Pergamon

Mathl. Comput. Modelling Vol. 27, No. 7, pp. 1–9, 1998

0895-7177/98 $19.00 + 0.00

PII: S0895-7177(98)00032-6

Statistical Analysis of Lyapunov Exponents from Time Series: A Jacobian Approach

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(Received April 1997; revised and accepted January 1998)

Abstract—In this paper, we provide a statistical analysis for the Lyapunov exponents estimated from time series. Through the Jacobian estimation approach, the asymptotic distributions of the estimated Lyapunov exponents of discrete-time dynamical systems are studied and characterized based on the time series. Some new results under weak conditions are obtained. The theoretical results presented in the paper are illustrated by numerical simulations.

Keywords—Central limit theorem, Dynamical system, Jacobian, Simulation, Statistical test.

1. INTRODUCTION

During the recent years there have been increasing interest and research work on chaos, particularly chaos in time series, from statistical points of view (see, for example, the special journal issue [1], survey-type articles [2,3], and books [4,5]).

Two commonly used characteristics for classifying and quantifying the chaoticity of a dynamical system are

(1) fractal dimensions [3] and
(2) Lyapunov exponents [5].

Different methods used for estimating fractal dimensions and Lyapunov exponents in a noisy environment have recently been proposed and discussed [4,6–8], among many others (see, for example, a brief summary given by the present authors in [9]). This paper is to further study the asymptotic behavior of the estimated Lyapunov exponents from time series data, by employing the Jacobian method, under some standard statistical criteria.

Consider the dynamical system

$$y_{t+1} = f(Y_t) + u_t, \tag{1}$$

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We would like to thank J. Wiorkowski and X. Wang for their helpful comments and suggestions.

where u_t is a random innovation, $Y_t := (y_t, y_{t-1}, \dots, y_{t-d+1})$ for some integer $d \geq 1$ is a data vector, f is a function (usually nonlinear) of d variables, and $t = 1, 2, \dots$. We will assume that f is continuously differentiable with respect to each of its variables.

Let

$$J_t = Df(Y_t) = \begin{pmatrix} f_1' & f_2' & \dots & f_{d-1}' & f_d' \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{2}$$

be the (companion) Jacobian matrix of f evaluated at Y_t, where f_i' denotes the partial derivative of f with respect to the i^{th} variable. Also, set

$$T_m = J_m J_{m-1}, \dots, J_1 = Df^m(Y_1), \qquad m = 1, 2, \dots, \tag{3}$$

and use $\|\cdot\|$ for the spectral (or any equivalent) norm of a matrix. Then, as is well known, the following limit

$$\lambda_1 = \lim_{m \to \infty} \frac{1}{m} \ln \|T_m\|, \tag{4}$$

if exists, is called the dominant (largest) *Lyapunov exponent* of the dynamical system. Furthermore, if we denote by $a_i(m, Y)$ the i^{th} largest eigenvalue of $Df^m(Y)$, then

$$\lambda_i(Y) = \lim_{m \to \infty} \frac{1}{m} \ln |a_i(m, Y)|, \qquad i = 1, 2, \dots, d, \tag{5}$$

are called the *Lyapunov spectrum* of the dynamical system.

It can be shown that when $i = 1$, equation (5) gives the same λ_1 as that calculated by equation (4) [10].

Under some technical conditions pertaining to the ergodic theory, the dominant Lyapunov exponent and the Lyapunov spectrum are both independent of the initial conditions almost surely with respect to the underlying invariant measure induced by f [11], where a measure ρ is said to be *invariant* induced by a measurable function f if for any measurable set B in the domain of f, we have $\rho(f^{-1}(B)) = \rho(B)$. In this paper, we restrict our attention on f being continuously differentiable.

The dominant Lyapunov exponent is the most commonly used indicator for the chaoticity of a time series, which can also be expressed as the expectation value of $\ln \|Df(Y)\|$ with respect to the invariant measure ρ in some cases.

There are two kinds of methods used for estimating Lyapunov exponents. The "direct methods", for example, proposed in [12], works as follows. Given a time series $\{y_t\}$ and a time instant t_1, we search for the next instant t_2 such that $|y_{t_1} - y_{t_2}|$ is sufficiently small. Then we record the growth of $|y_{t_1+j} - y_{t_2+j}|$ for some integer $j \geq 1$, until the trajectory diverges to beyond a given limit. We then replace (y_{t_1}, y_{t_2}) by (y_{t_1+j}, y_{t_2+j}) and repeat the same procedure. The average diverging rate over the entire time series is the estimate of the dominant Lyapunov exponent λ_1. The other kind of methods are "Jacobian methods", which are based on the state-space reconstructions of the time series. The key step in this approach is to estimate the Jacobians. If the Jacobian can be computed at different data points, then we can use them to estimate the dominant Lyapunov exponent, as well as, the Lyapunov spectrum. In this regard, several different methods for finding the Jacobian have been proposed, including for instance the linear Taylor series approximation [13], the higher-order Taylor series methods [14], and the nonparametric regression method [10,15]. The consistency in the estimation of the Lyapunov exponents using local linear regression [16] has been proved in [8], which well support the various statistical methods existing in the literature.

In this paper, we propose and characterize a Jacobian method for estimating Lyapunov exponents of dynamical systems. The proposed approach quantifies the variations of the estimated

Lyapunov exponents statistically via central limit theorem which, to our knowledge, is a new method in the existing literature. The asymptotic distribution of the estimated Lyapunov exponents facilitates statistical inferences on the estimated Lyapunov exponents. One advantage of our approach is that we can estimate the Lyapunov exponents using a relative short time series generated by the unknown dynamical systems.

This paper is organized as follows. In Section 2, we review a lemma (Lemma 2.1) from [17]. Using this lemma, we then derive some theoretical results in Section 3. Theorem 3.2, a consequence of Theorem 3.1 that is based on Lemma 2.1, is the main result in this paper. Theorem 3.1 provides asymptotic distribution of the estimated Lyapunov exponents; while Corollary 3.3, as a special case of Theorem 3.2, provides a central limit theorem for the largest Lyapunov exponent. Numerical simulations are included in Section 4, showing consistency with the theoretical results. In the numerical simulations, we use a least-squares regression method segmentally over the observed time series, so as to estimate the elements of the Jacobian matrix via a Volterra series expansion.

2. ASYMPTOTIC DISTRIBUTION OF EIGENVALUES

This paper is devoted to a study of the asymptotic distribution of the estimated Lyapunov exponents of the discrete-time dynamical system (1) using time series data. For this purpose, we first discuss a result given in [17] on the asymptotic distribution of the eigenvalues of symmetric random matrices.

Let $RS^{p\times p}$ be the family of $p\times p$ real symmetric random matrices. For any $M\in RS^{p\times p}$, let $\varphi(M)=(\varphi_1(M),\dots,\varphi_p(M))$, in which $\varphi_1(M)\geq\cdots\geq\varphi_p(M)$, be the ordered eigenvalues of M. Also, use $\mathcal{L}(X_n)\to\mathcal{L}(X)$, or $X_n\to^d X$, to mean that elements of the sequence of random matrices $\{X_n\}$ converge weakly to the corresponding elements of the matrix X.

The following result can be found in [17].

LEMMA 2.1. *Let Σ and the sequence $\{S_n\}$ be all in $RS^{p\times p}$, and set*

$$W_n=n^{1/2}(S_n-\Sigma),\qquad n=1,2,\dots.$$

Let $\sigma_1>\cdots>\sigma_k$ be the distinct eigenvalues of Σ with multiplicity p_i, $i=1,\dots,k$, where $p_1+\cdots+p_k=p$. Suppose, moreover, that

$$\mathcal{L}(W_n)\to\mathcal{L}(W),\qquad (n\to\infty)$$

for some $p\times p$ matrix W, where W and S_n are both block matrices of the form

$$W=(W_{ij}),\qquad W_{ij}\in RS^{p_i\times p_j},\qquad S_n=(S_{n,ij}),\qquad S_{n,ij}\in RS^{p_i\times p_j},$$

with $i,j=1,\dots,k$. Then we have

$$n^{1/2}(\varphi(S_n)-\varphi(\Sigma))=H\left(\widetilde{W_n}\right)+R_n,\tag{6}$$

where

$$H\left(\widetilde{W_n}\right)=n^{1/2}\begin{pmatrix}\varphi(S_{n,11}-\sigma_1I_1)\\ \varphi(S_{n,22}-\sigma_2I_2)\\ \vdots\\ \varphi(S_{n,kk}-\sigma_kI_k)\end{pmatrix}\to^d\begin{pmatrix}\varphi(W_{11})\\ \varphi(W_{22})\\ \vdots\\ \varphi(W_{kk})\end{pmatrix},$$

and $R_n=O_p(n^{-1/2})$.

In the following, we denote

$$H(\widetilde{W})=\begin{pmatrix}\varphi(W_{11})\\ \varphi(W_{22})\\ \vdots\\ \varphi(W_{kk})\end{pmatrix},$$

and study the asymptotic distribution of the estimated Lyapunov exponents.

3. ASYMPTOTIC DISTRIBUTION OF LYAPUNOV EXPONENTS

As is clear, generally we cannot calculate the Jacobians from the time series data if the function f in the dynamical system (1) is unknown. For this reason, one needs a more practical method to estimate the Jacobians. In addition, the asymptotic distribution of the Lyapunov exponents based on the estimated Jacobians is needed for making statistical inference.

Our main result concerning with the asymptotic distribution of the Lyapunov exponents using only time series data is the following.

THEOREM 3.1. *Suppose that the observed time series is generated from the dynamical system (1). Let T_m be the true but unknown Jacobian of the system and $\widehat{T}_m$ be its estimate from the data, such that*

$$\mathcal{L}\left(m^{1/2}\left(\widehat{\Sigma}_m - \Sigma\right)\right) \to \mathcal{L}(W), \qquad (m \to \infty),$$

where

$$\Sigma = \lim_{m\to\infty} \Sigma_m, \qquad \Sigma_m = \left(T_m^\top T_m\right)^{1/2m}, \qquad \widehat{\Sigma}_m = \left(\widehat{T}_m^\top \widehat{T}_m\right)^{1/2m},$$

in which we assume that Σ has only simple eigenvalues and that the elements in the diagonal of W are normally distributed. Then we have

$$\mathcal{L}\left(m^{1/2}\left(\varphi\left(\widehat{\Sigma}_m\right) - \varphi(\Sigma)\right)\right) \to \mathcal{L}\left(H\left(\widetilde{W}\right)\right), \qquad (m \to \infty),$$

where $H(\widetilde{W})$ is a $p \times 1$ vector defined above with normally distributed components.

The basic idea for a proof of this result is that if Σ has simple eigenvalues then by equation (6) in Lemma 2.1 one has

$$\mathcal{L}\left(m^{1/2}\left(\varphi\left(\widehat{\Sigma}_m\right) - \varphi(\Sigma)\right)\right) \to \mathcal{L}\left(H\left(\widetilde{W}\right)\right). \tag{7}$$

We remark that if the conditional least-squares regression is used in the estimation of the elements of J_t, then by Theorem 8.2.1 of [18], the central limit theorem holds for the estimated parameters. Hence, if the number of J_t's is large, the assumption of the $\sqrt{m}$-convergence of $\widehat{\Sigma}_m$ in distribution to a normal random variable is achievable by the conditional least-squares regression. Hence, the convergence assumption on the estimated Jacobian matrix made in the above theorem is reasonable. It is noteworthy that the convergence rate of the estimate of the J_t depends on the length of the particular segment of the time series used in estimating it elements, which may differ from $\sqrt{m}$.

We also remark that in Lemma 2.1, if the assumption on simple eigenvalues does not hold, then $\widetilde{W}$ therein will be the eigenvalues of the corresponding principal block of the matrix. Since we are generally interested in the largest eigenvalue (the dominant Lyapunov exponent), the assumption of a simple eigenvalue structure is indeed not too restrictive.

THEOREM 3.2. *Under the assumption of Theorem 3.1, we have*

$$\mathcal{L}\left(m^{1/2}\left(\widehat{\Lambda}_m - \Lambda\right)\right) \to \mathcal{L}(X), \tag{8}$$

where

$$\begin{aligned} \Lambda_m &= \log(\varphi(\Sigma_m)), \\ \Lambda &= \lim_{m\to\infty} \Lambda_m, \\ \widehat{\Lambda}_m &= \log\left(\varphi\left(\widehat{\Sigma}_m\right)\right), \end{aligned}$$

and X is a $p \times 1$ vector with normally distributed zero-mean random variables.

The significance of this result is that it establishes the central limit theorem, as well as, the $\sqrt{m}$-consistence of the estimated Lyapunov exponents by the Jacobian method described above.

To give an outline of the proof, we only point out that Theorem 3.1 yields

$$\mathcal{L}\left(m^{1/2}\left(\varphi\left(\widehat{\Sigma}_m\right)-\varphi(\Sigma)\right)\right) \to \mathcal{L}\left(H\left(\widetilde{W}\right)\right).$$

Hence, by the differentiability of the log function, we can verify that

$$\mathcal{L}\left(m^{1/2}\left(\widehat{\Lambda}_m-\Lambda\right)\right) \to \mathcal{L}(X),$$

where X is a $p \times 1$ vector, which consists of normally distributed zero-mean random variables.

A few important remarks are in order.

First, a comparison of Theorem 3.2 with Theorem 3 given in [19] is interesting. It may appear that they have kind of similarity, but the result of [19] differs from Theorem 3.2 stated above in both conditions and conclusions. Theorem 3 of [19] requires that the random matrices in the product be statistically independent, which is very difficult (if not impossible) to satisfy in the case of estimating Lyapunov exponents from time series data by employing the Jacobian method. The asymptotic distribution of the conclusion given in [19] is about the eigenvalues of the product of the J_t's, not the product of the $\widehat{J}_t's$, while the latter was discussed in this paper. It is also worth mentioning that if the assumption of $\mathcal{L}(\widehat{\Sigma}_m - \Sigma) \to \mathcal{L}(W)$ is changed to $\mathcal{L}(\Sigma_m - \Sigma) \to \mathcal{L}(W)$ in Theorem 3.2, then we obtain the same conclusion as that given in Theorem 3 of [19], yet with weaker assumptions (namely, the independence of the Jacobians J_t's is not required).

COROLLARY 3.3. *Consider only the largest Lyapunov exponent λ of the time series. Under the assumption of Theorem 3.2 and, moreover, suppose that λ_1 is unique, we have*

$$\mathcal{L}\left(m^{1/2}\left(\widehat{\lambda}_{1m}-\lambda_1\right)\right) \to \mathcal{N}\left(0,\sigma_1^2\right), \qquad (m \to \infty), \tag{9}$$

where $\widehat{\lambda}_{1m}$ is the estimate of λ_1 obtained from the Jacobian method.

We note that this is actually a special case of Theorem 3.2, since λ_1 is one of the elements in Λ and $\widehat{\lambda}_{1m}$ is the corresponding element in $\widehat{\Lambda}_m$.

4. A NUMERICAL EXAMPLE AND DISCUSSION

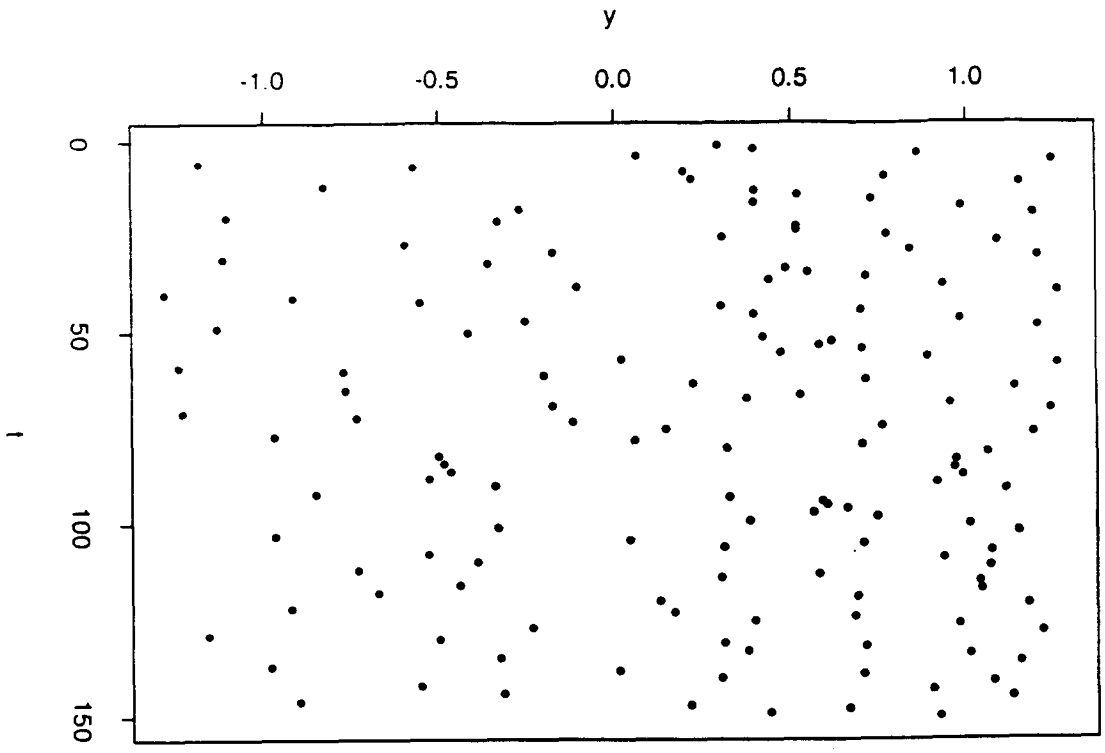
In this section, we study a numerical example which illustrates the asymptotic results presented in Section 3.

We are interested in computing the Lyapunov exponents of an observed time series obtained from an underlying smooth discrete-time dynamical system. Since any nonlinear system can be represented as a nonlinear AR Volterra series of infinite order [20], we applied the least-squares method segmentally over the observed time series in searching the proper order (q) and proper degree (p) of the nonlinear AR series. For example, in our simulation study, the observed time series is generated from the Hénon map:

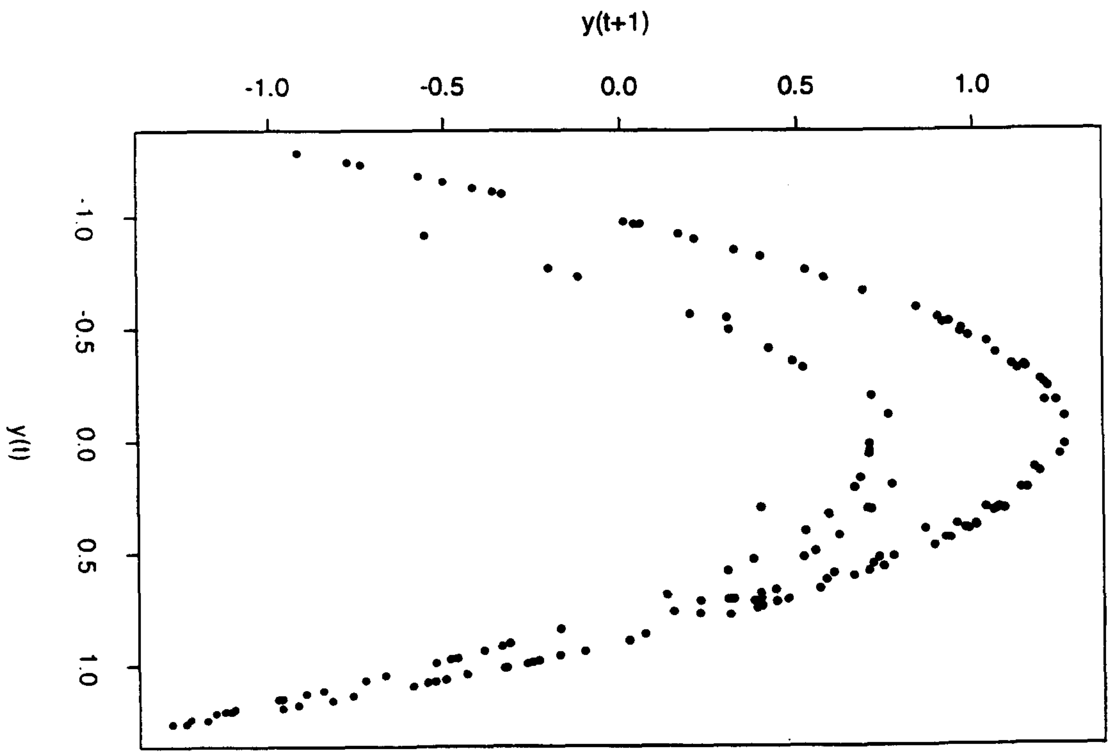
$$y_{t+1} = 1 + 0.3y_{t-1} - 1.4y_t^2, \qquad t = 1, 2, \dots, \tag{10}$$

where, compared with model (1), the stochastic term u_t degenerates to zero in the map.

To carry out a Monte Carlo simulation, we generated 200 time series from the Hénon map with length n=150 each, where the initial values for each time series were taken uniformly from



(a)



(b)

Figure 1. The time scatter plot (a) and the state space plot (b) of a typical realization of the Hénon map.

interval [0,0.5]. The time scatter plot and the state space plot of a typical realization of such time series are shown in Figure 1.

In practice, we usually do not know the underlying dynamical system that generates the time series. Hence, the least-squares method discussed above is used to identify the order q and the degree p. In our example, the Volterra series with $q = 2$ and $p = 2$ was obtained as follows:

$$y_{t+1} = \mu_0 + \mu_1 y_t + \mu_2 y_{t-1} + \mu_3 y_t^2 + \mu_4 y_{t-1}^2 + \mu_5 y_t y_{t-1}. \tag{11}$$

This Volterra series has the best fit to the observed time series generated by the Hénon map (via the ordinary least-squares method). The residual sum of squares from this regression is extremely small. In fact, the residuals were generated only by random rounded errors. For the 200 time series, in estimating the Jacobians, we used a segment of length $\ell = 50$, which resulting in one hundred ($m = 100$) estimated Jacobians J_t. For each subsequence of the generated time series, the elements in J_t were estimated from the estimated coefficients in equation (11), that is,

$$f_1' = \mu_1 + 2\mu_3 + \mu_5 y_{t-1}, \qquad f_2' = \mu_2 + 2\mu_4 + \mu_5 y_t,$$

for $t = 1, 2, \ldots, 100$. In our simulation studies, the estimates of μ_0, μ_2, and μ_3 are very close to 1, 0.3, and -1.4, respectively. In fact, the differences are less than 10^{-100} in absolute value. The estimates of other coefficients in equation (11) are less than 10^{-13} in absolute value.

The estimated $\hat{T}_m$ is thus equal to $\hat{J}_m, \ldots, \hat{J}_1$. It is noteworthy that a direct computation of $\hat{T}_m$ is numerically impossible when m is large (say $m \geq 30$). Since our goal is to estimate the Lyapunov exponents, we can compute the Lyapunov exponents without directly computing $\hat{T}_m$. The technique used here is adapted from [13] by using sequential QR decomposition of matrix $\hat{J}_{j+1}Q_j$:

$$\hat{J}_{j+1}Q_j = Q_{j+1}R_{j+1}, \qquad j = 1, 2, \ldots, m-1,$$

where $Q_0 = I$ is the identity matrix, Q_j is an orthogonal matrix, and R_j is an upper triangular matrix with positive diagonal elements. Note that

$$\hat{J}_m, \ldots, \hat{J}_1 = Q_m R_m R_{m-1}, \ldots, R_1.$$

Hence, the i^{th} estimated Lyapunov exponent (the i^{th} element of $\hat{\Lambda}_m$) is

$$\lambda_i = \frac{1}{m} \sum_{j=1}^{m} \ln R_{j(i)},$$

where $R_{j(i)}$ is the i^{th} diagonal element of R_j.

It follows from the simulation results that the estimated Lyapunov exponents have mean values $\lambda_1 = 0.4143$ and $\lambda_2 = -1.6183$, and the standard deviations are $\sigma_1 = 0.0391$ and $\sigma_2 = 0.0403$, respectively. The histogram of the Lyapunov exponents are presented in Figure 2. The histogram shows that the simulation results support the theoretical analysis given in Section 3.

Using the asymptotic distribution of the estimated Lyapunov exponents, we construct the 95% confidence intervals for the underlying Lyapunov exponents λ_1: (0.4066,0.4220) and λ_2: $(-1.6262, -1.6104)$. The confidence intervals are consistent with the Lyapunov exponents 0.42 and -1.62 calculated from other studies [21]. The statistical properties can then lead to performing further statistical inferences about the estimates of the Lyapunov exponents.

In the Monte Carlo simulation study of the distribution of the estimated Lyapunov exponents, we need to know the underlying dynamic system, which is usually impractical. Hence, the bootstrap method can be used to assess the means and standard deviations of the estimated Lyapunov exponents, and then the confidence intervals for the Lyapunov exponents can be constructed from these estimates [9].

The theoretical properties and numerical methods studied in this paper are also valid for time series observed from dynamical systems with measurement errors. This topic is still under investigation and will be reported elsewhere.

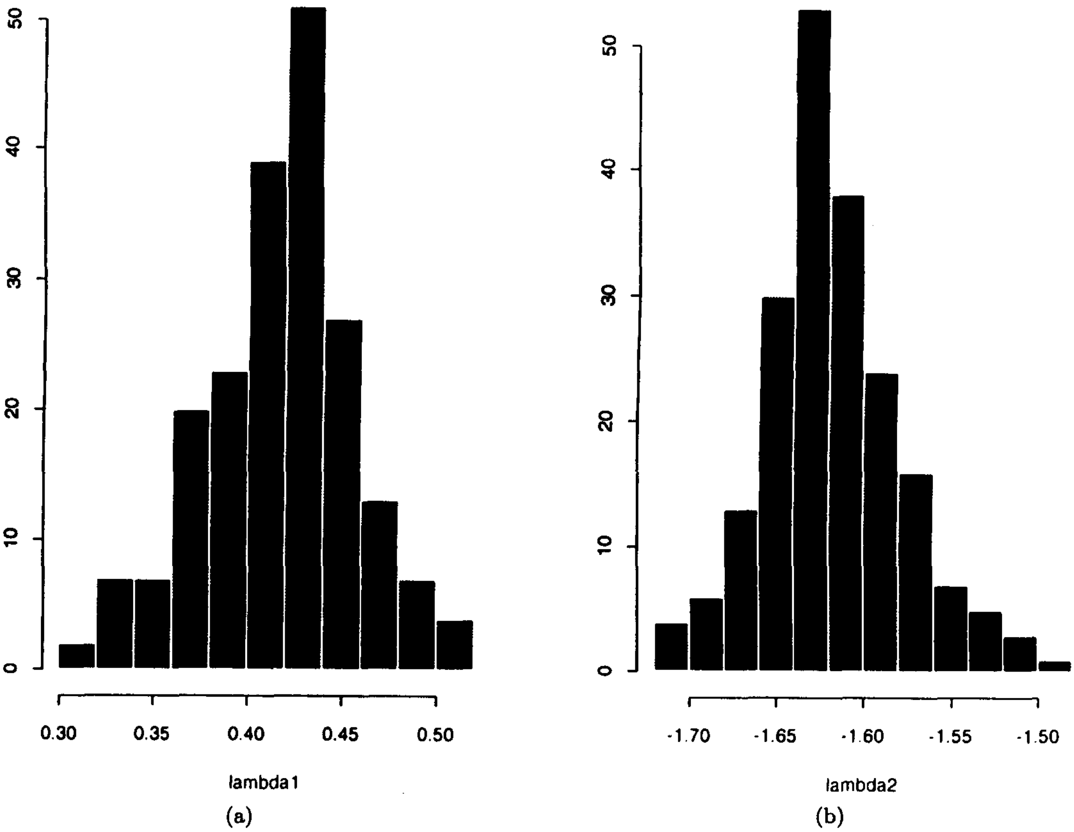


Figure 2. The histograms of the estimated first Lyapunov exponent (a) and the second Lyapunov exponent (b) of the simulated Hénon map.

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