A Sequential Importance Sampling for Estimating the Multi-Period Market Risk

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Objective

- We propose a sequential importance sampling for estimating the multi-period market risk.
- We compare the performance of proposed method to that of the crude Monte Carlo simulation.

Backgrounds

Definitions & Notations

- Let P_t, $t = 1, 2, \ldots$, be the price of a portfolio at the end of the t-th time period.
- Let $R_t = \log(P_t/P_{t-1})$ be the single period log return and $\Phi_t = \{R_t, R_{t-1}, \ldots, R_1\}$.
- The log return of the portfolio over k time periods is represented as $R_t(k) = \sum_{i=0}^{k-1} R_{t+i}$.
- The value-at-risk of the portfolio over k periods with confidence level q is defined as

$$\mathrm{VaR}_t^q(k) = \text{the q-quantile of } -R_t(k) \text{ given } \Phi_{t-1}. \tag{1}$$

- The expected shortfall of the portfolio over k periods with confidence level q is defined as

$$\mathrm{ES}_t^q(k) = -E[R_t(k)|R_t(k) \le -\mathrm{VaR}_t^q(k)]. \tag{2}$$

GJR-GARCH Model

- Let $\sigma_t^2 = V[R_t|\Phi_{t-1}]$ be the conditional volatility of R_t.
- In GJR-GARCH(p,q) model,

$$\sigma_t^2 = \omega + \sum_{i=1}^{q}(\alpha_i + \gamma_i I_{t-i})R_{t-i}^2 + \sum_{i=1}^{p}\beta_i\sigma_{t-i}^2, \tag{3}$$
$$R_t = \sigma_t z_t,\ z_t \overset{i.i.d.}{\sim} \pi(z),$$

where $\pi(z)$ is a pdf with mean 0 and variance 1, and I_t is the indicator function such that

$$I_t = \begin{cases} 1, & \text{if } R_t < 0, \\ 0, & \text{if } R_t \ge 0. \end{cases}$$

- $\theta = (\omega, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$ is estimated by fitting model (3) to Φ_t; denote it by $\hat\theta_t$.

Crude Monte Carlo simulation

- We define $H_t = \{(\sigma_t^2, R_t), (\sigma_{t-1}^2, R_{t-1}), \ldots\}$, and rewrite equation (3) as

$$\sigma_t^2 = \psi(H_{t-1}; \theta), \quad t = 1, 2, \ldots. \tag{4}$$

- Let $\tilde\pi_{t-1}(z)$ be the approximate distribution of $\pi(z)$ given Φ_{t-1}.
- Let $\{\tilde Z_t, \tilde Z_{t-1}, \ldots, \tilde Z_{t+k-1}\}$ be random samlpes from $\tilde\pi_{t-1}(z)$: for $i = 0, 1, \ldots, k-1$, we generate

$$\begin{aligned} \tilde\sigma_{t+i}^2 &= \psi(\tilde H_{t+i-1}; \hat\theta_{t-1}), \\ \tilde R_{t+i} &= \tilde\sigma_{t+i}\tilde Z_{t+i}, \end{aligned} \tag{5}$$

- A simulated log return over k periods from period t is as follows:

$$\tilde R_t(k) = \sum_{i=0}^{k-1}\tilde R_{t+i}. \tag{6}$$

- Repeating the above procedure N times independently, we obtain $\tilde R_t^{(1)}(k), \ldots, \tilde R_t^{(N)}(k)$.
- $\widehat{\mathrm{VaR}}_t^q(k)$ is estimated as the $100q$-th percentile of the negated log returns over k days, i.e.

$$\widehat{\mathrm{VaR}}_t^q(k) = \mathrm{Percentile}(\{-\tilde R_t^{(1)}(k), \ldots, -\tilde R_t^{(N)}(k)\}, 100q), \tag{7}$$

and that

$$\widehat{\mathrm{ES}}_t^q(k) = -\frac{\sum_{j=1}^N I(\tilde R_t^{(j)}(k) \le \widehat{\mathrm{VaR}}_t^q(k))\tilde R_t^{(j)}(k)}{\sum_{j=1}^N I(\tilde R_t^{(j)}(k) \le \widehat{\mathrm{VaR}}_t^q(k))}. \tag{8}$$

Proposed scheme

Sequential Importance Sampling

- Suppose that residuals $\{\hat Z_{t-i} = R_{t-i}/\hat\sigma_{t-i}, i = 1, 2, \ldots, m\}$ are obtained at the end of period $t-1$ for a sufficiently large m.
- We trim the residuals : if the absolute value of a residual is larger than 4, then it is replaced by 4 or -4.
- We approximate $\pi(z)$ by $\pi_{t-1}(z) = \frac{1}{m}\sum_{j=1}^m \phi_\delta(z - \hat Z_{t-j})$, where $\phi_\delta(z)$ is the pdf of $N(0, \delta^2)$.
- We define the importance sampling pdf of $\tilde Z_{t+i}$ for $i = 0, \ldots, k-1$ as follows: for $\lambda \in (-\infty, \infty)$,

$$g_{t-1}(z; \lambda) \propto \exp\{\lambda z\}\pi_{t-1}(z), \quad -\infty < z < \infty. \tag{9}$$

- We generate N processes of $\{\tilde Z_t^{(j)}, \tilde Z_{t-1}^{(j)}, \ldots, \tilde Z_{t+k-1}^{(j)}\}$ from $g_{t-1}(z;\lambda)$, $j = 1, \ldots, N$.
- Applying equation (5), we obtain $\tilde R_{t:(t+k-1)}^{(j)}$, the log return process corresponding to $\tilde Z_{t:(t+k-1)}^{(j)}$.
- The unnormalized likelihood ratio of $\tilde R_{t:(t+k-1)}^{(j)}$ with respect to $\pi_{t-1}(z)$ is given by

$$w^{(j)} = \exp\left\{-\lambda\sum_{i=0}^{k-1}\tilde Z_{t+i}^{(j)}\right\}, \quad j = 1, \ldots, N. \tag{10}$$

- Let $\tilde R_t^{(j)}(k)$ be the k-period log return corresponding to $\tilde R_{t:(t+k-1)}^{(j)}$. If we let $W^{(j)} = w^{(j)}/\sum_{j=1}^N w^{(j)}$, then we have that

$$\widehat{\Pr}_{\pi_{t-1}}\{\tilde R_t(k) \le x\} = \sum_{j=1}^N I(\tilde R_t^{(j)}(k) \le x)W^{(j)} = \sum_{j=1}^N I(r_j \le x)W_j, \tag{11}$$

where $\{r_1, \ldots, r_k\}$ be the order statistic of $\{\tilde R_{t:(t+k-1)}^{(1)}, \ldots, \tilde R_{t:(t+k-1)}^{(N)}\}$ and W_j be the likelihood ratio corresponding to r_j.

- We define $j^* = \max\{J : \sum_{j=1}^J W_j \le 1-q\}$. It follows from equation (11) that

$$\widehat{\mathrm{VaR}}_t^q(k) = -\frac{r_{j^*} + r_{j^*+1}}{2}, \ \widehat{\mathrm{ES}}_t^q(k) = -\frac{\sum_{i=1}^{j^*} r_j W_j}{\sum_{i=1}^{j^*} W_j}. \tag{12}$$

The optimal twisting parameter

- Given H_{t-1} and θ, $R_{t:(t+k-1)}$ is determined by $Z_{t:(t+k-1)}$. i.e. for a function $r(\mathbf{z})$, $\mathbf{z} \in \mathbb{R}^k$,

$$R_t(k) = r(Z_{t:(t+k-1)}; H_{t-1}, \theta). \tag{13}$$

- The joint pdf of $Z_{t:(t+k-1)}$ is denoted by $\pi(z) = \prod_{i=1}^k \pi(z_i)$ for $\mathbf{z} \in \mathbb{R}^k$.
- Since $r(Z_{t:(t+k-1)}; H_{t-1}, \theta)$ is less than $-\mathrm{VaR}_t^q(k)$ with probability p, equation (2) says that

$$\mathrm{ES}_t^q(k) = E_\pi\left[\tau_t(Z_{t:(t+k-1)})\right], \tag{14}$$

where

$$\tau_t(z) = \frac{-r(z; H_{t-1}, \theta)I(r(\mathbf{z}; H_{t-1}, \theta) \le -\mathrm{VaR}_t^q(k))}{p}, \quad \mathbf{z} \in \mathbb{R}^k.$$

- Since $\tau_t(Z_{t:(t+k-1)})$ is nonnegative almost surely, the optimal importance sampling pdf of $Z_{t:(t+k-1)}$ for the estimation of the k period expected shortfall is as follows :

$$g^*(z; H_{t-1}, \theta, \mathrm{VaR}_t^q(k)) = \frac{\tau_t(\mathbf{z})\pi(\mathbf{z})}{\mathrm{ES}_t^q(k)}, \quad \mathbf{z} \in \mathbb{R}^k. \tag{15}$$

- In the proposed sequential importance sampling, the pdf of an importance sample $\mathbf{z}$ is

$$g_{t-1}(\mathbf{z}; \lambda) = \prod_{i=1}^k g_{t-1}(z_i; \lambda), \quad \mathbf{z} \in \mathbb{R}^k. \tag{16}$$

- We want to determine the value of the twisting parameter λ so that the cross entropy of $g_{t-1}(\mathbf{z}; \lambda)$ relative to $g^*(\mathbf{z}; H_{t-1}, \theta)$ is minimized. Then, the desired value of λ is given by

$$\lambda^* = \operatorname{argmin} E_{g^*}\left[\log\frac{g^*(\mathbf{Z}; H_{t-1}, \theta, \mathrm{VaR}_t^q(k))}{g_{t-1}(\mathbf{Z}; \lambda)}\right] = \operatorname{argmax} E_\pi[\tau_t(\mathbf{Z})\log g_{t-1}(\mathbf{Z}; \lambda)]. \tag{17}$$

Finding the optimal twisting parameter

- The approximate value of λ^* can be found by the stochastic approximation.
- Let $\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(L)} \overset{i.i.d.}{\sim} \pi_{t-1}(\mathbf{z})$. Then,

$$E_\pi[\tau_t(\mathbf{Z})\log g_{t-1}(\mathbf{Z}; \lambda)] \approx \frac{1}{L}\sum_{l=1}^L \tau_t(\mathbf{Z}^{(l)})\log g_{t-1}(\mathbf{Z}^{(l)}; \lambda). \tag{18}$$

- For $R^{(L)} = r(z^{(l)}; H_{t-1}, \hat\theta_{t-1})$, we let $\mathcal{S} = \{l : R^{(l)} \le -\mathrm{VaR}_t^q(k))\}$. The above equation is rewritten as

$$E_\pi[\tau_t(\mathbf{Z})\log g_{t-1}(\mathbf{Z}; \lambda)] \approx -\frac{1}{pL}\sum_{l \in \mathcal{S}} R^{(l)}\log g_{t-1}(\mathbf{Z}^{(l)}; \lambda). \tag{19}$$

- Maximizing the above equation gives the pseudo-optimal vaule of λ^*.

Numerical Results

Simulation for 10 days 99% VaR

- The data contain the S&P 500 Index for the 13365 days from January 4, 1971 to December 29, 2023 with a window size of 750 days.
- At each 10th days we simulate the stochastic process of daily log returns over next 10 days using the fitted GJR-GARCH(1,1) model.
- In crude Monte Carlo (CMC), $\pi(t)$ is approximated by the standardized Student t:

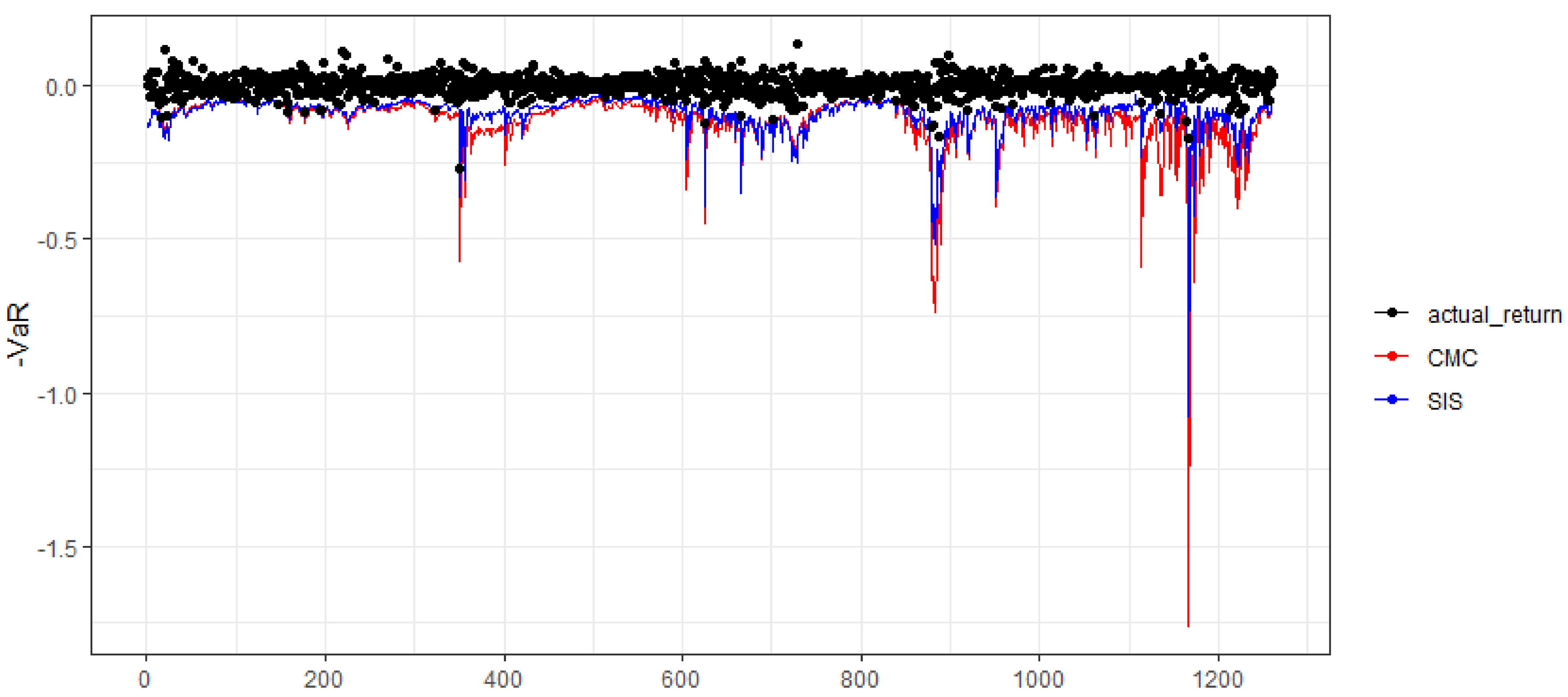
$$R_t = \sigma_t Z_t, \quad Z_t \overset{i.i.d.}{\sim} \text{Standardized Student t}, \tag{20}$$

- We use parallel processing in CMC and SIS.
- To estimate the standard error, we use a naive method.

Simulation Results

- The result for the described simulation is given in the following table and figure.

Risk measure	Method	δ	confidence level	standard error	relative error	Time(sec.)	excessive loss
VaR	CMC	0.25	0.99	0.00329	0.02574	613.8214	9
	SIS			0.00262	0.02842	476.9906	13
ES	CMC	0.25	0.99	0.00685	0.03518	616.3564	
	SIS			0.00287	0.02319	486.1364	



Conclusion

- Our proposed estimator takes much less time than the CMC and shows the standard errors lower than the CMC.
- Our proposed method is more efficient than CMC.