2019-1 Discrete Math Final Exam Solutions

Prob 1 Since $(x+\sqrt{xy}+y)(x-\sqrt{xy}+y)(x+y)^5$ is equal to
$((x+y)^2-xy)(x+y)^5 = (x+y)^7 - xy(x+y)^5$, (5 pts)
the coefficient of x^3y^4 is $\binom{7}{3}-\binom{5}{2}=25$ (5 pts).

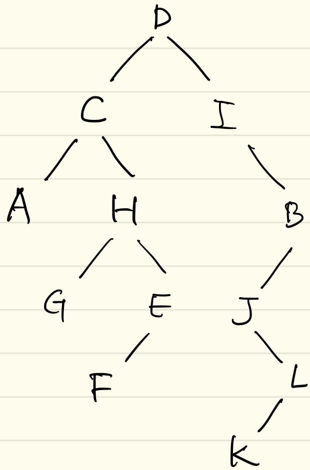
Prob 2 For each i, there are two choices for A_{ii}. (2 pts)
For each $1 \le i < j \le n$, there are 5 choices for A_{ij}, A_{ji}. (5 pts)
$((A_{ij}, A_{ji}) = (0,0), (0,1), (0,2), (1,0), (2,0).)$
Therefore the number of such matrices is $2^n 5^{\binom{n}{2}}$. (3 pts)

Prob 3 The characteristic polynomial is x^2-2x-1. (5 pts)
The solutions to $x^2-2x-1=0$ are $x=1\pm\sqrt{2}$.
Thus $a_n = \alpha(1+\sqrt{2})^n + \beta(1-\sqrt{2})^n$ (5 pts)
Since $a_0 = \alpha+\beta = 0$ and $a_1 = a(1+\sqrt{2}) + b(1-\sqrt{2}) = 1$,
$\alpha = \frac{1}{2\sqrt{2}}$, $\beta = -\frac{1}{2\sqrt{2}}$ (5 pts).
Thus $a_n = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^n - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^n$.

Prob 4

1. Let a_i = # edges (v_i, v_j) s.t. $i<j$.
   Then $a_1 = 18-1 = 17$
   $a_2 = \lfloor 18/2 \rfloor - 1 = 8$
   $a_3 = \lfloor 18/3 \rfloor - 1 = 5$
   $a_4 = \lfloor 18/4 \rfloor - 1 = 3$
   $a_5 = \lfloor 18/5 \rfloor - 1 = 2$
   $a_6 = \lfloor 18/6 \rfloor - 1 = 2$
   $a_7 = a_8 = a_9 = 1$
   $a_{10} = \cdots = a_{18} = 0$. Thus $e = a_1 + \cdots + a_{18} = 38$. (5 pts)
2. $v_1, v_2, v_4, v_8, v_{16}$ are all connected $\Rightarrow K_5$. (5 pts)
3. v_1, v_2, v_3 are connected to v_6, v_{12}, v_{18} $\Rightarrow K_{3,3}$. (5 pts)
   (or v_1, v_2, v_4 and v_8, v_{12}, v_{16})

Prob 5 The tree is



(5 pts)

$\Rightarrow$ postorder = AGFEHCKLJBID. (5 pts)

Prob 6

1. ACE (3 pts)
2. CAFE (4 pts)
3. BEEF (4 pts)

Prob 7 It is enough to show that if G is bipartite, then $\overline{G}$ is nonplanar. (5 pts)

If G is bipartite we can divide the set of vertices of G into two sets A and B such that no vertices in A or in B are connected.

By the pigeonhole principle A or B contains at least 5 vertices. (5 pts)

Then $\overline{G}$ contains K_5 since the vertices in A and in B are all connected in $\overline{G}$. (5 pts).

Prob 8. Let $s_i = b_1 + \cdots + b_i$. Then $1 \le s_i \le 2019$.

Consider the sequence $s_1+100, \ldots, s_{1060}+100$. Then $1 \le s_i + 100 \le 2119$. (5pts)

Since $\{s_i \mid 1 \le i \le 1060\}$ and $\{s_i + 100 \mid 1 \le i \le 1060\}$ are 2120 numbers in $\{1, \ldots, 2119\}$, by the pigeonhole principle, we must have $s_j = s_i + 100$. (5 pts)

Then $j > i$ and $b_{i+1} + \cdots + b_j = s_j - s_i = 100$. (5 pts).