Discrete Mathematics

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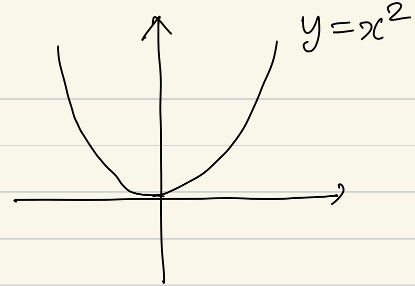
Textbook : Discrete mathematics 8th ed
by Johnsonbaugh
(Chapters 1 ~ 9)

Grade :
Participation : 10
Homework : 20 ~> WEBWORK
Midterm : 30
Final : 40

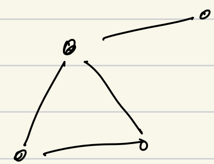
What is Discrete Math?

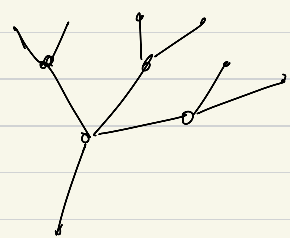
- Calculus is the study of continuous objects.



- Disc. Math " discrete objects.

$\{1, 2, 3\}$.





We will cover

- sets, logic
- proofs, proof techniques.
- functions, sequences, relations
- algorithms.
- number theory.
- counting
- graph theory.
- trees.

Ch 1. Sets and Logics

§1.1. Sets

집합

- A set is a collection of objects. 원소
- The objects of a set are call elements. (members)

ex) $\{1,2,3,4\}$ is a set with 4 elements, 1, 2, 3, 4.

$\{0,1\}$, $\{a,b,c\}$, $\{\{1\},\{1,2\},\{2,3,4\}\}$

- A set is determined by its elements. The order of the elements and their multiplicities do not matter. 순서 상관 X

ex) $\{1,2,3,4\}=\{3,4,1,2\}$

$=\{1,1,2,2,3,4,4,4\}$ 중복 제외

- A set can be described by listing properties of its elements.

ex) $A=\{x \mid x \text{ is a positive integer}\}$ 같은 set

$=\{1,2,3,\dots\}$

실수

$B=\{x \mid x \text{ is a real number } \underbrace{\text{s.t.}}_{\text{such that}}\ 0<x<1\}$

$=\{0,1,2,3,\dots,-1,-2,\dots$

$\frac{1}{2},\frac{1}{3},\frac{4}{3},-\frac{2}{5},\dots$

$\sqrt{2},\pi,-\sqrt{26},\dots\}$

※ Frequently used sets

정수

$\mathbb{Z}=\{x \mid x \text{ integer}\}$

유리수

$\mathbb{Q}=\{x \mid x \text{ rational number}\}$

실수

$\mathbb{R}=\{x \mid x \text{ real number}\}$.

$\mathbb{Z}^-=$ set of negative integers

$=\{x \mid x \text{ neg int}\}$

$\mathbb{Z}^+,\mathbb{Q}^+,\mathbb{R}^+$ are defined similarly

$\mathbb{Z}^{\text{nonneg}}=\{0,1,2,3,\dots\}$.

* Notation

If x is an element of a set A,
we write $x \in A$.
(otherwise, we write $x \notin A$).

집합의 크기

If A is a finite set, the cardinality of A is the number of elements in A, denoted by $|A|$.

ex) $A = \{1,2,3,4\}$.
$3 \in A$, $2 \in A$, $5 \notin A$, $|A| = 4$.

The set with no elements is called the empty set, denoted ϕ 공집합

Two sets A and B are equal $(A=B)$ if they have the same elements.

ex) $\{1,2,3\} = \{x \mid x \text{ integer s.t. } 1 \le x \le 3\}$
$\{1,2\} \neq \{1,2,3\}$.

ex). $\phi = \{\ \}$, oh! $\phi \neq \{\phi\}$. no!

- For sets A and B, we say
  A contains B (or B is contained in A)
  if every element of B is also an elt of A.
  → we write $B \subseteq A$ or $A \supseteq B$
  → We also say B is a subset of A. 부분집합

ex) $\{1,2,3\} \subseteq \mathbb{Z}$.
$\{1,2,3\} \subseteq \{1,2,3\}$.
$\{1,2,3\} \not\subseteq \{1,2\}$.

$B \subseteq A \iff$ B is a subset of A
$\iff$ B is contained in A
$\iff$ A contains B
$\iff$ if $x \in B$, then $x \in A$.

ex). $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, $\mathbb{Z} \not\subseteq \mathbb{Q}^+$

$$\underset{-1}{\mathbb{Z}\ni} \qquad \underset{-1}{\mathbb{Q}^+ \not\ni}$$

진부분집합

- $A \subset B$ means A is a proper subset of B, that is, A is a subset of B and $A \neq B$.
  $A \subset B \iff A \subseteq B$ and $A \neq B$. 같은경우제외!
  Sometime we write $A \subsetneq B$.

ex) $\{1,2,3\} \subseteq \{1,2,3\}$.
$\{1,2,3\} \not\subset \{1,2,3\}$.
$\{1,2\} \subset \{1,2,3\}$.

$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

Question T or F?
① If $A \subset B$, then $A \subseteq B$ (T)
② If $A \subseteq B$, then $A \subset B$. (F)
(If $A = B$, then $A \subseteq B$ but $A \not\subset B$).

- For a set X, the power set $\mathcal{P}(X)$ of X is the set of all subsets of X.

모든부분집합

ex) $A = \{1,2,3\}$.
$\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

ex) $\mathcal{P}(\phi) = \{\phi\}$. $\qquad \phi \subseteq \phi$

Note ϕ is a subset of every set.

Def) X, Y: sets
$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$ union 합
$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$ intersection 교
$X - Y = \{x \mid x \in X \text{ and } x \notin Y\}$ difference 차

ex) $A = \{1,2,3\}$, $B = \{2,3,4\}$.
$A \cup B = \{1,2,3,4\} = B \cup A$
$A \cap B = \{2,3\} = B \cap A$.
$A - B = \{1\}$
$B - A = \{4\}$
Note $A - B \neq B - A$ in general.
$A \cup B = B \cup A$, $A \cap B = B \cap A$.

Def) If $X \cap Y = \phi$, then we say X and Y are disjoint. 서로소

If S is a collection of sets, then "S is pairwise disjoint" means for any two sets X, Y in S, we have $X \cap Y = \phi$.

ex) $\{-1,-2,-3\}$, $\{1,2\}$ are disjoint.
$\{-1,-2,-3\}$, $\{-1,0,1\}$ are not disjoint

ex) $S = \{\phi, \{1\}, \{2\}, \{3,4\}\}$
S is pairwise disjoint.

ex) $S = \{\{1\}, \{2\}, \{1,3\}\}$
is not pairwise disjoint.

When we consider sets which are contained in a given set U, then U is called a universal set or a universe.
전체 집합

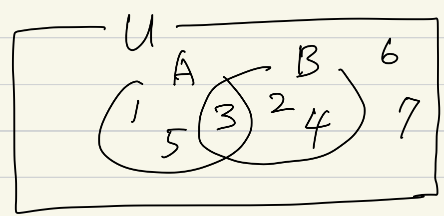
Given a universe U and its subset X, $U - X$ is the complement of X, written $\overline{X}$ (or X^c)
여집합

ex) $U = \{1,2,3,4,5\}$. $A = \{2,4\}$.
$\overline{A} = U - A = \{1,3,5\}$.

ex) If $U = \mathbb{R}$, then $\mathbb{Q} \subseteq \mathbb{R}$.
유리수 실수
$\overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q}$ = the set of irrational numbers.
무리수

* Venn diagram

ex) $U = \{1,2,\dots,7\}$. $A = \{1,3,5\}$. $B = \{2,3,4\}$.



Theorem

Let U be a universe and $A, B, C \subseteq U$.

1. *Associative Laws*

   결합

$$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$$

2. *Commutative Laws*

   교환

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

3. *Distributive laws*

   분배

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. *identity laws*

$$A \cup \emptyset = A \quad A \cap U = A$$

5. *Complement laws*

$$A \cup \overline{A} = U, \quad A \cap \overline{A} = \emptyset$$

6. *Idempotent Laws*

$$A \cup A = A, \quad A \cap A = A$$

7. *Bound Laws*

$$A \cup U = U, \quad A \cap \emptyset = \emptyset$$

8. *Absorption Laws*

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

9. *Involution Laws*

$$\overline{\overline{A}} = A$$

10. $0/1$ *Laws*

$$\overline{\emptyset} = U, \quad \overline{U} = \emptyset$$

11. *De Morgan's Laws for sets*

    드모르간

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Pf of ⑪ Note: $X = Y \iff X \subseteq Y$ and $Y \subseteq X$.

Let's prove $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. done!

This means $x \in \overline{A \cup B} \Rightarrow x \in \overline{A} \cap \overline{B}$.

$x \in \overline{A \cup B} \Rightarrow x \in U - (A \cup B). \Rightarrow x \notin A \cup B.$

$\Rightarrow x \notin A$ and $x \notin B$. $\Rightarrow x \in \overline{A}$ and $x \in \overline{B}$.

$\Rightarrow x \in \overline{A} \cap \overline{B}$.

Let's prove $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

$x \in \overline{A} \cap \overline{B}. \Rightarrow x \in \overline{A}$ and $x \in \overline{B}$.

$\Rightarrow x \notin A$ and $x \notin B \Rightarrow x \notin A \cup B$

$\Rightarrow x \in \overline{A \cup B}$.

Since $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

we have $\overline{A \cup B} = \overline{A} \cap \overline{B}$. $\square$.

If S is a collection of sets, then we write

$\bigcup S = \{x \mid x \in X \text{ for some } X \in S\}$.

$\bigcap S = \{x \mid x \in X \text{ for all } X \in S\}$.

If $S = \{A_1, A_2, \ldots, A_n\}$, then

$$\bigcup S = A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

$$\bigcap S = A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i .$$

ex) For $i \geq 1$, let $A_i = \{i, i+1, i+2, \ldots\}$

Let $S = \{A_1, A_2, A_3, \ldots\}$.

Question: What are $\bigcup S$ and $\bigcap S$?

① $\bigcup S = A_1 \cup A_2 \cup \cdots = \bigcup_{i=1}^{\infty} A_i = A_1$

$= \{1, 2, 3, \ldots\}$.

$A_1 = \{1, 2, 3, \ldots\}$ $\qquad A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$

$A_2 = \{2, 3, 4, \ldots\}$

$A_3 = \{3, 4, 5, \ldots\}$

② $\bigcap S = A_1 \cap A_2 \cap \cdots = \bigcap_{i=1}^{\infty} A_i$.

Answer: $\bigcap S = \phi$.

Suppose $\bigcap S \neq \phi$.

There is some element $x \in \bigcap S$.

Then $x \in A_1$, $x \in A_2$, … $x \in A_i$.

Since $x \in A_1 = \{1, 2, \ldots\}$,

x is an integer.

Consider A_{x+1}.

Since $A_{x+1} \in S$, and $x \in \bigcap S$,

we have $x \in A_{x+1}$.

But $A_{x+1} = \{x+1, x+2, x+3, \ldots\}$

does not have x. (contradiction).

Thus, $\bigcap S = \phi$. $\square$.

Def) A (set) partition of a set X is a collection of nonempty and nonintersecting subsets of X whose union is X.

In other words, P is a partition of X if

① $P = \{B_1, B_2, \dots B_n\}$, $B_i \subseteq X$

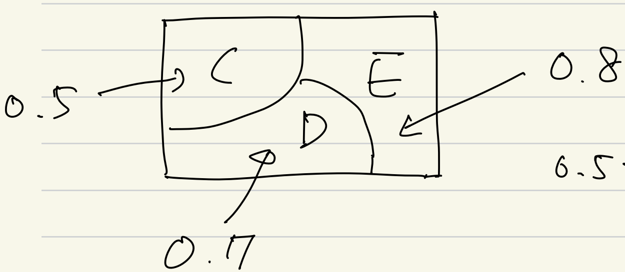
② $B_i \neq \phi$ for all i 공집합X

③ $B_i \cap B_j = \phi$ for all $i \neq j$. 두집합이 서로소

④ $B_1 \cup B_2 \cup \cdots \cup B_n = X$. 집합은 합집합 → 전체집합

Let's consider your computer.

suppose disk of size 2 TB



$0.5 + 0.7 + 0.8$
$= 2$

Def) An ordered pair is a pair (a, b) of elements whose order is important.
$((a,b) \neq (b,a)$, if $a \neq b)$.

An n-tuple is an ordered sequence of n elements $(a_1, a_2, \dots, a_n)$.

데카르트 곱셈 (곱집합)

The Cartesian product of sets X and Y is

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

ex) $X = \{1, 2\}$, $Y = \{a, b\}$.

$X \times Y = \{(1,a), (1,b), (2,a), (2,b)\}$.

$Y \times X = \{(a,1), (a,2), (b,1), (b,2)\}$

$X \times Y \neq Y \times X$ in general.

We always have $|X \times Y| = |X| \cdot |Y|$.

We also define

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, \dots, a_n) \mid a_i \in A_i, i = 1, \dots, n\}.$$

We have

$$|A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n|.$$