Ch 6. Counting methods and the Pigeonhole Principle.

§ 6.1. Basic Principles

Question: How many different ways are there to order a sandwich at Subway?

5step

- Choose bread : 10
- size : 2
- meat : 30
- vegetable : 2^{10}
- source : 20

total : $10 \cdot 2 \cdot 30 \cdot 2^{10} \cdot 20$.

곱셈 원칙

Multiplication Principle

If an activity is constructed in l successive ways,

step 1 is done in n_1 ways
step 2 " n_2 "
⋮
step l " n_l ",

then the number of different activities is

$n_1 n_2 \cdots n_l$.

ex) Subway sandwich problem.

ex) # subsets of $\{1, 2, \ldots, n\}$ is 2^n.

number

A subset is constructed by choosing

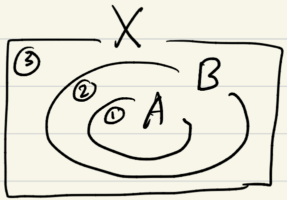
n step

whether 1 is in this subset or not ⇝ 2 ways
" 2 " 있거나 없거나 ⇝ 2 "
⋮
" n " ⇝ 2 ".

$$\# \text{ subsets} = \overbrace{2 \cdot 2 \cdots 2}^{n} = 2^n.$$

ex) # pairs (A,B) of subsets of $X=\{1,2,\dots,n\}$ such that $A \subseteq B$.

sol)



① A, ② $B-A$, ③ $X-B$.

There are 3 regions in this Venn diagram.

nstep

1 can be put in A, $B-A$, $X-B$. 3개 중 하나에 존재하므로

2 ″ ″

⋮

n ″ ″.

Answer $= 3^n$.

ex) Let X be a set with n elements. relation on X의 가능한 경우의 수

What's the # of relations on X?

sol) # relations on X = # subsets of $X \times X$.

$|X \times X| = n^2$.

Ans $= 2^{n^2}$. $\square$

ex) # reflexive relations on X?

sol) Suppose $X=\{x_1,\dots,x_n\}$.

Then a relation R on X can be considered as an $n \times n$ matrix. A.

$(x_i,x_j) \in R \iff A_{ij}=1$
$(x_i,x_j) \notin R \iff A_{ij}=0$. 2가지선택지 $\Rightarrow 2^{n^2}$ (전체 경우의 수)

요소가 n^2개 있음

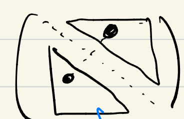
$A = \begin{pmatrix} 1 & & \square \\ & \ddots & \\ & & 1 \end{pmatrix}$ n개 must have 1s in diag. reflexive

The remaining entries can be 0 or 1.

$\Rightarrow 2^{n^2-n}$. $\square$

ex) # symmetric relations on X.

sol)

$A=$ 

We can determine (i,j) entry for $i \le j$ freely. → 한쪽이 결정되면 나머지도 결정됨

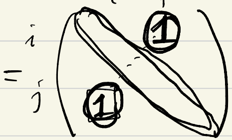
Then the remaining part is determined.

Ans $= 2^{\#(i,j) \text{ s.t. } 1 \le i \le j \le n} = 2^{n+\frac{1}{2}(n^2-n)}$ 대각성분 △ 하나의 값

$= 2^{\frac{1}{2}(n^2+n)}$.

ex) # anti-symmetric relations on X?

sol) $(x,y) \in R$, $(y,x) \in R \Rightarrow x = y$.

$A =$ 

$A_{ij} = 1$ and $A_{ji} = 1$
$\Rightarrow i = j$.

Except the diagonal entries, we can pair (i,j) and (j,i) entries. $(i \neq j)$.

For these two entries, we have 3 possibilities.
$(A_{ij}, A_{ji}) = (0,0), (0,1), (1,0)$. (가정이 (1,1) 이라)

For each A_{ii}, there are 2 possibilities.

Ans $= 2^n \cdot 3^{\frac{1}{2}(n^2-n)}$ □

transitive relations is more difficult.

Addition Principle

교집합 요소가 없는경우

$X_1, \dots, X_\ell$: pairwise disjoint sets. $|X_i| = n_i$

The number of ways to $\boxed{\text{select one element}}$ from one of $X_1, \dots, X_\ell$ is $n_1 + n_2 + \cdots + n_\ell$. 서로소인 각집합에서 하나씩

(In other words, $|X_1 \cup \cdots \cup X_\ell| = |X_1| + \cdots + |X_\ell|$ if $X_1, \dots, X_\ell$ are disjoint.)

서로소

ex) # ways to take 2 classes of different fields among

5 distinct courses from computer science
4 " math
3 " physics?

sol) Select 1 from cs and 1 from m $= 5 \cdot 4$
" cs " p $= 5 \cdot 3$ 덧셈
" m " p $= 4 \cdot 3$ 덧셈

각 1개씩 고르는것이므로 덧셈

total $= 5 \cdot 4 + 5 \cdot 3 + 4 \cdot 3 = 47$.

합집합 크기 구하는 공식

* Inclusion-Exclusion Principle

For finite sets X, Y,

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$

For X, Y, Z,

$$|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|.$$

For $X_1, X_2, \dots, X_n$,

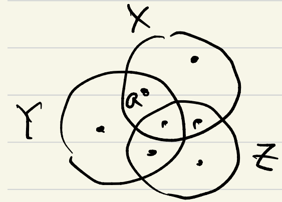
$$|X_1 \cup \cdots \cup X_n| = |X_1| + \cdots + |X_n|$$
$$- |X_1 \cap X_2| - \cdots - |X_{n-1} \cap X_n| \quad (\text{all of } |X_i \cap X_j|,\ 1 \le i < j \le n)$$
$$+ |X_1 \cap X_2 \cap X_3| + \cdots + |X_{n-2} \cap X_{n-1} \cap X_n| \quad (\text{all of } |X_i \cap X_j \cap X_k|,\ 1 \le i < j < k \le n)$$
$$\vdots$$
$$+ (-1)^{n-1} |X_1 \cap \cdots \cap X_n|.$$

Pf) ① Can be done by induction on n.

Or, ② can be proved by considering 디테일한건 생략
the contribution of each element of $X_1 \cup \cdots \cup X_n$
in both sides. $\square$

②예시

Let's consider the contributions when $n=3$.



The contribution of a in

LHS $= 1$, ($|X \cup Y \cup Z|$ 에 포함되므로 1)

RHS $= \underset{+|X|}{1} + \underset{+|Y|}{1} - \underset{-|X \cap Y|}{1} = 1$

ex) # integers relatively prime to 6 서로소
in $\{1, 2, \ldots, 100\}$.

sol) $U = \{1, 2, \ldots, 100\}$.
A = set of multiples of 2 in U 배수
B = 〃 3 〃.
Then $x \in U$ is relatively prime to 6 ⇒ x는 6과 서로소
iff $x \notin A \cup B$.

$$\begin{aligned} \text{Ans} &= |U - A \cup B| \\ &= |U| - |A \cup B| \\ &= |U| - |A| - |B| + |A \cap B|. \end{aligned}$$

$|A|$ = # multiples of 2 in $U = \lfloor \frac{100}{2} \rfloor = 50$
$|B|$ = 〃 3 $= \lfloor \frac{100}{3} \rfloor = 33$

$|A \cap B|$ = 〃 6 $= \lfloor \frac{100}{6} \rfloor = 16$

Ans $= 100 - 50 - 33 + 16 = 67$.

순열과 조합

§6.2. Permutations and Combinations.

순열

Def) A permutation of $x_1, x_2, \dots, x_n$ is an arrangement of them.

ex) There are 6 permutations of a, b, c.

$abc, acb, bac, bca, cab, cba$.

Thm There are $n!$ permutations of n elements.

Pf) Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a permutation of $x_1, x_2, \dots, x_n$.

자리

choices for $\pi_1 = n$

" $\pi_2 = n-1$

$\vdots$ $\vdots$

" $\pi_n = 1$

ways to construct $\pi = n!$ □

ex) # permutations of $1, 2, \dots, 9$ such that 6 and 8 are adjacent 인접한

Sol) We must have 68 or 86 in such a permutation.

Let $x = 68$ or 86. → 1개의 문자로 지정

$$\text{Ans} = \left(\begin{array}{l}\#\text{ permutations of elements in}\\ \underbrace{(\{1,\dots,9\}-\{6,8\})\cup\{x\}}_{8\text{개의 원소}}\end{array}\right) \cdot (\#\text{ choices of } x)$$

$$= 8! \cdot 2.$$ □

Def) An r-permutation of $x_1, \dots, x_n$ is an arrangement of the elements in an r-subset of $\{x_1, \dots, x_n\}$.

r 개 자리만

ex) 2-permutations of $1, 2, 3, 4$.

12, 21, 13, 31, 14, 41, 23, 32, 24, 42, 34, 43.

Thm # r-permutations of $x_1, \ldots, x_n$ is

$$P(n,r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

전체원소수 자리개수

Pf) let $\pi = \pi_1 \pi_2 \cdots \pi_r$ be an r-permutation of $x_1, \ldots, x_n$.

\# choices for $\pi_1 = n$

〃 $\pi_2 = n-1$

⋮

〃 $\pi_r = n-r+1$

r-1 개를 이미 사용

$\Rightarrow$ $n(n-1)\cdots(n-r+1)$ ways to construct π. □

ex) # ways to make a waiting list for 3 men and 8 women such that no two men are adjacent.

sol) Arrange 8 women in $8!$ ways.

$$\vee \; w_1 \; \vee \; w_2 \; \vee \; \cdots \; \vee \; w_8 \; \vee$$

There are 9 positions between the 8 women.

We need to put 3 men there.

There are $P(9,3)$ ways to do this

$9 \cdot 8 \cdot 7$.

Answer $= 8! \cdot 9 \cdot 8 \cdot 7$. □

Def) An <u>r-combination</u> of a set X is an r-subset of X.

조합

(It is a selection of r elements <u>without considering the order</u>.) 순서와 관계없이

\# r-combinations of an n-set is denoted by

$\binom{n}{r}$, or $C(n,r)$, ${}_nC_r$. ← 다양한 표현

$\uparrow$

n choose r

ex) 2-combinations of $\{1,2,3,4\}$

$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$.

12, 13, 14, 23, 24, 34.

Thm $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ if $0 \le r \le n$.

Pf) If $|X| = n$, r-permutations of X is obtained by
first selecting r elements from X
and arranging them.

$\Rightarrow P(n,r) = \binom{n}{r} \cdot r!$

$$\Rightarrow \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}. \qquad \square$$

Note If $r > n$, then it is common to define

$\binom{n}{r} = 0$.

Thm $\binom{n}{r}$ is computed as follows.

계산 방법 3가지

① # r-subsets of $\{1, 2, \ldots, n\}$. (r개의 원소)

② # binary sequences of length n with r 1's and $n-r$ 0's.

③ # shortest paths from $(0,0)$ to $(n-r, r)$.

Pf) ① is just the def.

② r-subsets $\xleftrightarrow{\text{일대일 대응 (bijection)}}$ binary sequences.

$A \longleftrightarrow a_1 a_2 \cdots a_n$, $a_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{o.w.} \end{cases}$

e.g. 3-subset of 5

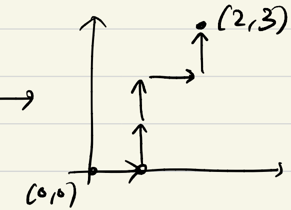
$\{2,3,5\} \longleftrightarrow$

1	2	3	4	5
0	1	1	0	1

③ binary sequences $\xleftrightarrow{\text{일대일대응}}$ shortest paths.

$a_1 a_2 \cdots a_n$

$01101 \longleftrightarrow$



ith step is
up if $a_i = 1$
right if $a_i = 0$

Cor) # paths from $(0,0)$ to (a,b) is $\binom{a+b}{a}$.

카탈란 수

* Catalan number

Thm # paths from $(0,0)$ to (n,n) which never go below $y=x$ is $\frac{1}{n+1}\binom{2n}{n}$.

Pf) total # paths from $(0,0)$ to $(n,n) = \binom{2n}{n}$

G_n = # good paths (never go below $y=x$)

B_n = # bad paths (go below $y=x$).

We need to show $G_n = \frac{1}{n+1}\binom{2n}{n}$.

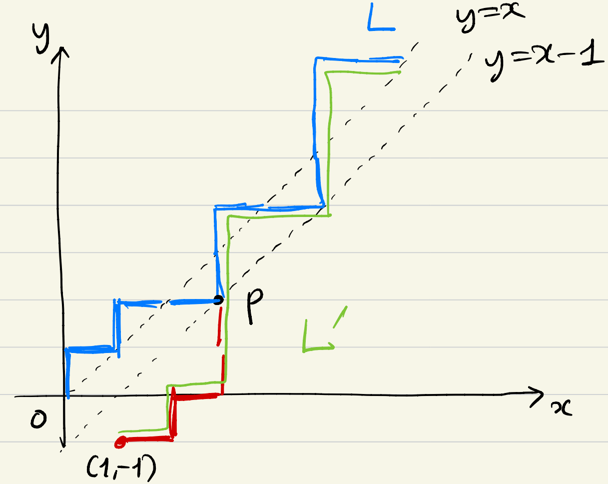
Clearly, $\binom{2n}{n} = G_n + B_n$.

We will compute B_n instead.

무조건 → / ↑ 방향으로가게끔

Every bad path L (B_n) touches the line $y=x-1$.

Let p be the first time L touches $y=x-1$.



Reflect the part of L from O to p about the line $y=x-1$.

Let L' be the resulting path.

Then L' is from $(1,-1)$ to (n,n).

→로 p가 생김
그 후 reflect 하면
→가 ↓로 바뀜
$(0,0) \Rightarrow (1,-1)$

Any path from $(1,-1)$ to (n,n) can be obtained in this way.

모든경로 (p가 x-1 직선위 어느지점에 찍히든 상관없음)

So, B_n = # paths from $(1,-1)$ to (n,n) ↓ 쉬프트

$=$ 〃 $(0,0)$ to $(n-1, n+1)$

$= \binom{2n}{n-1}$

Thus $G_n = \binom{2n}{n} - B_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$. □