Discrete Mathematics Midterm Exam (GEDB007-41, Spring 2022)

Problem 1. For a set X, let $\mathcal{P}(X)$ be the set of all subsets of X. Let $\emptyset$ be the empty set. Determine whether each statement is true or false. You don't have to explain your answer.

(1) $\emptyset \subseteq \{\emptyset\}$.

(2) $\emptyset \in \{\emptyset\}$.

(3) $(1,2) \in \{1,(1,2)\} \times \{1,\{1,2\}\}$.

(4) $\emptyset$ is a relation on $\mathcal{P}(\emptyset)$.

(5) $(\emptyset,\emptyset)$ is a relation on $\mathcal{P}(\emptyset)$.

(6) $\{\emptyset\}$ is a relation on $\mathcal{P}(\emptyset)$.

(7) $\{(\emptyset,\emptyset)\}$ is a relation on $\mathcal{P}(\emptyset)$.

(8) $\mathcal{P}(\emptyset) \times \mathcal{P}(\emptyset)$ is a relation on $\mathcal{P}(\emptyset)$.

(9) $\mathcal{P}(\emptyset) \times \emptyset$ is a relation on $\mathcal{P}(\emptyset)$.

(10) $\mathcal{P}(\emptyset)$ is a relation on $\mathcal{P}(\emptyset)$.

Problem 2. Prove or disprove each statement.

(1) $\forall x \forall y((x^2 > y^2) \to (x < y))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(2) $\forall x \exists y((x^2 > y^2) \to (x < y))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(3) $\forall x \exists y(x^2 > y^2)$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(4) $\exists x \forall y(x < y)$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(5) $\forall y \exists x(x < y)$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

Problem 3. Prove that there is no positive integer solution to the equation $2x^3 + 3y^2 = 130$.

Problem 4. Let R be the relation on $X = \{1,2,3,4\}$ determined by the following conditions.

- R is reflexive.
- R is antisymmetric.
- R is transitive.
- $(1,2),(2,4),(3,1) \in R$.

Write the matrix of relation for R with respect to the ordering $1,2,3,4$. Explain your answer.

Problem 5. Let $X = \{1, 2, 3, \ldots, 2022\}$. We define the relation R on the set of all subsets of X as follows. For two subsets $A, B \subseteq X$,

$$(A, B) \in R \quad \Leftrightarrow \quad \{x \in A : x \text{ is divisible by } 3\} = \{x \in B : x \text{ is divisible by } 3\}.$$

(1) Show that R is an equivalent relation.

(2) Find the number of elements in the equivalence class containing $\{1, 3, 5, \ldots, 2021\}$.

Problem 6. Consider the following algorithm.

Input : s, n (s is a sequence of n numbers: $s = (s_1, s_2, \ldots, s_n)$ and n is a positive integer)
Output: an integer

Alice$(s, n)\{$
    if $(n = 1)$
        return s_1
    $m = 1$
    for $i = 2$ **to** n
        if $(s_i < s_m)$
            $m = i$
    if $(m = 1)$
        $s' = (s_2 - 1, s_3 - 1, \ldots, s_n - 1)$
    if $(2 \leq m \leq n - 1)$
        $s' = (s_1, \ldots, s_{m-1}, s_{m+1} - 1, s_{m+2} - 1 \ldots, s_n - 1)$
    if $(m = n)$
        $s' = (s_1, \ldots, s_{n-1})$
    return $s_m +$ **Alice**$(s', n - 1)$
$\}$

(1) What is the output of the algorithm **Alice** for the input $s = (1, 2, 3, 4, 5)$ and $n = 5$? Explain your answer.

(2) What is the output of the algorithm **Alice** for the input $s = (1, 2, 3, 4, 5, 1, 2, 3, 4, 5)$ and $n = 10$? Explain your answer.

Problem 7. Prove or disprove:

$$\sum_{k=1}^{n} \log k^{n-k+1} = \Theta(n^2 \log n).$$

Problem 8. Let $f_0, f_1, f_2, \ldots$ be the sequence defined by $f_0 = 1, f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Show that for $n \geq 1$,

$$\sum_{i=1}^{2n} f_{i-1} f_i = f_{2n}^2.$$