§1.3 Conditional Propositions and Logical Equivalences.

Def) p, q : propositions

The proposition "if p then q" is called a conditional proposition (조건명제) and denoted by $p \to q$ (we say "p implies q")

ex) If $x=-1$, then $x^2=1$.

Def) The truth value of $p \to q$ is defined as follows.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p가 참인데 q가 거짓인 경우만 거짓

"$p \to q$" is false $\Leftrightarrow$ p is true and q is false.

ex) If you do your homework, then you will get an A.

p: you do your homework
q: you will get an A

The only case that you can claim that the instructor lied is when you did homework but didn't get A

p : true, q : false.

Note $p \to q$ is true whenever p is false

This is said to be true by default or vacuously true.

vacuosly true

q가 뭐든 True!
(무시가능)

ex) p: T, q: F, r: T

① $\underbrace{p \wedge q}_{F} \to \underbrace{r}_{T}$: T

② $\underbrace{p \vee q}_{T} \to \underbrace{\neg r}_{F}$: F

③ $\underbrace{p}_{T} \wedge \underbrace{(q \to r)}_{T}$: T

④ $\underbrace{p}_{T} \to \underbrace{(q \to r)}_{T}$: T.

$\wedge$: and
$\vee$: or
$\neg$: not

Def) The converse 역 of $p \to q$ is the proposition $q \to p$

ex) The converse of "if $x=2$ then $x^2=4$" T is "if $x^2=4$ then $x=2$." F

Note $p \to q$ and $q \to p$ may have different truth values. 대개 다른 값을 가질 것임

Def) "p if and only if q" is denoted by $p \leftrightarrow q$.
The truth value of $p \leftrightarrow q$ is

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

둘다 같아야 T

Note "$p \leftrightarrow q$" is the same as "$p \to q$ and $q \to p$"
$(p \to q) \wedge (q \to p)$.

ex) " $1<5$ (T) if and only if $2<8$ (T) " : T
" $1>5$ (F) if and only if $2>8$ (F) " : T

Def) P, Q : propositions made of propositions $p_1, p_2, \ldots, p_n$.
We say that P and Q are (logically) equivalent if for any truth values of $p_1, \ldots, p_n$, either P and Q are both true or both false. 둘다 참 아니면 거짓
"$P \equiv Q$" means P and Q are equivalent.
→ 논리적 동치

ex) de Morgan's laws 드모르간

① $\neg(p \lor q) \equiv \neg p \land \neg q$

② $\neg(p \land q) \equiv \neg p \lor \neg q$

We can prove this using truth tables.

p	q	$\neg(p \lor q)$	$\neg p \land \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

ex)

"$p \to q$" is false $\Leftrightarrow$ p is true and q is false.

not

"$p \to q$" true $\Leftrightarrow \neg((p \text{ true}) \land (q : F))$

이게 참일려면 으로 바꿔야함

$(p(참) \land \neg q(참))$

$\Leftrightarrow \neg(p \land \neg q)$

$\equiv \neg p \lor (\neg\neg q)$

$\equiv \neg p \lor q$

$$\boxed{p \to q \equiv \neg p \lor q}$$

ex) $p \leftrightarrow q \equiv (p \to q) \land (q \to p)$.

Def) The contrapositive 대우 (or transposition) of $p \to q$ is $\neg q \to \neg p$.

Thm Every conditional proposition is equivalent to its contrapositive.

$p \to q \equiv \neg q \to \neg p$.

§1.4 Arguments and Rules of Inference

The answer is either A or B.
B is not the answer.
Therefore A is the answer.

추론 : 주어진 명제로 새로운 명제가 참임을 증명

Def) An argument is a sequence of propositions

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array} \quad \text{or} \quad p_1, p_2, \dots, p_n \,/\, \therefore q.$$

"$\therefore$" reads "therefore".

$p_1, \dots, p_n$: hypotheses 가정
q : conclusion 결론

The argument is valid provided if $p_1, \dots, p_n$ are true then q is true.

Otherwise it is invalid.

ex)
$$\begin{array}{c} p \to q \\ p \\ \hline \therefore q \end{array}$$

valid

가정 T → 결론 T

: Invalid

To show that this is valid, we need to consider all possible values of p and q (for which the hypotheses are true)

p	q	$p \to q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

← this is the only case that all hyp. are true.

ex)
$$\begin{array}{c} p \to q \\ q \\ \hline \therefore p \end{array}$$

invalid because if $p = F$, $q = T$ then hyp are true but con is false.

✱ Rules of Inference.

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array} \qquad \begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array} \qquad \begin{array}{l} p \\ \hline \therefore p \vee q \end{array} \qquad \begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array} \qquad \begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array} \qquad \begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

ex) If $2=3$, then $\sqrt{2}$ is an integer.
$\sqrt{2}$ is an integer
$\therefore$ $2=3$

의미는 보지말고 구조만

Let's write this using p, q.

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

Invalid.

ex)
If $\underbrace{\text{I study hard}}_{p}$ or $\underbrace{\text{I get rich}}_{q}$, then $\underbrace{\text{I get an A}}_{r}$.
I get an A.
$\therefore$ If I don't study hard, then I get rich.

$$\begin{array}{ll} p \vee q \to r & T \\ r & T \\ \hline \therefore \underbrace{\neg p}_{T} \to \underbrace{q}_{F} & F \end{array}$$

가정T, 결론F 인 예시를 찾자

So $p=F$, $q=F$, $r=T$. ~> counter example.

Invalid.

ex) If I study hard or I get rich, then I get an A.
(p: I study hard, q: I get rich, r: I get an A)

I get an A

∴ If I don't study hard, then I get an A.

$$\begin{array}{ll} p \vee q \to r & T \\ r & T \\ \hline \therefore \neg p \to r & F \end{array}$$

이래야 Invalid인데
r=T이면 결론이 무조건 T라서 Valid하다!
(반례를 찾을 수X)

$r=T,\ \neg p = T,\ r = F$
cannot happen

Since there is no contradiction, arg is valid.

→ 무조건 F일 수밖에

ex)
$$\begin{array}{ll} p \wedge \neg p & T \\ \hline \therefore q & F \end{array}$$
valid.

가정 결론
$T \to F$ 이 경우만 False(Invalid)이기에

Note: $p \wedge \neg p$ is always false.

T 가정
F 결론
이 경우만 피하자!

ex)
$$\begin{array}{ll} p \to (q \to r) & T \\ q \to (p \to r) & T \\ \hline \therefore (p \vee q) \to r & F \end{array}$$

$(p \vee q) \to r$: F
$\Rightarrow p \vee q = T,\ r = F$
If $p = T$ then $(q \to r) = T$, so $q = F$.
In this case, $p=T,\ q=F,\ r=F$.
↳ counterexample. 반례 찾음 → Invalid
invalid.

ex)
$$\begin{array}{ll} (p \to q) \wedge (r \to s) & T \\ p \vee s & T \\ \hline \therefore q \vee s & F \end{array}$$

$q \vee s = F \Rightarrow q = F,\ s = F$ 둘다 F
$p \vee s = T \Rightarrow p = T$
$(p \to q) = F \Rightarrow (p \to q) \wedge (r \to s) = \boxed{F}$ 무조건 F
False
We cannot find a counterexample.
valid.

§1.5 Quantifiers 한정자 ($\forall$, $\exists$ 등..)

Recall : "x is an integer" is not a proposition. However, once x is x가 정해지면! determined, then this statement is is a proposition.

Def) $P(x)$: statement involving a variable x.

D : a set.

명제함수

We say that P is a propositional function with respect to D if for each $x \in D$, $P(x)$ is a proposition.

D is called the domain of discourse of P.

논의 영역

Note : $P : D \to \{T, F\}$.

ex) $P(n) =$ "n is an integer".

$D = \mathbb{R}$. 실수

P is a propositional function with domain of discourse D.

$P(3) =$ "3 is an integer" (true)

$P(\sqrt{2}) =$ "$\sqrt{2}$ is an integer" (false).

Def) P : a propositional function with domain of discourse D.

"$\forall x\, P(x)$" means for all x, $P(x)$.

$\forall x\, P(x)$ $\qquad$ $x \in D$

ex) $\forall x\, (x^2 \geq 0)$, $D = \mathbb{R}$. true.

ALL $\qquad$ P $\qquad$ x가 Real number 안에서 P를 판별

Note : $\forall x\, P(x)$ is false if for at least one $x \in D$, $P(x)$ is false.

A value of x for which $P(x)$ is false is called a counterexample to $\forall x\, P(x)$

반례

ex) $D = \mathbb{R}$ $\qquad$ $\forall x\, (x^2 - 1 > 0)$.

False. $\quad$ $x = 0$ is a counterexample.

$x^2 - 1 = 0^2 - 1 = -1 > 0$ (false).

$\forall x\, P(x)$: 모든 x에 대해 P(x)가 성립한다.

<u>Note</u> We also say

for all x, $P(x)$

= for arbitrary x, $P(x)$

= for every x, $P(x)$

= for any x, $P(x)$

= $\forall x\ P(x)$.

Exist

Def) " $\exists x\ P(x)$ " means

" there exists x, $P(x)$ ".

This is true if there is at least one

$x \in D$ for which $P(x)$ is true.

$\exists x\ P(x)$

x가 하나라도 존재하면 $P(x)$ 성립

ex) $\exists x \left(\frac{x}{x^2+1} = \frac{2}{5} \right)$ $D = \mathbb{R}$.

If $x=2$, then $\frac{x}{x^2+1} = \frac{2}{2^2+1} = \frac{2}{5}$.

True.

ex) $\exists x \in \mathbb{R} \left(\frac{1}{x^2+1} > 1 \right)$

If $x \in \mathbb{R}$, then $x^2 \geq 0$

So $x^2+1 \geq 1$. Then $\frac{1}{x^2+1} \leq 1$.

Therefore, there is no $x \in \mathbb{R}$ s.t. $\frac{1}{x^2+1} > 1$.

하나도 없기에

$\Rightarrow$ False.

ex) For some prime p, $p+1$ is also prime.

True. If $p=2$, then $p=2$ prime

$p+1=3$ prime.

괄호로 들어가면 $\forall \leftrightarrow \exists$ 서로 바뀜

Thm ① $\neg(\forall x\, p(x)) \equiv \exists x \neg p(x)$

② $\neg(\exists x\, p(x)) \equiv \forall x \neg p(x)$.

pf) ① Suppose $\neg(\forall x\, p(x))$ is true.

Then $\forall x\, p(x)$ is false. 모든 x에 대해 성립

There exists at least one x s.t. $p(x)$ is false, i.e., $\neg p(x)$. 성립하지 않게 하는 x가 하나 존재

Thus $\therefore$ $\boxed{\exists \neg p(x)}$ is true.

p(x)가 거짓이 되도록 하는 x가 존재

True $\updownarrow$ False

Suppose $\neg(\forall x\, p(x))$ is false.

Then $\forall x\, p(x)$ is true.

For all x, $p(x)$ is true.

$\Rightarrow$ There is no x, s.t. $p(x)$ false.

$\Rightarrow$ $\exists x \neg p(x)$ is false.

② Exercise. 연습해보자! $\square$

Note 모든 X에 대해 성립의 반대

Not all x satisfy $p(x)$.

$\neq$ All x do not satisfy $p(x)$. 성립하는 것도 있을 수 있기에

$\neg(\forall x\, p(x)) \not\equiv \forall x(\neg p(x))$

모든 성립하지 않는다 (X)

ex) Not all integers are positive. : T

All integers are not positive. : F

ex) Not everything that glitters is gold.

$p(x)$: x glitters

$Q(x)$: x is gold.

$\neg(\forall x\,(p(x) \to Q(x)))$

$\equiv \exists x(\neg(p(x) \to Q(x)))$

$\equiv \exists x\,(p(x) \wedge \neg Q(x))$

There exists something that glitters but is not gold.

same meaning

ex) $\exists x \in \mathbb{R}$ 실수 $(x^2 - 2x + 1 = 0)$.

True. $x = 1$.

ex) $\exists x \in \mathbb{R}$ 실수 $(\underbrace{x^2 + x + 1}_{\text{doesnot have real roots.}} = 0)$.

False.

doesnot have real roots. 실근

§1.6. Nested quantifiers
중첩 한정자

We can have $P(x,y)$, a propositional function with 2 variables.
We can use $\forall x$, $\exists x$, $\forall y$, $\exists y$ together.

$\forall x \forall y\, P(x,y)$ means for all x, for all y, $P(x,y)$.

$\forall x \exists y\, P(x,y)$ means for all x, there exists y, $P(x,y)$.

ex) $\forall m \exists n\, (m<n)$. $D = \mathbb{Z}\times\mathbb{Z}$
for all m, there is n s.t. $m<n$.
True, if $n=m+1$, then $m<n$.
n을 이렇게 설정하면 모든 m에 대해 성립하는 n이 적어도 하나 존재

ex) $\exists n \forall m\, (m<n)$. $D=\mathbb{Z}\times\mathbb{Z}$.
False. For every n, if $m=n$, then $m<n$ false.

The order of quantifiers is important.
중첩 순서가 중요하다!

ex) $\forall x \forall y\, ((x>0)\wedge(y>0) \rightarrow (x+y>0))$ $\mathbb{R}\times\mathbb{R}$
True.

ex) $\forall x \forall y\, ((x>0)\vee(y>0) \rightarrow (x+y>0))$ $\mathbb{R}\times\mathbb{R}$.
False. $x=1$, $y=-2$. $x+y=-1$.

ex) $\forall x \exists y\, (x+y=0)$ $\mathbb{R}\times\mathbb{R}$.
True. Take $y=-x$.

ex) $\forall x \exists y\, (x>y)$ $\mathbb{Z}^+\times\mathbb{Z}^+$
False. $x=1$. There is no positive integer y s.t. $x>y$.

ex) $\exists x \exists y\, ((x>1)\wedge(y>1)\wedge(xy=6))$ $\mathbb{Z}\times\mathbb{Z}$.
True. $x=2$, $y=3$.

ex) $\exists x \exists y\, ((x>1)\wedge(y>1)\wedge(xy=7))$ $\mathbb{Z}\times\mathbb{Z}$.
False. 7 is prime.
소수 (곱해서 나올 수 X)

ex) $\lim_{x \to a} f(x) = L$

임의의 작은 값 δ 에 대해 적절한 ε을 찾을 수 있다

$\Leftrightarrow$ For every $\delta > 0$, there exists $\varepsilon > 0$ s.t.

$0 < |x-a| < \varepsilon \rightarrow |f(x) - L| < \delta$

$\Leftrightarrow$ $\forall \delta > 0 \; \exists \varepsilon > 0$ $\left(0 < |x-a| < \varepsilon \rightarrow |f(x) - L| < \delta \right)$

ex) No matter what you imagine this movie will show you more than that.

x : imagination

y : a scene in the movie.

$p(x,y)$: y is more than x.

$\forall x \exists y \; P(x,y)$.

ex) $\mathbb{R} \times \mathbb{R}$.

① $\forall x \forall y \; ((x<y) \rightarrow (x^2<y^2))$

② $\forall x \exists y$ ($p \rightarrow q$)

③ $\exists x \forall y$ (〃)

④ $\exists x \exists y$ (〃)

sol) ① False. $x=-1$, $y=0$. 반례

p → q 가 T 면 됨 (p가 F면 무조건 T)

② True. For all x, we can take $y = x$.

③ True. Take $x = 0$.

④ True Take $x=0$, $y=1$.