nonlinear wave equation with a "null form" by choosing the initial data at past null infinity, so they can even obtain global existence for large energy data, see [20].

In the current work, there are no "null form" in the nonlinearity, but instead, the nonlinearity depends only on the solution itself, and we only commute the "bad" vectorfield once with the operator $\Box$ when we do the bootstrap argument, so the nonlinearity will not cause trouble. Moreover, since the nonlinearity involves only the solution itself, we need one derivative less to close the bootstrap than the work of Wang and Yu. Our method for 3+1 semi-linear wave equations is also valid in the 2+1 case, and we shall talk about this briefly at the end of the paper.

2 Preliminaries

2.1 Basic Geometric Construction

This part is quite similar to [21], the only difference is that we are in the 3-space dimensional Minkowski spacetime $\mathbb{R} \times \mathbb{R}^3$. We shall use the same notations as in [21]. We have the optical functions:

$$u = \frac{1}{2}(t - r) \quad \underline{u} = \frac{1}{2}(t + r)$$

null vectorfields:

$$L = \partial_t + \partial_r \quad \underline{L} = \partial_t - \partial_r$$

as well as the rotation fields:

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i, 1 \leq i, j \leq 3$$

and also the relation between the operators Ω and $\slashed{\nabla}$:

$$|\Omega^k \phi| = |r|^k |\slashed{\nabla}^k \phi| \tag{3}$$

In section 3 and section 4, the parameter u will be confined in the interval $[u_0, -1]$ where $u_0 \sim -T_0$ (then we can see that the life-span will be automatically $\sim T_0$). The parameter $\underline{u}$ is confined in $[0, \delta]$ where δ is a small parameter which will be determined later. As in [21], the corresponding cones are pictured as follows (See 2.1). When we derive estimates in section 3 and 4, $\underline{u} \in [0, \delta]$ where δ will be sufficiently small. Since T_0 and u_0 are fixed numbers, in the region where $(\underline{u}, u] \in [0, \delta] \times [u_0, -1]$, the parameter $r \sim 1$. In particular, we have

$$|\Omega^k \phi| \sim |\slashed{\nabla}^k \phi|$$

2.2 Energy Identity

Let f be a solution for the following non-homogenous wave equation on $\mathbb{R}^{3+1}$:
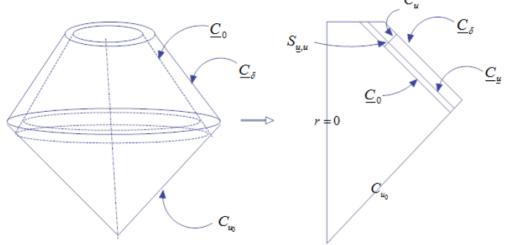
$$\Box f = \Phi \tag{4}$$

Figure 1: null cones (This figure is borrowed from [21].)

The energy momentum tensor associated to f is

$$T_{\mu\nu}[f] = \partial_\mu f \partial_\nu f - \frac{1}{2} g_{\mu\nu} |\nabla f|^2 \tag{5}$$

Obviously, it is symmetric and satisfies the following identity:

$$\nabla^\mu T_{\mu\nu}[f] = \Phi \cdot \nabla_\nu f \tag{6}$$

Given a vectorfield X, which will be used as a *multiplier vector field*, the associated energy currents are defined as follows

$$P_\alpha^X[f] = T_{\alpha\mu}[f] X^\mu, \quad K^X[f] = \frac{1}{2} T^{\mu\nu}[f] {}^{(X)}\pi_{\mu\nu}$$

where the deformation tensor ${}^{(X)}\pi_{\mu\nu}$ is defined by

$$^{(X)}\pi_{\mu\nu} = \mathcal{L}_X g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu \tag{7}$$

By (6), we easily obtain:

$$\nabla^\mu P_\mu^X[f] = K^X[f] + \Phi \cdot Xf \tag{8}$$

We can express T in terms of null frames $\{e_1 = r^{-1}\partial_\theta, e_2 = (r\sin\theta)^{-1}, e_3 = \underline{L}, e_4 = L\}$:

$$T(L, L) = |Lf|^2 \quad T(\underline{L}, \underline{L}) = |\underline{L}f|^2 \quad T(L, \underline{L}) = |\slashed{\nabla}f|^2$$

where we express the Minkowski metric in polar coordinates:

$$g = -dt^2 + dr^2 + r^2(d\theta^2 + (\sin\theta)^2 d\varphi^2)$$

We shall use $X = L$ and $\underline{L}$ as multiplier vectorfields, the corresponding deformation tensors and currents are:

$$^{(L)}\pi = \frac{1}{r}\slashed{g} \quad {}^{(\underline{L})}\pi = -\frac{1}{r}\slashed{g} \tag{9}$$

$$K^L = \frac{1}{2r} Lf\underline{L}f, \quad K^{\underline{L}} = -\frac{1}{2r} Lf\underline{L}f$$

where $\slashed{g}$ is the restriction of the Minkowski metric to the sphere $S_{\underline{u},u}$.
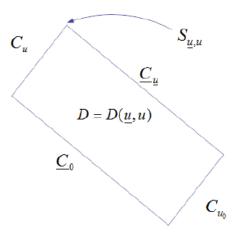
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Figure 2: integral domain (This figure is borrowed from [21].)

We use $\mathcal{D}(u,\underline{u})$ to denote the space-time slab enclosed by the hypersurfaces C_{u_0}, $\underline{C}_0$, C_u and $\underline{C}_{\underline{u}}$ as pictured above (see 2.2). We integrate (8) on $\mathcal{D}(u,\underline{u})$ to obtain:

$$\int_{C_u} T[f](X,L) + \int_{\underline{C}_{\underline{u}}} T[f](X,\underline{L}) = \int_{C_{u_0}} T[f](X,L) + \int_{\underline{C}_0} T[f](X,\underline{L})$$
$$+ \iint_{\mathcal{D}(u,\underline{u})} K^X[f] + \Phi \cdot Xf$$

where L and $\underline{L}$ are corresponding normals of the null hypersurfaces C_u and $\underline{C}_{\underline{u}}$.

In applications, the data on $\underline{C}_0$ is always vanishing, thus, we have the following formula:

$$\int_{C_u} T[f](X,L) + \int_{\underline{C}_{\underline{u}}} T[f](X,\underline{L}) = \int_{C_{u_0}} T[f](X,L) + \iint_{\mathcal{D}(u,\underline{u})} K^X[f] + \Phi \cdot Xf \qquad (10)$$

2.3 Gronwall and Sobolev Inequalities

We need the following Sobolev inequalities which can be derived from *isoperimetric inequalty*. The proof can be found in [3].

Lemma 2.1 *Let $(S,\not{g})$ be a compact 2-dimensional Riemannian manifold and ϕ a smooth function on S, which is square-integrable and with square-integrable first derivatives. Then for $2 < p < \infty$, $\phi \in L^p(S)$ and we have:*

$$|S|^{-1/p}\|\phi\|_{L^p(S)} \le C_p\sqrt{I'(S)}\|\phi\|_{W_1^2(S)}$$

Here C_p is a numerical constant depending only on p, and $|S|$ is the area of the sphere $S_{\underline{u},u}$.

$$I'(S) = \max\{I(S), 1\}$$

where $I(S)$ is the isoperimetric constant of S, and we define:

$$\|\phi\|_{W_1^2(S)} = \|\not{\nabla}\phi\|_{L^2(S)} + |S|^{-1/2}\|\phi\|_{L^2(S)}$$

Lemma 2.2 *Let $(S,\not{g})$ be a compact 2-dimensional Riemannian manifold and ϕ a smooth function on S, which belongs to $L^p(S)$ and with first derivatives which also belong to $L^p(S)$, for some $p > 2$. Then $\phi \in L^\infty(S)$ and we have*

$$\sup|\phi| \le C_p\sqrt{I'(S)}|S|^{(1/2)-(1/p)}\|\phi\|_{W_1^p(S)}$$

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Here C_p is a numerical constant depending only on p, and we define:

$$\|\phi\|_{W_1^p(S)} = \|\overline{\nabla}\phi\|_{L^p(S)} + |S|^{-1/2}\|\phi\|_{L^p(S)}$$

Also by *isoperimetric inequality*, we can deduce:

Lemma 2.3 Let ϕ be a smooth function on C_u vanishing on $S_{0,u}$ then with the same condition as Lemma 2.1, we have:

$$\int_{S_{\underline{u},u}}|\phi|^6 d\mu_{\slashed{g}} \leq C(\int_{S_{\underline{u},u}}|\phi|^4 d\mu_{\slashed{g}})(|u|^{-2}\int_{S_{\underline{u},u}}|\phi|^2 d\mu_{\slashed{g}} + \int_{S_{\underline{u},u}}|\overline{\nabla}\phi|^2 d\mu_{\slashed{g}})$$

where C is an absolute constant.

Obviously, from Lemma 2.1 and Lemma 2.2, we obtain:

$$\sup_S |\phi| \leq C_p[|S|^{-1/2}\|\phi\|_{L^2(S)} + \|\overline{\nabla}\phi\|_{L^2(S)} + |S|^{1/2}\|\overline{\nabla}^2\phi\|_{L^2(S)}] \tag{11}$$

With the same condition as Lemma 2.3, we have:

Lemma 2.4 Let ϕ be a smooth C_u-function vanishing on $\underline{C}_0$, we have:

$$\sup_{\underline{u}}(|u|^{1/2}\|\phi\|_{L^4(S_{\underline{u},u})}) \leq C_p\|L\phi\|_{L^2(C_u)}^{1/2}[\|\phi\|_{L^2(C_u)} + |u|\|\overline{\nabla}\phi\|_{L^2(C_u)}]^{1/2}$$

and by Gronwall's inequality, we have:

$$\|\phi\|_{L^2(S_{\underline{u},u})} \lesssim \|L\phi\|_{L^2(C_u)}^{1/2}\|\phi\|_{L^2(C_u)}^{1/2}$$

Also from Lemma 2.3, we have:

Lemma 2.5 Let ϕ be a smooth function on $\underline{C}_{\underline{u}}$, the following estimates hold:

$$\sup_u(|u|^{1/2}\|\phi\|_{L^4(S_{\underline{u},u})}) \leq C_p\{|u_0|^{1/2}\|\phi\|_{L^4(S_{\underline{u},u_0})} + \||u|^{1/2}\underline{L}\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}\cdot$$
$$[\||u|^{-1/2}\phi\|_{L^2(\underline{C}_{\underline{u}})}^2 + \||u|^{1/2}\overline{\nabla}\phi\|_{L^2(\underline{C}_{\underline{u}})}^2]^{1/4}\}$$

also:

$$\|\phi\|_{L^2(S_{\underline{u},u})} \lesssim \|\phi\|_{L^2(S_{\underline{u},u_0})} + \|\underline{L}\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}\|\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}$$

From the above lemmas, we can easily obtain the following L^∞ estimates:

$$\|\phi\|_{L^\infty(S_{\underline{u},u})} \lesssim |u|^{-1/2}\|L\phi\|_{L^2(C_u)}^{1/2}\|\phi\|_{L^2(C_u)}^{1/2}$$
$$+\|L\overline{\nabla}\phi\|_{L^2(C_u)}^{1/2}\|\overline{\nabla}\phi\|_{L^2(C_u)}^{1/2} + |u|^{1/2}\|L\overline{\nabla}^2\phi\|_{L^2(C_u)}^{1/2}\|\overline{\nabla}^2\phi\|_{L^2(C_u)}^{1/2}$$

and also:

$$\|\phi\|_{L^\infty(S_{\underline{u},u})} \lesssim |u|^{-1/2}\|\phi\|_{L^2(S_{\underline{u},u_0})} + \|\overline{\nabla}\phi\|_{L^2(S_{\underline{u},u_0})} + |u|^{1/2}\|\overline{\nabla}^2\phi\|_{L^2(S_{\underline{u},u_0})}$$
$$+|u|^{-1/2}\|\underline{L}\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}\|\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2} + \|\underline{L}\overline{\nabla}\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}\|\overline{\nabla}\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2} + |u|^{1/2}\|\underline{L}\overline{\nabla}^2\phi\|_{\underline{L}^2(\underline{C}_{\underline{u}})}^{1/2}\|\overline{\nabla}^2\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}$$

In all of the above lemmas, we always assume that ϕ vanishes on $\underline{C}_0$.

Actually, in the following, we only use another version of the above Sobolev inequality, with $\overline{\nabla}$ substituted by Ω. In this case, the weight of $|u|$ will change, but it doesn't matter, because in our case, $|u|$ is more or less like a constant.
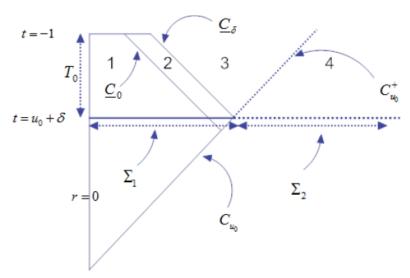
We also need the standard Gronwall's inequality:

Figure 3: construction of initial data (This figure is borrowed from [21].)

Lemma 2.6 *Let $f(t)$ be a non-negative function defined on an interval I with initial point t_0. If f satisfies*:

$$\frac{d}{dt}f \le a \cdot f + b$$

where two non-negative functions $a, b \in L^1(I)$, then for all $t \in I$, we have:

$$f(t) \le e^{A(t)}\left(f(t_0) + \int_{t_0}^{t} e^{-A(\tau)}b(\tau)d\tau\right) \tag{12}$$

where $A(t) = \int_{t_0}^{t} a(\tau)d\tau$.

2.4   Outline of the Proof

We will follow the main steps of [21]. The Cauchy data will be finally given on $t = u_0 + \delta$ and the solution will exist at least for $t \in [u_0 + \delta, -1]$. This can be shown in the above picture (see 2.4): First, we give initial data on the null hypersurface C_{u_0} where $u_0 \le \underline{u} \le \delta$. When $u_0 \le \underline{u} \le 0$, the data is trivial, therefore the solution in Region 1 is zero. When $0 \le \underline{u} \le \delta$, the data will be chosen as follows:

$$\phi(\underline{u}, u_0, \theta) = \delta^{1/2}\psi_0(\frac{\underline{u}}{\delta}, \theta)$$

where the energy of ψ_0 is larger than E_0. We then show that we can construct a solution in Region 2. Consequently, we take the restriction of the solution constructed to the surface $\Sigma_1 \subset \{t = u_0 + \delta\}$ as the first part of the Cauchy data.

Second, we extend the Cauchy data on Σ_1 to $\Sigma_2 \subset \{t = u_0 + \delta\}$ such that the energy is small. By small data theory, we can construct a solution in Region 4.

Third, From previous two steps, we can show that the restriction of the solution already constructed to $\underline{C}_\delta$ and $C_{u_0}^+$ (where $\underline{u} \ge \delta$) are small. We use them as initial data and we can solve this small data problem to construct solution on Region 3. We finally combines the solutions in Region 1,2,3 and 4 to finish the construction.

3   Characteristic Initial Data

First, we require that the data $\phi(\underline{u}, u_0, \theta)$ to satisfy

$$\phi(\underline{u}, u_0, \theta) = 0 \quad \text{for all} \quad \underline{u} \le 0$$

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Therefore, according to the Huygens principle, the solution ϕ of (1) satisfies

$$\phi \equiv 0 \quad \text{in Region 1} \quad = \{\underline{u}(x) \leq 0, u_0 \leq u(x) \leq 0\}$$

Secondly, we choose

$$\phi(\underline{u}, u_0, \theta) = \delta^{1/2} \psi_0(\frac{u}{\delta}, \theta) \tag{13}$$

where ψ_0 is a smooth function supported in $(0, 1)$ with respect to its first variable.

The data given in the above form is called a *short pulse*, a name invented by Christodoulou in [3].

In order to derive the energy estimates, as in [21], we need the following commutators:

$$[L, \Omega] = 0 \quad [\underline{L}, \Omega] = 0 \quad [\Box, \Omega] = 0 \tag{14}$$

$$[\Box, L] = \frac{1}{r^2}(L - \underline{L}) + \frac{2}{r^3}\slashed{\Delta} \quad [\Box, \underline{L}] = \frac{1}{r^2}(\underline{L} - L) - \frac{2}{r^3}\slashed{\Delta}$$

Here the operator $\slashed{\Delta}$ is the Laplacian on standard sphere $\mathbb{S}^2$.

On the initial hypersurface C_{u_0}, we have the following bounds on data:

$$\|L\phi\|_{L^\infty(C_{u_0})} \lesssim \delta^{-1/2} \quad \|\Omega\phi\|_{L^\infty(C_{u_0})} \lesssim \delta^{1/2}$$

and for higher order derivatives, we have:

$$\|L\Omega^k\phi\|_{L^\infty(C_{u_0})} \lesssim_k \delta^{-1/2}$$
$$\|\Omega^{k+1}\phi\|_{L^\infty(C_{u_0})} \lesssim_k \delta^{1/2}$$
$$\|L^2\Omega^{k-1}\phi\|_{L^\infty(C_{u_0})} \lesssim_k \delta^{-3/2}$$

We also need a bound for $\underline{L}$-derivatives. To do this, we write the equation in null frames:

$$-L\underline{L}\phi + \frac{1}{r^2}\slashed{\Delta}\phi + \frac{1}{r}(L\phi - \underline{L}\phi) = \pm|\phi|^{k-1}\phi \tag{15}$$

We can write the above as a propagation equation for $\underline{L}\phi$ along C_{u_0}:

$$L(\underline{L}\phi) = a \cdot \underline{L}\phi + b$$

where

$$a = -\frac{1}{r} \quad b = \frac{1}{r}L\phi + \frac{1}{r^2}\slashed{\Delta}\phi \mp |\phi|^{k-1}\phi$$

Obviously,

$$\|a\|_{L^\infty(C_{u_0})} \lesssim 1, \quad \|b\|_{L^\infty(C_{u_0})} \lesssim \delta^{-1/2}$$

Then by Gronwall's inequality, we easily obtain:

$$\|\underline{L}\phi\|_{L^\infty(C_{u_0})} \lesssim \delta^{1/2} \tag{16}$$

Similarly, by using the commutator, we obtain

$$\|\underline{L}\Omega\phi\|_{L^\infty(C_{u_0})} \lesssim \delta^{1/2} \quad \|\underline{L}\Omega^2\phi\|_{L^\infty(C_{u_0})} \lesssim \delta^{1/2} \tag{17}$$

To obtain a long time existence theorem for (1), we have to derive estimates on ϕ as well as its derivatives. it's very natual that these estimates should be compatible with the bounds for ϕ on initial

hypersurface. However, as stated in [21], a *relaxed* estimate, which is easier to derive, is enough. That is, we just need the following bounds on $\Omega^k \phi$:

$$\|\Omega^{k+1}\phi\|_{L^\infty(C_{u_0})} \lesssim_k 1 \tag{18}$$

Summarizing, we have the following bounds on initial data:

$$\|L\Omega^k\phi\|_{L^\infty(C_{u_0})} \lesssim \delta^{-1/2} \tag{19}$$
$$\|\Omega^{k+1}\phi\|_{L^\infty(C_{u_0})} \lesssim 1$$
$$\|\underline{L}\Omega^k\phi\|_{L^\infty(C_{u_0})} \lesssim \delta^{1/2}$$

for $k = 0, 1, 2$, and

$$\|L^2\phi\|_{L^\infty(C_{u_0})} \lesssim \delta^{-3/2} \tag{20}$$
$$\|L^2\Omega\phi\|_{L^\infty(C_{u_0})} \lesssim \delta^{-3/2}$$

From these L^∞ bounds, we obtain easily the L^2 bounds:

$$\|L\Omega^k\phi\|_{L^2(C_{u_0})} \lesssim 1 \tag{21}$$
$$\|\Omega^{k+1}\phi\|_{L^2(C_{u_0})} \lesssim \delta^{1/2}$$

for $k = 0, 1, 2$, and

$$\|L^2\phi\|_{L^2(C_{u_0})} \lesssim \delta^{-1} \tag{22}$$
$$\|L^2\Omega\phi\|_{L^2(C_{u_0})} \lesssim \delta^{-1}$$

We shall show that (21) and (22) will hold on all later outgoing null hypersurfaces C_u where $-1 > u > u_0$ provided the solution of (1) can be constructed up to C_u.

4   A priori Estimates

We start by defining a family of energy norms. For this purpose, we slightly abuse the notations: we use C_u to denote $C_u^{[0,\underline{u}]}$ and $\underline{C}_{\underline{u}}$ to denote $\underline{C}_{\underline{u}}^{[u_0,u]}$, by definition,

$$C_u^{[0,\underline{u}]} = \{p \in C_u | 0 \le \underline{u}(p) \le \underline{u}\} \quad \underline{C}_{\underline{u}}^{[u_0,u]} = \{p \in \underline{C}_{\underline{u}} | u_0 \le u(p) \le u\}$$

We define the following norms which are the same order as in [21], but remember, we are now in $\mathbb{R}^{3+1}$ other than $\mathbb{R}^{2+1}$. So essentially, we use one less derivative than [21].

$$E_1(u, \underline{u}) = \|L\phi\|_{L^2(C_u)} + \delta^{-\frac{1}{2}}\|\Omega\phi\|_{L^2(C_u)}, \tag{23}$$
$$\underline{E}_1(u, \underline{u}) = \|\Omega\phi\|_{L^2(\underline{C}_{\underline{u}})} + \delta^{-\frac{1}{2}}\|\underline{L}\phi\|_{L^2(\underline{C}_{\underline{u}})},$$
$$E_2(u, \underline{u}) = \|L\Omega\phi\|_{L^2(C_u)} + \delta^{-\frac{1}{2}}\|\Omega^2\phi\|_{L^2(C_u)},$$
$$\underline{E}_2(u, \underline{u}) = \|\Omega^2\phi\|_{L^2(\underline{C}_{\underline{u}})} + \delta^{-\frac{1}{2}}\|\underline{L}\Omega\phi\|_{L^2(\underline{C}_{\underline{u}})},$$
$$E_3(u, \underline{u}) = \|L\Omega^2\phi\|_{L^2(C_u)} + \delta^{-\frac{1}{2}}\|\Omega^3\phi\|_{L^2(C_u)},$$
$$\underline{E}_3(u, \underline{u}) = \|\Omega^3\phi\|_{L^2(\underline{C}_{\underline{u}})} + \delta^{-\frac{1}{2}}\|\underline{L}\Omega^2\phi\|_{L^2(\underline{C}_{\underline{u}})}$$

We also need another family of norms which involves at least two null derivatives. They are defined as follows:

$$F_2(u, \underline{u}) = \delta \| L^2 \phi \|_{L^2(C_u)}, \tag{24}$$

$$\underline{F}_2(u, \underline{u}) = \| \underline{L}^2 \phi \|_{L^2(\underline{C}_{\underline{u}})},$$

$$F_3(u, \underline{u}) = \delta \| L^2 \Omega \phi \|_{L^2(C_u)},$$

$$\underline{F}_3(u, \underline{u}) = \| \underline{L}^2 \Omega \phi \|_{L^2(\underline{C}_{\underline{u}})}.$$

We shall prove as in [21]:

Main A priori Estimates. If δ is sufficiently small, for all initial data of (1) and all $I \in \mathbb{R}_{>0}$ which satisfy

$$E_1(u_0, \delta) + E_2(u_0, \delta) + E_3(u_0, \delta) + F_2(u_0, \delta) + F_3(u_0, \delta) \leq I, \tag{25}$$

there is a constant $C(I)$ depending only on I (in particular, not on δ), so that

$$\sum_{i=1}^{3} [E_i(u, \underline{u}) + \underline{E}_i(u, \underline{u})] + \sum_{j=2}^{3} [F_j(u, \underline{u}) + \underline{F}_j(u, \underline{u})] \leq C(I), \tag{26}$$

for all $u \in [u_0, u^*]$ and $\underline{u} \in [0, \underline{u}^*]$ where $u_0 \leq u^* \leq -1$ and $0 \leq \underline{u}^* \leq \delta$.

We consider the set $\mathcal{A} \subset \mathcal{U} := \{(u, \underline{u}) : u \in [u_0, u^*], \underline{u} \in [0, \underline{u}^*]\}$, in which the following holds:

$$M := \sum_{i=1}^{3} [E_i(u, \underline{u}) + \underline{E}_i(u, \underline{u})] + \sum_{j=2}^{3} [F_j(u, \underline{u}) + \underline{F}_j(u, \underline{u})] \leq AI, \tag{27}$$

where $A > 1$ is a sufficiently large constant depending only on initial data. Obviously, $\mathcal{A}$ is not empty. Here $\mathcal{U}$ is the set where the solution exists. We shall prove that actually $\mathcal{A} = \mathcal{U}$.

4.1 Preliminary Estimates

Under the bootstrap assumption (27), we first derive L^∞ for one derivatives of ϕ. We will also obtain the L^4 estimates for derivatives of ϕ up to the second order.

We start with $L\phi$. According to Sobolev inequalities, we have (we shall omit the weights on $|u|$)

$$\| L\phi \|_{L^4(S_{\underline{u}, u})} \lesssim \| L^2 \phi \|_{L^2(C_u)}^{1/2} (\| L\phi \|_{L^2(C_u)}^{1/2} + \| L\Omega\phi \|_{L^2(C_u)}^{1/2})$$

$$\lesssim (\delta^{-1} M)^{\frac{1}{2}} M^{\frac{1}{2}}$$

Hence,

$$\| L\phi \|_{L^4(S_{\underline{u}, u})} \lesssim \delta^{-1/2} M \tag{28}$$

Similarly, we have

$$\| L\Omega\phi \|_{L^4(S_{\underline{u}, u})} \lesssim \delta^{-1/2} M \tag{29}$$

Now we consider $\Omega\phi$. According to Sobolev inequalities, we have

$$\| \Omega\phi \|_{L^4(S_{\underline{u}, u})} \lesssim \| L\Omega\phi \|_{L^2(C_u)}^{1/2} (\| \Omega\phi \|_{L^2(C_u)}^{1/2} + \| \Omega^2\phi \|_{L^2(C_u)}^{1/2})$$

$$\lesssim M^{1/2} ((\delta^{1/2} M)^{1/2} + (\delta^{1/2} M)^{1/2})$$

Thus,

$$\| \Omega\phi \|_{L^4(S_{\underline{u}, u})} \lesssim \delta^{1/4} M \tag{30}$$

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Similarly,

$$\|\Omega^2\phi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{1/4}M \tag{31}$$

Finally, we turn to the estimates on $\underline{L}\phi$.

$$\|\underline{L}\phi\|_{L^4(S_{\underline{u},u})} \lesssim \|\underline{L}\phi\|_{L^4(S_{\underline{u},u_0})} + \|\underline{L}^2\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}(\|\underline{L}\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2} + \|\Omega\underline{L}\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2})$$
$$\lesssim \delta^{1/2} + \delta^{1/4}M$$

If δ is sufficiently small, we obtain

$$\|\underline{L}\phi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{1/4}M \tag{32}$$

Similarly, we also obtain

$$\|\underline{L}\Omega\phi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{1/4}M \tag{33}$$

Finally we need an L^∞ bound for ϕ. If we set

$$\|\phi\|_{L^\infty(S_{\underline{u},u})} \leq N$$

then by the first L^∞ Sobolev inequality, we have:

$$N \lesssim \delta^{1/4}N^{1/2}M^{1/2} + \delta^{1/4}M$$

So if δ is sufficiently small, we have:

$$\|\phi\|_{L^\infty(S_{\underline{u},u})} \lesssim \delta^{1/4}M \tag{34}$$

We summarize all the estimates in the following proposition.

Proposition 4.1 Under the bootstrap assumption (27), if δ is sufficiently small, we have

$$\delta^{-1/4}\|\phi\|_{L^\infty(S_{\underline{u},u})} + \delta^{1/2}\|L\Omega\phi\|_{L^4(S_{\underline{u},u})} + \delta^{-1/4}\|\Omega^2\phi\|_{L^4(S_{\underline{u},u})} + \delta^{-1/4}\|\underline{L}\Omega\phi\|_{L^4(S_{\underline{u},u})}$$
$$+\delta^{1/2}\|L\phi\|_{L^4(S_{\underline{u},u})} + \delta^{-1/4}\|\Omega\phi\|_{L^4(S_{\underline{u},u})} + \delta^{-1/4}\|\underline{L}\phi\|_{L^4(S_{\underline{u},u})} \lesssim M$$

4.2 Estimates on E_k and $\underline{E}_k$

For simplicity, we shall assume that $k = 2$, because if $k > 2$, one just need to bound the extra power by L^∞ norm.

We commute Ω^i (for $i = 1, 2$) with (1), we have [2]

$$\Box\Omega^i\phi = \phi\Omega^k\phi + \sum_{|p|>0,|q|>0,p+q=i}\Omega^p\phi\Omega^q\phi$$

Now we use the basic energy identity for this equation where we take $f = \Omega^i\phi$ ($i = 0, 1, 2$) and $X = L$, then we have:

$$\int_{C_u}|L\Omega^i\phi|^2 + \int_{\underline{C}_{\underline{u}}}|\not\nabla\Omega^i\phi|^2 = \int_{C_{u_0}}|L\Omega^i\phi|^2 + \tag{35}$$
$$\iint_{\mathcal{D}}(\phi\Omega^i\phi)L\Omega^i\phi + \sum_{|p|>0,|q|>0,p+q=i}\iint_{\mathcal{D}}(\Omega^p\phi\Omega^q\phi)L\Omega^i\phi + \iint_{\mathcal{D}}\frac{1}{2r}\underline{L}\Omega^i\phi L\Omega^i\phi$$
$$= \int_{C_{u_0}}|L\Omega^i\phi|^2 + S_1 + S_2 + S_3$$

---

[2]For simplicity, we omit the constant coefficients and the sign for nonlinearity.

where S_j are defined in the obvious way. We also recall that:

$$\|\Omega^i\phi\|_{L^p(S_{\underline{u},u})} \sim \|\slashed{\nabla}^i\phi\|_{L^p(S_{\underline{u},u})}$$

By Proposition 4.1 and bootstrap assumption,

$$S_1 \lesssim \|\phi\|_{L^\infty(S_{\underline{u},u})} \int_{u_0}^u \|\Omega^i\phi\|_{L^2(C_{u'})}\|L\Omega^i\phi\|_{L^2(C_{u'})}du' \lesssim (\delta^{1/4}M)(\delta^{1/2}M)M \lesssim \delta^{3/4}M^3$$

also

$$S_2 \lesssim \int_{u_0}^u \|\Omega^p\phi\|_{L^4(C_{u'})}\|\Omega^q\phi\|_{L^4(C_{u'})}\|L\Omega^i\phi\|_{L^2(C_{u'})}du' \lesssim \int_{u_0}^u (\delta^{1/2}M)(\delta^{1/2}M)Mdu' \lesssim \delta M^3$$

and

$$S_3 \lesssim \|L\Omega^i\phi\|_{L^2(\mathcal{D})}\|\underline{L}\Omega^i\phi\|_{L^2(\mathcal{D})}$$
$$\lesssim (\int_{u_0}^u \|L\Omega^i\phi\|_{L^2(C_{u'})}^2 du')^{1/2}(\int_0^{\underline{u}} \|\underline{L}\Omega^i\phi\|_{L^2(\underline{C}_{\underline{u}'})}^2 d\underline{u}')^{1/2} \lesssim M(\delta M) \lesssim \delta M^2$$

Put all these in (35), we obtain:

$$\sum_{i=0}^2 (\|L\Omega^i\varphi\|_{L^2(C_u)} + \|\slashed{\nabla}\Omega^i\varphi\|_{L^2(\underline{C}_u)}) \lesssim I + \delta^{3/8}M^{3/2} \tag{36}$$

We still consider, for $i = 0, 1, 2$,

$$\Box\Omega^i\phi = \phi\Omega^k\phi + \sum_{|p|>0,|q|>0,p+q=i} \Omega^p\phi\Omega^q\phi$$

But now we take $X = \underline{L}$ in the energy identity:

$$\int_{C_u} |\slashed{\nabla}\Omega^i\phi|^2 + \int_{\underline{C}_{\underline{u}}} |\underline{L}\Omega^i\phi|^2 = \int_{C_{u_0}} |\slashed{\nabla}\Omega^i\phi|^2 \tag{37}$$
$$+ \iint_{\mathcal{D}} \phi\Omega^i\phi\underline{L}\Omega^i\phi + \sum_{|p|>0,|q|>0,p+q=i} \iint_{\mathcal{D}} \Omega^p\phi\Omega^q\phi(\underline{L}\Omega^i\phi) - \iint_{\mathcal{D}} \frac{1}{2r}(\underline{L}\Omega^i\phi)(L\Omega^i\phi)$$
$$= \int_{C_{u_0}} |\slashed{\nabla}\Omega^i\phi|^2 + T_1 + T_2 + T_3$$

where T_j are defined in the obvious way.

As before, by Proposition 4.1 and bootstrap assumption, we have:

$$T_1 \lesssim \|\phi\|_{L^\infty(S_{\underline{u},u})} \int_0^{\underline{u}} \|\Omega^i\phi\|_{L^2(\underline{C}_{\underline{u}'})}\|\underline{L}\Omega^i\phi\|_{L^2(\underline{C}_{\underline{u}'})}d\underline{u}' \lesssim (\delta^{1/4}M)(M)(\delta^{1/2}M)\delta \lesssim \delta^{7/4}M^3$$

also

$$T_2 \lesssim \int_0^{\underline{u}} \|\Omega^p\phi\|_{L^4(\underline{C}_{\underline{u}'})}\|\Omega^q\phi\|_{L^4(\underline{C}_{\underline{u}'})}\|\underline{L}\Omega^i\phi\|_{L^2(\underline{C}_{\underline{u}'})}d\underline{u}' \lesssim (\delta^{1/4}M)(\delta^{1/4}M)(\delta^{1/2}M)\delta \lesssim \delta^2M^3$$

Similar to S_3, we have:

$$T_3 \lesssim \delta M^2$$

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Put these in (37), we obtain:

$$\sum_{i=0}^{2}(\|\slashed{\nabla}\Omega^i\varphi\|_{L^2(C_u)} + \|\underline{L}\Omega^i\varphi\|_{L^2(\underline{C}_u)}) \lesssim I + \delta^{1/2}M^{3/2} \tag{38}$$

Combining (36) and (38) we obtain:

$$\sum_{i=1}^{3}(E_i(u,\underline{u}) + \underline{E}_i(u,\underline{u})) \lesssim I + \delta^{3/8}M^{3/2} \tag{39}$$

4.3   Estimates on $F_2(u,\underline{u})$ and $\underline{F}_2(u,\underline{u})$

We first consider the bound of $\|\underline{L}^2\varphi\|_{L^2(\underline{C}_u)}$. We commute $\underline{L}$ with (1), we obtain

$$\Box \underline{L}\phi = \phi\underline{L}\phi + \frac{1}{r^2}(\underline{L}\phi - L\phi) - \frac{2}{r^3}\slashed{\Delta}\phi$$

We use the basic energy identity where we take $f = \underline{L}\phi$ and $X = \underline{L}$, therefore,

$$\int_{C_u}|\slashed{\nabla}\underline{L}\phi|^2 + \int_{\underline{C}_u}|\underline{L}^2\phi|^2 = \int_{C_{u_0}}|\slashed{\nabla}\underline{L}\phi|^2 + \iint_{\mathcal{D}}\phi\underline{L}\phi\underline{L}^2\phi \tag{40}$$

$$+ \iint_{\mathcal{D}}\frac{1}{r^2}(\underline{L}\phi - L\phi)\underline{L}^2\phi + \iint_{\mathcal{D}}\frac{1}{r^3}\slashed{\Delta}\phi\underline{L}^2\phi \tag{41}$$

$$= \int_{C_{u_0}}|\slashed{\nabla}\underline{L}\phi|^2 + S_1 + S_2 + S_3 \tag{42}$$

For S_1, we have, by Proposition 4.1 and bootstrap assumption:

$$S_1 \lesssim \|\phi\|_{L^\infty(S_{\underline{u},u})}\int_0^{\underline{u}}\|\underline{L}\phi\|_{L^2(\underline{C}_{u'})}\|\underline{L}^2\phi\|_{L^2(\underline{C}_{u'})}d\underline{u}' \lesssim (\delta^{1/4}M)(\delta^{1/2}M)(M)\delta \lesssim \delta^{5/4}M^3$$

also,

$$S_2 \lesssim \int_0^{\underline{u}}\|\underline{L}\phi\|_{L^2(\underline{C}_{u'})}\|\underline{L}^2\phi\|_{L^2(\underline{C}_{u'})}d\underline{u}' + (\int_{u_0}^u\|L\phi\|_{L^2(C_{u'})}^2 du')^{1/2}(\int_0^{\underline{u}}\|\underline{L}^2\phi\|_{L^2(\underline{C}_{u'})}^2 d\underline{u}')^{1/2}$$

$$\lesssim \delta^{3/2}M^2 + \delta^{1/2}M^2 \lesssim \delta^{1/2}M^2$$

The estimates for S_3 is similar,

$$S_3 \lesssim \int_0^{\underline{u}}\|\Omega^2\phi\|_{L^2(\underline{C}_{u'})}\|\underline{L}^2\phi\|_{L^2(\underline{C}_{u'})}d\underline{u}' \lesssim \delta M^2$$

So we obtain:

$$\underline{F}_2(u,\underline{u}) \lesssim \delta I + \delta^{1/4}M^{3/2} \tag{43}$$

Next, we consider the bound for $\|L^2\phi\|_{L^2(C_u)}$, we commute L with (1):

$$\Box L\phi = \phi L\phi - \frac{1}{r^2}(\underline{L}\phi - L\phi) + \frac{2}{r^3}\slashed{\Delta}\phi$$

We use the energy identity with $f = L\phi$ and $X = L$,

$$\int_{C_u}|L^2\phi|^2 + \int_{\underline{C}_u}|\Omega L\phi|^2 = \int_{C_{u_0}}|L^2\phi|^2 + \iint_{\mathcal{D}}\phi L\phi L^2\phi$$

$$+ \iint_{\mathcal{D}}\frac{1}{r^2}(\underline{L}\phi - L\phi)L^2\phi + \iint_{\mathcal{D}}\frac{1}{r^3}\slashed{\Delta}\phi L^2\phi$$

$$= \int_{C_{u_0}}|L^2\phi|^2 + T_1 + T_2 + T_3$$

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As usual, by Proposition 4.1 and bootstrap assumption, we have:

$$T_1 \lesssim \|\phi\|_{L^\infty(S_{\underline{u},u})} \int_{u_0}^u \|L\phi\|_{L^2(C_{u'})}\|L^2\phi\|_{L^2(C_{u'})}du' \lesssim (\delta^{1/4}M)M(\delta^{-1}M) \lesssim \delta^{-3/4}M^3$$

also,

$$T_2 \lesssim (\int_0^{\underline{u}} \|\underline{L}\phi\|^2_{L^2(\underline{C}_{u'})}d\underline{u}')^{1/2}(\int_{u_0}^u \|L^2\phi\|^2_{L^2(C_{u'})}du')^{1/2} + \int_{u_0}^u \|L\phi\|_{L^2(C_{u'})}\|L^2\phi\|_{L^2(C_{u'})}du'$$
$$\lesssim \delta^{-1}M^2$$

Estimates for T_3 is similar,

$$T_3 \lesssim \int_{u_0}^u \|\Omega^2\phi\|_{L^2(C_{u'})}\|L^2\phi\|_{L^2(C_{u'})}du' \lesssim (\delta^{1/2}M)(\delta^{-1}M) \lesssim \delta^{-1/2}M^2$$

So we have:

$$\int_{C_u} |L^2\phi|^2 \lesssim \delta^{-2}I^2 + \delta^{-1}M^3$$

thus,

$$F_2(u,\underline{u}) = \delta\|L^2\phi\|_{L^2(C_u)} \lesssim I + \delta^{1/2}M^{3/2} \tag{44}$$

Summarizing, we obtain:

$$F_2(u,\underline{u}) \lesssim I + \delta^{1/2}M^{3/2} \tag{45}$$
$$\underline{F}_2(u,\underline{u})) \lesssim \delta I + \delta^{1/4}M^{3/2}$$

4.4 Estimates on $F_3(u,\underline{u})$ and $\underline{F}_3(u,\underline{u})$

We first estimate $F_3(\underline{u},u)$, we commute L and Ω with (1) to derive

$$\Box L\Omega\phi = \phi L\Omega\phi + \Omega\phi L\phi - \frac{1}{r^2}(\underline{L}\Omega\phi - L\Omega\phi) + \frac{2}{r^3}\not\Delta\Omega\phi$$

Applying the energy identity with $f = L\Omega\phi$, and $X = L$, we obtain:

$$\int_{C_u} |L^2\Omega\phi|^2 + \int_{\underline{C}_u} |\not\nabla L\Omega\phi|^2 = \int_{C_{u_0}} |L^2\Omega\phi|^2 + \iint_{\mathcal{D}} \phi L\Omega\phi L^2\Omega\phi + \iint_{\mathcal{D}} \Omega\phi L\phi L^2\Omega\phi$$
$$+ \iint_{\mathcal{D}} \frac{1}{r^2}(\underline{L}\Omega\phi - L\Omega\phi)L^2\Omega\phi + \iint_{\mathcal{D}} \frac{1}{r^3}\not\Delta\Omega\phi L^2\Omega\phi$$
$$= \int_{C_{u_0}} |L^2\Omega\phi|^2 + S_1 + S_2 + S_3 + S_4$$

As before, by Proposition 4.1 and bootstrap assumption, we have:

$$S_1 \lesssim \|\phi\|_{L^\infty(S_{\underline{u},u})} \int_{u_0}^u \|L\Omega\phi\|_{L^2(C_{u'})}\|L^2\Omega\phi\|_{L^2(C_{u'})}du' \lesssim (\delta^{1/4}M)M(\delta^{-1}M) \lesssim \delta^{-3/4}M^3$$

also

$$S_2 \lesssim \int_{u_0}^u \|\Omega\phi\|_{L^4(C_{u'})}\|L\phi\|_{L^4(C_{u'})}\|L^2\Omega\phi\|_{L^2(C_{u'})}du' \lesssim (\delta^{1/2}M)(\delta^{1/2}M)(\delta^{-1}M) \lesssim M^3$$

For S_3, we have:

$$S_3 \lesssim (\int_0^{\underline{u}} \|\underline{L}\Omega\phi\|_{L^2(C_{\underline{u}'})}^2 d\underline{u}')^{1/2}(\int_{u_0}^u \|L^2\Omega\phi\|_{L^2(C_{u'})}^2 du')^{1/2} + \int_{u_0}^u \|L\Omega\phi\|_{L^2(C_{u'})}\|L^2\Omega\phi\|_{L^2(C_{u'})}du'$$

$$\lesssim (\delta M)(\delta^{-1}M) + M(\delta^{-1}M) \lesssim \delta^{-1}M^2$$

Similarly,

$$S_4 \lesssim \int_{u_0}^u \|\not{\Delta}\Omega\phi\|_{L^2(C_{u'})}\|L^2\Omega\phi\|_{L^2(C_{u'})}du' \lesssim (\delta^{1/2}M)(\delta^{-1}M) \lesssim \delta^{-1/2}M^2$$

So we obtain:

$$\int_{C_u} |L^2\Omega\phi|^2 \lesssim \delta^{-2}I^2 + \delta^{-1}M^3$$

i.e.

$$F_3(u,\underline{u}) \lesssim I + \delta^{1/2}M^{3/2} \tag{46}$$

For $\underline{F}_3(\underline{u},u)$, we commute $\underline{L}$ and Ω with (1) to derive

$$\Box\underline{L}\Omega\phi = \phi\underline{L}\Omega\phi + \underline{L}\phi\Omega\phi + \frac{1}{r^2}(\underline{L}\Omega\phi - L\Omega\phi) - \frac{2}{r^3}\not{\Delta}\Omega\phi$$

We use the energy identity with $f = \underline{L}\Omega\phi$ and $X = \underline{L}$,

$$\int_{C_u} |\not{\nabla}\underline{L}\Omega\phi|^2 + \int_{\underline{C}_{\underline{u}}} |\underline{L}^2\Omega\phi|^2 = \int_{C_{u_0}} |\not{\nabla}\underline{L}\Omega\phi|^2 + \iint_{\mathcal{D}} \phi\underline{L}\Omega\phi\underline{L}^2\Omega\phi + \iint_{\mathcal{D}} \underline{L}\phi\Omega\phi\underline{L}^2\Omega\phi$$

$$+ \iint_{\mathcal{D}} \frac{1}{r^2}(\underline{L}\Omega\phi - L\Omega\phi)\underline{L}^2\Omega\phi + \iint_{\mathcal{D}} \frac{1}{r^3}\not{\Delta}\Omega\phi\underline{L}^2\Omega\phi$$

$$= \int_{C_{u_0}} |\not{\nabla}\underline{L}\Omega\phi|^2 + T_1 + T_2 + T_3 + T_4$$

By Proposition 4.1 and bootstrap assumption,

$$T_1 \lesssim \|\phi\|_{L^\infty(S_{\underline{u},u})} \int_0^{\underline{u}} \|\underline{L}\Omega\phi\|_{L^2(\underline{C}_{\underline{u}'})}\|\underline{L}^2\Omega\phi\|_{L^2(\underline{C}_{\underline{u}'})}d\underline{u}'$$

$$\lesssim (\delta^{1/4}M)\delta(\delta^{1/2}M)(M) \lesssim \delta^{7/4}M^3$$

$$T_2 \lesssim \int_0^{\underline{u}} \|\underline{L}\phi\|_{L^4(\underline{C}_{\underline{u}'})}\|\Omega\phi\|_{L^4(\underline{C}_{\underline{u}'})}\|\underline{L}^2\Omega\phi\|_{L^2(\underline{C}_{\underline{u}'})}d\underline{u}' \lesssim \delta(\delta^{1/4}M)(\delta^{1/4}M)M \lesssim \delta^{3/2}M^3$$

also

$$T_3 \lesssim \int_0^{\underline{u}} \|\underline{L}\Omega\phi\|_{L^2(\underline{C}_{\underline{u}'})}\|\underline{L}^2\Omega\phi\|_{L^2(\underline{C}_{\underline{u}'})}d\underline{u}' + (\int_{u_0}^u \|L\Omega\phi\|_{L^2(C_{u'})}^2 du')^{1/2}(\int_0^{\underline{u}} \|\underline{L}^2\Omega\phi\|_{L^2(\underline{C}_{\underline{u}'})}^2 d\underline{u}')^{1/2}$$

$$\lesssim \delta^{3/2}M^2 + \delta^{1/2}M^2 \lesssim \delta^{1/2}M^2$$

Similarly,

$$T_4 \lesssim \int_0^{\underline{u}} \|\not{\Delta}\Omega\phi\|_{L^2(\underline{C}_{\underline{u}'})}\|\underline{L}^2\Omega\phi\|_{L^2(\underline{C}_{\underline{u}'})}d\underline{u}' \lesssim \delta M^2$$

So we obtain:

$$\int_{\underline{C}_{\underline{u}}} |\underline{L}^2\Omega\phi|^2 \lesssim I^2 + \delta^{1/2}M^3$$

i.e.

$$\underline{F}_3(u,\underline{u}) \lesssim I + \delta^{1/4}M^{3/2} \tag{47}$$

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4.5 End of the Bootstrap Argument

Combining (39), (45), (46) and (47) we obtain:

$$M \lesssim I + \delta^{1/4} M^{3/2} \tag{48}$$

Choosing δ sufficiently small depending on the quantity I together with the bootastrap assumption:

$$M \leq AI$$

we obtain:

$$M \leq \frac{A}{2} I \tag{49}$$

Therefore $\mathcal{A}$ is both an open and closed subset of $\mathcal{U}$, then is $\mathcal{U}$ itself.

This completes the proof of **Main A priori Estimates**.

4.6 Higher Order Estimates

For the estimates of higher order derivatives, we just use the induction to prove it, since we have established the estimates for the lower derivatives up to the 3rd order. With the definitions:

$$E_k(u,\underline{u}) = \|L\Omega^{k-1}\phi\|_{L^2(C_u)} + \delta^{-1/2}\|\Omega^k\phi\|_{L^2(C_u)}$$
$$\underline{E}_k(u,\underline{u}) = \|\Omega^k\phi\|_{L^2(\underline{C}_{\underline{u}})} + \delta^{-1/2}\|\underline{L}\Omega^{k-1}\phi\|_{L^2(\underline{C}_{\underline{u}})}$$

and

$$F_k(u,\underline{u}) = \delta\|L^2\Omega^{k-2}\phi\|_{L^2(C_u)} \quad \underline{F}_k(u,\underline{u}) = \|\underline{L}^2\Omega^{k-2}\phi\|_{L^2(\underline{C}_{\underline{u}})}$$

Then if the initial data satisfy

$$\sum_{i=1}^{n+2} E_i(u_0,\delta) + \sum_{j=2}^{n+2} F_j(u_0,\delta) \leq I$$

we have:

$$E_{n+2}(u,\underline{u}) + \underline{E}_{n+2}(u,\underline{u}) + F_{n+2}(u,\underline{u}) + \underline{F}_{n+2}(u,\underline{u}) \lesssim_n I$$

Similar as Proposition 4.1,

$$\delta^{-1/4}\|\Omega^{n-1}\phi\|_{L^\infty(S_{\underline{u},u})} + \delta^{1/2}\|L\Omega^n\phi\|_{L^4(S_{\underline{u},u})} + \delta^{-1/4}\|\Omega^{n+1}\phi\|_{L^4(S_{\underline{u},u})} + \delta^{-1/4}\|\underline{L}\Omega^n\phi\|_{L^4(S_{\underline{u},u})}$$
$$+ \delta^{1/2}\|L\Omega^{n-1}\phi\|_{L^4(S_{\underline{u},u})} + \delta^{-1/4}\|\Omega^n\phi\|_{L^4(S_{\underline{u},u})} + \delta^{-1/4}\|\underline{L}\Omega^{n-1}\phi\|_{L^4(S_{\underline{u},u})} \lesssim_n I$$

5 Existence of Solutions

By the a priori estimates, we can show that (1) with data prescribed on C_{u_0} where $u_0 \leq \underline{u} \leq \delta$ can be solved all the way up to $t = -1$.

We use the local existence result of Alan. D. Rendall [14], which states that there exists a solution around S_{0,u_0}, say, defined in the region enclosed by $\underline{C}_0$, C_{u_0} and $t = u_0 + \epsilon$ with $\epsilon \ll \delta$. Thanks to the a priori Estimates, the solution and its derivatives are bounded on $t = u_0 + \epsilon$ by the initial data. Therefore, we can solve a Cauchy problem with data prescribed on $t = u_0 + \epsilon$ to construct a solution in the future domain of dependence of $t = u_0 + \epsilon$ whose boundary contains two null hypersurfaces $C_{u_0+\epsilon}$ and $\underline{C}_\epsilon$. Now we have two characteristic problem: for the first one, the data is prescribed on $\underline{C}_0$ and

$C_{u_0+\epsilon}$; for the second one, the data is prescribed on C_{u_0} and $\underline{C}_\epsilon$. We can use Rendall's local existence result again to solve them around $S_{0,u_0+\epsilon}$ and S_{ϵ,u_0}. In this way, we can actually push the solution to $t = u_0 + \epsilon + \epsilon'$ with another small ϵ'. Then we can repeat the above process to push the solution all the way to $t = u_0 + \delta$, and then to $t = -1$. Actually, from the second step, since we the main A priori estimate, the the length of the interval where the solution exists is the same. Therefore we can finally push the solution to $t = -1$.

6 Construction of Cauchy Data, Final Conclusions

Proposition 6.1 *Assume we have bound on $E_i(u_0)$ with $i = 1, 2, ..., n + 2$ and $F_j(u_0)$ with $j = 2, 3, ..., n + 2$ for some fixed $n \geq 10$. Then for $k = 4, ..., n$, we have*

$$\|\Omega^{k-4}\phi\|_{L^\infty(\underline{C}_\delta} + ... + \|\Omega^{k-1}\phi\|_{L^\infty(\underline{C}_\delta)} + \|\underline{L}\Omega^{k-3}\phi\|_{L^\infty(\underline{C}_\delta)} + \|\underline{L}^2\Omega^{k-4}\phi\|_{L^\infty(\underline{C}_\delta)} \lesssim \delta^{1/4}$$

$$\|L\Omega^{k-3}\phi\|_{L^\infty(\underline{C}_\delta)} + \|L^2\Omega^{k-4}\phi\|_{L^\infty(\underline{C}_\delta)} \lesssim \delta^{1/4}$$

Proof. The estimate for $\|\Omega^{k-1}\phi\|_{L^\infty(\underline{C}_\delta)}$ comes from the property before section 5. For the L and $\underline{L}$ derivatives, we just consider $L\phi$ and $\underline{L}\phi$, the higher order derivatives are similar. To start, we write equation (15) in two different forms:

$$L(\underline{L}\phi) = a(\underline{L}\phi) + \bar{b} + \frac{1}{r}(L\phi) \tag{50}$$

and also

$$\underline{L}(L\phi) = -a(L\phi) + \bar{b} - \frac{1}{r}(\underline{L}\phi) \tag{51}$$

where

$$\bar{b} = \frac{1}{r^2}\mathcal{\Delta}\phi \mp |\phi|^{k-1}\phi$$

By Gronwall's inequality, we have, since $L\phi$ vanishes near S_{δ,u_0}:

$$\|L\phi\|_{L^\infty(S_{\underline{u},u})} \lesssim \int_{u_0}^u (\|\underline{L}\phi\|_{L^\infty(S_{\underline{u},u'})} + \|b\|_{L^\infty(S_{\underline{u},u'})})du'$$

(Here $\underline{u}$ is very close to δ) and also

$$\|\underline{L}\phi\|_{L^\infty(S_{\underline{u},u})} \lesssim \int_0^{\underline{u}}(\|L\phi\|_{L^\infty(S_{\underline{u}',u})} + \|b\|_{L^\infty(S_{\underline{u}',u})})d\underline{u}'$$

Defining

$$A(u, \underline{u}) = \sup_{\underline{u}' \in [0,\underline{u}]} \|L\phi\|_{L^\infty(S_{\underline{u}',u})}$$

$$B(u, \underline{u}) = \sup_{u' \in [u_0,u]} \|\underline{L}\phi\|_{L^\infty(S_{\underline{u},u'})}$$

Since

$$\|b\|_{L^\infty(S_{\underline{u},u})} \lesssim \delta^{1/4}$$

we obtain:

$$A(u, \underline{u}) \lesssim B(u, \underline{u}) + \delta^{1/4} \tag{52}$$

$$B(u, \underline{u}) \lesssim \delta A(u, \underline{u}) + \delta^{5/4} \tag{53}$$

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where we have used the fact that

$$A(u, \underline{u}') \lesssim A(u, \underline{u})$$
$$B(u', \underline{u}) \lesssim B(u, \underline{u})$$

Substituting (53) in (52),

$$A(u, \underline{u}) \lesssim \delta A(u, \underline{u}) + \delta^{1/4}$$

So if we choose δ sufficiently small, we obtain:

$$A(u, \underline{u}) \lesssim \delta^{1/4}$$

Back to (53),

$$B(u, \underline{u}) \lesssim \delta^{5/4}$$

Then the proposition follows. $\square$

So we obtain from the above proposition that the data on $\underline{C}_\delta$ induced from the solution are small in energy norms. Note also that we lose one derivative when we integrate the propagation equation because of the term $\not{\!\!\Delta}\phi$.

Now we can construct our Cauchy data, which is similar to [21], and whose energy is larger than E_0

We now choose a Cauchy hypersurface $\Sigma = \{t = u_0 + \delta\}$. Let $\Sigma_1 = \Sigma \bigcap (\text{Region } 1 \bigcup \text{Region } 2)$ and $\Sigma_2 = \Sigma - \Sigma_1$. By the above proposition, we know that there is an an annular region E bounded by S_{δ, u_0} and a smaller sphere near S_{δ, u_0} in Σ_1, on which the solution is small. Then by a Whitney extension theorem established by Fefferman [4] and using a cut off function, we can extend the data $(\phi_1^{(0)}, \phi_1^{(1)})$ on Σ_1 to the whole Σ with the following properties:

$$(\phi^{(0)}, \phi^{(1)})|_{\Sigma_1} = (\phi_1^{(0)}, \phi_1^{(1)})$$
$$(\phi^{(0)}, \phi^{(1)})|_{\{x \in \Sigma_2 | \text{dist}(x, \Sigma_1) \geq 1\}} = (0, 0)$$
$$\|\phi^{(0)}\|_{L^\infty(\{x \in \Sigma_2 | \text{dist}(x, \Sigma_1) \leq 1\})},$$
$$\|(\partial^{k-1}\phi^{(0)}, \partial^{k-2}\phi^{(1)})\|_{L^\infty(\{x \in \Sigma_2 | \text{dist}(x, \Sigma_1) \leq 1\})} \lesssim \delta^{1/4} \quad \text{for} \quad k = 2, 3, ..., n$$

where we denote by $\partial^{k-1}\phi_0$ and $\partial^{k-2}\phi_1$ the derivatives appearing in Proposition 6.1.

Therefore, according to the small data theory, we obtain a solution ϕ in Region 4. See [7]. In particular, the energy flux on $C_{u_0}^+$ induced from the solution in Region 4 are small.

We now have the data on $\underline{C}_\delta$ and $C_{u_0}^+$. They are past boundaries of Region 3. We can then solve this small data problem in Region 3. Together with the solutions constructed in other regions, this completes the construction of the whole solution.

Next, we must show that the energy of the initial data is larger than E_0, provided that δ is suitably small. Recall the definition:

$$E(\phi(t)) = \frac{1}{2} \int_{\Sigma_t} |\partial_t \phi|^2 + |\nabla_x \phi|^2 \pm \frac{2}{k+1} |\phi|^{k+1} dx$$

By Proposition 4.1 we have an L^∞ bound for ϕ, and note also that the solution on $\Sigma_{u_0+\delta}$ is compactly supported in an annular domain of size δ. So the potential energy will be very small. We must prove that the kinetic energy is large.

To do this, we use the energy identity on the domain bounded by $\underline{C}_0$, C_{u_0} and $\{t = u_0 + \delta\}$. Since by (9), the spacetime integral is clearly small (depending on δ), and the solution vanishes on $\underline{C}_0$, so the energy considered is comparable to the energy on C_{u_0}, which can be larger than E_0, if we choose ψ_0 properly. This completes the proof of the main conclusion.

7 2-D Case–A Sketch

Our method can also be used to deal with the equation in $\mathbb{R} \times \mathbb{R}^2$. We will use the equation:

$$\Box \phi = -\phi e^{\phi^2} \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R} \tag{54}$$

where

$$\Box = -\partial_{tt}^2 + \Delta_x$$

as an example, the general case can be dealt similarly. The energy associated to (54) is

$$E(\phi) = \int_{\mathbb{R}^3} (\frac{1}{2}|\partial_t \phi(t,x)|^2 + \frac{1}{2}|\nabla_x \phi(t,x)|^2 - e^{\phi(t,x)^2} + 1)dx \tag{55}$$

The equation (54) is the focusing, energy super-critical nonlinear wave equation, see [6] and [17] for a reference. There are few results about this equation in the focusing case.

Now in $\mathbb{R}^{2+1}$, we have the rectangular coordinates (t, x_1, x_2) as well as null-polar coordinates $(u, \underline{u}, \theta)$. The geometric setting and the energy identity is almost the same as in the case $\mathbb{R}^{3+1}$. Now the rotation vectorfield is

$$\Omega = \partial_\theta = x_1 \partial_2 - x_2 \partial_1$$

$S_{\underline{u},u}$ is a circle, and the energy currents associated to the deformation tensors are

$$K^\Omega = 0 \quad K^L = \frac{1}{2r}(|\nabla\!\!\!\!/\,\phi|^2 + L\phi\underline{L}\phi) \quad K^{\underline{L}} = -\frac{1}{2r}(|\nabla\!\!\!\!/\,\phi|^2 + L\phi\underline{L}\phi) \tag{56}$$

Since now we in $\mathbb{R}^2 \times \mathbb{R}$, the Sobolev inequalities will be different. Actually, we have:

Lemma 7.1 *For a smooth function f on the circle $S_{\underline{u},u}$, we have*

$$\sup_{S_{\underline{u},u}} |f| \le |u|^{1/2}(\int_{S_{\underline{u},u}} |\nabla\!\!\!\!/\,f|^2 d\mu_{g\!\!\!/})^{1/2} + |u|^{-1/2}(\int_{S_{\underline{u},u}} |f|^2 d\mu_{g\!\!\!/})^{1/2}$$

$$\int_{S_{\underline{u},u}} |f|^6 d\mu_{g\!\!\!/} \le (\int_{S_{\underline{u},u}} |f|^4 d\mu_{g\!\!\!/})\{|u| \int_{S_{\underline{u},u}} |\nabla\!\!\!\!/\,f|^2 d\mu_{g\!\!\!/} + |u|^{-1} \int_{S_{\underline{u},u}} |f|^2 d\mu_{g\!\!\!/}\}$$

This lemma can be proved by using the isoperimetric inequality on circles.

Also we have:

Lemma 7.2 *Let ϕ be a smooth function on C_u vanishing on $S_{0,u}$, then we have*

$$|u|^{1/4}\|\phi\|_{L^4(S_{\underline{u},u})} \lesssim \|L\phi\|_{L^2(C_u)}^{1/2}(\|\phi\|_{L^2(C_u)}^{1/2} + |u|^{1/2}\|\nabla\!\!\!\!/\,\phi\|_{L^2(C_u)}^{1/2})$$

$$\|\phi\|_{L^2(S_{\underline{u},u})} \lesssim \|L\phi\|_{L^2(C_u)}^{1/2}\|\phi\|_{L^2(C_u)}^{1/2}$$

Lemma 7.3 *Let ϕ be a smooth function on $\underline{C}_{\underline{u}}$, we have the following estimates*:

$$|u|^{1/4}\|\phi\|_{L^4(S_{\underline{u},u})} \lesssim |u_0|^{1/4}\|\phi\|_{L^4(S_{\underline{u},u_0})} + \|\underline{L}\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}(\|\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2} + \||u'|\nabla\!\!\!\!/\,\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2})$$

$$\|\phi\|_{L^2(S_{\underline{u},u})} \lesssim \|\phi\|_{L^2(S_{\underline{u},u_0})} + \|\underline{L}\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}\|\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}$$

The proof of Lemma 7.2 and Lemma 7.3 can be found in [21].

We construct the characteristic initial data in the same way as 3-D case. The main difference in 2-D case is that we only need the first and the second derivatives of the solution to close the bootstrap,

this is because the Sobolev inequalities involve one less derivative in 2-D case. Namely, we define the following norms which has one less derivative than [21]:

$$E_1(u, \underline{u}) = \|L\phi\|_{L^2(C_u)} + \delta^{-\frac{1}{2}}\|\Omega\phi\|_{L^2(C_u)} \tag{57}$$

$$\underline{E}_1(u, \underline{u}) = \|\Omega\phi\|_{L^2(\underline{C}_{\underline{u}})} + \delta^{-\frac{1}{2}}\|\underline{L}\phi\|_{L^2(\underline{C}_{\underline{u}})},$$

$$E_2(u, \underline{u}) = \|L\Omega\phi\|_{L^2(C_u)} + \delta^{-\frac{1}{2}}\|\Omega^2\phi\|_{L^2(C_u)},$$

$$\underline{E}_2(u, \underline{u}) = \|\Omega^2\phi\|_{L^2(\underline{C}_{\underline{u}})} + \delta^{-\frac{1}{2}}\|\underline{L}\Omega\phi\|_{L^2(\underline{C}_{\underline{u}})},$$

and also

$$F_2(u, \underline{u}) = \delta\|L^2\phi\|_{L^2(C_u)}, \tag{58}$$

$$\underline{F}_2(u, \underline{u}) = \|\underline{L}^2\phi\|_{L^2(\underline{C}_{\underline{u}})}$$

then we obtain the following result which is similar to that of 3-D case:

Theorem 7.1 *If δ is sufficiently small, for all characteristic initial data of (54) and all positive real number I which satisfy*

$$E_1(u_0, \delta) + E_2(u_0, \delta) + F_2(u_0, \delta) \leq I \tag{59}$$

there is a constant $C(I)$ depending only on I, so that

$$\sum_{i=1}^{2}[E_i(u, \underline{u}) + \underline{E}_i(u, \underline{u})] + F_2(u, \underline{u}) + \underline{F}_2(u, \underline{u}) \leq C(I) \tag{60}$$

Once we establish the above result, the following steps are exactly the same as 3-D case.

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