Discrete Mathematics Final Exam (Spring 2021)

Scores: Highest 10 (27) means 27 students got the highest score 10.

	P1	P2	P3	P4	P5	P6	P7	P8	Total
Average	5.9	2.6	3.6	12.5	10.3	5.1	7.8	3.4	46.6
Highest	10 (27)	10 (12)	10 (20)	15 (46)	20 (7)	15 (15)	10 (44)	10 (10)	100 (1)

Problem 1. Ch 6 [**10 points**] Let n be a positive integer. Find the number of all $n \times n$ matrices $A = (A_{i,j})_{1 \le i,j \le n}$ satisfying the following two conditions:

- $A_{i,j} \in \{-1, 0, 1\}$ for all $1 \le i, j \le n$, and
- $A_{i,j}A_{j,i} = 0$ for all $1 \le i, j \le n$ with $i \ne j$.

Solution. For each i, the entry $A_{i,i}$ can be any element in $\{-1, 0, 1\}$, so there are 3^n choices for the diagonal entries [**2 points**]. For each i, j with $i < j$, there are $9 - 4 = 5$ choices for the non-diagonal entries $A_{i,j}$ and $A_{j,i}$ [**2 points**]. Since there are $\binom{n}{2} = \frac{n^2-n}{2}$ pairs (i, j) with $i < j$ [**2 points**], we have $5^{\frac{n^2-n}{2}}$ possibilities for the non-diagonal entries [**2 points**]. Therefore the answer is $3^n 5^{\frac{n^2-n}{2}}$ [**2 points**]. □

Problem 2. Ch 6 [**10 points**] Find the number of positive integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 2021$ such that $x_i \equiv i \pmod 2$ for all $i = 1, 2, \dots, 5$.

Solution. Let $x_i = 2y_i$ if i is even and $x_i = 2y_i - 1$ if i is odd [**3 points**]. Then the equation becomes $y_1 + y_2 + y_3 + y_4 + y_5 = 1012$ [**3 points**]. The number of positive solutions to this equation is $\binom{1011}{4}$ [**4 points**]. □

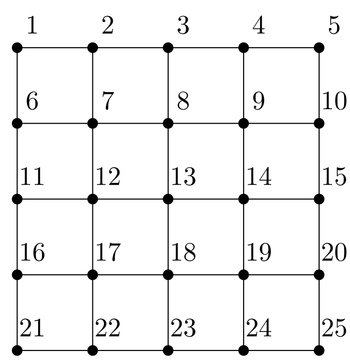
Problem 3. Ch 6 [**10 points**] Suppose that $a_1, \dots, a_{1312}$ are distinct integers in $\{1, 2, \dots, 2021\}$. Prove that there are two integers i, j with $1 \le i, j \le 1312$ such that $a_i - a_j = 601$.

Solution. Let $b_k = a_k + 601$, for $k = 1, 2, \dots, 1312$ [**2 points**]. Since $b_k \le 2021 + 601 = 2622$, the 2624 numbers $a_1, \dots, a_{1312}, b_1, \dots, b_{1312}$ are contained in $\{1, 2, \dots, 2622\}$ [**2 points**]. By the pigeonhole principle, there are two equal numbers [**2 points**]. Since a_k's are all distinct and b_k's are all distinct, we must have $a_i = b_j$ for some i, j [**2 points**]. Then $a_i = a_j + 601$, which is the same as $a_i - a_j = 601$ [**2 points**]. □

Problem 4. Ch 7 [**15 points**] Let $a_0, a_1, a_2, \dots$ be the sequence defined by $a_0 = 2, a_1 = -9$, and $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \ge 2$. Find a general formula for a_n.

Solution. The characteristic polynomial is $x^2 - 6x + 9$ [**3 points**]. This has double root 3 [**3 points**]. Thus we can write $a_n = r3^n + snr^n$ for some constants r and s [**3 points**]. Since $a_0 = r = 2$ and $a_1 = 3r + 3s = -9$ [**3 points**], we obtain $r = 2$ and $s = -5$ [**3 points**]. Therefore $a_n = 2 \cdot 3^n - 5 \cdot nr^n$. □

Problem 5. Ch 8 [**20 points**] Let G be the following graph and let A be the adjacency matrix of G with respect to the vertex ordering $1, 2, \dots, 25$.



(1) Evaluate the sum of all entries of A, that is, $\sum_{i=1}^{25}\sum_{j=1}^{25} A_{i,j}$.

(2) Evaluate the sum of the diagonal entries of A^2, that is, $\sum_{i=1}^{25}(A^2)_{i,i}$.

(3) Evaluate the $(1, 25)$-entry $(A^8)_{1,25}$ of A^8.

(4) Evaluate the sum of the entries in row 13 of A^3, that is, $\sum_{i=1}^{25}(A^3)_{13,i}$.

Solution. (1) By definition this is equal to the number of edges times 2 [**2 points**]. Thus the answer is $2\cdot(4\times5+4\times5) = 80$ [**2 points**].

(2) Since $(A^2)_{i,i}$ is the degree of vertex i, $\sum_{i=1}^{25}(A^2)_{i,i}$ is equal to the sum of the degrees of the vertices [**2 points**], which is equal to the number of edges times 2 [**2 points**]. Therefore by (1) the answer is 80.

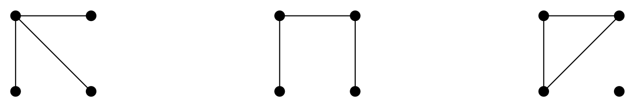
(3) This is the number of paths from vertex 1 to vertex 25 of length 8 [**2 points**]. The answer is $\binom{8}{4}$ [**2 points**].

(4) This is the number of paths starting from vertex 13 to any vertex using 3 edges [**2 points**]. At vertex 13 there are 4 ways to visit its neighbor x [**2 points**]. For each x there are also 4 ways to visit its neighbor y. For each y there are 4 ways to visit its neighbor unless $y \in \{3, 11, 15, 23\}$ [**2 points**], in which case there are 3 ways. Thus the answer is $4^3 - 4 = 60$ [**2 points**].

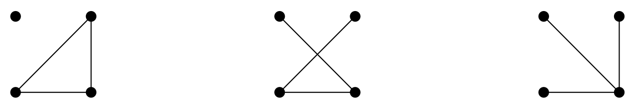
$\square$

Problem 6. Ch 8 [**15 points**] The *complement* of a simple graph G is the simple graph $\overline{G}$ with the same vertices as G such that an edge exists in $\overline{G}$ if and only if it does not exist in G. Find the number of simple graphs G with vertex set $\{1, 2, 3, 4\}$ such that G and $\overline{G}$ are isomorphic.

Solution. Since G and $\overline{G}$ must have the same number of edges, there are 3 edges in G [**3 points**]. There are 3 non-isomorphic graphs with 4 vertices and 3 edges as follows [**3 points**].



There complements are respectively as follows [**3 points**].



Only the middle ones are isomorphic. Therefore G must be a path of length 3 [**3 points**]. There are $\frac{4!}{2} = 12$ such paths with vertices $\{1, 2, 3, 4\}$ [**3 points**]. $\square$

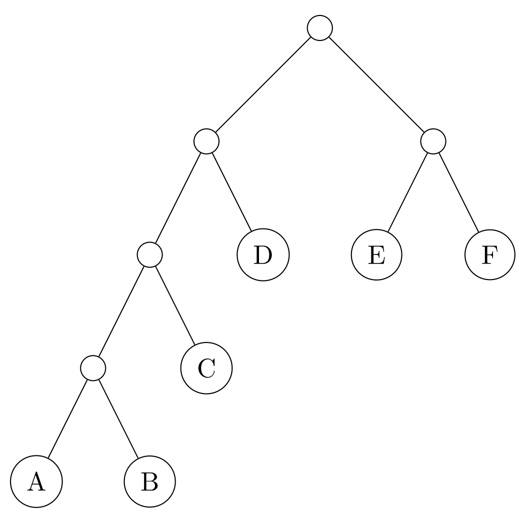
Problem 7. Ch 9 [**10 points**] Consider the set of letters in the following table.

letter	A	B	C	D	E	F
frequency	1	2	4	6	9	10

(1) Find an optimal Hoffman code for these letters. (Draw a tree and write a 0-1 sequence for each letter. There are many possible optimal Hoffman codes, and you need to find just one of them.)

(2) Encode "CAFE" using your Hoffman code.

Solution. (1) A possible answer is this [**3 points**].



Assign 1 to each left edge and 0 to each right edge. Then we obtain the following code [**4 points**].

letter	A	B	C	D	E	F
string	1111	1110	110	10	01	00

(2) Using the above code the answer is 11011110001 [**3 points**].

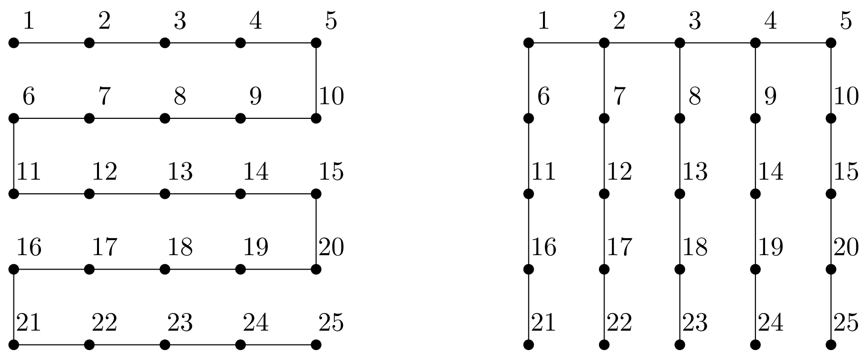
□

Problem 8. Ch 9 [**10 points**] Let G be the graph in Problem 5.

(1) Draw the spanning tree of G obtained by the depth-first search with the vertex ordering $1, 2, \ldots, 25$.

(2) Draw the spanning tree of G obtained by the breadth-first search with the vertex ordering $1, 2, \ldots, 25$.

Solution. The answers are as follows ([**5 points**] each).



□