Linear Independence Visualization

Figure 1: Linear Independence Visualization

Linear Algebra Day 3 — Basis and Dimension

☒ Objectives

- Understand what a **basis** is in a vector space
- Learn the meaning of **dimension** and how it relates to the basis
- Identify **linearly independent** vs **dependent** sets
- Use visuals and examples to build intuition

1☒ Concept Summary

☒ Definition

A **basis** of a vector space is a set of vectors that: 1. **Span** the space
2. Are **linearly independent**

☒ Intuition

- A basis is like the **coordinate frame** of a space
- Every vector in the space can be expressed **uniquely** as a combination of basis vectors
- The **dimension** of a space is the number of vectors in any basis for that space

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☒ Visual: Independence vs Dependence

Below, blue and red vectors point in the same direction → linearly dependent.
Blue and green vectors point in different directions → linearly independent.

2☒ Key Formulas and Rules

☒ Linear independence

A set of vectors (v_1, v_2, ⋯, v_k) is linearly independent if:

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

only when:

$$a_1 = a_2 = \cdots = a_k = 0$$

☒ Dimension

The **dimension** of a vector space is:

> the number of vectors in a basis of that space

Examples:

- ☒² → dimension 2
- ☒³ → dimension 3
- A line through the origin in ☒² → dimension 1

3☒ Worked Examples

☒ Example 1: Check if vectors form a basis

Are these a basis of ☒²?

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Check: is one a multiple of the other?

$$v_2 = 3 \cdot v_1$$

→ Yes → **Linearly dependent** → ☒ **Not a basis**

☒ **Example 2: Standard basis in ☒^{3}**

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

→ These are linearly independent and span ☒^{3}
☒ **They form a basis**

4☒ Practice Problems

1. Do the vectors form a basis of ☒^{2}?

$$\begin{bmatrix}1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\-1\end{bmatrix}$$

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2. Are these vectors linearly independent?

$$\begin{bmatrix}2\\1\\0\end{bmatrix}, \quad \begin{bmatrix}-1\\3\\1\end{bmatrix}, \quad \begin{bmatrix}1\\2\\3\end{bmatrix}$$

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3. What's the dimension of the span of:

$$\begin{bmatrix}1\\2\end{bmatrix}, \quad \begin{bmatrix}2\\4\end{bmatrix}$$

5☒ Metacognition Check

☐ Can I test independence with the zero vector condition?

☐ Can I explain dimension in terms of basis count?

☐ Can I visualize basis vs non-basis examples?

6☒ Real-World Applications

- **Robotics**: Robot movement spaces depend on basis and dimensionality

- **Data Science**: Dimensionality reduction = choosing a new basis
- **Physics**: Vectors like forces and velocities live in vector spaces with defined bases

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☒ Tomorrow's Preview

Day 4: Matrix Representation & Linear Transformations
We'll use matrices to represent how vectors move, rotate, stretch, or shrink.