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Problem 1. (5 points) Determine whether each proposition is true or false.

(1) For any sets X, Y, Z of integers, we have $(X \times Z) - Y = X \times Z$.

(2) For any sets X, Y, Z, we have $(X \times Z) - Y = X \times Z$.

(3) For any set X, we have $(X \times X) - X = X \times X$.

(4) If R is a symmetric relation on a set X, then R is not antisymmetric.

(5) If R is an antisymmetric relation on a set X, then R is not symmetric.

Problem 2. (10 points) For each algorithm, answer the question. You don't have to explain your answer.

(1) Input : a, b, c (integers)
Output: k (integer)

```
Alice(a, b, c) {
    k = a
    if (b < k)
        k = b + 7
    if (c < k)
        k = c + 1
    return k
}
```

What is the output of the algorithm **Alice** for the input $a = 5$, $b = 2$, $c = 8$?

(2) Input : s, n (s is a sequence of n numbers: $s = (s_1, s_2, \dots, s_n)$)
Output: k (integer)

```
Bob(s, n)
    k = s_1
    for i = 2 to n
        if (s_i > k)
            k = s_i
    return k
}
```

What is the output of the algorithm **Bob** for the input $s = (8, 9, 12, 5, 3, 5, 4, 7)$ and $n = 8$?

(3) Input : s, n (s is a sequence of n numbers: $s = (s_1, s_2, \dots, s_n)$)
Output: k (integer)

```
Chris(s, n)
    k = 0
    for i = 1 to n - 1
        for j = i + 1 to n
            if (s_i > s_j)
                k = k + 1
    return k
}
```

What is the output of the algorithm **Chris** for the input $s = (8, 9, 12, 5, 3, 5, 4, 7)$ and $n = 8$?

Problem 3. (10 points) Let $X = \{1, 2, \dots, 2019\}$. Suppose that R is a relation on X satisfying the following conditions:

- R is symmetric.
- R is transitive.
- In the matrix of the relation R, every row has at least one nonzero entry.

Prove that R is an equivalence relation.

Problem 4. (15 points) Prove that a base b integer m has $\lfloor 1 + \log_b m \rfloor$ digits.

Problem 5. (15 points) Find the inverse of 283 modulo 2019.

Problem 6. (15 points) Prove that

$$\sum_{i=1}^{n} i^2 \lg i = \Theta(n^3 \lg n).$$

.

Problem 7. (15 points) Let L be the set of all strings, including the null string, that can be constructed by repeated application of the following rules:

- If $\alpha \in L$, then $a\alpha b \in L$ and $b\alpha a \in L$.
- If $\alpha \in L$ and $\beta \in L$, then $\alpha\beta \in L$.

Prove that if $\alpha \in L$, then α has equal numbers of a's and b's.

Problem 8. (15 points) Let $X = \{1, 2, \ldots, 9\}$. Suppose that f is a function from X to X given by

$$f = \{(1,6), (2,6), (3,9), (4,2), (5,8), (6,9), (7,4), (8,1), (9,7)\}.$$

Find the values $f^2(1)$, $f^{20}(3)$, $f^{201}(5)$, and $f^{2019}(8)$, where

$$f^n = f \circ f \circ \cdots \circ f$$

is the n-fold composition of f.

.

2019-1. Discrete Math Midterm Solutions.

P1. T, F, F, F, F. (1 point each)

P2. (1) 9 (2 pts) (2) 12 (3 pts) (3) 18 (5 pts)

P3. Let $x \in X$. Since the matrix of relation has a nonzero entry in the row x, we have at least one $y \in X$ such that $(x,y) \in R$. (2 pts)
Since R is symmetric and $(x,y) \in R$, we have $(y,x) \in R$. (2 pts)
Since R is transitive and $(x,y), (y,x)$, we have $(x,x) \in R$. (2 pts).
Therefore R is reflexive. (2 pts)
Thus R is an equivalence relation. (2 pts).

P4. Let d be the number of digits.
Then $m = a_d b^d + a_{d-1} b^{d-1} + \cdots + a_0 b^0$
for $0 \le a_i \le b-1$, $(0 \le i \le d-1)$ and $1 \le a_d \le b-1$ (5 pts)
Thus $b^d \le m \le (b-1) \cdot (b^d + b^{d-1} + \cdots + 1) = b^{d+1} - 1 < b^{d+1}$ (5 pts)
By taking $\log_b$, we obtain

$$d \le \log_b m < d+1$$ (5 pts).

Thus $\lfloor \log_b m \rfloor = d$.

P5.

$$2019 = 7 \cdot 283 + 38$$
$$283 = 7 \cdot 38 + 17$$
$$38 = 2 \cdot 17 + 4$$
$$17 = 4 \cdot 4 + 1$$ (5 pts)

So,

$$\begin{aligned} 1 &= 17 - 4 \cdot 4 = 17 - 4(38 - 2 \cdot 17) \\ &= -4 \cdot 38 + 9 \cdot 17 = -4 \cdot 38 + 9(283 - 7 \cdot 38) \\ &= 9 \cdot 283 - 67 \cdot 38 = 9 \cdot 283 - 67(2019 - 7 \cdot 283) \\ &= -67 \cdot 2019 + 478 \cdot 283. \end{aligned}$$ (5 pts)

Thus the answer is 283. (5 pts)

P6. $\sum_{i=1}^{n} i^2 \lg i \le n \cdot n^2 \log n \Rightarrow O(n^3 \log n)$. (5 pts)

$$\sum_{i=1}^{n} i^2 \lg i \ge \lceil \tfrac{n}{2} \rceil^2 \lg \lceil \tfrac{n}{2} \rceil + \cdots + n^2 \lg n \quad (3\text{ pts})$$

$$\ge \left(\frac{n^2}{4} \lg \frac{n}{2}\right) \cdot \frac{n}{2} \quad (2\text{ pts})$$

$$= \frac{n^3}{8} \lg \frac{n}{2} \ge \frac{1}{8} \cdot n^3 \lg n. \quad (3\text{ pts})$$

So $\sum_{i=1}^{n} i^2 \lg i = \Omega(n^3 \lg n)$. (2 pts).

Therefore $\sum_{i=1}^{n} i^2 \lg i = \Theta(n^3 \lg n)$.

P7 We prove by induction on the number n of letters in α.

If $n=0$, then α is the null string, so $\#a\text{'s} = \#b\text{'s} = 0$. (5 pts)

Suppose that the statement is true for all strings in L with less than n letters and that $\alpha \in L$ has n letters.

By the construction of L, we have either

① $\alpha = a\beta b$ or $b\beta a$ for some $\beta \in L$

② $\alpha = \beta\gamma$ for some $\beta, \gamma \in L$. with $\beta, \gamma \ne \alpha$.

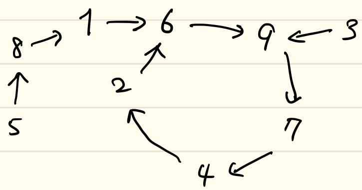
In the first case ①, β has the same numbers of a's and b's, then so does α. (5 pts)

In case ②, β, γ have smaller number of letters.

Thus by induction hypothesis, β, γ have the same number of a's and b's, and so does α. (5 pts)

Therefore by induction the statement is true for all n.

P8 Let's represent f by the following diagram with an arrow from i to $f(i)$:



Then for every $i \in \{2, 6, 9, 7, 4\}$ we have $f^5(i) = i$.

Thus $f^2(1) = 9$. (2 pts)

$f^{20}(3) = f^{19}(9) = f^4(9) = 6$ (3 pts)

$f^{201}(5) = f^{198}(6) = f^3(6) = 4$ (5 pts)

$f^{2019}(8) = f^{2017}(6) = f^2(6) = 7$. (5 pts)