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Reconstruction of Phase Space of Dynamical Systems Using Method of Time Delay

B. Kliková and A. Raidl

Charles University in Prague, Faculty of Mathematics and Physics, Prague, Czech Republic.

Abstract. Selected elements of dynamical system (DS) theory approach to nonlinear time series analysis are introduced. Key role in this concept plays a method of time delay. The method enables us reconstruct phase space trajectory of DS without knowledge of its governing equations. Our variant is tested and compared with well-known TISEAN package for Lorenz and Hénon systems.

Introduction

There are number of methods of nonlinear time series analysis (e.g. nonlinear prediction or noise reduction) that work in a phase space (PS) of dynamical systems. We assume that a given time series of some variable is generated by a dynamical system. A specific state of the system can be represented by a point in the phase space and time evolution of the system creates a trajectory in the phase space. From this point of view we consider our time series to be a projection of trajectory of DS to one (or more – when we have more simultaneously measured variables) coordinates of phase space. This view was enabled due to formulation of embedding theorem [1], [2] at the beginning of the 1980s. It says that it is possible to reconstruct the phase space from the time series.

One of the most frequently used methods of phase space reconstruction is the method of time delay. The main task while using this method is to determine values of time delay τ and embedding dimension m. We tested individual steps of this method on simulated data generated by Lorenz and Hénon systems. We compared results computed by our own programs with outputs of program package TISEAN created by R. Hegger, H. Kantz, and T. Schreiber [3].

Method of time delay

The most frequently used method of PS reconstruction is the method of time delay. If we have a time series of a scalar variable we construct a vector

$$x(t_i), \quad i = 1, \ldots, N,$$

in phase space in time t_i as following:

$$\mathbf{X}(t_i) = \left[x(t_i), x(t_i + \tau), x(t_i + 2\tau), \ldots, x(t_i + (m-1)\tau)\right],$$

where i goes from 1 to $N - (m - 1)\tau$, τ is time delay, m is a dimension of reconstructed space (embedding dimension) and $M = N - (m - 1)\tau$ is number of points (states) in the phase space. According to embedding theorem, when this is done in a proper way, dynamics reconstructed using this formula is equivalent to the dynamics on an attractor in the origin phase space in the sense that characteristic invariants of the system are conserved. The time delay method and related aspects are described in literature, e.g. [4]. We estimated the two parameters—time delay and embedding dimension—using algorithms below.

Choosing a time delay

To determine a suitable time delay we used average mutual information (AMI), a certain generalization of autocorrelation function. Average mutual information between sets of measurements A and B is defined [5]:

$$I_{AB} = \sum_{a_i b_j} P_{AB}(a_i, b_j) \log_2 \left[\frac{P_{AB}(a_i, b_j)}{P_A(a_i) P_B(b_j)} \right]$$

where $P_A(a_i)$ is probability of occurence of a_i in set A, $P_B(b_i)$ probability of occurence of b_i in set B and $P_{AB}(a_i, b_j)$ associated probability of co-occurence of a_i in set A and b_j in B. When we use $x(t_1)$, $x(t_2)$, …,

$x(t_i)$ as set A and $x(t_1+\tau)$, $x(t_2+\tau)$, …, $x(t_i+\tau)$ as set B, then the quantity

$$I(\tau)=\sum_{x(t_i),\,x(t_i+\tau)} P(x(t_i),x(t_i+\tau))\log_2\left[\frac{P(x(t_i),x(t_i+\tau))}{P(x(t_i))P(x(t_i+\tau))}\right]$$

represents an average information content that we know about value x in time $t+\tau$ from value x in time t. According to [6] suitable value of τ is location of the first local minimum of $I(\tau)$. Decrease of AMI to $I(\tau)/I(0) = 0.2$ [7] or $I(\tau)/I(0) = 1/e$ [4] can be used in situations of monotonous decrease of $I(\tau)$.

Choosing a dimension of reconstructed phase spase

For estimating embedding dimension we used a method of false nearest neighbours (FNN), described e.g. in [4]. This method is based on idea that when the trajectory is projected to the space of too little dimension, trajectory crosses itself and so called false neighbour states occur. When dimension of the phase space reconstruction increases, number of trajectory self-crossings and false neighbours decreases. When the dimension is large enough, both should disappear completely. To determine if the neighbours are false or not, two criteria, mentioned in [4], were used. We computed a fraction $R(t_i)$, defined as [8]

$$R(t_i)=\frac{\left|x(t_i+m\tau)-x^{NN}(t_i+m\tau)\right|}{\left\|X_m(t_i)-X_m^{NN}(t_i)\right\|}.$$

As a threshold R_T we used values of $R_T = 10$ and 15, recommended in literature [5, 8]. If $R(t_i) \geq R_T$, the states are considered false neighbours. As the second criterion of falsity of the neighbours we used fraction

$$\frac{\left|x(t_i+m\tau)-x^{NN}(t_i+m\tau)\right|}{R_A}\geq A_T$$

where R_A is radius of the attractor

$$R_A^2=\frac{1}{N}\sum_{i=1}^{N}\left[x(t_i)-\bar{x}\right]^2,\quad \bar{x}=\frac{1}{N}\sum_{i=1}^{N}x(t_i).$$

We used several values of A_T around $A_T = 2$ [5].

Embedding window

We also computed a so called embedding window τ_w, defined as $\tau_w = (m - 1)\tau$ [4]. It determines a length of segment of time series needed for reconstruction of single point in m-dimensional phase space using time-delayed coordinates. For assessment of embedding window we used a formula proposed by Gibson et al. [9]

$$\tau_w\approx\sqrt{\frac{3\left\langle x^2(t_i)\right\rangle}{\left\langle \dot{x}^2(t_i)\right\rangle}}. \tag{1}$$

Correlation dimension

We estimated a correlation dimension of an attractor using correlation integral, defined as

$$C^{(m)}(r)=\frac{2}{M(M-1)}\sum_{i=1}^{M}\sum_{j=i+1}^{M}\theta\left(r-\left\|\mathbf{X}(t_i)-\mathbf{X}(t_j)\right\|\right)$$

where $\theta(x)$ is Heaviside function, M is number of points (states) in the phase space and m embedding dimension. In analysis of time series data with finite precision and length, we assumed that in certain range of finite scales $r \in \langle r_{min};r_{max}\rangle$ a relation $C^{(m)}(r) \sim r^{d2}$ is applicable [4]. We used a local slopes method to estimate a value of correlation dimension, described e.g. in [4]. Correlation dimension d_2 was determined as a value of local slope $d(r)$ of the graph of dependence of ln $C^{(m)}(r)$ on ln(r) in a section of plateau, corresponding to interval of r between some r_{min} and r_{max}.

Used data

The methods were tested on time series generated by Lorenz and Hénon systems. Lorenz system is given by equations [10]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -xz + rx - y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz.$$

Parameters σ, b and r were chosen so that the behaviour of the system is chaotic: $\sigma = 16$, $b = 4$, $r = 45.92$. The time series had N = 10,000 values, sampling step $\Delta t = 0.01$. In our computations variable x was used.

The Hénon map is a time-discrete map generated by equations [11]

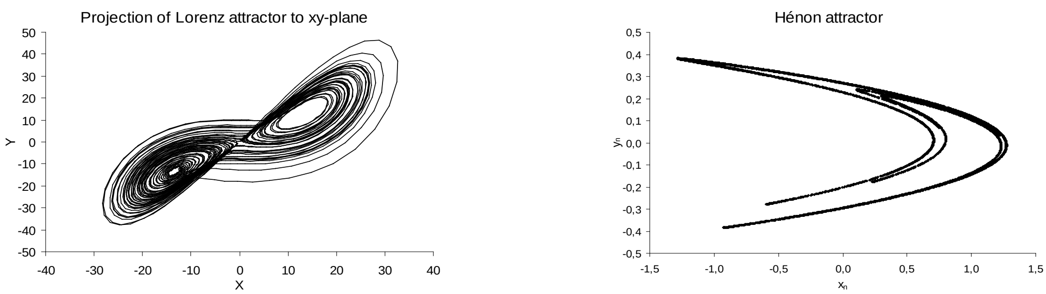
$$x_{n+1} = 1 + y_n - a x_n^2,$$
$$y_{n+1} = b x_n,$$

where $a = 1.4$ and $b = 0.3$. Time series of variable x_n of 10,000 values was used in our computations.

Results and discussion

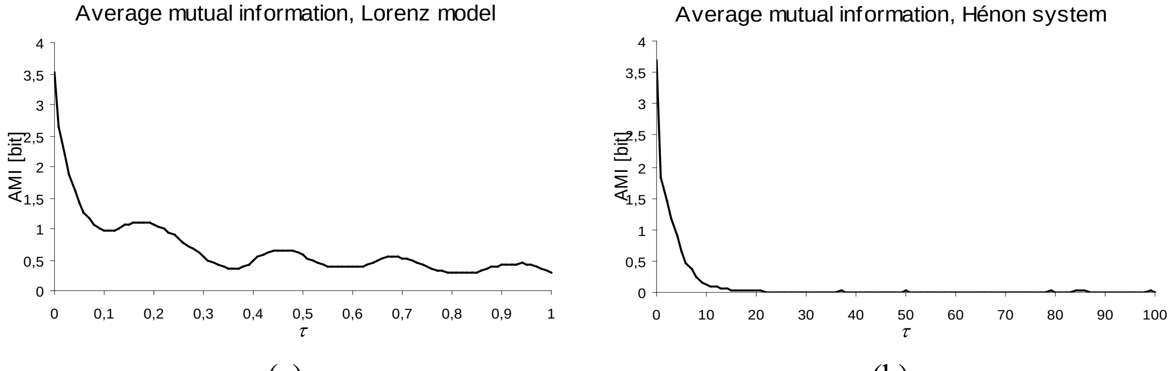
We tried to find convenient combinations of time delay and dimension of reconstructed phase space using AMI, method of FNN and by computing a relation (1). In most cases program package TISEAN gave the same or comparable results as our own programs.

Suitable time delay for Lorenz system was determined as a value where AMI had its first local minimum. It was $\tau = 0.11$. For Hénon system situation was more complicated as for Hénon system AMI is a monotonously decreasing function of τ. Value of fraction $I(\tau)/I(0) \approx 0.2$ was reached for $\tau = 4$ or 5 and $I(\tau)/I(0) \approx 1/e$ was reached for $\tau = 2$. However other computations showed that a suitable value of time delay for Hénon system for a good phase space reconstruction is $\tau = 1$. Both TISEAN and our programs gave the same results. Average mutual information for Lorenz and Hénon systems computed by our own program is in Figure 2.



(a) (b)

Figure 1. (a) Projection of Lorenz attractor to xy-plane. (b) Hénon attractor.

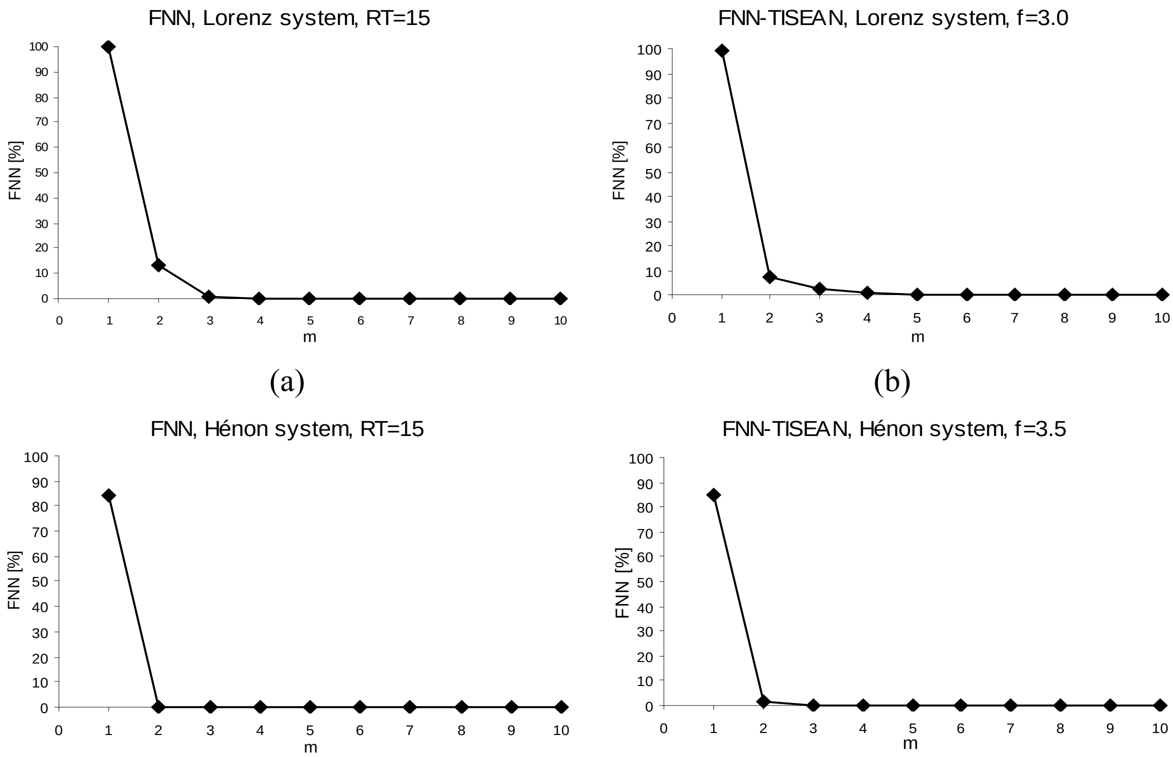


(a) (b)

Figure 2. Average mutual information for (a) Lorenz and (b) Hénon systems computed by our program.

To choose an embedding dimension we used a method of FNN. We know that the proper dimension for reconstruction of attractor of Lorenz system is 3 [4]. This result was more visible using the first criterion for value of R_T equal to both 10 and 15. For the second criterion the most suitable value of A_T was 0.8 or 0.5. TISEAN uses the first criterion but in this case it was good to choose R_T (= f) = 3. For Hénon system a suitable embedding dimension appeared to be 2. It was clearly visible for R_T =10 and 15, A_T = 0.5 and f = 3 or 3.5 for TISEAN program. Program TISEAN gave comparable results as our own program (see Figure 3). It remains a problem which values of thresholds to choose for both criteria to get proper results.

Combinations of embedding dimension and time delay that satisfied relation (1) best are in Table 1. For Lorenz system for dimension 3 the most suitable time delay seemed to be between 0.08 and 0.12. This is in agreement with what we found using AMI and it corresponds to embedding window τ_w around 0.20. For Hénon system the suitable embedding window turned out to be 1 or 2. For dimension 2 it corresponds to suitable time delays 1 and 2, for higher embedding dimensions (3 or 4) to suitable time delay 1. That is why we used value τ = 1 in other computations (method of FNN, correlation integral). However, optimal choice of the parameters has to follow purpose for which the reconstructed vector in PS will be used [3].

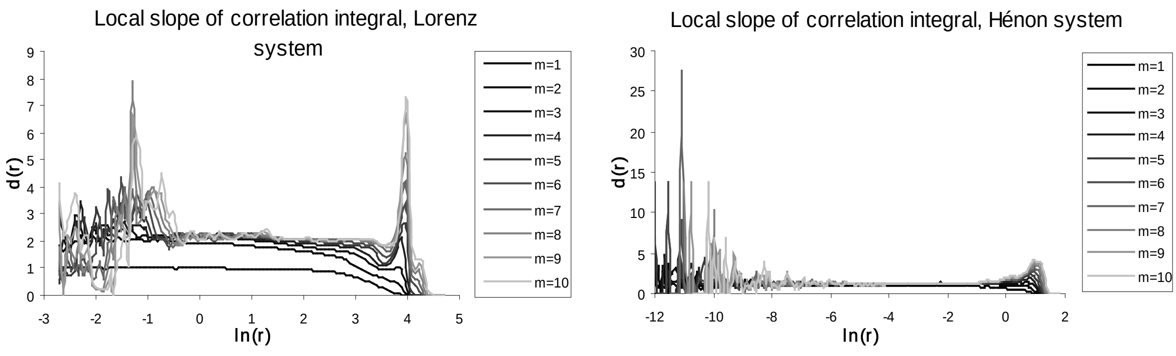


(a) (b) (c) (d)

Figure 3. Fraction of false nearest neighbours for (a, b) Lorenz and (c, d) Hénon systems computed (a, c) by our program and (b, d) by TISEAN.

Table 1. Combinations of embedding dimension and time delay that satisfy relation (1) for Lorenz and Hénon systems.

Lorenz system						Hénon system	
m	τ	m	τ	m	τ	m	τ
3	0.08	4	0.06	5	0.04	2	1
3	0.09	4	0.07	5	0.05	2	2
3	0.10	4	0.08	5	0.06	3	1
3	0.11					4	1
3	0.12						



(a) (b)

Figure 4. Local slope of ln $C^{(m)}(r)$ for (a) Lorenz and (b) Hénon systems computed by our program.

Figure 4 shows a local slope $d(r)$ of the graph of dependence of ln $C^{(m)}(r)$ on ln(r) for Lorenz and Hénon systems computed by our program. Correlation dimension came out to be slightly over 2.0 for Lorenz system. This was in agreement with the correct value of d_2 2.06 [5]. For Henon system estimate of correlation dimension was around 1.21. This is in a good agreement with values from literature, which range from 1.21 [12] to 1.27 [13]. We obtained similar results using TISEAN.

Conclusion

Programs for estimation of basic parameters needed in the frame of phase space signal analysis were developed. Results were compared with outputs provided by software package TISEAN. In most cases TISEAN and our versions gave comparable results but it remains a problem to determine suitable values of several thresholds to get proper results. Embedding window τ_w was estimated by not so commonly used generalization of nonlinear correlation "function" (proposed by Gibson et al., [9]). Some degree of independence was confirmed in embedding parameters choice for noise-free data.

In our next work it would be convenient to assess relevance of the methods for more complex systems, e.g. so called Lorenz climate system [14]. It will serve us as a clue for analysis of more complex series of real measurements. We are going to apply the methods to real atmospheric data, special attention will be devoted to aerological data sets analysis. We will try to answer the question if high atmospheric data, e.g. above 200 hPa, are describable by low- or high-dimensional dynamics.

References

[1] Packard N., Crutchfield J., Farmer D., Shaw R.: Geometry from a time series. *Phys. Rev. Lett.*, 45 (1980), p. 712.
[2] Takens F.: Detecting strange attractors in turbulence. *Lecture Notes in Math.*, 898, (1981).
[3] Hegger R., Kantz H., and Schreiber T.: Practical implementation of nonlinear time series methods: The TISEAN package, CHAOS 9, 413 (1999)
[4] Horák J., Krlín L., Raidl A.: Deterministický chaos a jeho fyzikální aplikace. *Academia* 2003.
[5] Abarbanel H. D. I.: Analysis of observed chaotic data. *Springer, New York* 1996.
[6] Fraser A. M., Swinney H. L.: Independent coordinates for strange attractors from mutual information. *Phys. Rev. A*, 33 (1986), p. 1134.
[7] Abarbanel H. D. I.: The analysis of observed chaotic data in physical systems. *Rev. Mod. Phys*., 65 (1993), p. 1331
[8] Kennel M., Brown R., Abarbanel H.: Determining embedding dimension for phase-space reconstruction using geometrical construction. *Phys. Rev. A*, 45 (1992), p. 3403.
[9] Gibson J. F., Farmer J. D., Casdagli M. C., Eubank S.: An analytical approach to practical phase spase reconstruction. *Physica D*, 57 (1992), p. 1.
[10] Lorenz E. N.: Deterministic non-periodic flow. *J. Atmos. Sci*., 20 (1963), p. 20.
[11] Hénon M.: A two-dimensional mapping with a strange attractor. *Comun. Math. Phys*., 50 (1976), p. 69.
[12] Grassberger P., Procaccia I.: Characterization of strange attractors. *Phys. Rev. Lett*., 50 (1983), p. 346.
[13] Alligood K. T., Sauer T. D., Yorke J. A.: Chaos – an introduction to dynamical systems. *Springer, New York* 1997.
[14] Lorenz E. N.: Dimension of weather and climate attractors. *Nature*, 353 (1991), p. 241.