Geometric Brownian Motion and its application in simulation of Stock Price

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Abstract

Probability and statistics are widely applied in the modern financial market, and it contributes and prompts the study in stock market analysis. The Brownian motion particularly plays a key role in the scenario on the prediction of stock volatility. This paper is going to discuss the fundamental mathematical and statistical theories applied in the simulation of stock volatility. It involves the introduction to geometric Brownian motion, stochastic process, random walk, and some important statistical theorems. The most frequent model for modeling stock prices is geometric Brownian motion (GBM). GBM posits that random shocks accompany a continuous drift. While period returns are normally distributed under GBM, multi-period price levels (for example, ten days) are lognormally distributed. Forecasting of stock prices acts as an important challenge in nowadays stock market decision.

Keywords: Geometric Brownian Motion, Stochastic Process, Stock volatility, Random Walk.

1.INTRODUCTION

Scottish Biologist Robert Brown [11] discovered Brownian motion when he studied pollen particles floating in water through the microscope. He found out that minute particles in the pollen grains executing the jittery motion. Even though Robert Brown confirmed that this kind of motion was due to pollen being alive, the definition was given by Louis Bachelier [12] in 1900. Louis Bachelier stated and define the theory in his paper "The theory of speculation". Later then, the theory is sufficiently explained by Albert Einstein [13] with the application of a probabilistic model. It became the keystone of a fully probabilistic formulation of statistical mechanics and a well-established subject of physical investigation. He made a point of molecules moved randomly in water, if the statement regarding kinetic energy of fluids is right.
Therefore, in a very short period of time from random directions, a small particle will be affected by numerous random strength. Regarding the observation from Robert, this randomness will form a movement of a sufficiently small particle.

Brownian motion has contributed a lot in thermodynamics foundations and it is also one of the key component of statistical physics. Following the paper in 1900 published by Albert Einstein [13], he published his paper in 1905 regarding the content of Brownian motion and molecular-kinetic theory of heat. In this paper, he provides new views of microscopic dynamics and macroscopic observable phenomena. During that time, the discovery of Brownian motion inspired many scholars. Callen and Welton [14] contributed to response function and other theorem, for example, the quantum fluctuation-dissipation. In other areas such as quantum mechanics and mathematics, Brownian motion also played a key part in it. The Gauss-Markov processes with diffusion theory by Onsager and Machlup and mathematicians such as Cauchy, Khintchine are all influenced.

Brownian motion is also widely used in finance when modeling random behavior that evolves over time. One of the very important men in addressing and building the foundation of financial market is Louis Bachelier who is a French mathematician born in the Normandy. He indicated in his report that he was using nascent Brownian motion formulations to depict stock and option market price variations. The primary premise in his work is that if asset prices display an identifiable pattern in the short term, speculators would locate it, exploit it, and eradicate it.
After fifty years, Paul Samuelson proposed his paper regarding a new model of option pricing based on what Louis Bachelier found, which is included in his two publications. In *Rational Theory of Warrant Pricing* (Samuelson, 1965b), he stressed the speculation that asset prices fully reflect all available market information which provides the basis for the

later effective market assumption defined in 1965 by Fema. Furthermore, in another paper of *Proof That Properly Anticipated Prices Fluctuate Randomly* (Samuelson, 1965a), he showed that martingale is formed by speculative price fluctuations around a price reflecting the expected rate of. While it has been widely implemented in financial economics, Brownian's motion still has three points to clarify. The first one is whether price is dependent or independent. The second is whether price changes are stationary. The last one is if normal distributed price movements is easily applied for application in real markets.

2.METHODS

Assume a particle moves a unit of distance to the right with probability p, and 1−p if it moves to the left. Suppose the distance +1 if the particle moves to right and −1 if the particle moves to left.

2.1 Measure theory

We begin by introducing the measure theory, which provides the essential definition and theorems that are useful for the following Brownian Motion and Random Walk.

Definition 2.1.1

We define a probability space as a triple (Ω,F,P). Ω is a defined as a group of "outcomes", F is defined as a group of "events", and P is defined as a function which assigns probabilities to events, for F→[0,1]

Theorem 2.1.2

We assume F is a σ-field if it satisfies the following properties:

i. for a collection of subsets of Ω, if A∈ F then $A^c \in$ F

ii. if $A_i \in$ F is a countable sequence of sets then $\cup_i A_i \in F$. *Countable means finite or countably infinite.

(Ω,F) is called a measurable space which defines that it is a space on which we can put a measure, without P. A measure is a nonnegative countably additive set function: a function $\mu: F \rightarrow R$ with:

i. $\mu(A) \geq \mu(\emptyset) = 0$ *for all* $A \in$ F

ii. If $A_i \in$ F is a countable sequence of disjoint sets, then

$$\mu(\cup_i A_i) = \sum_i \mu(A_i) \qquad (1)$$

If $\mu(\Omega) = 1$, μ is the probability measure.

Remark 2.1.3

We defined S as a Semialgebra by the following(S is a collection of sets):

i. $S, T \in S$ *implies* $S \cap T \in S$

ii. If S ∈ S, then S^c is a finite disjoint union of sets in S.

A random variable is a real-valued function X defined on Ω if for every Borel set B $\subset R$ we have $X^{-1}(B) = \{\omega: X(\omega) \in B\} \in F$.

The indicator function of a set A ∈ F is :

$$1\_A\ (\omega) = \begin{cases} 1, \omega \in A \\ 0, \omega \notin A \end{cases} \qquad (2)$$

Definition 2.1.4

Suppose X is a R.V. We say X is a probability measure on R, for Borel sets A which has μ(A) = P (X ∈ A).

Remark 2.1.5

The followings are all the properties of distribution F:

i. Nondecreasing function.

ii. $lim_{x\rightarrow\infty} F(x) = 1, lim_{x\rightarrow-\infty} F(x) = 0$

iii. F is right continuous: $lim_{y \downarrow x} F(y) = F(x)$

iv. $If\, F(x-) = lim_{y \uparrow x} F(y),\ then\ F(x-) = P(X < x)$

v. $P(X = x) = F(x) - F(x-)$

Definition 2.1.6

A measurable map is a function $X: \Omega \to S\ from\ (\Omega, F)\ to\ (S, S)\ if\ X^{-1}(B) \equiv \{\omega : X(\omega) \in F\}$ A random vector is a variable X which satisfies $(S, S) = (R^d, R^d)$ *and* $d > 1$. *In this case, if* $d = 1$, *X is called a random variable, in short of r. v.*

Theorem 2.1.7

$If\{\omega: X(\omega) \in A\} \in F$ for all $A \in F$ and A generates S, then random variable X is measurable.

(S is in fact the smallest σ-field that contains A)

Theorem 2.1.8

If $X: (\Omega, F)(S, S)$ *and* $f: (S, S)\ (T, T)$ *are measurable maps*, then f(x) is a measurable map from (Ω, F) *to* (T, T).

If $X_1, \dots X_n$ *are random variables and* $f: (R^n, R^n) \to$ (R, R) is measurable, then $f(X_1, \dots, X_n)$ *is a random variable.*

If $X_1, \dots X_n$ *are random variables*, then $X_1 + \dots + X_n$ *is a random variable.*

Theorem 2.1.9

A standard normal variable density function of R.V. X is defined as:

$$p_x(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \qquad (3)$$

The complex Laplace transform of X is defined as, regarding to Lebesgue measure,:

$$E[e^{zX}] = e^{\frac{z^2}{2}}, \forall z \in C \qquad (4)$$

The characteristic function of X is (by taking z = iξ):

$$E[e^{i\xi X}] = e^{-\frac{\xi^2}{2}},\ where\ \xi \in R \qquad (5)$$

Theorem 2.1.10

Assume that Y follows the $N(m, \sigma^2) - distribution$, Y' follows the $N(m', \sigma'^2)$ *distribution, and Y and Y' are independent.* Then Y + Y' follows the $N(m + m', \sigma^2 + \sigma'^2)$ – distribution, we call this as the law of Sums of independent Gaussian variables

Definition 2.1.11

Absolutely continuous is the measure μ with respect to ν if

$$\forall A \in F, \mu(A) = 0 \qquad (6)$$

then

$$\nu(A) = 0 \qquad (7)$$

Symbolically:

$$\nu \ll \mu \qquad (8)$$

Definition 2.1.12

For ∃ A∈ F, $\mu(A) = 0$ *and* $\nu(A^c) = 0$, we say measure μ and measure ν are mutually singular with each other, denoted by:

$$\mu \perp \nu \qquad (9)$$

Definition 2.1.13

Radom-Nikodym theorem: Assume u and v be two $\sigma - finite$ measure on (Ω, F),

i. *if* $\nu \ll \mu$, then there exist a $\mu - a.s$ unique measurable function f: $\Omega \to R$, such that

$$\forall A \in F, \nu(A) = \int_A f du \qquad (10)$$

ii. There exist a $\mu - a.s$ unique measurable function f such that:

$$\sigma = \nu - \int f du,\ \sigma(A) = \nu(A) - \int_A f du \qquad (11)$$

Then $\sigma \perp \nu$

2.2 Gaussian process

The Gaussian random process is really important not only in theory basis but also in some real applications models. We first introduce random variables and vectors. Then, we move our discussion to spaces and we evaluate some fundamental properties such as independence and conditioning in the Gaussian process.

Definition 2.2.1

A collection of random variables $\{X_t\}, t\in I$ is said to be jointly Gaussian if for

$$\forall t_1, t_2, \ldots, t_n \in I,\ \forall a_1, a_2, \ldots, a_n \in R,\ \sum_{i=1}^{n} a_i X_{ti} \textit{ is a Gaussian variable}$$

Remark 2.2.2: If $E(X_t) = 0, for\ \forall t$, It is said to be centered. For any random vector $(X_{t1}, \ldots, X_{tn})$, it is a normal distribution on R^n

Definition 2.2.3

The characteristic function of random variable X is $\Phi(t) = Ee^{itX} = Ecos(tX) + isin(tX)$.

Proposition 2.2.4: The characteristic function is uniquely determined by its distribution law. In other words, if $\Phi_X(\theta) = \Phi_Y(\theta), for\ \forall\theta,$ then X shares the same distribution with Y.

Remark 2.2.5: For non-negative random variable, its generating moment function is uniquely determined.

Definition 2.2.6

Moment Generating Function: Suppose a random variable X, its MGF(moment generating function) is $M(t) = E(e^{tX})$.

Theorem 2.2.7

Convergence in law: A sequence of random variables X_n converges in distribution to the random variable X, denoted$X_n \xrightarrow{d} X$, if $F_n \to F$for all points at which F is continuous.

Remark2.2.8: There exist $X_n \to X \Leftrightarrow \Phi_{Xn}(\theta) \to \Phi_X(\theta), \forall\theta \in R$

Definition 2.2.9

The characteristic function for random vector, $X = (X_1, \ldots, X_n), X_i: (\Omega, F, P) \to (R, B(R), \mu$, is $\Phi_X(\theta_1, \ldots, \theta_n) = E(e^{i\sum_{i=1}^{n}\theta_i x_i})$.

Definition 2.2.10

Gaussian variable: A standard Gaussian variable of random variable X has density function as followed:

$$p_x(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \quad (12)$$

Gaussian vector: Assume *E* is a *d*-dimensional Euclidean space and *(u, v)* is the inner product in the space; A Gaussian vector is defined as for every $u\in E, (u, X)$ *is a real gaussian variable*, if random variable X in E.

Theorem 2.2.11

Let $\{X_i\}$ be a collection of random variables, for $X_t \sim N(at, \sigma_t^2)$, $\{X_i\}$ are independently with each other, we have

$$\sum_{i=1}^{j} b_i X_i \sim N\left(\sum_{i=1}^{j} b_i a_i,\ \sum_{i=1}^{j} b_i^2 \sigma_i^2\right), for\ \forall b_i \in R. \quad (13)$$

2.3 Brownian Motion

We then move the discussion to next part: The Brownian Motion. It is one of the most important part in the content of Stock market. In this paper, we particularly construct the Geometric Brownian Motion within the definitions investigate some of its properties.

Definition 2.3.1

Pre-Brownian motion is the random process $(B_t), t\in R+$ defined by $B_t = G(1_{[0,t]})$. G is the white noise. Pre-Brownian motion is a Gaussian process s.t

$$K(s, t) = \min\{s, t\} = S \wedge t. \quad (14)$$

Theorem 2.3.2

The properties of the (real-valued) random process $(X_t), t \geq 0$ are as followed:

i. (X_t) is a pre-Brownian motion, for $t \geq 0$;

ii. (X_t) is a centered Gaussian process, for $t \geq 0$;

iii. $X_0 = 0$ a.s., and, for every $0 \leq s \leq t$, the random variable $X_t - X_s$ is independent of $\sigma(X_r, r \leq s)$;

iv. $X_0 = 0$, and for $0 = t_0 < t_1 < \dots < t_p$, the variables $X_{t_i} - X_{t_{i-1}}$, $1 \leq i \leq p$ are said to be independent and the variable $X_{t_i} - X_{t_{i-1}}$ is distributed according to N(0, $t_i - t_{i-1}$), assuming every , $1 \leq i \leq p$.

Theorem 2.3.3

Let (B_t), $t \geq 0$ be a pre-Brownian motion. Then, we define $0 < t_0 < t_1 < \dots < t_n$, the law of the vector $(B_{t1}, B_{t2}, \dots, B_{tn})$ has density

$$p(X_1, \dots, X_n) = \frac{1}{(2\pi)^{n/2}\sqrt{t_1(t_2-t_1)\cdots(t_n-t_{n-1})}} \exp\left(-\sum_{i=1}^{n} \frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right) \quad (15)$$

where by convention $x_0 = 0$.

Theorem 2.3.4

Assume B is a pre-Brownian motion. It has the following property:

i. Based on the symmetry property, -B is a pre-Brownian motion;

ii. $B_t^{\lambda} = \frac{1}{\lambda} B_{t\lambda^2}$ is a pre-Brownian motion for $\forall \lambda > 0$

iii. $B_t^{(s)} = B_{s+t} - B_s$ is a pre-Brownian motion.

iv. For $\forall s \geq 0$, independent of $\sigma(B_r, r \leq s)$.

Definition 2.3.5

Let (X_t), $t \in T$ be a random process with values in E. The mappings $T \in t \rightarrow X_t(\omega)$ obtained when fixed $\omega \in \Omega$. A collection of mappings from T into E indexed by $\omega \in \Omega$ is formed by the sample paths of X, is called the sample paths of X.

Definition 2.3.6

Assume there are two random process (X_t), $t \in T$ and (x), $t \in T$. We could define X is a modification of x if it satifies

$$\forall t \in T, P(Xt = X_t) = 1 \quad (16)$$

Definition 2.3.7

Indistinguishable is called and defined as if there exists a negligible subset N of Ω such that

$$\forall \omega \in \Omega \setminus N, \forall t \in T, X_t(\omega) = X_t(\omega) \quad (17)$$

Definition 2.3.8

Kolmogorov's lemma: Suppose $X = (X_t)$, $t \in I$ as a random process which is indexed by a bounded interval I of R, and taking values in E, D. Suppose there exists reals q, ε, C>0 such that, for each s, t $\in I$,

$$E[d(X_s, X_t)^q] \leq C|t-s|^{1+\varepsilon} \quad (18)$$

X has sample paths which are Holder continuous with exponent $\alpha \in (0, \frac{\omega}{q})$. Then, there exists a modification X of X. So, for every $\omega \in \Omega$ and every $\alpha \in (0, \frac{\omega}{q})$, there is a finite constant $C_{\alpha}(\omega)$ s.t, for every s,t $\in I$,

$$d(X_s(\omega), X_t(\omega)) \leq C_{\alpha}(\omega)|t-s|^{\alpha}. \quad (19)$$

Theorem 2.3.9

Let $B = (B_t)$, $t \geq 0$ be a pre-Brownian motion. We say that the process B has a modification. The modification has sample paths which are continuous, and even locally Holder continuous with exponent $\frac{1}{2} - \delta$ for every $\delta \in (0, \frac{1}{2})$.

Theorem 2.3.10

A process (B_t), $t \geq 0$ is a Brownian motion if:

i. (B_t), $t \geq 0$ is a pre-Brownian motion.

ii. All sample paths of B are continuous.

Theorem 2.3.11
Wiener measure is the image of the probability measure $P(d\omega)$ under this mapping, denoted by $W(dw)$ which is a probability measure on $C(R_+, R)$ and $W(A) = P(B. \in A)$ for every measurable subset A of $C(R_+, R)$.
Definition 2.3.12
Blumenthal's zero-one law defines that the $\sigma - field\ F_{0+}$ is trivial, in the sense that P(A) = 0 or 1 for every $A \in F_{0+}$.
Theorem 2.3.13
For every $a \in R$, let $T_a = \inf\{t \geq 0: B_t = a\}$, then $a.s., \forall a \in R, T_a < \infty$.
It leads to $\lim_{t \to \infty} \sup B_t = +\infty, \lim_{t \to \infty} \inf B_t = -\infty$.

2.4 Important theorems
Theorem 2.4.1
The Brownian Motion is Gaussian process, as proved by followed:
For a sequence of time $0 \leq t_1 \leq t_2 \leq ... \leq t_n = t$, $\{B(t_1) = x_1, ..., B(t_n) = x_n\}$ can be rewritten in terms of independent increment events: $\{B(t_1) = x_1, B(t_2) - B(t_1) = x_2 - x_1 .., B(t_n) - B(t_{n-1}) = x_n - x_{n-1}\}$.
Thus $f(x_1, ..., x_n) = f_{t_1}(x_1) ... f_{t_n - t_{n-1}}(x_n)$, where f is a probability density function.
According to the Uniqueness Theorem of moment-generating function, there is

$$f(x_1, ..., x_n) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad (20)$$

which is the density for a $N(0, t)$ distribution.
Definition 2.4.2
A stochastic process S(t) is a Geometric Brownian Motion (GBM) if

$$S(t) = S(0) e^{\sigma B(t) + (\mu - \frac{\sigma^2}{2})t} \quad (21)$$

where μ is a constant, σ >0 is a constant.
Remark 2.4.3 According to Ito's formula, the differential form of Geometric Brownian Motion is

$$dS(t) = \mu S(t) dt + \sigma S(t) dB(t). \ (22)$$

Integral form is

$$S(t) = S(0) + \int_0^t \mu S(u) du + \int_0^t \sigma S(u) dB(u). \ (23)$$

Theorem 2.4.4
Volatility of Geometric Brownian Motion is defined as: Let $0 \leq T_1 \leq T_2$; $\{t_0, .., t_n\}$ is a partition of $[T_1, T_2]$, there is

$$\frac{1}{T_2 - T_1} \sum_0^{n-1} [S(t_{k+1}) - S(t_k)]^2 \approx \sigma^2 S^2(T_1) \cdot \quad (24)$$

Let $0 \leq T_1 \leq T_2$; $\{t_0, .., t_n\}$ is a partition of the interval $[T_1, T_2]$.
Define that $Y_i \equiv \frac{S(t_i)}{S(t_{i-1})} = \exp[\sigma(B(t_i) - B(t_{i-1})) + (\mu - \frac{\sigma^2}{2})(t_i - t_{i-1})]$. (25)
Therefore, $S(T_2) = S(T_1) \prod_1^n Y_i$. Particularly, set $T_1 = 0, T_2 \equiv T$, there is

$$S(T) = S(0) \prod_1^n Y_i \cdot \quad (26)$$

Theorem 2.4.5
Geometric Brownian Motion is a Markov Process.
Proof. This conclusion can be directly deduced from the theorem that the Brownian Motion is a Markov Process since Geometric Brownian Motion is a function of Brownian Motion.

Let $X(t) \equiv \sigma B(t) + (\mu - \frac{\sigma^2}{2})t. \quad (27)$

$$\begin{aligned}&S(t+\triangle)\\&=S(0)\exp(X(t+\triangle))\\&=S(0)\exp(X(t))\exp(X(t+\triangle)-X(t))\\&=S(t)\exp(X(t+\triangle)-X(t)).\end{aligned}$$

Thus $S(t+\triangle)$ (given S(t)) is related only to a future increment of Brownian Motion.

The binomial lattice model is a modified version of a series of Bernoulli trials. This research is going to apply the Binomial Lattice Model to approximate the Geometric-Brownian Motion. Consider a sequence of independent and identically distributed random variables $\{X_n : n \geq 1\}$, each assuming only two values. Two changes have been made: first, it allows the two possible values of X_n to be general $P(X_n = u) = p = 1 - P(X_n = d), n \geq 1$.

Second, this research considers a multiplicative model instead of an additive one. Let $S_n = S_0 * X_1 * X_2 ... * X_n, n \geq 1$.

There S_n intends to represent the stock price at stage n, and X_n is the changing factor $S_n = S_{n-1} X_n, n \geq 1$.

The stochastic process $\{S_n : n \geq 0\}$ is still a Markov chain, namely the probability of future states conditional on the past and present depends only on the present state. The possible values of S_n are determined by binomial (n.p) probabilities

$$P(S_n = S_0 u^k d^{n-k}) = C_n^k p^k (1-p)^{n-k}, 0 \leq k \leq n. (28)$$

It considers a financial market consisting of a bond $B_{\Delta t} = e^{r\Delta t}, B_t = B(t)$, a stock $S_t = S(t)$, and a call-option $C_t = C(t)$, where the trade is only possible at the time $t = 0$ and $t = \Delta t$. The following assumptions have been made. There is a fixed interest rate $r > 0$ on the bond with initial value $B_0 = 1$. Taking proportional yield into account, at $t = \Delta t$ there holds $B_{\Delta t} = e^{r\Delta t}$. There are two possibilities for the price $S_{\Delta t}$ of the stock with an initial value $S = S_0$ at time $t = \Delta t$:

$S_{\Delta t} = uS$ (+) with probability P(+)=q, $q \in [0,1]$, or

$S_{\Delta t} = dS$ (-) with probability P(-)=1-q, where $u > d > 0$.

The price of the call option is K and the maturity date is T.

There is no arbitrage and short sellings are allowed. There are no transaction costs and no dividends on the stocks. Consider a bond B_t with a risk-free interest rate $r > 0$ and proportional yield. Then, it follows that $B_t = B_0 e^{rt}$. Equivalently, it can be rewritten as

$$\ln(B_t) = \ln(B_0) + rt. \quad (29)$$

Taking into account the uncertainty of the stock market, for the value S_t of the stock it assumes $\ln(S_t) = \ln(S_0) + bt +$ 'uncertainty'.

As for the uncertainty concerned, assuming that it has expectation 0 and is normal distributed: Let the random variable of uncertainty be U, then $U \sim N(0, \sigma^2 t)$,(that is $Var(\sigma W_t) = \sigma^2 t$). Thus there is

$$\ln(S_t) = \ln(S_0) + bt + \sigma W_t. \ (30)$$

Setting $\mu := b + \frac{\sigma^2}{2}$, it can be deduced that

$$S_t = S_0 e^{\mu t + \sigma W_t - \frac{\sigma^2 t}{2}}. \ (31)$$

It follows that S_t is a geometric Brownian motion. Note that S_t is log-normally distributed.

For the geometric Brownian motion S_t, it follows that

$$E(S_t) = S_0 e^{\mu t} \ (32)$$

$$Var(S_t) = S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1). \ (33)$$

Proof.

Since W_t is N(0,t)-distributed, there is

$$E(e^{\sigma W_t})$$
$$= \frac{1}{\sqrt{2\pi t}} \int_R e^{\sigma x - \frac{x^2}{2t}} dx$$
$$= \frac{1}{\sqrt{2\pi t}} e^{\frac{\sigma^2 t}{2}} \int_R e^{-\frac{(x-\sigma t)^2}{2t}} dx$$
$$= e^{\frac{\sigma^2 t}{2}}$$

Moreover, we obtain

$$Var(S_t)$$
$$= E(S_t^2) - E(S_t)^2$$
$$= S_0^2 e^{(2\mu - \sigma^2)t} E(e^{2\sigma W_t}) - S_0^2 e^{2\mu t}$$
$$= S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$$

3. Conclusion

Geometric Brownian motion applies a selected method to model the analysis of the stock price, with the relation to the volatility. In the above discussion, this paper provides the fundamental explanation of the important theorems utilized in the analysis. Numerous scholars are looking towards developing statistical models that can accurately predict future stock prices. In this paper,the fundamentals of the stochastic process are described, as well as a basic definition of geometric Brownian motion. The philosophy behind using geometric Brownian motion to represent stock prices is also explored. My future work will focus on GARMA model, which is a time series model with special properties. The long memory model has a high correlation between observations that are far apart in time. By further studying in such model, I will compare each model and research on more optimized analysis model.

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