§6.3. Generalized Permutations and Combinations

ex) What's the number of ways to arranges MISSISSIPPI ?

sol) M : 1, I : 4, S : 4, P : 2 — total 11 letters

If we consider all letters as distinct letters then there are $11!$ ways.

$4!$ arrangements of I, I, I, I are the same.

$4!$ 〃 S, S, S, S 〃

$2!$ 〃 P, P 〃

$$\Rightarrow \frac{11!}{4!\,4!\,2!}$$

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1 2 3 4 5 6 7 8 9 10 11

I M S P P I I S I S S

Select M in pos 2 $\binom{11}{1}$ ways.

〃 I^4 〃 1, 6, 7, 9. $\binom{10}{4}$ $\binom{6}{4}$

〃 S^4 〃 3, 8, 10, 11

〃 P^2 〃 4, 5. $\binom{2}{2}$.

같은것이 있는 순열

Thm The number of arrangements of

n_1 identical objects of type 1

n_2 〃 2

⋮

n_k 〃 k

is $$\frac{(n_1+\cdots+n_k)!}{n_1!\cdots n_k!}$$

pf) Let $n = n_1+\cdots+n_k$.

1~n의 object의 위치를 정하는 것과 같기 때문에

There are n positions for the objects.

위치

We select the positions of the n_1 objects of type 1 in $\binom{n}{n_1}$ ways.

Then select the pos of n_2 objs of type 2 in $\binom{n-n_1}{n_2}$ ways.

⋮

Finally select 〃 n_k 〃 〃 k in $\binom{n-n_1-\cdots-n_{k-1}}{n_k}$ ways.

$$\text{Ans} = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\,\cancel{(n-n_1)!}} \cdot \frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}} \cdots \frac{\cancel{(n-n_1-\cdots-n_{k-1})!}}{n_k!\,\underbrace{(n-n_1-\cdots-n_k)!}_{=1}}$$

$$= \frac{n!}{n_1!\cdots n_k!}.$$

□

중복순열(중복허용)

Thm The number of ways to select r elements from an n-set with repetition allowed is $\binom{n+r-1}{r}$. $\quad ({}_nH_r)$

Pf) We can consider selecting r elements from $1, 2, \ldots, n$.

Since we don't consider the order of the selected elements, we can say the selected elements are

$i_1, i_2, \ldots, i_r$, $\quad 1 \le i_1 \le i_2 \le \cdots \le i_r \le n$.

그냥 ≤ 대신 < 라면 조합 $= \binom{n}{r}$

Ans = # r-tuples $(i_1, \ldots, i_r)$, $1 \le i_1 \le \cdots \le i_r \le n$.

Let's define

$j_1 = i_1$

$j_2 = i_2 + 1$ $\qquad\qquad i_k \le i_{k+1}$

$j_3 = i_3 + 2$

$$\begin{array}{ccc} j_k & < & j_{k+1} \\ \| & & \| \\ i_k + k - 1 & \le & i_{k+1} + k \end{array}$$

$\vdots$

n이하여

$j_r = i_r + r - 1$

에의하여

Then $\underbrace{1 \le j_1 < j_2 < \cdots < j_r \le n+r-1}$.

Ans = # $(j_1, \ldots, j_r)$ satisfying $\uparrow$

$= \binom{n+r-1}{r}$. $\square$

분배

* Distribution Problems.

ex) # ways to distribute 10 identical candies to 5 children?

Sol) This is equal to # ways to select 10 children from 5 children with repetition allowed.

↳ 10개의 사탕 ⇒ 5아이들 나눠줌

$C_1, C_2, \ldots, C_5$: 5 children. (distinct)

$$\begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 3 & 0 & 4 & 2 & 1 \end{array}$$

$$\Rightarrow {}_5H_{10} = \binom{5+10-1}{10} = \binom{14}{10}.$$

$= {}_{14}C_{10}$

Note. This is also equal to the number of nonnegative integer solutions to

$x_1 + \cdots + x_5 = 10$. ← 이것과 같다.

(x_i = # candies child i gets)

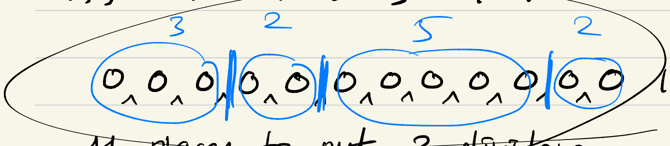
* Integer Solutions.

Thm # positive integer solutions to 양수

$x_1+\cdots+x_k=n$

is $\binom{n-1}{k-1}$. ${}_{n-1}C_{k-1}$

Pf) e.g. $x_1+x_2+x_3+x_4=12$.



12 balls

11 places to put 3 dividers.

In general, if there are n balls we have $n-1$ spots. ← 양끝제외

To create k regions, we need to put $k-1$ divisors.

$$\Rightarrow \binom{n-1}{k-1}.$$ □

Thm # nonnegative integer solutions to 음이아닌정수

$x_1+\cdots+x_k=n$

is $\binom{n+k-1}{k-1}$.

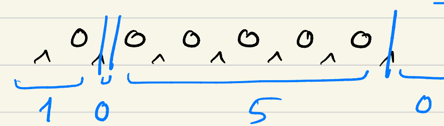
Pf) We can change this to the previous problem by letting $y_i=x_i+1$. 변환

$\Rightarrow y_1+\cdots+y_k=n+k$ (y_i : pos.)

Ans $=\binom{n+k-1}{k-1}$.

Pf2). We can simply arrange n balls and $k-1$ dividers, but we may have more than one divisor in each spot, and also at the left or right end. 양끝허용,

$\Rightarrow n+1$ spots ← 중복허용



$x_1+x_2+x_3+x_4=6$

$1+0+5+0=6$.

$$\Rightarrow {}_{n+1}H_{k-1} = \binom{n+1+k-1-1}{k-1} = \binom{n+k-1}{k-1}.$$

§6.7. Binomial Coefficients and Combinatorial Identies.

Thm (Binomial Theorem). 이항정리

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

↳ 이항계수

Pf) When we expand

$(x+y)^n = (x+y)(x+y)\cdots(x+y)$

we select one term in each factor and multiply the selected terms, which gives one term in the expansion.

e.g. $(x+y)(x+y) = x\cdot x + x\cdot y + y\cdot x + y\cdot y$

$= x^2 + 2xy + y^2$.

In $(x+y)^n = (x+y)\cdots(x+y)$, if we select z_i (either x or y) in the ith factor then we get $z_1 z_2 \cdots z_n$.

If there are k x's in $z_1, \dots z_n$, then $z_1\cdots z_n = x^k y^{n-k}$.

So the term equal to $x^k y^{n-k}$ appears $\binom{n}{k}$ times $(0\le k\le n)$.

$\Rightarrow (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. ∎

Cor $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. $\cdots$ ①

$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$. ($x=1$인 경우)

$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0$. ($x=-1$인 경우)

$\binom{n}{n+1}$ 이런거는 0이라 지장없음

or $\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots$

By taking the derivative of ①

$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}$. ← 미분값

If we substitute $x=1$,

$n\cdot 2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$.

※ $\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$

파스칼 항등식

Thm (Pascal's Identity) $(0 \le k \le n)$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Pf) $\binom{n}{k} = (\#\ k\text{-subsets of } \{1, \dots, n\})$

$= (\#\ k\text{-subsets } A \text{ of } \{1, \dots, n\},\ n \notin A)$

$\quad + ($ 〃 $,\ n \in A)$

$= (\#\ k\text{-subsets of } \{1, \dots, n-1\})$ n이 없으니까

$\quad + (\#\ (k-1)\text{-subsets of } \{1, \dots, n-1\})$ n이 있으니까

$= \binom{n-1}{k} + \binom{n-1}{k-1}.$ $\square$

파스칼 삼각형

Pascal triangle.

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}$$

$$\begin{array}{ccc}
\binom{n-1}{k-1} & & \binom{n-1}{k} \\
\searrow & & \swarrow \\
 & \binom{n}{k} &
\end{array}$$

$$\begin{array}{ccccccccccccc|c}
 & & & & & & 1 & & & & & & & 1 \\
 & & & & & 1 & & 1 & & & & & & 2 \\
 & & & & 1 & & 2 & & 1 & & & & & 4 \\
 & & & 1 & & 3 & & 3 & & 1 & & & & 8 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & & 16 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 &
\end{array}$$

특징①

특징②

n번째 행 → 합: 2^n

하키스틱

하키스틱 증명

ex). $\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$

Pf 1) Use induction on n.

Base case : $n=k$.

$$\binom{k}{k}=\binom{k+1}{k+1}=1.$$

Inductive step. Assume for n.

$$\underbrace{\binom{k}{k}+\binom{k+1}{k}+\cdots+}_{\text{ind hyp}}\binom{n+1}{k}\overset{?}{=}\binom{n+2}{k+1}$$

ind hyp

$=\binom{n+1}{k+1}$

$$\text{LHS} \Rightarrow \binom{n+1}{k+1}+\binom{n+1}{k}=\binom{n+2}{k+1}$$

by Pascal identity.

True for $n+1$.

We are done by induction. $\square$

Pf 2).

$\binom{n+1}{k+1}=\#(i_1,\ldots,i_{k+1}),\quad 1\le i_1<\cdots<i_{k+1}\le n+1.$

The last integer i_{k+1} can be one of

$k+1, k+2, \ldots, n+1$

If $i_{k+1}=j$, $(k+1\le j\le n+1)$,

then $\#(i_1,\ldots,i_k)$ such that $1\le i_1<\cdots<i_k\le i_{k+1}-1 = j-1$

is $\binom{j-1}{k}$.

So, $\binom{n+1}{k+1}=\sum_{j=k+1}^{n+1}\binom{j-1}{k}$

$$=\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}. \quad \square$$

§6.8. Pigeonhole Principle.

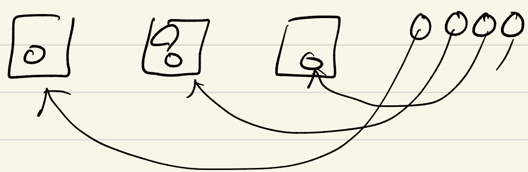
비둘기집 원리 → 쉽지만 적용하기 쉽지않다

Pigeonhole Principle

n>m일때 n개의 비둘기 ↓ m개의 집에 넣는다면 최소 2마리는 같은집에!

If n pigeons fly into m pigeonholes with $n > m$, then there are at least two pigeons in the same pigeonhole.

ex).



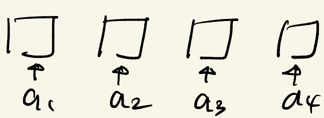
Pigeonhole Principle (General version)

If there are n pigeons, m pigeonholes

$\Rightarrow \left\lceil \frac{n}{m} \right\rceil$ pigeons in one hole. 최소 1개에는

ex). 15 pigeons and 4 pigeonholes

then one pigeonholes has at least $\left\lceil \frac{15}{4} \right\rceil = 4$.



$a_1 + \cdots + a_4 = 15$

If $a_i < \frac{15}{4}$ $\forall i$

$\Rightarrow \Sigma a_i < 15$. 모순!! ⇒ 1개의 a_i는 $\frac{15}{4}$보다 크다

ex) Show that any $(n+1)$-subset of $\{1, \ldots, 2n\}$ contains integers relative prime to each other.

$(\gcd(a,b) = 1)$

Pf) Let $X = \{a_1, \ldots, a_{n+1}\} \subseteq \{1, \ldots, 2n\}$.

pigeons : $a_1, \ldots, a_{n+1}$. n+1개

pigeonholes : $\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}$ n개

피존홀을 설정하는게 관건!

pigeon a_i flies into pigeonhole B if $a_i \in B$.

By Pigeonhole Principle,

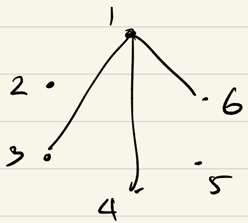
↳ $a_i, a_j \in \{2k-1, 2k\}$. for some $i \neq j$, $1 \leq k \leq n$.

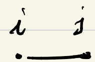
This means $a_i, a_j = 2k-1, 2k$.

$\Rightarrow \gcd(a_i, a_j) = \gcd(2k-1, 2k) = 1$. $\square$

ex) If there are 6 people, there exist 3 mutual friends or 3 mutual strangers.

pf). $P_1, \dots P_6$: 6 people.

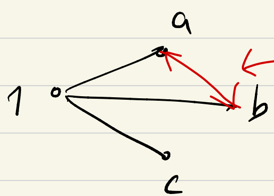


We connect 

if P_i, P_j are friends.

Among 2,3,4,5,6, there are 3 friends of 1 or 3 strangers for 1.

① Suppose 1 knows 3 people, a, b, c.



이렇게되면
a, b, 1은 친구

If two of a, b, c know each other, then including 1, there are 3 mutual friends.

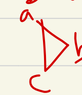
Otherwise, a, b, c are mutual strangers.

a-b, b-c, a-c 모두 낯선관계 ⇒ 3명의 낯선 친구

② Suppose 1 do not know any of a, b, c.

a-b가 서로 알지못함 ⇒ a, b, 1은 낯선관계

↳ 그렇지 않으면 ⇒ a, b, c는 친구



□

This is an example of Ramsey theory.

ex). X is a collection of subsets of $\{1, \dots, n\}$. 집합

such that

$\forall A, B \in X, \quad A \cap B \neq \phi.$ 겹치는 원소가 있다

What's the maximum size of X?

Pf) let $X = \{A_1, \dots, A_k\}$. $\quad U = \{1, \dots, n\}$.

Pigeons : $A_1, \dots, A_k$.

Pigeonholes : $\{B, B^c\}$, $\quad B \subseteq U$. → 경우의 수 2^n

(There are 2^{n-1} pigeonholes.) B, B^c 가 한 쌍 이므로 2^{n-1}

Then every subset of U is in one pigeonhole.

If $k > 2^{n-1}$ then 2개 뽑았을 때 같은 비둘기집에 들어간다

we have $A_i, A_j \in \{B, B^c\}$.

Then $A_i \cap A_j = B \cap B^c = \phi$, contradiction.

So, $k \leq 2^{n-1}$ → 실제 k의 값을 구해야함

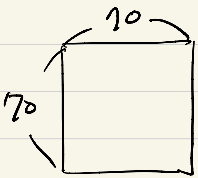
If $X = \{A : 1 \in A, A \subseteq U\}$

then X satisfies the condition

and $|X| = 2^{n-1}$

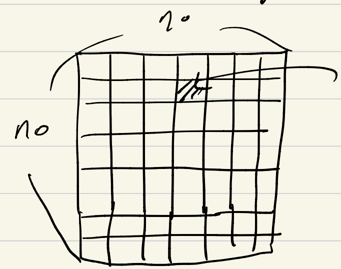
The answer is 2^{n-1} □

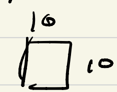
ex). There are 50 points inside



Show that there are 2 points whose distance is < 15.

Pf) Divide the square by 10×10 squares.



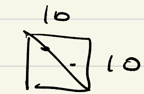


we have 49 $\square$ squares.

Pigeons : 50 points.

Pigeonholes : 49 $\square$ squares.

There are two points in on



max distance of these two points

is $10\sqrt{2} = 14.xx < 15$. □

ex). An inventory consists of a list of 80 items. Each item is marked "available" or "unavailable." There are 45 available items. Show that there are two available items exactly 9 items apart. 9만큼 떨어진 아이템이 존재 (e.g. at positions 13 and 22).

Pf) let $a_1, \dots, a_{45}$ be the positions of available items.

$$1 \le a_1 < \cdots < a_{45} \le 80.$$

We want $a_i = a_j + 9$ for some i, j.

Let $b_i = a_i + 9$, $i = 1, \dots, 45$.

Then

$$1 \le b_1 < b_2 < \cdots < b_{45} \le 89.$$

Pigeons : $a_1, \dots, a_{45}, b_1, \dots b_{45}$ 90개

pigeonholes : $1, 2, \dots 89$. 89개

By pigeonhole principle, we have two integers in $a_1, \dots a_{45}, b_1, \dots b_{45}$ that are equal.

But $a_1, \dots a_{45}$ are all distinct

$b_1, \dots b_{45}$ 〃 .

So, $a_i = b_j$ for some i, j.

Then $a_i = a_j + 9$, so we are done. □.