Linear Independence Visualization

Figure 1: Linear Independence Visualization

Linear Algebra Day 3 — Basis and Dimension

☐ Objectives

- Understand what a basis is in a vector space
- Learn the meaning of dimension and how it relates to the basis
- Identify linearly independent vs dependent sets
- Use visuals and examples to build intuition

1☐ Concept Summary

▪ Definition

A basis of a vector space is a set of vectors that: 1. Span the space
2. Are linearly independent

▪ Intuition

- A basis is like the coordinate frame of a space
- Every vector in the space can be expressed uniquely as a combination of basis vectors
- The dimension of a space is the number of vectors in any basis for that space

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▪ Visual: Independence vs Dependence

Below, blue and red vectors point in the same direction → linearly dependent. Blue and green vectors point in different directions → linearly independent.

2☐ Key Formulas and Rules

▪ Linear independence

A set of vectors (v_1, v_2, …, v_k) is linearly independent if:

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

only when:

$$a_1 = a_2 = \cdots = a_k = 0$$

▪ Dimension

The dimension of a vector space is:

the number of vectors in a basis of that space

Examples:

- $\mathbb{R}^2$ → dimension 2
- $\mathbb{R}^3$ → dimension 3
- A line through the origin in $\mathbb{R}^2$ → dimension 1

3☐ Worked Examples

☐ Example 1: Check if vectors form a basis

Are these a basis of $\mathbb{R}^2$?

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Check: is one a multiple of the other?

$$v_2 = 3 \cdot v_1$$

→ Yes → Linearly dependent → ☐ Not a basis

☐ Example 2: Standard basis in $\mathbb{R}^3$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

→ These are linearly independent and span $\mathbb{R}^3$
☐ They form a basis

4☐ Practice Problems

1. Do the vectors form a basis of $\mathbb{R}^2$?

$$\begin{bmatrix}1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\-1\end{bmatrix}$$

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2. Are these vectors linearly independent?

$$\begin{bmatrix}2\\1\\0\end{bmatrix}, \quad \begin{bmatrix}-1\\3\\1\end{bmatrix}, \quad \begin{bmatrix}1\\2\\3\end{bmatrix}$$

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3. What's the dimension of the span of:

$$\begin{bmatrix}1\\2\end{bmatrix}, \quad \begin{bmatrix}2\\4\end{bmatrix}$$

5☐ Metacognition Check

☐ Can I test independence with the zero vector condition?

☐ Can I explain dimension in terms of basis count?

☐ Can I visualize basis vs non-basis examples?

6☐ Real-World Applications

- Robotics: Robot movement spaces depend on basis and dimensionality

- Data Science: Dimensionality reduction = choosing a new basis
- Physics: Vectors like forces and velocities live in vector spaces with defined bases

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☐ Tomorrow's Preview

Day 4: Matrix Representation & Linear Transformations
We'll use matrices to represent how vectors move, rotate, stretch, or shrink.