Discrete Mathematics Final Exam (Spring 2021)

Problem 1. Let n be a positive integer. Find the number of all $n \times n$ matrices $A = (A_{i,j})_{1 \le i,j \le n}$ satisfying the following two conditions:

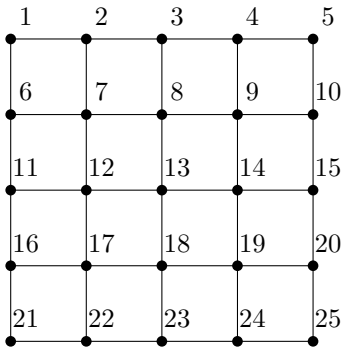
- $A_{i,j} \in \{-1, 0, 1\}$ for all $1 \le i, j \le n$, and
- $A_{i,j}A_{j,i} = 0$ for all $1 \le i, j \le n$ with $i \ne j$.

Problem 2. Find the number of positive integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 2021$ such that $x_i \equiv i \pmod 2$ for all $i = 1, 2, \ldots, 5$.

Problem 3. Suppose that $a_1, \ldots, a_{1312}$ are distinct integers in $\{1, 2, \ldots, 2021\}$. Prove that there are two integers i, j with $1 \le i, j \le 1312$ such that $a_i - a_j = 601$.

Problem 4. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by $a_0 = 2, a_1 = -9$, and $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \ge 2$. Find a general formula for a_n.

Problem 5. Let G be the following graph and let A be the adjacency matrix of G with respect to the vertex ordering $1, 2, \ldots, 25$.



(1) Evaluate the sum of all entries of A, that is, $\sum_{i=1}^{25} \sum_{j=1}^{25} A_{i,j}$.

(2) Evaluate the sum of the diagonal entries of A^2, that is, $\sum_{i=1}^{25} (A^2)_{i,i}$.

(3) Evaluate the $(1, 25)$-entry $(A^8)_{1,25}$ of A^8.

(4) Evaluate the sum of the entries in row 13 of A^3, that is, $\sum_{i=1}^{25} (A^3)_{13,i}$.

Problem 6. The *complement* of a simple graph G is the simple graph $\overline{G}$ with the same vertices as G such that an edge exists in $\overline{G}$ if and only if it does not exist in G. Find the number of simple graphs G with vertex set $\{1, 2, 3, 4\}$ such that G and $\overline{G}$ are isomorphic.

Problem 7. Consider the set of letters in the following table.

letter	A	B	C	D	E	F
frequency	1	2	4	6	9	10

(1) Find an optimal Hoffman code for these letters. (Draw a tree and write a 0-1 sequence for each letter. There are many possible optimal Hoffman codes, and you need to find just one of them.)

(2) Encode "CAFE" using your Hoffman code.

Problem 8. Let G be the graph in Problem 5.

(1) Draw the spanning tree of G obtained by the depth-first search with the vertex ordering $1, 2, \ldots, 25$.

(2) Draw the spanning tree of G obtained by the breadth-first search with the vertex ordering $1, 2, \ldots, 25$.