Student ID		Name		Instructor	Jang Soo Kim

Problem 1 (10 points)**.** Find the coefficient of x^3y^4 in the expansion of

$$(x+\sqrt{xy}+y)(x-\sqrt{xy}+y)(x+y)^5.$$

Problem 2 (10 points)**.** Find the number of $n \times n$ matrices A satisfying the following three conditions:

- For all $1 \le i, j \le n$, we have $A_{i,j} \in \{0, 1, 2\}$.
- For all $1 \le i \le n$, we have $A_{i,i} \ne 0$.
- For all $1 \le i, j \le n$, if $A_{i,j} \ne 0$ and $A_{j,i} \ne 0$, then $i = j$.

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Problem 3 (15 points)**.** Let $\{a_n\}_{n\ge 0}$ be the sequence given by $a_0 = 0$, $a_1 = 1$ and for $n \ge 2$,

$$a_n = 2a_{n-1} + a_{n-2}.$$

Find a general formula for a_n.

Problem 4 (15 points)**.** Let G be the simple graph with vertices $v_1, v_2, \dots, v_{18}$ such that for $1 \le i < j \le 18$, v_i and v_j are adjacent if and only if j is a multiple of i.

1. Find the number of edges of G.
2. Show that G contains a subgraph isomorphic to K_5.
3. Show that G contains a subgraph isomorphic to $K_{3,3}$.

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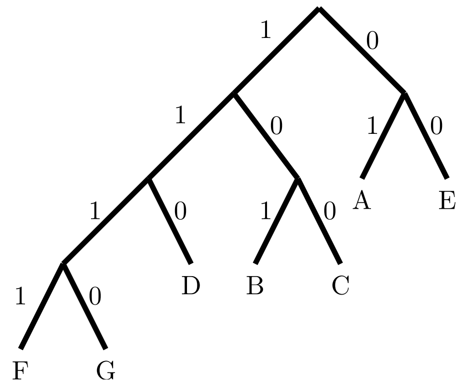
Problem 5 (10 points)**.** Suppose that T is a binary tree with vertices $A, B, C, D, E, F, G, H, I, J, K, L$ such that

- the preorder listing of the vertices of T is $DCAHGEFIBJLK$,
- the inorder listing of the vertices of T is $ACGHFEDIJKLB$.

What is the postorder listing of the vertices of T?

Problem 6 (10 points)**.** Decode each bit string using the following Huffman code:



1. 0110000
2. 10001111100
3. 10100001111

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Problem 7 (15 points)**.** For a simple graph G, the *complement* $\overline{G}$ of G is the graph with the same vertex set such that (u, v) is an edge of $\overline{G}$ if and only if (u, v) is not an edge of G. Prove or disprove the following statement.

- If G is a simple graph with at least 9 vertices, then either G is non-bipartite or $\overline{G}$ is non-planar.

Problem 8 (15 points)**.** Suppose that $b_1, b_2, \ldots, b_{1060}$ are positive integers satisfying $b_1 + b_2 + \cdots + b_{1060} = 2019$. Prove that there are consecutive integers $i, i+1, \ldots, j$ such that $b_i + b_{i+1} + \cdots + b_j = 100$.

2019-1 Discrete Math Final Exam Solutions

Prob 1 Since $(x+\sqrt{xy}+y)(x-\sqrt{xy}+y)(x+y)^5$ is equal to
$((x+y)^2-xy)(x+y)^5 = (x+y)^7 - xy(x+y)^5$, (5 pts)
the coefficient of x^3y^4 is $\binom{7}{3}-\binom{5}{2}=25$ (5 pts).

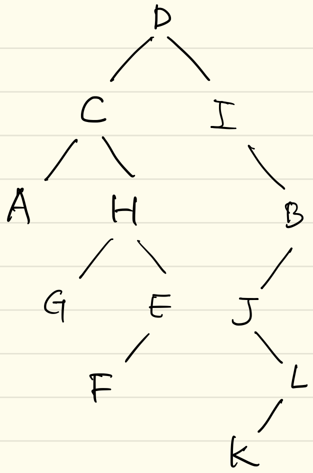
Prob 2 For each i, there are two choices for A_{ii}. (2 pts)
For each $1 \le i < j \le n$, there are 5 choices for A_{ij}, A_{ji}. (5 pts)
$((A_{ij}, A_{ji}) = (0,0), (0,1), (0,2), (1,0), (2,0).)$
Therefore the number of such matrices is $2^n 5^{\binom{n}{2}}$. (3 pts)

Prob 3 The characteristic polynomial is x^2-2x-1. (5 pts)
The solutions to $x^2-2x-1=0$ are $x=1\pm\sqrt{2}$.
Thus $a_n = \alpha(1+\sqrt{2})^n + \beta(1-\sqrt{2})^n$ (5 pts)
Since $a_0 = \alpha+\beta = 0$ and $a_1 = a(1+\sqrt{2}) + b(1-\sqrt{2}) = 1$,
$\alpha = \frac{1}{2\sqrt{2}}$, $\beta = -\frac{1}{2\sqrt{2}}$ (5 pts).
Thus $a_n = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^n - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^n$.

Prob 4

1. Let a_i = # edges (v_i, v_j) s.t. $i<j$.
   Then
   $a_1 = 18-1 = 17$
   $a_2 = \lfloor 18/2 \rfloor - 1 = 8$
   $a_3 = \lfloor 18/3 \rfloor - 1 = 5$
   $a_4 = \lfloor 18/4 \rfloor - 1 = 3$
   $a_5 = \lfloor 18/5 \rfloor - 1 = 2$
   $a_6 = \lfloor 18/6 \rfloor - 1 = 2$
   $a_7 = a_8 = a_9 = 1$
   $a_{10} = \cdots = a_{18} = 0$. Thus $e = a_1 + \cdots + a_{18} = 38$. (5pts)
2. $v_1, v_2, v_4, v_8, v_{16}$ are all connected $\Rightarrow K_5$. (5 pts)
3. v_1, v_2, v_3 are connected to v_6, v_{12}, v_{18} $\Rightarrow K_{3,3}$. (5 pts)
   (or v_1, v_2, v_4 and v_8, v_{12}, v_{16})

Prob 5 The tree is



(5 pts)

$\Rightarrow$ postorder = AGFEHCKLJBID. (5 pts)

Prob 6
1. ACE (3 pts)
2. CAFE (4 pts)
3. BEEF (4 pts)

Prob 7 It is enough to show that if G is bipartite, then $\overline{G}$ is nonplanar. (5 pts)

If G is bipartite we can divide the set of vertices of G into two sets A and B such that no vertices in A or in B are connected.

By the pigeonhole principle A or B contains at least 5 vertices. (5 pts)

Then $\overline{G}$ contains K_5 since the vertices in A and in B are all connected in $\overline{G}$. (5 pts).

Prob 8. Let $S_i = b_1 + \cdots + b_i$. Then $1 \le S_i \le 2019$.

Consider the sequence $S_1 + 100, \dots, S_{1060} + 100$. Then $1 \le S_i + 100 \le 2119$. (5pts)

Since $\{S_i \mid 1 \le i \le 1060\}$ and $\{S_i + 100 \mid 1 \le i \le 1060\}$ are 2120 numbers in $\{1, \dots, 2119\}$, by the pigeonhole principle, we must have $S_j = S_i + 100$. (5 pts)

Then $j > i$ and $b_{i+1} + \cdots + b_j = S_j - S_i = 100$. (5 pts).