Discrete Mathematics Midterm Exam (Spring 2021)

Scores: Highest 10 (20) means 20 students got the highest score 10.

	P1	P2	P3	P4	P5	P6	P7	P8	Total
Average	6.8	5.3	7.8	3.6	3.7	7.0	5.9	4.9	41.6
Highest	10 (20)	10 (5)	15 (24)	10 (1)	15 (4)	15 (18)	10 (30)	15 (15)	81 (2)

Problem 1. Ch 1 [**10 points**] Prove or disprove each statement.

(1) The following argument is valid:

$$\begin{array}{l} q \to (r \wedge p) \\ (p \vee q) \to (r \wedge \neg p) \\ \neg r \wedge q \\ \hline \therefore r \vee \neg r \end{array}$$

(2) The following argument is valid:

$$\begin{array}{l} q \to (r \wedge p) \\ p \\ \neg p \\ \hline \therefore \neg q \wedge \neg r \end{array}$$

(3) The following argument is valid:

$$\begin{array}{l} \neg r \to \neg p \\ r \to (\neg p \vee \neg q) \\ \hline \therefore p \to (q \wedge r) \end{array}$$

(4) $(p \to q) \equiv (\neg p \vee q)$.

(5) $(p \to q) \wedge (q \to r) \equiv (p \to r)$.

Solution. (1) Since the conclusion $r \vee \neg r$ is always true, the argument is valid. [**2 points**]

(2) Since the two assumptions p and $\neg p$ cannot be satisfied, the argument is valid. [**2 points**]

(3) Since $p = T, q = F, r = T$ is a counterexample, it is invalid. [**2 points**]

(4) The following truth table shows that it is true. [**2 points**]

p	q	$p \to q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

(5) $p = F, q = T, r = F$ is a counterexample, so it is false. [**2 points**]

□

Problem 2. Ch 1 [**10 points**] Prove or disprove each statement.

(1) $\forall x \forall y((x < y) \to (x^2 > y^2))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(2) $\forall x \exists y((x < y) \to (x^2 > y^2))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(3) $\exists x \forall y((x < y) \to (x^2 > y^2))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(4) $\exists x \exists y((x < y) \to (x^2 > y^2))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

Solution. (1) Since $x = -1, y = 1$ is a counterexample, it is false. [**2 points**]

(2) For every $x \in \mathbb{R}$, if we take $y = x - 1$, then since the assumption $x < y$ is not satisfied, the statement is true. [**2 points**]

(3) If $x \geq 0$, then $(x < y) \to (x^2 < y^2)$ for all y, so there is no y satisfying the condition. [**2 points**] If $x < 0$, then taking $y = -x$ gives $x < y$ but $x^2 \not> y^2$. Therefore the statement is false. [**2 points**]

(4) If we take $x = 1, y = 0$, then since the assumption $x < y$ is not satisfied, the statement is true. [**2 points**]

$\square$

Problem 3. Ch 2 [**15 points**]

Prove that $\forall n \exists a \exists b((n \geq 12) \to (n = 3a + 7b))$, where the domain of discourse is $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

Solution. We use strong induction on n. If $n = 12, 13, 14$, then it is true because $12 = 3 \cdot 4 + 7 \cdot 0$, $13 = 3 \cdot 2 + 7 \cdot 1$, and $14 = 3 \cdot 0 + 7 \cdot 2$. [**5 points**]

Let $k \geq 15$ and assume that it is true for all $12 \leq n < k$. Consider the case $n = k$. Then since $12 \leq n - 3 < k$, by the induction hypothesis we have $n - 3 = 3a + 7b$ for some $a, b \in \mathbb{Z}^+$. [**5 points**] Then $n = 3(a + 1) + 7b$, so it is also true when $n = k$. By the strong induction it is true for all $n \geq 12$. [**5 points**] $\square$

Problem 4. Ch 3 [**10 points**]

Let X be the set of all functions from $\{1, 2, \ldots, 2021\}$ to $\{1, 2, \ldots, 2021\}$. Define a relation R on X by $(f, g) \in R$ if and only if $f \circ g = g \circ f$. Prove or disprove each statement.

(1) R is reflexive.

(2) R is symmetric.

(3) R is antisymmetric.

(4) R is transitive.

Solution. (1) Since $f \circ f = f \circ f$, R is reflexive. [**2 points**]

(2) Since $f \circ g = g \circ f$ implies $g \circ f = f \circ g$, R is symmetric. [**2 points**]

(3) Let I be the identity function and let f be the function given by $f(i) = 1$ for all $i \in \{1, 2, \ldots, 2021\}$. Then $f \circ I = I \circ f$ and $I \circ f = f \circ I$, so $(f, I), (I, f) \in R$. But since $I \neq f$, R is not antisymmetric. [**2 points**]

(4) Let I be the identity function. Let f and g be the functions given by $f(i) = 1$ and $g(i) = 2$ for all $i \in \{1, 2, \ldots, 2021\}$. Then $f \circ I = I \circ f = f$ and $I \circ g = g \circ I = g$, so $(f, I), (I, g) \in R$. [**2 points**] However, $f \circ g = f$ and $g \circ f = g$ are different, so R is not transitive [**2 points**]

$\square$

Problem 5. Ch 3 [**15 points**]

Let X be the set of pairs (A, B) of subsets $A, B \subseteq \{1, 2, 3\}$ such that $B \neq \emptyset$. Define a relation R on X by $((A, B), (A', B')) \in R$ if and only if $|A| \cdot |B'| - |A'| \cdot |B| = 0$.

(1) Show that R is an equivalence relation.

(2) Find the equivalent class containing $(\emptyset, \{1\})$.

Solution. (1) The condition $|A| \cdot |B'| - |A'| \cdot |B| = 0$ can be written as $\frac{|A|}{|B|} = \frac{|A'|}{|B'|}$. Since $\frac{|A|}{|B|} = \frac{|A|}{|B|}$, R is reflexive. [**3 points**] Since $\frac{|A|}{|B|} = \frac{|A'|}{|B'|}$ is the same as $\frac{|A'|}{|B'|} = \frac{|A|}{|B|}$, R is symmetric. [**3 points**] If $\frac{|A|}{|B|} = \frac{|A'|}{|B'|}$ and $\frac{|A'|}{|B'|} = \frac{|A''|}{|B''|}$, then $\frac{|A|}{|B|} = \frac{|A''|}{|B''|}$. Thus R is transitive. [**3 points**] Therefore R is an equivalent relation.

(2) Since $\frac{|\emptyset|}{|\{1\}|} = 0$, the equivalent class containing $(\emptyset, \{1\})$ is the set of pairs (A, B) of subsets $A, B \subseteq \{1, 2, 3\}$ such that $B \neq \emptyset$ and $A = \emptyset$. [**3 points**] Thus the answer is

$$\{(\emptyset, \{1\})\}, \{(\emptyset, \{2\})\}, \{(\emptyset, \{3\})\}, \{(\emptyset, \{1, 2\})\}, \{(\emptyset, \{1, 3\})\}, \{(\emptyset, \{2, 3\})\}, \{(\emptyset, \{1, 2, 3\})\}. \qquad \textbf{[3 points]}$$

$\square$

Problem 6. Ch 4 [**15 points**]

Write an algorithm that receives (s, n), where $s = (s_1, \ldots, s_n)$ is a sequence n distinct integers $(n \geq 2)$, and returns the second largest integer in s. (You may assume that $s_1, \ldots, s_n$ are all distinct.)

Solution. lines 2,3,4: 3 pts, lines 5,6,7: 3 pts, lines 9,10,11: 3 pts, lines 12,13: 3 pts, lines 12,14: 3 pts.

```
1   max2(s, n) {
2       if (s_1 > s_2)
3           m_1 = s_1
4           m_2 = s_2
5       else
6           m_1 = s_2
7           m_2 = s_1
8       for i = 3, ..., n
9           if (s_i > m_1)
10              m_2 = m_1
11              m_1 = s_i
12          if (m_1 > s_i > m_2)
13              m_2 = s_i
14      return m_2
15  }
```

□

Problem 7. Ch 4 [**10 points**]

For a positive integer n, let $f(n) = 1^{2021} + 2^{2021} + \cdots + (2n)^{2021}$. Prove or disprove: $f(n) = \Theta(n^{2022})$.

Solution. Since $f(n) \leq 2n \cdot (2n)^{2021} = 2^{2022}n^{2022}$, we have $f(n) = O(n^{2022})$ [**5 points**]

Since $f(n) \geq (n+1)^{2021} + \cdots + (2n)^{2021} > n \cdot n^{2021} = n^{2022}$, we have $f(n) = \Omega(n^{2022})$ [**5 points**]

Therefore $f(n) = \Theta(n^{2022})$. □

Problem 8. Ch 5 [**15 points**]

(1) Find the inverse of 44 mod 2021.

(2) Find the inverse of 2021 mod 44.

Solution. (1) Since

$$\begin{aligned} 2021 &= 45 \cdot 44 + 41 \\ 44 &= 1 \cdot 41 + 3 \\ 41 &= 13 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1, \end{aligned}$$

we can write

$$\begin{aligned} 1 = 3 - 2 &= 3 - (41 - 13 \cdot 3) \\ &= -41 + 14 \cdot 3 = -41 + 14(44 - 41) \\ &= 14 \cdot 44 - 15 \cdot 41 = 14 \cdot 44 - 15(2021 - 45 \cdot 44) \\ &= -15 \cdot 2021 + 689 \cdot 44. \qquad \textbf{[5 points]} \end{aligned}$$

Thus the inverse of 44 mod 2021 is 689. [**5 points**]

(2) By the above computation, inverse of 2021 mod 44 is $-15 + 44 = 29$. [**5 points**]

□