Discrete Mathematics Final Exam (GEDB007-41, Spring 2022)

Problem 1. Let n be an odd integer with $n > 1$. Prove or disprove: n is a composite if and only if it has a divisor d with $1 < d \leq n/3$.

Problem 2. (1) Find the inverse of 567 mod 131.

(2) Find the inverse of 131 mod 567.

Problem 3. For an integer $n \geq 3$, let $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, 3, 4, 5\}$. Find the number of functions $f : X \to Y$ such that

$$|\{f(x) : x \in X\}| \leq 3.$$

Problem 4. Let $a_1 a_2 \ldots a_{1000}$ be a rearrangement of

$$\overbrace{1 \cdots 1}^{700}\overbrace{3 \cdots 3}^{200}\overbrace{5 \cdots 5}^{100}.$$

Prove that there are two integers i and j such that $1 \leq i < j \leq 1000$ and $a_i + a_{i+1} + \cdots + a_j = 199$.

Problem 5. Find a formula for the nth term a_n of the sequence $a_0, a_1, a_2, \ldots$, which satisfies the initial conditions $a_0 = 3$, $a_1 = 0$ and the recurrence relation given by

$$(n^2 - n)a_n = 8(n-1)a_{n-1} - 16a_{n-2}.$$

Problem 6. Show that every bipartite graph with 21 vertices has at most 110 edges.

Problem 7. Consider the set of letters in the following table.

letter	A	B	C	D	E	F	G
frequency	7	1	7	11	3	4	2

(1) Find an optimal Hoffman code for these letters. (Draw a tree and write a 0-1 sequence for each letter. There are many possible optimal Hoffman codes, and you need to find just one of them.)

(2) Encode “BAG” using the Hoffman code.

Problem 8. Let $A = 1, B = 2, C = 3, D = 4, E = 5, F = 6$. Compute each expression, which is either a prefix form or a postfix form. Write “invalid” if the expression is not valid as a prefix or postfix expression.

(1) $ABCD++++$

(2) $+++ABCD$

(3) $AB+CD++$

(4) $A+B+CD+$

(5) $A+B+C+D+$

(6) $AB+CDEF*+-*$