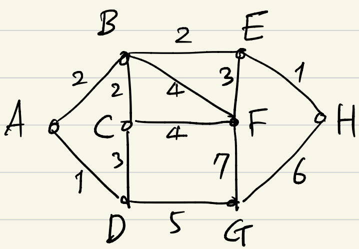
§8.4. A shortest path algorithm.



City map 가중치 비용(걸리는 시간)

The edge weights are the cost to move from one city to another.

Q: What is the minimum cost to move from A to H? total (weight)

In other words, find the length of a shortest path from A to B.

We assume that all edge weights are positive.

다익스트라

* Dijkstra's shortest path algorithm.

Input: A connected, weighted graph $G=(V,E)$ and edge weight w and two vertices a, z.

Output: the length of a shortest path from a to z.

Algorithm

① Label the vertices

$L(a)=0, \quad L(v)=\infty \quad \forall v \neq a.$ a가 아닌 vertex 모두 ∞

Set all vertices "active" 탐색하지 않은 노드

② Find an active vertex v with smallest label.

Make v inactive 이미 탐색한 노드

For each active vertex x adjacent to v replace $L(x)$ by $\min\{L(x), L(v)+w(v,x)\}$

③ Repeat ② until z becomes inactive.

④ Return $L(z)$.

copy

* Dijkstra's shortest path algorithm.

Input : A connected, weighted graph $G=(V,E)$ and edge weight w and two vertices a, z.

Output : the length of a shortest path from a to z.

Algorithm

① Label the vertices

$L(a)=0, \quad L(v)=\infty \quad \forall v \neq a.$

Set all vertices "active".

② Find an active vertex v with smallest label.

Make v inactive.

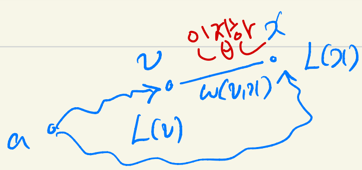
For each active vertex x adjacent to v (인접한)

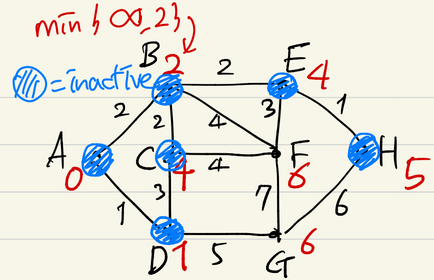
replace $L(x)$ by $\min\{L(x), L(v)+w(v,x)\}$

③ Repeat ② until z becomes inactive

④ Return $L(z)$.

$L(v)$ = the length of shortest path from a to v.





active한 애들 중 smallest value순 : $A \to D \to B \to C \to E \to H \to F \to G$ 순으로 탐색

$L(H)=5$ is the length of a shortest path from A to H.

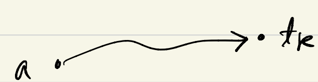
<u>Thm</u> Dijkstra's algorithm always finds the length of a shortest path.

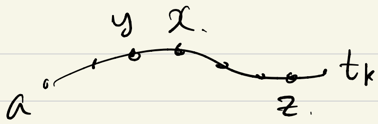
pf). Let $t_1, t_2, \dots, t_n$ be the sequence of inactive vertices.

We claim $L(t_k)$ is the len of shortest path from a to t_k by induction on k. (strong ind)

$k=1$: $t_1 = a$, and $L(a)=0$, clear.

Suppose it's true for $1, 2, \dots, k-1$. Suppose $L(t_k)$ is not len of shortest path.





shortest path. p

v를 Inactive로 만드는 시점에서 L(v)는 최소라는걸 증명

For $v \in V$ let $S(v)$ = length of shortest path from a to v.
Then $L(v) \geq S(v)$ because by construction there is a path from a to v of length $L(v)$.
It remains to show $L(v) \leq S(v)$ by the time that v becomes inactive.
For contradiction suppose not, i.e., $L(v) > S(v)$.
This means there is a path p from a to v whose length is smaller than $L(v)$. 더 작은 p존재

z를 안거치고 갈수도 있어서 ㄴ!!

Claim: every vertex in p must be inactive.
Suppose not. Find the first active vertex, x.

active 정점이 있다고 가정

Then the prev vertex y is inactive, so

이미 선택된 v

$L(x) \leq L(y) + w(y,x) \leq$ len of path $< L(t_k)$.
Then x must be selected instead of t_k

L(x)< L(tk)면 다익스트라가 x를 골랐어야 한다는 것 => 모순
안골랐으니!

Thus all vertices in p are inactive.
Let z be the last vertex in p before t_k.
Then by ind hyp, $L(z)$ is the len of shortest path from a to z. So, $L(t_k) \leq L(z) + w(z, t_k)$
$\leq w(a \to z) + w(z, t_k)$
$= \cancel{wt(p)}$.

∴ L(tk)=wt(p) 최단거리다!
weight total(p)

Thus $L(t_k) \leq wt(p)$.
But $L(t_k) \geq wt(p)$ because

다익스트라가 가장 작은 값들로 업데이트해왔기에!

for any v, there is a path from a to v whose length is $L(v)$.

§8.5. Representations of graphs

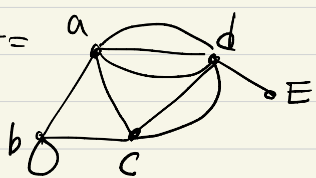
We can represent graphs using diagrams.
We can also represent graphs using matrices.

Def) $G=(V,E)$ graph with $V=\{v_1,\dots,v_n\}$.
The adjacency matrix (인접 행렬) of G is the $n\times n$ matrix $A=(a_{ij})_{i,j=1}^{n}$ defined by

$$a_{ij} = \#\text{ edges from } i \text{ to } j. \text{ (directed)}$$
(between i and j if undirected)

The rows and columns of A are indexed by the vertices of G.

ex) $G=$



$V=\{a,b,c,d,e\}$

a→d가 3개 경로라서

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 3 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}\end{array}$$

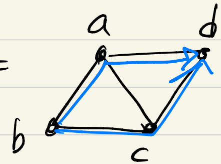
루프, 멀티엣지 존재 X

Note If G is a simple graph then A is a 0-1 matrix.

$$(i,j)\text{-entry} = \begin{cases} 1 & \text{if } (v_i, v_j)\in E \\ 0 & \text{otherwise.} \end{cases}$$

ex)

$G =$



$$A = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 1 & 0 & 1 \\ d & 1 & 0 & 1 & 0 \end{array}$$

$$A^2 = \begin{array}{c} \\ b \\ \\ \\ \end{array} \begin{pmatrix} 0 & 1 & 1 & 1 \\ \boxed{1 \quad 0 \quad 1 \quad 0} \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{array}{c} a \\ b \\ c \\ \\ \end{array} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

(column labels of the first factor: a b c d; of the second factor: d over the last column)

$$= \begin{array}{c} \\ b \\ \\ \\ \end{array} \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & \boxed{2} \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

(column label d over the last column)

$$2 = \boxed{1 \cdot 1} + 0 \cdot 0 + \boxed{1 \cdot 1} + 0 \cdot 0$$

$b \;\cancel{\cdot}\; b \;\cancel{\cdot}\; d$

path $b - a - d$ 1개

$b - c - d$ 2개

$b \;\cancel{\cdot}\; d \;\cancel{\cdot}\; d$

Thm $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$

A : adj mat of G. $(k \geq 1)$.

The (i,j)-entry of A^k is # paths of length k from v_i to v_j. 귀납법으로 증명하는데 생각~

Def) $G=(V,E)$.

$V=\{v_1,\dots,v_n\}$

$E=\{e_1,\dots,e_m\}$.

always zero-one matrix이다!

The incidence matrix of G is the 입사행렬

$n\times m$ matrix $M=(m_{ij})_{1\le i\le n,\ 1\le j\le m}$

$n\times m$

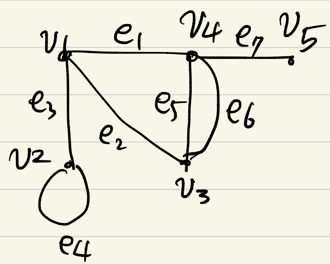
$$m_{ij}=\begin{cases}1 & \text{if } v_i \text{ is incident to } e_j\\ 0 & \text{otherwise.}\end{cases}$$

연결된

$$M=\begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7\\ \hline v_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\ v_2 & 0 & 0 & 1 & 1 & 0 & 0 & 0\\ v_3 & 0 & 1 & 0 & 0 & 1 & 1 & 0\\ v_4 & 1 & 0 & 0 & 0 & 1 & 1 & 1\\ v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{array}$$

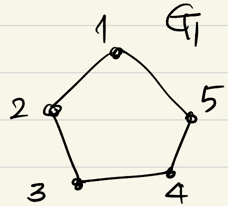
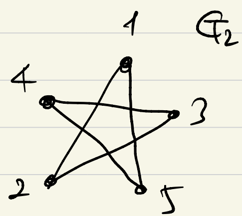
vertice, edge에 대한 정보를 얻을 수 있다

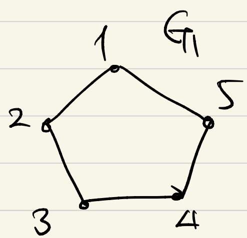
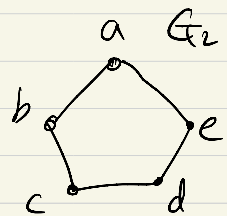
ex)



§8.6. Isomorphisms of Graphs
동형사상

How do we distinguish two graphs?

①  $\overset{?}{=}$  Yes!

②  $\overset{?}{=}$  No!

$G=(V,E)$

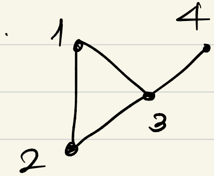
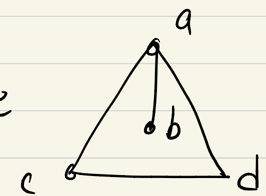
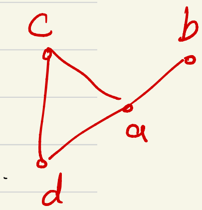
For ①, G_1 and G_2 have same V, E. ⇒ Isomorphism

$G_1 = G_2$ exactly the same graph.

For ②, G_1 and G_2 have different vertices.

Def) $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$: simple graphs. 간단하게 하기 위해

G_1 and G_2 are isomorphic if there is a 1-1 (one to one) & onto function $f: V_1 \to V_2$ (bijection) such that $(u,v)\in E_1 \iff (f(u),f(v))\in E_2$.

ex).  $\approx$  

$f: \{1,2,3,4\} \to \{a,b,c,d\}$ (Vertices)

$f(1)=c$, $f(2)=d$, $f(3)=a$, $f(4)=b$ (bijection)

1−2 연결됨 → c−d 연결됨? 이런식으로 대응되는 모든 엣지가 둘다 존재하면 Isomorphic !!

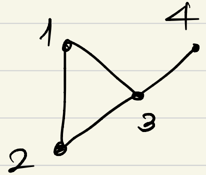
If G_1 and G_2 are isomorphic we write $G_1 \approx G_2$.

Otherwise, $G_1 \not\approx G_2$.

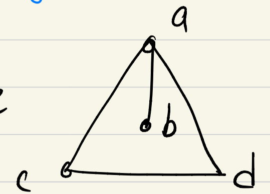
Thm G_1 and G_2 are isomorphic iff there are some orderings of V_1 and V_2 s.t. their adjacency matrices are equal.

인접행렬

ex)



$\approx$



$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array} \quad \overset{\text{행렬이}}{\underset{\text{같다}}{=}} \quad \begin{array}{c|cccc} & c & d & a & b \\ \hline c & 0 & 1 & 1 & 0 \\ d & 1 & 0 & 1 & 0 \\ a & 1 & 1 & 0 & 1 \\ b & 0 & 0 & 1 & 0 \end{array}$$

효율적으로!

Q: How can we show that two graphs are isomorphic?

A: We really need to find a proper bijection.

Q: How can we show that two graphs are $\boxed{\text{NOT}}$ isomorphic?

Not임을 증명하자!

Def) G: a graph.

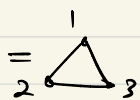
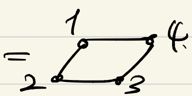
불변

A property of G is called an invariant if every graph isomorphic to G also has this property.

ex) # vertices is an invariant.

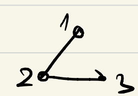
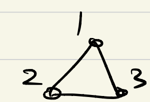
(More precisely the property "G has n vertices" is invariant.)

edges is an invariant.

ex) $G_1 =$  $G_2 =$ 

Invariant위반!

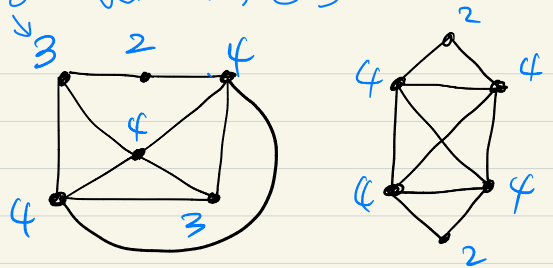
G_1, G_2 are not isomorphic because # vertices different.

ex) $G_1 =$  $\not\cong$ G_2 

$|E_1| = 2 \neq |E_2| = 3.$

degree vertex: 6개, edge: 10개

ex)

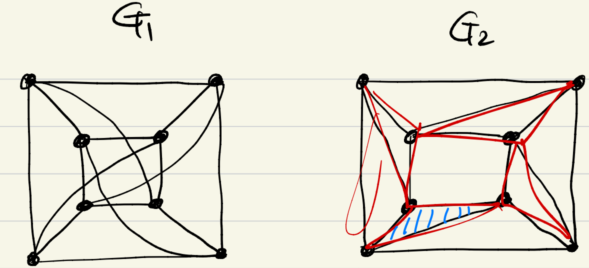


degree sequences are different.

invariants

So, the two graphs are not isomorphic.

ex)



G_1 has 0 triangles.

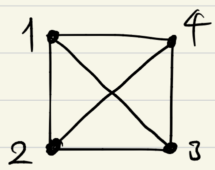
Invariant : 삼각형 개수

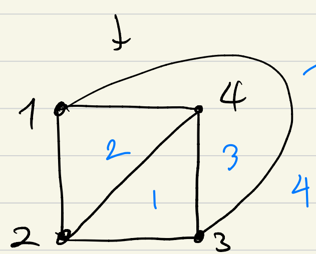
G_2 〃 many 〃 .

$G_1 \ncong G_2$

§8.7. Planar Graphs

Def) A graph is <u>planar</u> (평면그래프) if it can be drawn on a plane (평면) without crossing edges.

ex)  is planar

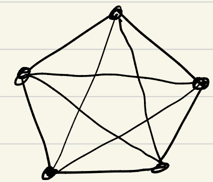


→ crossing 없이그리기

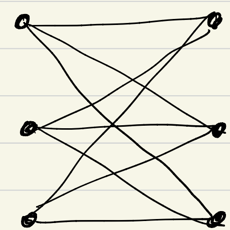
$v=4, \quad e=6, \quad f=4$

$v-e+f=4-6+4=2.$

Q: K_5 planar?



$K_{3,3}$ planar?



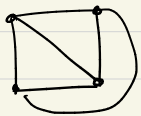
오일러 공식

Thm (Euler's formula)

G: a connected planar graph with v vertices, e edges and f faces.
나눠진 구역

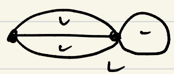
Then $v-e+f=2$.

ex)



$v=4$, $e=6$, $f=2$.

ex)



$v=2$, $e=4$, $f=4$

$2-4+4=2$.

Pf) Induction on e.

If $e=0$, then $v=1$, $f=1$.

$v-e+f=1-0+1=2$. (base)

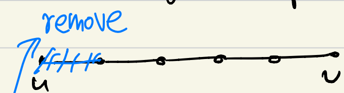
Suppose that the theorem is true for $e=n$.

Consider a connected planar graph G with $e=n+1$.

Case I: G has no cycles. Then $f=1$

사이클이 생겨야 f가 증가!

Find a longest simple path. Let u and v be the ending points.



Then $\deg u = \deg v = 1$ because otherwise we get a cycle or a longer path. (그렇지 않으면 모순)

Let G' be the graph obtained from G by deleting u and the edge incident to it.

Then G' has $v'=v-1$, $e'=e-1$, $f'=f$.

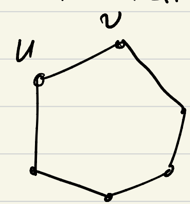
Since G' has $e'=n$ edges, by ind hyp,

$v'-e'+f'=2$.

$(v-1)-(e-1)+f=2 \Rightarrow v-e+f=2$.

Case II: G has a cycle.

Find a simple cycle. Let u, v be adjacent vertices in this cycle.



Let $G' = G$ with (v,u) removed.

Then $v'=v$, $e'=e-1$, $f'=f-1$

By ind hyp, $v'-e'+f'=2$.

$\Rightarrow v'-(e-1)+(f-1)=2 \Rightarrow v-e+f=2$.

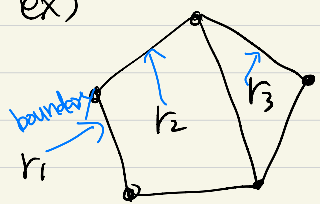
□

Def) G: planar graph drawn on a plane without crossing edges.

r is a face of G in this drawing.

The degree of r is the # steps needed to walk along the boundary of r (returning to the starting point).

ex)

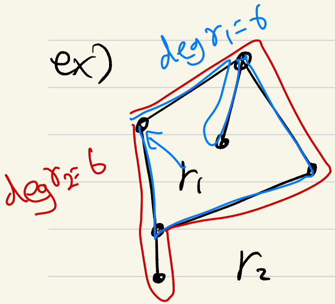


$\deg r_1 = 5$

$\deg r_2 = 4$

$\deg r_3 = 3$.

ex)



$\deg r_1 = 6$

$\deg r_2 = 6$.

Thm G: planar

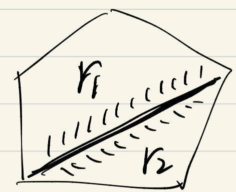
F: set of all faces in G.

$$\sum_{r \in F} \deg r = 2e.$$

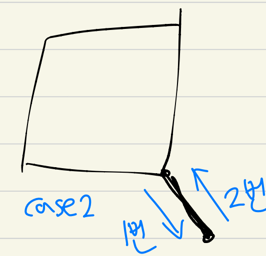
2개의 face가 1개의 엣지를 공유하므로 (혹은 ↘↗ 2번 이동하므로)

Pf) Every edge is counted twice when we compute LHS.

좌변



case 1



case 2

$\square$.

〈조건〉

Thm G: simple, connected, planar with at least 3 vertices 3개 정점이상

Then $e \leq 3v - 6$. ← 조건 만족하는데 $e \leq 3v-6$ 이 아니라면 not planar

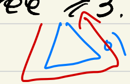
Pf) Suppose G is drawn on a plane.

By Euler's formula

$$v - e + f = 2.$$

Since G is simple, connected, $v \geq 3$,

every face has degree ≥ 3.


이게 최소 degree

By the above thm

$$2e = \sum_{r \in F} \deg r \geq \sum_{r \in F} 3 = 3f$$

$$2e \geq 3(2 - v + e)$$

$$\Rightarrow e \leq 3v - 6. \qquad \square$$

Cor K_5 is not planar.

Pf) K_5 is simple, connected and has at least 3 vertices.

If K_5 is planar, then $e \leq 3v - 6$.

But $e = 10$, $v = 5$.

$$10 \not\leq 3 \cdot 5 - 6 = 9$$

So K_5 is not planar. $\square$

For $K_{3,3}$, $v = 6$, $e = 9$.

⋮ ⋮

$$e \leq 3v - 6$$

$$9 \leq 3 \cdot 6 - 6 = 12$$ 조건을 만족시키지못함 → Not planar

조건을 만족함 → planar라고 단정짓는 것은 위험

추가

Thm G : connected, simple, bipartite, planar with $v \geq 4$.

Then $e \leq 2v - 4$.

Pf) G : drawn on a plane without crossing edges.

Since G is bipartite, G does not have 3-cycles. △모양 (왜냐, 삼각형은 bipartite 위반)

Then every face has $\deg \geq 4$. ($\deg = 3$가 될 수 없기에)

$$2e = \sum_{r \in F} \deg r \geq 4f$$

$$e \geq 2f = 2(2 - v + e)$$

$$\Rightarrow \quad e \leq 2v - 4. \qquad \square$$

Cor $K_{3,3}$ is not planar.

Pf) $v = 6, \quad e = 9.$

$$e \leq 2v - 4$$

$$9 \nleq 2 \cdot 6 - 4 = 8$$

$\Rightarrow$ not planar. $\square$

Interestingly, K_5, $K_{3,3}$ are the only essentially nonplanar graphs.

$\Rightarrow$ K_5 or $K_{3,3}$을 포함한 그래프는 non-planar이다.

쿠라토스키 이론

Thm (Kuratowski's Thm).

G is planar iff G does not "contain" K_5 or $K_{3,3}$.