Discrete Mathematics Final Exam (GEDB007-43, Fall 2021)

Scores: Highest 10 (27) means 27 students got the highest score 10.

	P1	P2	P3	P4	P5	P6	P7	P8	Total
Average	4.8	7.4	2.3	8.5	1.6	6.0	10.7	4.9	42.5
Highest	10 (13)	15 (12)	10 (8)	10 (44)	15 (5)	15 (1)	15 (23)	10 (11)	96 (1)

Problem 1. Ch 6. **[10 points]** Find the number of integers n such that $5000 < n < 10000$ and $\gcd(n, 2021) = 1$. (Hint: $45^2 = 2025$.)

Solution. We have $2021 = 2025 - 4 = 45^2 - 2^2 = 43 \cdot 47$. Let $U = \{5001, 5002, \dots, 9999\}$, $A = \{n \in U : 43|n\}$, and $B = \{n \in U : 47|n\}$. Then the answer is

$$|U - (A \cup B)| = |U| - |A| - |B| + |A \cap B|. \qquad \textbf{[2 points]}$$

Since

$$\begin{aligned}
|U| &= 4999,\\
|A| &= \left\lfloor \frac{9999}{43} \right\rfloor - \left\lfloor \frac{5000}{43} \right\rfloor = 232 - 116 = 116, \qquad \textbf{[2 points]}\\
|B| &= \left\lfloor \frac{9999}{47} \right\rfloor - \left\lfloor \frac{5000}{47} \right\rfloor = 212 - 106 = 106, \qquad \textbf{[2 points]}\\
|A \cap B| &= \left\lfloor \frac{9999}{2021} \right\rfloor - \left\lfloor \frac{5000}{2021} \right\rfloor = 4 - 2 = 2, \qquad \textbf{[2 points]}
\end{aligned}$$

the answer is $4999 - 116 - 106 + 2 = 4779$ **[2 points]**. □

Problem 2. Ch 6. **[15 points]** Let $X = \{1, 2, \dots, n\}$ and $Y = \{1, 2, 3\}$. Find the number of onto functions $f : X \to Y$.

Solution. Let U be the set of all functions $f : X \to Y$. For $i = 1, 2, 3$, let A_i be the set of functions $f \in U$ such that i is not in the image of f. Then the number of onto functions $f : X \to Y$ is $|U - (A_1 \cup A_2 \cup A_3)|$ **[2 points]**.

By the principle of inclusion and exclusion, this number is equal to

$$|U| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|. \qquad \textbf{[2 points]}$$

Now we will compute each term in the above formula. Since $|U|$ is the number of functions $f : X \to Y$, we have $|U| = 3^n$ **[2 points]**. Since we can consider each function $f \in A_i$ as a function from X to $Y - \{i\}$, we have $|A_i| = 2^n$ **[2 points]**. Similarly, for distinct i, j, we have that $|A_i \cap A_j|$ is equal to the number of functions $f : X \to Y - \{i, j\}$, which is $1^n = 1$ **[2 points]**.

Finally, $|A_1 \cap A_2 \cap A_3|$ is equal to the number of functions $f : X \to Y - \{1, 2, 3\}$, which is $0^n = 0$ **[2 points]**.

Therefore, the answer is $3^n - 3 \cdot 2^n + 3$ **[3 points]**. □

Problem 3. Ch 7. **[10 points]** For a positive integer n, let a_n be the maximum number of regions that can be created by drawing n squares on a plane. For example, $a_1 = 2$ and $a_2 = 10$. Compute a_{25}.

Solution. Suppose $n \ge 2$ and there are $n - 1$ squares already drawn with a_{n-1} regions. Observe that two squares can intersect at most 8 points **[2 points]**. Therefore, if we add one more square, there will be at most $8n - 8$ intersections with the new square and the existing squares **[2 points]**. Then this new square has $8n - 8$ segments (or L-shapes) each of which will divide one region into two. Therefore $a_n = a_{n-1} + 8(n - 1)$ **[2 points]**. Using the recurrence we obtain

$$\begin{aligned}
a_n &= 8(n-1) + a_{n-1} = 8(n-1) + 8(n-2) + a_{n-2} = \cdots\\
&= 8(n-1) + 8(n-2) + \cdots + 8 \cdot 1 + a_1 = 4n(n-1) + 2. \qquad \textbf{[2 points]}
\end{aligned}$$

Thus the answer is $a_{25} = 2402$ **[2 points]**. □

Problem 4. Ch 7. **[10 points]** Find a formula for the nth term a_n of the sequence $a_0, a_1, a_2, \dots$, which satisfies the initial conditions $a_0 = -2$, $a_1 = \frac{7}{3}$ and the recurrence relation given by

$$12a_n = 4a_{n-1} + a_{n-2}.$$

Solution. Dividing both sides by 12, we get

$$a_n = \frac{a_{n-1}}{3} + \frac{a_{n-2}}{12}. \qquad \textbf{[2 points]}$$

The characteristic polynomial is $x^2 - \frac{1}{3}x - \frac{1}{12} = (x - \frac{1}{2})(x + \frac{1}{6})$ **[2 points]**. Thus $a_n = \alpha(\frac{1}{2})^n + \beta(-\frac{1}{6})^n$ for some α, β **[3 points]**. Using the initial conditions $a_0 = -2$ and $a_1 = \frac{7}{3}$, we obtain $\alpha = 3, \beta = -5$. Thus $a_n = 3(\frac{1}{2})^n - 5(-\frac{1}{6})^n$ **[3 points]**. □

Problem 5. Ch 8 [**15 points**] Let r be a positive integer. We say that a simple graph $G = (V, E)$ is *r-partite* if there exist r subsets $V_1, \dots, V_r$ of V satisfying the following conditions:

(1) $V_1 \cup \cdots \cup V_r = V$,

(2) $V_i \cap V_j = \emptyset$ for all $1 \le i < j \le r$,

(3) $(V_i \times V_i) \cap E = \emptyset$ for all $i = 1, 2, \dots, r$.

The *complement* of G is the simple graph $\overline{G}$ with the same vertices as G such that an edge exists in $\overline{G}$ if and only if it does not exist in G.

Find the largest integer n satisfying the following condition: there exists a 2021-partite graph G with n vertices such that $\overline{G}$ is planar.

Solution. Suppose that $G = (V, E)$ is a graph satisfying the given condition. Then $V = V_1 \cup \cdots \cup V_{2021}$ and there are no edges between two vertices in V_i for all i. Therefore $\overline{G}$ contains $K_{|V_i|}$ [**3 points**]. Since $\overline{G}$ is planar we must have $|V_i| \le 4$ for all i [**3 points**]. Then $n = |V| = |V_1| + \cdots + |V_{2021}| \le 4 \cdot 2021 = 8084$ [**3 points**]. If $|V_i| = 4$ for all i and G is the graph such that $(u, v) \in E$ for all $u \in V_i$ and $v \in V_j$ with distinct i, j, then $\overline{G}$ is the union of K_4's, so it is planar [**3 points**]. Therefore the largest integer n is 8084 [**3 points**]. □

Problem 6. Ch 8 [**15 points**] Let G be a simple graph on $\{1, 2, \dots, 10\}$. Suppose that G has 3 connected components and every vertex has degree 2.

(1) Find the number of possible graphs G.

(2) Let A be the adjacency matrix of G with respect to the vertex ordering $1, 2, \dots, 10$. Find the sum of the diagonal entries of $A^2 + A^3 + A^4$.

Solution. (1) Every connected component of G is a cycle [**2 points**]. Since G has 10 vertices and 3 connected components, it must have one 4-cycle and two 3-cycles [**2 points**]. The number of such graphs is

$$\binom{10}{4}\binom{6}{3} \cdot \frac{1}{2} \cdot \frac{(4-1)!}{2} \frac{(3-1)!}{2} \frac{(3-1)!}{2} = \binom{10}{4}\binom{6}{3}\frac{3}{2}. \qquad \textbf{[3 points]}$$

(2) Without loss of generality we may assume that G has cycles $(1, 2, 3)$, $(4, 5, 6)$, and $(7, 8, 9, 10)$. The sum of the diagonal entries of A^2 is 2 times the number of edges, which is $2 \cdot (3 + 3 + 4) = 20$ [**2 points**].

The sum of the diagonal entries of A^3 is the number of cycles (a, b, c) with a choice of the starting point. For each vertex x in a 3-cycle, there are two ways to make a 3-cycle staring at x. Thus the sum of the diagonal entries of A^3 is $6 \cdot 2 = 12$ [**2 points**].

The sum of the diagonal entries of A^4 is the number of 4-cycles (a, b, c, d) with a choice of the starting point (repeated vertices are allowed). For every vertex x, we can make a 4-cycle (x, y, x, z) in 4 ways because y and z can be any neighbors of x. There are $10 \cdot 4 = 40$ such 4-cycles. For every vertex x, we can also make a 4-cycle (x, y, z, y) with $x \ne z$ in 2 ways because y can be any neighbor of x and z is the unique neighbor of y not equal to x. There are $10 \cdot 2 = 20$ such 4-cycles [**2 points**]. A 4-cycle without repeated vertices is obtained by selecting a vertex in $\{7, 8, 9, 10\}$ and choosing a direction. Thus the number of 4-cycles without repeated vertices is $4 \cdot 2 = 8$ [**2 points**]. Thus the sum of the diagonal entries of A^4 is $60 + 8 = 68$.

By the above computations we obtain that the sum of the diagonal entries of $A^2 + A^3 + A^4$ is $20 + 12 + 68 = 100$. □

Problem 7. Ch 9 [**15 points**] Consider the set of letters in the following table.

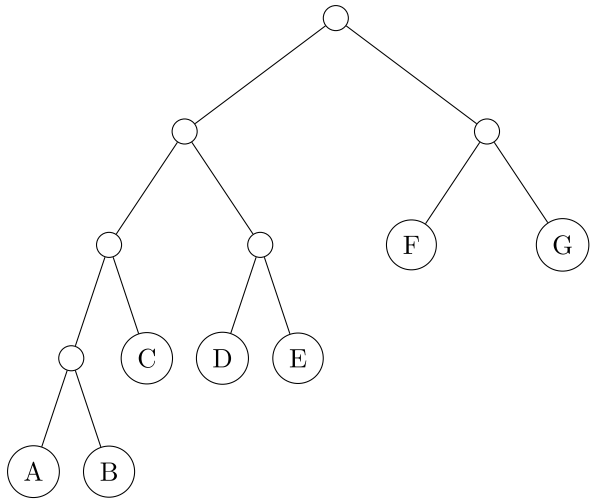
letter	A	B	C	D	E	F	G
frequency	1	1	2	3	3	6	9

(1) Find an optimal Hoffman code for these letters. (Draw a tree and write a 0-1 sequence for each letter. There are many possible optimal Hoffman codes, and you need to find just one of them.)

(2) Encode "CAGE" using the Hoffman code.

(3) Find the number of words with three letters (repetitions allowed) in A, B, C, D, E, F, G such that the length of the code using the optimal Hoffman code is 9. For example, if the letter A has code 10 and the letter B has code 01001, then the word ABA has code 100100110 whose length is 9.

Solution. (1) A possible answer is this [**3 points**].



Assign 1 to each left edge and 0 to each right edge. Then we obtain the following code [**3 points**].

letter	A	B	C	D	E	F	G
string	1111	1110	110	101	010	01	00

(2) Using the above code the answer is 110111100010 [**3 points**].

(3) Let XYZ be such a word. Since every letter has length $2, 3$, or 4, the sequence of lengths of X, Y, Z must be a permutation of $3, 3, 3$ or $2, 3, 4$. If it is $3, 3, 3$, then each of X, Y, Z is in C, D, E, so there are $3^3 = 27$ such words [**3 points**]. If it is a permutation of $2, 3, 4$, then word XYZ contains one letter in A, B, one letter in C, D, E, and one letter in F, G. The number of such words is $2 \cdot 3 \cdot 2 \cdot 3! = 72$ [**3 points**]. Thus the answer is $27 + 72 = 99$.

□

Problem 8. Ch 9. [**10 points**] Let $A = 1, B = 2, C = 3, D = 4, E = 5, F = 6, G = 7$. Compute each expression, which is either a prefix form or a postfix form. Write "invalid" if the expression is not valid as a prefix or postfix expression.

(1) $AB * C - DEF * +/$

(2) $AB + CD - /E * FG + -$

(3) $AB + CD - /EFG + *$

(4) $- * +ABC - D + EF$

Solution. (1) $AB * C - DEF * +/ = ((A * B) - C)/(D + (E * F)) = -\frac{1}{34}$ [**3 points**]

(2) $AB + CD - /E * FG + - = ((A + B)/(C - D)) * E - (F + G) = -28$ [**3 points**]

(3) $AB + CD - /EFG + *$ is invalid. [**1 points**]

(4) $- * +ABC - D + EF = ((A + B) * C) - (D - (E + F)) = 16$ [**3 points**]

□