Discrete Mathematics Midterm Exam (Spring 2021)

Problem 1. Prove or disprove each statement.

(1) The following argument is valid:

$$\begin{array}{l} q \to (r \wedge p) \\ (p \vee q) \to (r \wedge \neg p) \\ \neg r \wedge q \\ \hline \therefore r \vee \neg r \end{array}$$

(2) The following argument is valid:

$$\begin{array}{l} q \to (r \wedge p) \\ p \\ \neg p \\ \hline \therefore \neg q \wedge \neg r \end{array}$$

(3) The following argument is valid:

$$\begin{array}{l} \neg r \to \neg p \\ r \to (\neg p \vee \neg q) \\ \hline \therefore p \to (q \wedge r) \end{array}$$

(4) $(p \to q) \equiv (\neg p \vee q)$.

(5) $(p \to q) \wedge (q \to r) \equiv (p \to r)$.

Problem 2. Prove or disprove each statement.

(1) $\forall x \forall y((x < y) \to (x^2 > y^2))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(2) $\forall x \exists y((x < y) \to (x^2 > y^2))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(3) $\exists x \forall y((x < y) \to (x^2 > y^2))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(4) $\exists x \exists y((x < y) \to (x^2 > y^2))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

Problem 3. Prove that $\forall n \exists a \exists b((n \geq 12) \to (n = 3a + 7b))$, where the domain of discourse is $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

Problem 4. Let X be the set of all functions from $\{1, 2, \ldots, 2021\}$ to $\{1, 2, \ldots, 2021\}$. Define a relation R on X by $(f, g) \in R$ if and only if $f \circ g = g \circ f$. Prove or disprove each statement.

(1) R is reflexive.

(2) R is symmetric.

(3) R is antisymmetric.

(4) R is transitive.

Problem 5. Let X be the set of pairs (A, B) of subsets $A, B \subseteq \{1, 2, 3\}$ such that $B \neq \emptyset$. Define a relation R on X by $((A, B), (A', B')) \in R$ if and only if $|A| \cdot |B'| - |A'| \cdot |B| = 0$.

(1) Show that R is an equivalence relation.

(2) Find the equivalent class containing $(\emptyset, \{1\})$.

Problem 6. Write an algorithm that receives (s, n), where $s = (s_1, \ldots, s_n)$ is a sequence n distinct integers ($n \geq 2$), and returns the second largest integer in s. (You may assume that $s_1, \ldots, s_n$ are all distinct.)

Problem 7. For a positive integer n, let $f(n) = 1^{2021} + 2^{2021} + \cdots + (2n)^{2021}$. Prove or disprove: $f(n) = \Theta(n^{2022})$.

Problem 8. (1) Find the inverse of 44 mod 2021.

(2) Find the inverse of 2021 mod 44.