2019-1. Discrete Math Midterm Solutions.

P1. T, F, F, F, F. (1 point each)

P2. (1) 9 (2 pts) (2) 12 (3 pts) (3) 18 (5 pts)

P3. Let $x \in X$. Since the matrix of relation has a nonzero entry in the row x, we have at least one $y \in X$ such that $(x,y) \in R$. (2 pts)
Since R is symmetric and $(x,y) \in R$, we have $(y,x) \in R$. (2 pts)
Since R is transitive and (x,y), (y,x), we have $(x,x) \in R$. (2 pts).
Therefore R is reflexive. (2 pts)
Thus R is an equivalence relation. (2 pts).

P4. Let d be the number of digits.
Then $m = a_d b^d + a_{d-1} b^{d-1} + \cdots + a_0 b^0$
for $0 \le a_i \le b-1$, $(0 \le i \le d-1)$ and $1 \le a_d \le b-1$ (5 pts)
Thus $b^d \le m \le (b-1) \cdot (b^d + b^{d-1} + \cdots + 1) = b^{d+1} - 1 < b^{d+1}$ (5 pts)
By taking $\log_b$, we obtain

$$d \le \log_b m < d+1$$ (5 pts).

Thus $\lfloor \log_b m \rfloor = d$.

P5.
$$2019 = 7 \cdot 283 + 38$$
$$283 = 7 \cdot 38 + 17$$
$$38 = 2 \cdot 17 + 4$$
$$17 = 4 \cdot 4 + 1$$ (5 pts)

So,

$$1 = 17 - 4 4 = 17 - 4(38 - 2 \cdot 17)$$
$$= -4 \cdot 38 + 9 \cdot 17 = -4 \cdot 38 + 9(283 - 7 \cdot 38)$$
$$= 9 \cdot 283 - 67 \cdot 38 = 9 \cdot 283 - 67(2019 - 7 \cdot 283)$$
$$= -67 \cdot 2019 + 478 \cdot 283.$$ (5 pts)

Thus the answer is 283. (5 pts)

P6. $\sum_{i=1}^{n} i^2 \lg i \le n \cdot n^2 \log n \Rightarrow O(n^2 \log n)$. (5 pts)

$$\sum_{i=1}^{n} i^2 \lg i \geq \lceil \tfrac{n}{2} \rceil^2 \lg \lceil \tfrac{n}{2} \rceil + \cdots + n^2 \lg n \qquad (3\text{ pts})$$

$$\geq \left(\frac{n^2}{4} \lg \frac{n}{2}\right) \cdot \frac{n}{2} \qquad (2\text{ pts})$$

$$= \frac{n^3}{8} \lg \frac{n}{2} \geq \frac{1}{8} \cdot n^3 \lg n. \qquad (3\text{ pts})$$

So $\sum_{i=1}^{n} i^2 \lg i = \Omega(n^3 \lg n)$. (2 pts).

Therefore $\sum_{i=1}^{n} i^2 \lg i = \Theta(n^3 \lg n)$.

P7 We prove by induction on the number n of letters in α.

If $n=0$, then α is the null string, so $\# a\text{'s} = \# b\text{'s} = 0$. (5 pts)

Suppose that the statement is true for all strings in L with less than n letters and that $\alpha \in L$ has n letters.

By the construction of L, we have either

① $\alpha = a\beta b$ or $b\beta a$ for some $\beta \in L$

② $\alpha = \beta\gamma$ for some $\beta, \gamma \in L$. with $\beta, \gamma \neq \alpha$.

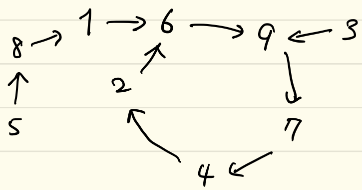
In the first case ①, β has the same numbers of a's and b's, then so does α. (5 pts)

In case ②, β, γ have smaller number of letters.

Thus by induction hypothesis, β, γ have the same number of a's and b's, and so does α. (5 pts)

Therefore by induction the statement is true for all n.

P8 Let's represent f by the following diagram with an arrow from i to $f(i)$:



Then for every $i \in \{2, 6, 9, 7, 4\}$ we have $f^5(i) = i$.

Thus $f^2(1) = 9$. (2 pts)

$f^{20}(3) = f^{19}(9) = f^4(9) = 6$ (3 pts)

$f^{201}(5) = f^{198}(6) = f^3(6) = 4$ (5 pts)

$f^{2019}(8) = f^{2017}(6) = f^2(6) = 7$. (5 pts)