Ch 3. Functions, sequences, and relations.

§3.1 Functions

함수

Def) A <u>function</u> $f: X \to Y$ is a way of assigning to each $x \in X$ exactly one $y \in Y$.

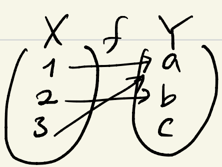
X: the <u>domain</u> of f 정의역

Y: the <u>codomain</u> of f. 공역

the <u>range</u> of f is $\{ y \mid f(x) = y \text{ for some } x \in X \}$
치역

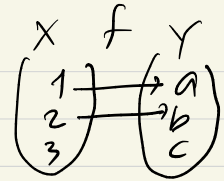
A function can be considered as a subset f of $X \times Y$ with the property that for each $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in f$.

ex) $X = \{1, 2, 3\}$, $Y = \{a, b, c\}$.
$f: X \to Y$. $f(1) = a$, $f(2) = b$, $f(3) = a$.



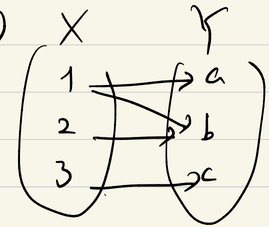
$f = \{(1, a), (2, b), (3, a)\} \subseteq X \times Y$ ~> 페어를 모두 저장한 집합

ex)



not a function.
정의역 중에 연결되지 X 요소가 있는 경우

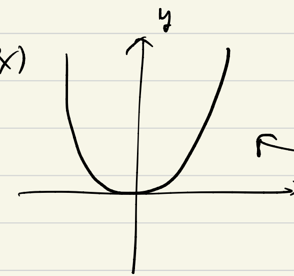
ex)



not a function.
정의역 < 두개랑 대응 되는 경우

If $f: X \to Y$ is a function with $X, Y \subseteq \mathbb{R}$ the <u>graph</u> of f is the set of points $(x, f(x))$ in $\mathbb{R}^2$.

ex)



$y = x^2$.

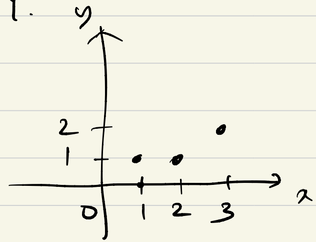
graph of $f(x) = x^2$.

ex) $X = \{1,2,3\}$, $Y = \{1,2,3\}$.

$f(1)=1$, $f(2)=1$, $f(3)=2$.

$f: X \to Y$.

graph



나머지를 의미

Def) If n, m integers, $m > 0$,

then n mod m is the remainder of n when divided by m.

That is, if $n = mq + r$, $0 \le r < m$

then n mod $m = r$.

ex). $7 \bmod 3 = 1$

$-2 \bmod 5 = 3$

$15 \bmod 5 = 0$.

Def) $x \in \mathbb{R}$.

바닥 내림

$\lfloor x \rfloor$ = floor of x

= largest integer $\le x$.

천장 올림

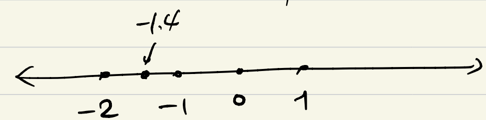
$\lceil x \rceil$ = ceiling of x

= smallest integer $\ge x$.

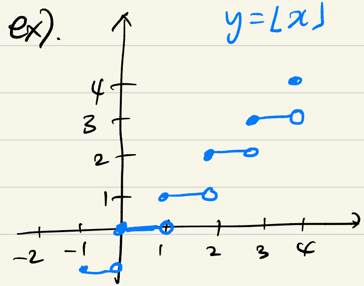
ex). $\lfloor 3.2 \rfloor = 3$. $\lceil 3.2 \rceil = 4$.

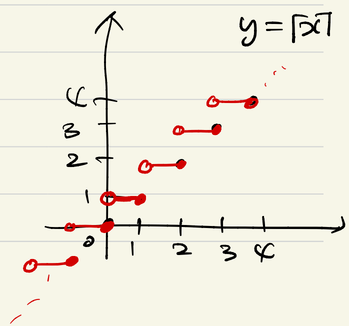
$\lfloor 3 \rfloor = 3$, $\lceil 3 \rceil = 3$.

$\lfloor -1.4 \rfloor = -2$, $\lceil -1.5 \rceil = -1$.



ex).





Def) $f: X \to Y$, function. 일대일 함수 단사함수

f is one-to-one (or injective) 최소

if for each $y \in Y$ there is at most one $x \in X$

공역=치역 값을 필요X with $f(x) = y$.

Equivalently, f is injective if

$f(x_1) = f(x_2)$ implies $x_1 = x_2$.

$\forall x_1 \forall x_2 \left((f(x_1) = f(x_2)) \to (x_1 = x_2) \right)$

ex) $f(n) = 2n+1$ is a one-to-one function

$(f: \mathbb{Z} \to \mathbb{Z})$.

ex) $f(n) = 2^n - n^2$ is not one-to-one.

$f(2) = 2^2 - 2^2 = 0$. but $2 \neq 4 \to$ not one to one

$f(4) = 2^4 - 4^2 = 0$.

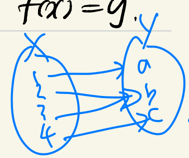
공역=치역 위로의 전사함수

Def) $f: X \to Y$ is onto (surjective)

if for every $y \in Y$ there is at least one $x \in X$

with $f(x) = y$.

$\forall y \in Y \; \exists x \in X \; (f(x) = y)$.



이런경우 일대일함수는 아니지만 onto함수

ex) $f(n) = 2n+1$ $f: \mathbb{Z} \to \mathbb{Z}$

is not onto.

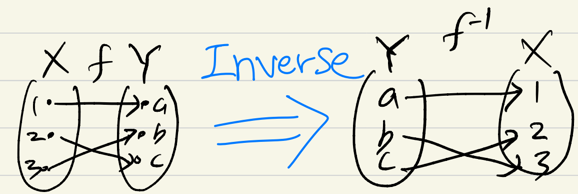
$f(n) \neq 2$ for all $n \in \mathbb{Z}$.

전단사함수 (일대일 대응)

Def) $f: X \to Y$ is a bijection if

it is one-to-one and onto.

ex)



ex) If $f: X \to Y$ is a bijection and X, Y finite then $|X| = |Y|$.

정의역 공역

Def) If $f: X \to Y$ is a bijection then

역함수

the Inverse of f is the function $f^{-1}: Y \to X$

such that $f^{-1}(y) = x$ where $f(x) = y$.

합성함수

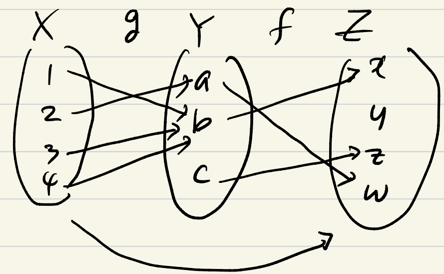
Def) $g: X \to Y$, $f: Y \to Z$.

The composition of f and g is the function

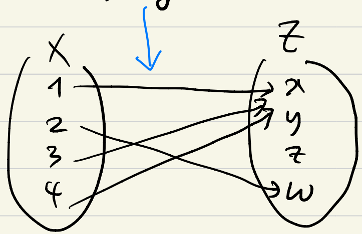
$f \circ g : X \to Z$

defined by $(f \circ g)(x) = f(g(x))$.

ex)



$f \circ g$



ex). $\sin^2(x) = (f \circ g)(x)$,

$g(x) = \sin x$, $f(x) = x^2$

$$(f \circ g)(x) = f(g(x)) = f(\sin x)$$
$$= \sin^2 x.$$

§3.2. Sequences and Strings.

Def) A sequence is a list of objects.
수열

ex) 1, 2, 3, 4, ...
2, 4, 6, 8.
a, a, b, x, z

If s is a sequence, s_i denotes the ith element in s.
That is, $s = s_1, s_2, s_3, \ldots$
We also write $s = \{s_n\}$.

ex) $1, 2, 3, 4, \ldots = \{n\}_{n=1}^{\infty}$
$2, 4, 6, 8 = \{2n\}_{n=1}^{4}$ 1~4 까지

We can also consider $\{s_n\}_{n=k}^{\infty} = s_k, s_{k+1}, \ldots$
or $\{s_n\}_{n=i}^{j} = s_i, s_{i+1}, \ldots, s_j$.

ex). s is a sequence given by
$$s_n = 2^n + 4\cdot 3^n \quad (n \geq 0).$$
$s_0 = 2^0 + 4\cdot 3^0 = 1 + 4 = 5.$
$s_1 = 2^1 + 4\cdot 3^1 = 2 + 12 = 14$
⋮

감소
Def) A sequence s is decreasing if $s_i > s_{i+1}$ $\forall i$
증가
" increasing if $s_i < s_{i+1}$ $\forall i$
" nondecreasing if $s_i \leq s_{i+1}$ $\forall i$
" nonincreasing if $s_i \geq s_{i+1}$ $\forall i$

ex) 1, 2, 3, 4, ... : increasing
−1, −2, −3, −4, ... : decreasing
1, 1, 2, 2, 3, 3, ... : nondecreasing
3, 3, 2, 2, 1, 1 : nonincreasing

Def) $\{a_i\}_{i=m}^{n}$

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

$$\prod_{i=m}^{n} a_i = a_m a_{m+1} \cdots a_n.$$

ex) $\sum_{i=1}^{n} 3^i$ $\qquad i = j+1$

$$= \sum_{j=0}^{n-1} 3^{j+1}$$

$$= \sum_{k=0}^{n-1} 3^{k+1} = \sum_{i=0}^{n-1} 3^{i+1}$$

If $A = \{a_1, a_2, \dots, a_n\}$ is a set,

$$\sum_{i \in A} i = a_1 + a_2 + \cdots + a_n$$

$$\prod_{i \in A} i = a_1 a_2 \cdots a_n.$$

Def) $\{s_n\}_{n=m}^{\infty}$

$n_1, n_2, \dots$: increasing sequence of integers in $\{m, m+1, \dots\}$

Then $\{s_{n_i}\}_{i=1}^{\infty}$ is called a <u>subsequence</u> of $\{s_n\}_{n=m}^{\infty}$.

부분 수열

ex) $s = 1, 2, 3, 4, \dots$

$2, 4, 6, \dots$ is a subsequence of s.

$1, 3, 5, \dots$ 〃

$1, 2, 4, 3, 5$ not 〃 순서 맞지 X

Def) X : a finite set

A <u>string</u> over X is a finite sequence of elements from X.

ex) $X = \{a, b, c\}$.

aabcc, abab, ccbab, ... : strings

We also write $aaa = a^3$

$aabbcccbba = a^2 b^2 c^3 b^2 a$.

The <u>empty string</u> is the string with no elements.
↳ denoted by λ. (also called <u>null string</u>)

Def) $X^* =$ set of all strings over X.
$X^+ =$ 〃 〃 except λ. $= X^* - \lambda$

ex) $X = \{a, b\}$.
X^* has $\lambda, a, b, aa, ab, ba, bb, \ldots$

Def). The <u>length</u> of a string α is the number of elements in α, denoted by $|\alpha|$.

ex) $\alpha = aaabba$ $\quad |\alpha| = 6$.

$\beta = a^{21} b^{10} a^{2} c^{10}$ $\quad |\beta| = 43$.

Def) α, β : strings.
$\alpha\beta$ is the string obtained by ~~contaterating~~ concatenating α, β.

연결

ex) $\alpha = abc$, $\beta = cba$
$\alpha\beta = abccba$, $\quad \beta\alpha = cbaabc$.

Def) A string β is a <u>substring</u> of a string α if $\alpha = \gamma\beta\delta$ for some strings γ, δ.

내부에서 발견 가능한

ex). $\alpha = a\underline{bca}bc$. $\quad \beta = bca$.

$\alpha = a\beta bc \Rightarrow \beta$ is a substring of α.

ex). $X = \{a, b\}$. $\quad f : X^* \to X^*$

$f(\alpha) = \alpha^R$
the reverse of α

$f(aaba) = abaa$. 거꾸로

Then f is a bijection.

Pf) one-to-one : $f(\alpha) = f(\beta) \Rightarrow (\alpha^R)^R = (\beta^R)^R \Rightarrow \alpha = \beta$.

onto : For any $\alpha \in X^*$, then $f(\alpha^R) = (\alpha^R)^R = \alpha$. 가 존재

공역 $\quad$ α에 대해 α^R 존재 $\quad$ 거꾸로

전, 공역 = 치역. $\square$

ex). $X = \{a, b\}$.

$f : X^* \times X^* \to X^*$

$f(\alpha, \beta) = \alpha\beta$.

Is f one-to-one? onto?

sol). $f(\lambda, a) = \lambda a = a = a\lambda = f(a, \lambda)$

not one-to-one.

For any $\alpha \in X^*$, $f(\lambda, \alpha) = \alpha$

$\Rightarrow$ onto.

§3.3. Relations

person	hobby
Alice	tennis
Bob	baseball
Chris	soccer
David	fishing
	video game.
	photo

Alice's hobbies are t, f, p.
Bob's " s, v.
Chris's " v
David has no hobbies.

Def) A relation from X to Y is a subset of $X \times Y$.
부분집합

ex) $X = \{A, B, C, D\}$.
$Y = \{t, b, s, f, v, p\}$.
$R \subseteq X \times Y$ is the relation defined by
$(x, y) \in R$ iff x has hobby y.
$R = \{(A,t), (A,f), (A,p), (B,s), (B,v), (C,v)\}. \subseteq X \times Y.$

ex) A function $f: X \to Y$ is a special relation such that $\forall x \in X$, $\exists$ unique $y \in Y$, $(x, y) \in f$.

If $X = Y$, then a relation from X to Y is called a relation on X.

ex) $X = \{2, 3, 5\}$, $Y = \{1, 2, 3, 4, 5, 6, 7\}$.
Relation R : $(x, y) \in R$ iff x divides y. 나눌수 있으면

$R = \{(2,2), (2,4), (2,6), (3,3), (3,6), (5,5)\}$.

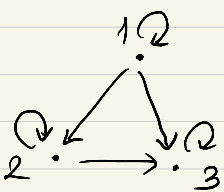
ex) $X = \{1, 2, 3\}$, $X=Y$

R : the relation on X

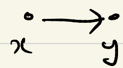
$(x,y) \in R$ iff $x \le y$.

$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$.

We can represent a relation on X using "digraphs" (directed graphs).

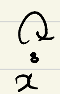


$(x,y) \in R$



directed edge

$(x,x) \in R$



loop

ex) $X = \{1, 2, 3, 4\}$.

$R = \{(1,1), (2,3), (3,4), (2,4), (4,2)\}$.

R is not reflexive because $(2,2) \notin R$.

R is not symmetric " $(2,3) \in R$ but $(3,2) \notin R$.

R is not antisymmetric " $(2,4) \in R$, $(4,2) \in R$ but $2 \ne 4$.

R is not transitive " $(2,4) \in R$, $(4,2) \in R$ but $(2,2) \notin R$.

Note R is antisym if the following condition holds.

$x \ne y \Rightarrow (x,y) \notin R$ or $(y,x) \cancel{\notin} R$ 아이오티

(contrapositive)

Edge에서 안됨

Def) R : relation on X.

not symmetric 이랑 다름X

R is reflexive if $(x,x) \in R \ \forall x \in X$. 모든원소 자기자신과 관계 반사성

R is symmetric if 모든 $\forall x, y \in X$, $(x,y) \in R \to (y,x) \in R$ 대칭적

R is antisymmetric if 모든 $x, y \in X$, $(x,y) \in R, (y,x) \in R \to x = y$. 반대칭적

같으면 반대칭 oh

다르면 한쪽만 존재해야함

R is transitive if $\forall x, y, z \in X$, $(x,y) \in R, (y,z) \in R \to (x,z) \in R$. 전이관계

$\phi \to$ reflexive ; False $\to$ If p ; ϕ은 p를 만족시키지 X $\to$ False

$\phi \to$ sym, antisym, transitive ; True $\to$ If x, y 있, $p \to q$; ϕ은 p를 거짓으로 만듦 $\to$ 무조건 True ex) R : relation on $\mathbb{R}$

$(x,y) \in R$ iff $x \le y$

R is reflexive $\quad x \le x$

R is not sym. $\quad 2 \le 3$ but $3 \not\le 2$.

R is antisym $\quad x \le y,\ y \le x \Rightarrow x = y$.

R is transitive $\quad x \le y,\ y \le z \Rightarrow x \le z$.

Def) A relation R on X is called a partial order if it is reflexive, antisymmetric and transitive.

부분순서

3개 만족하면 partial order! (sym는 신경 X)

ex). R : relation on $\mathbb{Z}^+$

$(x,y) \in R$ iff x divides y.

$(2,4) \in R$. $\quad (3,6) \in R$,

$(2,10) \in R$, ...

$(5,7) \notin R$

5랑 7은 비교하지 X (나눗셈 계산 불가)

$5/7 \ne 0$, $7/5 \ne 0$ → 부분적으로 비교가능 ⇒ 부분순서

계산불가

→ R is a partial order.

R : a partial order on X.

Let's write $x \precsim y$ if $(x,y) \in R$.

We say $x, y \in X$ are comparable if $x \precsim y$ or $y \precsim x$.

If $x \not\precsim y$ and $y \not\precsim x$ then x, y are incomparable.

둘다

If every two elements of X are comparable we say that R is a total order.

전순서

ex). let R be the relation on $\mathbb{R}$ given by

$(x,y) \in R$ iff $x \le y$.

비교가능

Then R is a total order.

This is not a total order.

4 and 6 are not comparable.

$4 \div 6 \ne 0$?

$(a,b) \in R, (b,c) \in R \rightarrow (a,c) \in R$

Def). If R is a relation from X to Y, the inverse of R is the relation 역관계 R^{-1} from Y to X such that

$$(y,x) \in R^{-1} \text{ iff } (x,y) \in R.$$

ex) $X = \{1,2,3\}$, $Y = \{a,b\}$.

$R = \{(1,a),(2,b),(3,a),(3,b)\}$

$R^{-1} = \{(a,1),(b,2),(a,3),(b,3)\}$

Def) $R_1 \subseteq X \times Y$, $R_2 \subseteq Y \times Z$

The composition of R_1 and R_2 is the relation

$R_2 \circ R_1 \subseteq X \times Z$

such that → 합성 함수 순서랑 같음

$(x,z) \in R_2 \circ R_1$ iff $\exists y \in Y$ such that $(x,y) \in R_1$, $(y,z) \in R_2$.

ex) $X = \{1,2,3\}$, $Y = \{a,b\}$, $Z = \{A,B,C\}$.

$R_1 = \{(1,a),(2,b),(3,a),(3,b)\}$.

$R_2 = \{(a,B),(a,C),(b,A)\}$.

$R_2 \circ R_1 = \{(1,B),(1,C),(2,A),(3,B),(3,C),(3,A)\}$.

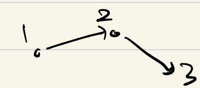
ex) Let R and S be transitive relations on X.

① Is $R \cup S$ transitive?

② 〃 $R \cap S$ 〃 ?

③ 〃 $R \circ S$ 〃 ?

Sol) ① No. Let $X = \{1,2,3\}$. $R = \{(1,2)\}$, $S = \{(2,3)\}$.



Then $R \cup S = \{(1,2),(2,3)\}$ not trans. No

② Yes. Let $(x,y),(y,z) \in R \cap S$. Then since $(x,y),(y,z) \in R$ & R is trans, $(x,z) \in R$.

〃 S & S 〃 $(x,z) \in S$.

Thus $(x,z) \in R \cap S$ and $R \cap S$ trans.

③ No.