Linear Algebra Day 3 — Basis and Dimension

🎯 Objectives

- Understand what a **basis** is in a vector space
- Learn the meaning of **dimension** and how it relates to the basis
- Identify **linearly independent** vs **dependent** sets
- Use visuals and examples to build intuition

1️⃣ Concept Summary

▪ Definition

A **basis** of a vector space is a set of vectors that: 1. **Span** the space
2. Are **linearly independent**

▪ Intuition

- A basis is like the **coordinate frame** of a space
- Every vector in the space can be expressed **uniquely** as a combination of basis vectors
- The **dimension** of a space is the number of vectors in any basis for that space

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▪ Visual: Independence vs Dependence

Below, blue and red vectors point in the same direction → linearly dependent.
Blue and green vectors point in different directions → linearly independent.

Linear Independence Visualization

2️⃣ Key Formulas and Rules

▪ Linear independence

A set of vectors ($v_1, v_2, ..., v_k$) is linearly independent if:

$$ a_1 v_1 + a_2 v_2 + \dots + a_k v_k = \begin{bmatrix} 0 \ \vdots \ 0 \end{bmatrix} $$

only when:

$$ a_1 = a_2 = \dots = a_k = 0 $$

▪ **Dimension**

The **dimension** of a vector space is:

> the number of vectors in a basis of that space

Examples:

- $\mathbb{R}^2 \to$ dimension 2
- $\mathbb{R}^3 \to$ dimension 3
- A line through the origin in $\mathbb{R}^2 \to$ dimension 1

3️⃣ Worked Examples

✅ Example 1: Check if vectors form a basis

Are these a basis of $\mathbb{R}^2$?

$$ v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\quad v_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} $$

Check: is one a multiple of the other?

$$ v_2 = 3 \cdot v_1 $$

→ Yes → **Linearly dependent** → ❌ **Not a basis**

✅ Example 2: Standard basis in ℝ³

$$ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

→ These are linearly independent and span $\mathbb{R}^3$
✅ **They form a basis**

4️⃣ Practice Problems

1. Do the vectors form a basis of $\mathbb{R}^2$?

$$ \begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} $$

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1. Are these vectors linearly independent?

$$ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},\quad \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix},\quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} $$

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1. What's the dimension of the span of:

$$ \begin{bmatrix} 1 \\ 2 \end{bmatrix},\quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} $$

5️⃣ Metacognition Check

- [ ] Can I test independence with the zero vector condition?
- [ ] Can I explain dimension in terms of basis count?
- [ ] Can I visualize basis vs non-basis examples?

6️⃣ Real-World Applications

- **Robotics**: Robot movement spaces depend on basis and dimensionality
- **Data Science**: Dimensionality reduction = choosing a new basis
- **Physics**: Vectors like forces and velocities live in vector spaces with defined bases

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📌 Tomorrow's Preview

Day 4: Matrix Representation & Linear Transformations
We'll use matrices to represent how vectors move, rotate, stretch, or shrink.