2019-2 DM midterm solutions

P1. F F F F T. (1 pt each)

P2. (1) 10 (2 pt)

(2) $(1+2+\cdots+7)+(1+2+3)+1=35$ (4 pt)

(3) $3+4+6=13$ (4 pt)

P3 (1) True.

Let $x \in X\times(Y-Z)$. Then $x=(a,b)$ for some $a\in X$, $b\in Y-Z$.

Since $b\in Y$ and $b\notin Z$, we have $(a,b)\in X\times Y$ and $(a,b)\notin X\times Z$.

Thus $x=(a,b)\in (X\times Y)-(X\times Z)$, and $X\times(Y-Z)\subseteq (X\times Y)-(X\times Z)$. (3 pt)

Let $x\in(X\times Y)-(X\times Z)$. Then $x=(a,b)$ for some $a\in X$, $b\in Y$ such that $(a,b)\notin X\times Z$. Since $a\in X$ and $(a,b)\notin X\times Z$ we have $b\notin Z$.

Thus $b\in Y-Z$ and we get $(a,b)\in X\times(Y-Z)$.

Therefore $(X\times Y)-(X\times Z)\subseteq X\times(Y-Z)$. (3 pt)

(2) False. A counterexample : $X=Y=Z=\{1\}$. (3 pt)

(3) False. A counterexample : $X=\{(1,1)\}$, $Y=Z=\{1\}$. (3 pt)

(4) False. A counterexample : $X=\{1,(1,1)\}$. (3 pt).

P4. Induction on n.

If $n=1$, both sides are 1. (2 pt)

Suppose that the statement is true for $n=k$.

Then for $n=k+1$,

$$\sum_{i=1}^{k+1}(-1)^{i-1}i^2=\sum_{i=1}^{k}(-1)^{i-1}i^2+(-1)^k(k+1)^2 \quad (2\text{ pt})$$

$$=(-1)^{k-1}\frac{k(k+1)}{2}+(-1)^k(k+1)^2 \quad (3\text{ pt})$$

$$=(-1)^k(k+1)\left(k+1-\frac{k}{2}\right)=(-1)^k\frac{(k+1)(k+2)}{2}. \quad (3\text{ pt}).$$

Thus it is also true for $n=k+1$.

By induction it is true for all $n\geq 1$.

P5 (1) reflexive: $\forall f \in X^X$, $(f,f) \in R$ because $f(1) = f(1)$. (3 pt)

symmetric: Let $(f,g) \in R$. Then $f(1) = g(1)$.

Since $g(1) = f(1)$ we get $(g,f) \in R$. (3 pt)

transitive: Let $(f,g), (g,h) \in R$. Then $f(1) = g(1)$, $g(1) = h(1)$.

Since $f(1) = h(1)$, we get $(f,h) \in R$. (3 pt)

(2) $[f] = \{ g : X \to X : g(1) = 1 \}$.

Since $g(i)$ can be any element in X for $2 \leq i \leq 2019$,

$[f]$ has 2019^{2018} elements. (3 pt)

(3) Each equivalence class is determined by the value $f(1)$ of any element f in the equivalent class.

Thus there are 2019 equivalence classes. (3 pt)

P6 (1) True. Suppose R is reflexive. Then $(x,x) \in R$ $\forall x \in X$.

Since $(x,x), (x,x) \in R$, $(x,x) \in R^2$. (3pt)

(2) False. A counterexample: $X = \{1,2\}$, $R = \{(1,2), (2,1)\}$ (3 pt)

Then $R^2 = \{(1,1), (2,2)\}$ is reflexive, but R is not.

(3) True. Suppose R is symmetric. Let $(x,y) \in R^2$.

Then $\exists z \in X$ s.t. $(x,z), (z,y) \in R$. (3 pt)

Since R is symmetric, $(z,x), (y,z) \in R$.

Since $(y,z), (z,x) \in R$ we get $(y,x) \in R^2$. (3 pt)

(4) False. Let $X = \{1,2\}$, $R = \{(1,2)\}$. (3 pt)

Then $R^2 = \emptyset$ is symmetric but R is not.

P7 (1) True. Suppose $f(n) = \Theta(g(n))$.

Then $\exists\, c_1, c_2 > 0$ s.t. $|f(n)| \le c_1 |g(n)|$ for sufficiently large n, (2 pt)

$|f(n)| \geq c_2 |g(n)|$ 〃 . (2 pt)

Then $|f(n)^{2019}| \le c_1^{2019} |g(n)^{2019}|$ (2 pt)

$|f(n)^{2019}| \geq c_2^{2019} |g(n)^{2019}|$. (2 pt)

Thus $f(n)^{2019} = \Theta(g(n)^{2019})$ (2 pt)

(2) False. A counterexample: $f(n) = n$, $g(n) = 2n$. (5 pt)

Then $n = \Theta(2n)$, but $2019^n \neq \Theta(2019^{2n})$.

P8 If $n = p_1^{e_1} \cdots p_k^{e_k}$, where p_i's are distinct primes, then

$\sigma(n) = (e_1+1) \cdots (e_k+1)$. (3 pt)

Thus $3 \nmid \sigma(n)$ if and only if $3 \nmid e_i + 1$ $\forall i = 1, \ldots, k$. (3 pt)

Since $12! = 2^{10} 3^5 5^2 7^1 11^1$, if n is a divisor of $12!$ then

$n = 2^a 3^b 5^c 7^d 11^e$, $0 \le a \le 10$, $0 \le b \le 5$, $0 \le c \le 2$, $0 \le d \le 1$, $0 \le e \le 1$. (3 pt)

Thus if n satisfies both $n \mid 12!$ and $3 \nmid \sigma(n)$, we must have

$a \in \{0, 1, \ldots, 10\} - \{2, 5, 8\}$,

$b \in \{0, 1, \ldots, 5\} - \{2, 5\}$,

$c \in \{0, 1, 2\} - \{2\}$. ... (3 pt)

$d \in \{0, 1\}$

$e \in \{0, 1\}$

Therefore the number of such n's is

$(11-3) \cdot (6-2) \cdot (3-1) \cdot 2 \cdot 2$

$= 8 \cdot 4 \cdot 2 \cdot 2$

$= 128$. (3 pt)