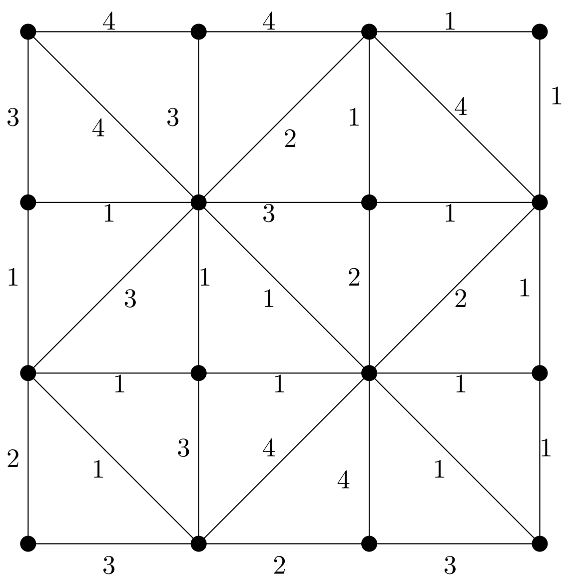
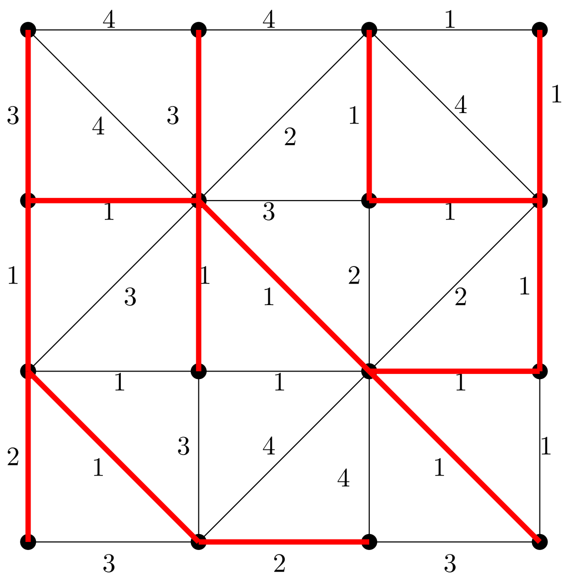
Student ID		Name		Instructor	Jang Soo Kim

Problem 1 (10 points)**.** Find a minimal spanning tree and compute the weight of it.



Solution. A possible answer is as follows: (5 pts)



The total weight is 19. (5 pts) □

Problem 2 (10 points)**.** Answer the following questions.

1. Construct an optimal Huffman code for the set of letters in the table.

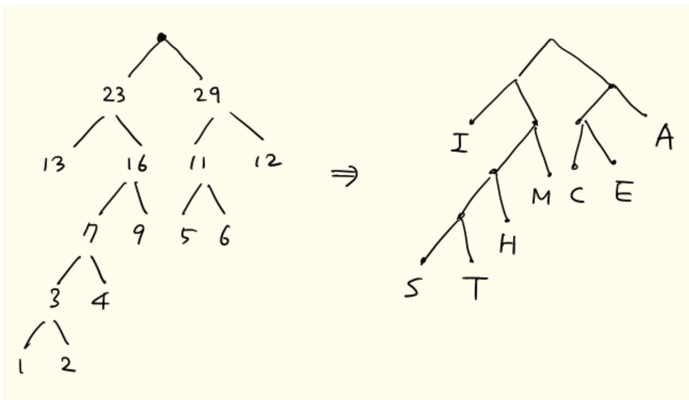
Letter	Frequency
A	12
C	5
E	6
H	4
I	13
M	9
S	1
T	2

2. How many binary digits are needed to encode "MATHEMATICS" using the Huffman code that you obtained above?

Solution. 1. We list the frequencies in increasing order and add the smallest two integers repeatedly:

$$1,2,4,5,6,9,12,13 \to (1+2),4,5,6,9,12,13$$
$$3,4,5,6,9,12,13 \to (3+4),5,6,9,12,13$$
$$5,6,7,9,12,13 \to (5+6),7,9,12,13$$
$$7,9,11,12,13 \to (7+9),11,12,13$$
$$11,12,13,16 \to (11+12),13,16$$
$$13,16,23 \to (13+16),23 \to 23,29$$

Thus we obtain the following tree (5 pts):



2. Since "MATHEMATICS" has 2 M's, 2 A's, 2 T's, and one of each I,H,E,C,S, the total number of digits is

$$2(3+2+5)+2+4+3+3+5 = 37. \qquad \text{(5 pts)}$$

□

Problem 3 (10 points)**.** Let A be the adjacency matrix of a simple graph G.

1. Prove or disprove: If A^2 has a diagonal entry equal to 0, then G is disconnected.
2. Prove or disprove: If G is disconnected, A^2 has a diagonal entry equal to 0.

Solution. 1. True. The diagonal entry $A_{i,i}$ is the number of edges incident to the vertex i. (3 pts) Therefore if $A_{i,i} = 0$, then the vertex i is an isolated vertex and G is disconnected. (4 pts)

2. False. If G is the graph on $\{1,2,3,4\}$ with two edges $(1,2)$ and $(3,4)$, then G is disconnected but A^2 has no zero diagonal entries. (3 pts)

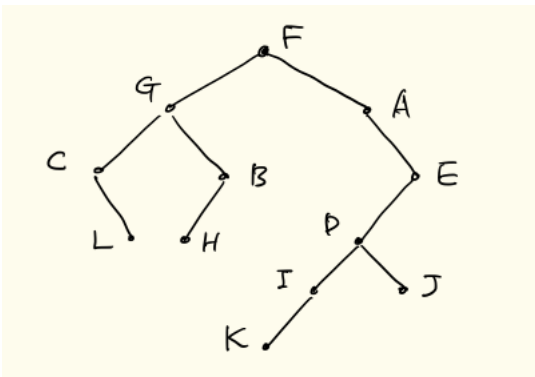
□

Problem 4 (10 points)**.** Suppose that T is a binary tree with vertices A,B,C,D,E,F,G,H,I,J,K,L such that

- the postorder listing of the vertices of T is $LCHBGKIJDEAF$, and
- the inorder listing of the vertices of T is $CLGHBFAKIDJE$.

Draw the binary tree T.

Solution. The binary tree T is drawn as follows (10 pts):



□

Problem 5 (10 points)**.** Find the number of sequences $(a_1, a_2, \ldots, a_{2019})$ satisfying the following three conditions:

- $a_i \in \{1, 2, 3, 4, 5\}$ for all $1 \le i \le 2019$,
- the number of integers $1 \le i \le 2019$ such that $a_i \le 2$ is 1000, and
- there is no integer $1 \le i \le 2018$ such that $a_i \le 2$ and $a_{i+1} \le 2$.

(Your answer must be as simple as possible without summation.)

Solution. In such a sequence $(a_1, \ldots, a_{2019})$, there are 1000 integers from $\{1, 2\}$ and 1019 integers from $\{3, 4, 5\}$. To construct such a sequence we can first arrange 1019 integers from $\{3, 4, 5\}$ in 3^{1019} ways (3 pts). We can then insert 1000 integers from $\{1, 2\}$ in $2^{1000}\binom{1020}{1000}$ ways. (4 pts) Therefore the answer is $3^{1019}2^{1000}\binom{1020}{1000}$. (3 pts) □

Problem 6 (20 points)**.** Let $\{a_n\}_{n\ge 0}$ be the sequence given by $a_0 = 0$, $a_1 = 1$ and for $n \ge 2$,

$$a_n = 4(a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1}.$$

Find a general formula for a_n for $n \ge 1$.

Solution. Let $b_i = a_0 + a_1 + \cdots + a_i$. (4 pts) Then $a_i = b_i - b_{i-1}$ and we have

$$b_n - b_{n-1} = 4b_{n-2} + (b_{n-1} - b_{n-2}),$$

which can be rewritten as

$$b_n = 2b_{n-1} + 3b_{n-2}. \qquad \text{(4 pts)}$$

The characteristic polynomial is $x^2 - 2x - 3 = (x - 3)(x + 1)$. Therefore

$$b_n = r3^n + s(-1)^n, \qquad \text{(4 pts)}$$

for some r, s. Since $b_0 = 0, b_1 = 1$, we get $r = 1/4$ and $s = -1/4$. Therefore

$$b_n = \frac{1}{4}\left(3^n - (-1)^n\right), \qquad \text{(4 pts)}$$

and for $n \ge 1$,

$$a_n = b_n - b_{n-1} = \frac{1}{4}\left(3^n - 3^{n-1} - (-1)^n + (-1)^{n-1}\right) = \frac{1}{2}\left(3^{n-1} + (-1)^{n-1}\right), \qquad \text{(4 pts)}$$

□

Problem 7 (15 points)**.** Suppose that P is a polyhedron satisfying the following conditions:

- At every vertex, there are 3 or 4 faces meeting at this vertex.
- Every face is a quadrilateral (4-gon).

Find the number of vertices of degree 3 in this polyhedron. Prove your answer.

Solution. Let a and b be the number of vertices of degree 3 and 4, respectively. Then

$$2e = 3a + 4b, \qquad \text{(4 pts)}$$
$$2e = 4f. \qquad \text{(4 pts)}$$

Thus

$$v = a + b, \qquad e = \frac{3a + 4b}{2}, \qquad f = \frac{3a + 4b}{4}.$$

Substituting this into Euler's formula $v - e + f = 2$ (3 pts), we obtain $a = 8$ (4 pts). □

Problem 8 (15 points)**.** Let X be a collection of subsets of $\{1, 2, \ldots, n\}$ such that for any two elements A, B in X we have $A \cap B \neq \emptyset$. What is the maximum size of X? Prove your answer.

Solution. If we define X to be the collection of subsets A of $\{1, 2, \ldots, n\}$ such that $1 \in A$, then X satisfies the condition and $|X| = 2^{n-1}$. (5 points)

We claim that 2^{n-1} is the maximum size. Suppose that X is a collection of subsets of $\{1, 2, \ldots, n\}$ with $|X| \ge 2^{n-1} + 1$. Now consider all sets $\{A, \overline{A}\}$ consisting of a subset $A \subset \{1, 2, \ldots, n\}$ and its complement $\overline{A} = \{1, 2, \ldots, n\} - A$. (5 points)

There are 2^{n-1} such sets $\{A, \overline{A}\}$. Since every element of X is contained in one of these subsets and X has more than 2^{n-1} elements, by the pigeonhole principle, there are two sets A and B in X such that $A, B \in \{C, \overline{C}\}$. Then $B = \overline{A}$ and we have $A \cap B = \emptyset$. (5 points)

Therefore X cannot satisfy the given condition. Therefore 2^{n-1} is the maximum size of X. □

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