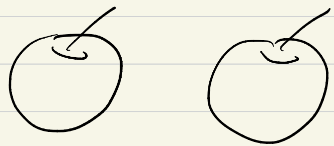
동치 관계

§3.4. Equivalence Relations.



① Apple 1 $\equiv$ Apple 1

② Apple 1 $\equiv$ Apple 2 $\Rightarrow$ Apple 2 $\equiv$ Apple 1.

③ Apple 1 $\equiv$ Apple 2 and Apple 2 $\equiv$ Apple 3
$\Rightarrow$ Apple $\equiv$ Apple 3.

$X \to X$

Def) A relation on X is called an equivalence relation if it is

① reflexive 반사성

② symmetric 대칭성

③ transitive. 전이성

ex) R : relation on $\mathbb{Z}$ a+b이면 $(a+b) \in R$

$(a,b) \in R$ iff $a+b$ is even.

Then R is an equiv. rel.   R이 동치관계임? 을 물어보는 문제

$\forall x \in \mathbb{Z}$

① reflexive : $x+x = 2x$ even $\Rightarrow (x,x) \in R$. T

② sym : $x+y = 2k \Rightarrow y+x = 2k$ T

③ trans : $x+y = 2k,\ y+z = 2l$

$\Rightarrow x+y+y+z = 2k+2l$ T

$x+z = 2(k+l-y)$.

ex) $\boxed{\le \text{ on } \mathbb{R}}$ is not an equiv rel. 동치가 아님

$\le$ is not sym. $1 \le 2$ but $2 \le 1$.

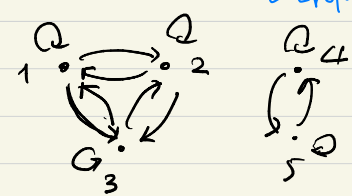
Def) R : equiv rel on X.

For $a \in X$, the equivalence class containing a is $[a] = \{x \in X \mid (a,x) \in R\}$.

동치류 class

다이어그래프 사용

ex)



$\boxed{R}$ eq. rel. on $\{1, \ldots, 6\}$.

The equivalence classes are

다이어그래프의 연결된 요소끼리

$\{1,2,3\}$, $\{4,5\}$, $\{6\} = [6]$

$[1] = [2] = [3]$ $[4] = [5]$

R의 다이어그래프 이렇게 주어졌을때 동치류 파악하기!

Def) A partition (파티션) of X is a collection of nonempty subsets $B_1, \dots, B_k$ (finite) of X such that

① $B_i \cap B_j = \phi$ if $i \neq j$

② $B_1 \cup \cdots \cup B_k = X$.

Thm X : a set. S : a partition of X.
Define a relation R on X by

$(x,y) \in R \iff x, y$ are in the same set $B \in S$.

Then R is an equivalence relation.

pf) Let $S = \{B_1, \dots, B_k\}$.

Then $B_i \cap B_j = \phi$ if $i \neq j$, $B_i \neq \phi$ $\forall i$.

$B_1 \cup \cdots \cup B_k = X$.

① reflexive : For every $x \in X$, $x \in B_i$ for some i.
Then $x \in B_i, x \in B_i \Rightarrow (x,x) \in R$.

② sym : Suppose $(x,y) \in R$. Then $x, y \in B_i$ for some i. Then $y, x \in B_i \Rightarrow (y,x) \in R$.

③ trans : Suppose $(x,y) \in R$, $(y,z) \in R$.
$x, y \in B_i$, $y, z \in B_j$ for some i, j.
Because $B_i \cap B_j$ contains y, $i = j$.
$x \in B_i$, $z \in B_i \Rightarrow (x,z) \in R$. □

ex) $S = \{\underset{B_1}{\{1,4\}}, \underset{B_2}{\{2,3,6\}}, \underset{B_3}{\{5\}}\}$ is a partition (Partition) of $X = \{1,2,3,4,5,6\}$.

$R = \{(1,1), (1,4), (4,1), (4,4),$
$(2,2), (2,3), (2,6), (3,2), (3,3), (3,6),$
$(6,2), (6,3), (6,6), (5,5)\}$.

같은 set (B)에 있는 (x, y) → R에 속함

ex (continued)

The equivalence classes of R are

$[1] = \{1, 4\}$

$[2] = \{2, 3, 6\}$.

$[5] = \{5\}$

(X : finite).

Thm R : an equiv rel on X.
Then the set of equivalence classes of R is a partition of X.
In other words, $S = \{[a] \mid a \in X\}$ is a partition of X. ∴ partition = equivalence class

$B_1 \quad B_2 \ldots \quad B_n$

Pf) let $S = \{[a_1], [a_2], \ldots [a_k]\}$, $[a_i] \neq [a_j]$.
We need to show
① $[a_i] \neq \phi$, $\forall i$
② $[a_i] \cap [a_j] = \phi$ if $i \neq j$.
③ $[a_1] \cup \cdots \cup [a_k] = X$.

a_i 라는 원소가 하나라도

①: Since $a_i \in [a_i]$ so $[a_i] \neq \phi$. 존재함

대우

∵ 존재하지 X → $[a_i]$ 자체가 없음

②: We will show that $[a_i] \cap [a_j] \neq \phi \Rightarrow [a_i] = [a_j]$.
Suppose $[a_i] \cap [a_j] \neq \phi$.
Then $x \in [a_i] \cap [a_j]$ for some $x \in X$.
Then $(a_i, x) \in R$, $(x, a_j) \in R$.

Since R is trans, $(a_i, a_j) \in R$.
let $y \in [a_i]$. $\Rightarrow (y, a_i) \in R$.
Since R is trans, $(y, a_j) \in R$.
$\Rightarrow y \in [a_j]$

부분집합

$\Rightarrow [a_i] \subseteq [a_j]$.
Similarly $[a_j] \subseteq [a_i]$, so $[a_i] = [a_j]$.

③ let $x \in X$.
Then $[x]$: an equiv class.
$[x] = [a_i]$ for some a_i.
$\Rightarrow x \in [x] = [a_i]$
$\Rightarrow x \in [a_1] \cup \cdots \cup [a_k]$ □

ex) $X = \{1, 2, \ldots, 10\}$.

divides

$(x, y) \in R \iff 3 \mid x - y$. $\frac{x-y}{3}$

Then R is an equiv rel.

pf). ① reflexive : $3 \mid x - x = 0$, $(x, x) \in R$.

② sym : $(x, y) \in R \Rightarrow 3 \mid x - y \Rightarrow 3 \mid y - x$

$\Rightarrow (y, x) \in R$.

③ trans : $(x, y) \in R$, $(y, z) \in R$.

Then $3 \mid x - y$, $3 \mid y - z$.

$\Rightarrow 3 \mid (x - y) + (y - z) \rightarrow 3 \mid x - z$

$\Rightarrow (x, z) \in R$.

equiv classes.

$\frac{x-y}{3}$의 값 나머지

$1-4/3 = -1$ 나눌수 있음

$[1] = \{1, 4, 7, 10\}$.

$[2] = \{2, 5, 8\}$

$[3] = \{3, 6, 9\}$.

$\{\{1, 4, 7, 10\}, \{2, 5, 8\}, \{3, 6, 9\}\}$

is a partition of X.

$x \mid y$ = x가 y를 나눈다 (y가 x의 배수이다

나눌수 있는지 없는지 파악!

행렬

§3.5. Matrices of relations.

Recall: A relation R from X to Y is ($\subset X \times Y$)
a set of pairs (x,y) where $x \in X$, $y \in Y$.

ex) R: relation from $X = \{1,2,3,4\}$
to $Y = \{a,b,c,d\}$ given by
$R = \{(1,a),(1,d),(2,c),(4,b),(4,d)\}$.

Def) R : relation from X to Y.
The matrix of R (relative to some orderings of $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_m\}$)
is the $n \times m$ matrix $M = (a_{ij})$
such that

$$a_{ij} = \begin{cases} 1 & \text{if } (x_i, y_j) \in R \\ 0 & \text{otherwise.} \end{cases}$$

관계에있으면 1
없다면 0

ex)

$$M = \begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 1 \end{array}$$

$X \times Y$ 가 전체

Note that M depends on the ordering of X, Y.
ordering

If we consider $X = \{1,3,4,2\}$
$Y = \{a,b,d,c\}$

$$M = \begin{array}{c|cccc} & a & b & d & c \\ \hline 1 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}$$

↓ may be I can give a detailed proof.

Thm R_1 : relation from X to Y

R_2 : 〃 Y to Z.

A_1 : matrix of R_1

A_2 : 〃 R_2.

The matrix of $R_2 \circ R_1$ has the same nonzero entries as $A_1 A_2$. 0이 아니면 $R_2 \circ R_1$ 관계에 있음

$$\boxed{A(R^{-1}) = A^T(R)}$$

ex) $X = \{1,2,3\}$ $Y = \{a,b\}$, $Z = \{x,y,z\}$.

$R_1 = \{(1,a),(2,b),(3,a),(3,b)\}$

$R_2 = \{(a,x),(a,y),(b,y),(b,z)\}$.

$R_2 \circ R_1 = \{(1,x),(1,y),(2,y),(2,z),$
$(3,x),(3,y),(3,z)\}$.

$$A_1 = \begin{array}{c|cc} & a & b \\ \hline 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{array} \qquad A_2 = \begin{array}{c|ccc} & x & y & z \\ \hline a & 1 & 1 & 0 \\ b & 0 & 1 & 1 \end{array}$$

$\therefore 3 \to a, b \longrightarrow a \to x,\ b \to x$

$$A_1 A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$\to \boxed{1 \cdot 1} + 1 \cdot 0 = 1$

The matrix of $R_2 \circ R_1$ is

$$\begin{array}{c|ccc} & x & y & z \\ \hline 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & 1 & 1 & 1 \end{array}$$

ex). $R = \{(a,a),(b,b),(c,c),(d,d),(a,c),(c,b)\}$.

R is a relation on $X = \{a,b,c,d\}$

$$A = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array} \qquad A^2 = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$a \to c \to b$

Question: Is R transitive? No.

R transitive $\Longleftrightarrow$ $(x,y),(y,z) \in R \Rightarrow (x,z) \in R$

Cor R : relation on $X = \{x_1, \dots, x_n\}$.

A : matrix of R.

동치

R is transitive if and only if the following holds

$(*)$ If (i,j)-entry of A^2 is nonzero ; $i \to j$ 까지 경유해서 도달가능

then (i,j)-entry of A is also nonzero

$i \to j$ 존재X

$\Downarrow$

전이성X

Pf). $(A^2)_{ij} \neq 0 \overset{\text{Thm}}{\Longleftrightarrow} (x_i, x_j) \in R \circ R \Longleftrightarrow (x_i, x_k), (x_k, x_j) \in R$ for some k.

transitive $\Longleftrightarrow$ $(x_i,x_k),(x_k,x_j) \in R \to (x_i,x_j) \in R$ $(A)_{ij} \neq 0$ □

ex) $R=\{(a,a),(b,b),(c,c),(d,d),(b,c),(c,b)\}$
is a relation on $X=\{a,b,c,d\}$

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad A^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ R is transitive.

Thm R : relation on X.
A : matrix of R.

대각성분만 모두 1이면!

R is reflexive $\Longleftrightarrow$ $A = \begin{pmatrix} 1 & & ? \\ & \ddots & \\ ? & & 1 \end{pmatrix}$

row→col col→row

R is symmetric $\Longleftrightarrow$ $A = A^T$
A is symmetric.

R is antisymmetric $\Longleftrightarrow$ A does not have

대각선 기준 대칭적으로 모두 1을 갖는 i, j가 없으면 antisym!

$$\begin{array}{c} \\ i \\ j \end{array} \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{pmatrix} \ddots & & 1 \\ & \ddots & \\ 1 & & \ddots \end{pmatrix} \end{array}$$

R is transitive $\Longleftrightarrow$ $\forall i,j \left((A^2)_{ij} \neq 0 \Rightarrow A_{ij} \neq 0 \right)$.

ex) $R=\{(a,b)\}$ on $X=\{a,b\}$.

$$A = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array} \qquad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

R is not reflexive
R is not sym $\quad A^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
R is antisym.
R is trans.

문제 많이 풀어보기!!

Ch 4. Algorithms.

§4.1 Introduction

An <u>algorithm</u> is something that receives an input and does something with it and returns an output.

pseudocode : Not an actual code like C, Java, Python, ...

<u>Algorithm</u> : Finding the max of three numbers.

Input : a, b, c

Output : large (the largest among a, b, c)

Input

$\{ \sim \}$: body

```
max3 (a, b, c) {
    large = a
    if (b > large)    // if b is larger than large
        large = b     //         update large.
    if (c > large)
        large = c
    return large
}
```

ex) let's compute $\max3(1, 5, 2)$.

$$\begin{array}{ccc} 1, & 5, & 2 \\ \shortparallel & \shortparallel & \shortparallel \\ a & b & c \end{array}$$

large = 1

because 5 > large, <u>large = 5</u>

because $2 \not> $ large, large is not updated.

returns <u>5</u>.

$(s = (s_1, s_2, \ldots, s_n))$

Alg : Finding the max in a sequence.

Input : s, n (s : sequence, n : integer)
Output : large.

수열

```
max (s, n) {
    large = s_1
    for i = 2 to n
        if (s_i > large)
            large = s_i
    return large.
}
```

들여쓰기 indentation

ex). $s = (5, 2, 1, 4, 6, 2, 2)$, $n = 7$.
max(s, n).

large = $\cancel{5}$. $\underline{\underline{6}}$

소수

Alg Determine whether n is prime.

Input : n (pos. int)
Output : True or False.

```
is_prime (n) {
    if n == 1
        return false.
    for i = 2 to n-1
        if (i divides n)  ←  (n % i == 0)
            return false
    return true.
}
```

actual code

Alg. Find the second largest number among three numbers.

Input: a, b, c (distinct real numbers)

Output: second largest number

1, 2번째 비교

```
max2(a,b,c) {
    if (a > b)
        m1 = a,  m2 = b.
    else
        m1 = b,  m2 = a
```

가장 큰 값과 3번째 비교

```
    if (c > m1)
        m2 = m1
        m1 = c
```

출력

```
    if (m2 < c < m1)
        m2 = c
    return m2
}
```

ex). max2(5, 4, 6).

$m_1 = \cancel{5}$, $m_2 = \cancel{4}$

$\quad\;\; 6 \qquad\quad 5$