Discrete Mathematics Final Exam (GEDB007-43, Fall 2021)

Problem 1. Find the number of integers n such that $5000 < n < 10000$ and $\gcd(n, 2021) = 1$. (Hint: $45^2 = 2025$.)

Problem 2. Let $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, 3\}$. Find the number of onto functions $f : X \to Y$.

Problem 3. For a positive integer n, let a_n be the maximum number of regions that can be created by drawing n squares on a plane. For example, $a_1 = 2$ and $a_2 = 10$. Compute a_{25}.

Problem 4. Find a formula for the nth term a_n of the sequence $a_0, a_1, a_2, \ldots$, which satisfies the initial conditions $a_0 = -2$, $a_1 = \frac{7}{3}$ and the recurrence relation given by

$$12a_n = 4a_{n-1} + a_{n-2}.$$

Problem 5. Let r be a positive integer. We say that a simple graph $G = (V, E)$ is *r-partite* if there exist r subsets $V_1, \ldots, V_r$ of V satisfying the following conditions:

(1) $V_1 \cup \cdots \cup V_r = V$,

(2) $V_i \cap V_j = \emptyset$ for all $1 \le i < j \le r$,

(3) $(V_i \times V_i) \cap E = \emptyset$ for all $i = 1, 2, \ldots, r$.

The *complement* of G is the simple graph $\overline{G}$ with the same vertices as G such that an edge exists in $\overline{G}$ if and only if it does not exist in G.

Find the largest integer n satisfying the following condition: there exists a 2021-partite graph G with n vertices such that $\overline{G}$ is planar.

Problem 6. Let G be a simple graph on $\{1, 2, \ldots, 10\}$. Suppose that G has 3 connected components and every vertex has degree 2.

(1) Find the number of possible graphs G.

(2) Let A be the adjacency matrix of G with respect to the vertex ordering $1, 2, \ldots, 10$. Find the sum of the diagonal entries of $A^2 + A^3 + A^4$.

Problem 7. Consider the set of letters in the following table.

letter	A	B	C	D	E	F	G
frequency	1	1	2	3	3	6	9

(1) Find an optimal Hoffman code for these letters. (Draw a tree and write a 0-1 sequence for each letter. There are many possible optimal Hoffman codes, and you need to find just one of them.)

(2) Encode "CAGE" using the Hoffman code.

(3) Find the number of words with three letters (repetitions allowed) in A, B, C, D, E, F, G such that the length of the code using the optimal Hoffman code is 9. For example, if the letter A has code 10 and the letter B has code 01001, then the word ABA has code 100100110 whose length is 9.

Problem 8. Let $A = 1, B = 2, C = 3, D = 4, E = 5, F = 6, G = 7$. Compute each expression, which is either a prefix form or a postfix form. Write "invalid" if the expression is not valid as a prefix or postfix expression.

(1) $AB * C - DEF * +/$

(2) $AB + CD - /E * FG + -$

(3) $AB + CD - /EFG + *$

(4) $- * +ABC - D + EF$