# Ch 7. Recurrence Relations

점화식

## §7.1. Introduction

ex) There is a pair of new born rabbits. Each pair of rabbits at least 2 months old reproduces another pair of rabbits every months. How can we compute # pairs of rabbits after $n$ months?

매달

2달 후

Sol) Let $a_n$ be # pairs of rabbits after $n$ months.

$\boxed{a_0 = 1, \ a_1 = 1}$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$,

$\vdots$


$a_n = $ (# pairs of rabbits already there) + (# pairs of newly born rabbits)

$= a_{n-1} + a_{n-2}$ for $n \geq 3$.

처음 낳게 2달 걸리니까

A recurrence relation for $a_n$ is

$\boxed{a_n = a_{n-1} + a_{n-2}.}$

$a_5 = 3 + 5 = 8$

$a_6 = 5 + 8 = 13$

$a_7 = 8 + 13 = 21$

$\vdots$

피보나치

**Def** A recurrence relation for the sequence $a_0, a_1, a_2, \ldots$ is an equation expressing $a_n$ in terms of previous terms $a_0, a_1, \ldots a_{n-1}$.

점화식

이전항

Initial conditions are explicit values of finitely many terms in this sequence.

many

ex) The Fibonacci sequence $f_0, f_1, f_2, \ldots$ has the recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2$$

and initial conditions $f_0 = 0$, $f_1 = 1$.

Note that the initial conditions and the recurrence relation completely determine the sequence.

수열을 결정함

ex) (Compounded Interest)

A person invests \$1000 at 12% interest annually compounded. What is the amount of balance after $n$ years.

sol) $A_n$ = balance after $n$ years.

$$A_0 = 1000.$$

$$A_n = A_{n-1} + 0.12\, A_{n-1} = 1.12\, A_{n-1}. \quad (n \geq 1).$$

$$A_n = 1.12\, A_{n-1} = (1.12)^2 A_{n-2}$$
$$= \cdots = (1.12)^n A_0 = (1.12)^n \cdot 1000.$$

"점화식을 푼다" 는 말은 n에 대해 식으로 나타낸다는 말

ex) Let's show that # subsets of $\{1, 2, \ldots, n\}$ is $2^n$.

Let $s_n$ be this number.

$s_0 = 1$, $s_1 = 2$.

For $n \geq 2$,

$s_n$ = # subsets of $\{1, \ldots, n\}$.

Any subset of $\{1, \ldots, n\}$ is obtained from a subset of $\{1, \ldots, n-1\}$ by adding $n$ or not.

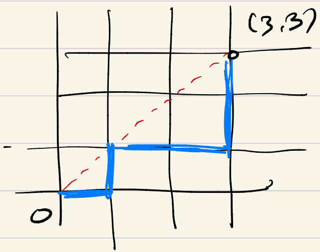
$$\Rightarrow \quad s_n = s_{n-1} \cdot 2$$

$$\Rightarrow \quad s_n = 2 s_{n-1} = 2^2 s_{n-2} = \cdots = 2^n s_0 = 2^n. \quad \square$$

ex) Recall that the Catalan number

$C_n$ = # paths from $(0,0)$ to $(n,n)$

that never go above $y=x$.




Show that for $n \geq 1$ $\quad$ $(C_0 = 1)$ emptypath라 1임

$$C_n = \sum_{k=1}^{n} \underset{\sim}{C_{k-1}} C_{n-k}.$$

ex). $C_1 = C_0 C_0 = 1$.

$C_2 = C_0 C_1 + C_1 C_0 = 1+1 = 2$.

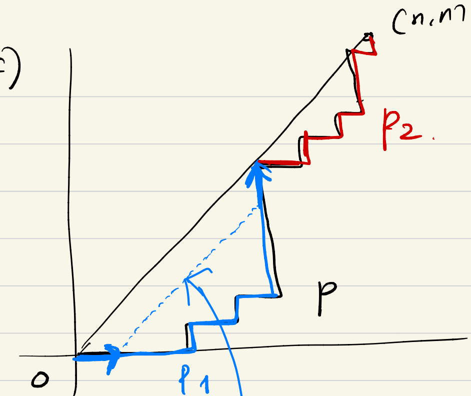
$C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0 = 2+1+2 = 5$.

$C_4 = C_0 C_3 + C_1 C_2 + C_2 C_1 + C_3 C_0$

$= 5 + 2 + 2 + 5 = 14$

$\vdots$

pf)




Any path counted by $C_n$ must start with east step $\rightarrow$.
$p$ 동쪽 스텝

Let $(k,k)$ be the first time $p$ returns to $y=x$.

Then $1 \leq k \leq n$. y=x에 닿는 첫 순간

Then $p$ is divided into two paths $p_1, p_2$.

$p_1 : (0,0) \to (k,k)$, $\quad p_2 : (k,k) \to (n,n)$.

$\# p_2 = C_{n-k}$ $\rightarrow$ $p_2 : (0,0) \to (n-k, n-k)$ 과 같음

$\# p_1 = C_{k-1}$ k-1인 이유

($\because$ The part of $p_1$ from $(1,0)$ to $(k,k-1)$ never goes above $y=x-1$.)

$\hookrightarrow$ $(0,0)$ to $(k-1,k-1)$ 경로와 같음

$$\Rightarrow \quad C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}.$$

ex) Let $a_n$ be # n-bit strings that do not contain 111.
Find a rec. rel. for $a_n$ and initial conditions.

공집합 포함

$0, 1$ $\quad$ $00, 01, 10, 11$

sol) $a_0 = 1$. $a_1 = 2$, $a_2 = 4$, $a_3 = 7$.

Let $n \geq 3$.

Let $s_1 s_2 \cdots s_n$ be an n-bit string not containing 111.

If $s_1 = 0$, then # possibilities for $s_2 \cdots s_n$ is $a_{n-1}$.

$a_{n-1}$에서 결정

If $s_1 s_2 = 10$ then # 〃 〃 $s_3 \cdots s_n$ is $a_{n-2}$.

$a_{n-2}$에서 결정

If $s_1 s_2 s_3 = 110$, then 〃 $s_4 \cdots s_n$ is $a_{n-3}$.

$a_{n-3}$에서 결정

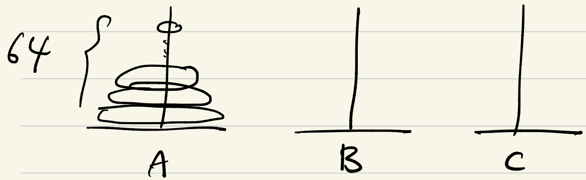
These three are all possible cases.

$\Rightarrow$ $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. $(n \geq 3)$.

$(A \to C)$.

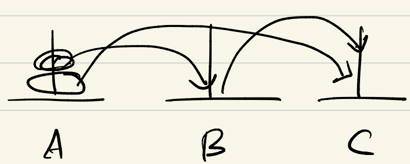
ex) (Tower of Hanoi)

There was an old man in Hanoi claiming the following.
There are 3 pegs and 64 disks of different sizes.




If you move all disks from A to C with the following rules then our world will be destroyed.

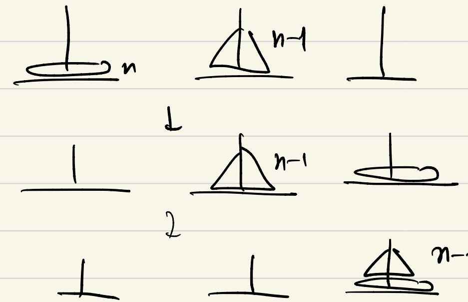
① You move one disk at a time.
② You can put one disk on top of a larger one.




let $a_n$ be # required moves to move $n$ disks.

$a_1 = 1$. $a_2 = 3$. . . .

If we move disk $n$ from A to C, C must be empty.
So $n-1$ disks must first be moved from A to B. $\Rightarrow$ $a_{n-1}$ moves




$a_{n-1}$ moves

1 move.

$a_{n-1}$ moves.

$$\Rightarrow a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1. \quad (n \geq 2)$$

$a_n = 2a_{n-1} + 1$

Let's compute the first few $a_n$.

$a_1 = 1, \quad a_2 = 3, \quad a_3 = 7, \quad a_4 = 15,$

$a_5 = 31, \ \ldots$

Guess: $a_n = 2^n - 1$.

점화식 푼다 (하지만, 이게 쉬운건 아님)

We can prove this using induction.

If $n=1$, true.

If true for $n$, then for $n+1$,

$$a_{n+1} = 2a_n + 1$$
$$= 2(2^n - 1) + 1$$
$$= 2^{n+1} - 1 \qquad \text{true!}$$

$\Rightarrow \quad a_n = 2^n - 1, \quad n \geq 1.$

If $n = 64$,

$$a_n = 2^{64} - 1 \approx 2^{64} = 2^4 \cdot (2^{10})^6$$
$$\approx 16 \cdot (1000)^6$$

1번에 1초 걸린다 가정

$$= 16 \cdot 10^{18} = 1.6 \times 10^{19}. \quad \text{(seconds)}$$

$$= \frac{1.6 \times 10^{19}}{3600 \times 24 \times 365} \quad \text{(years)}$$

$$\approx 5 \times 10^{11} \text{ years.}$$

5.8조년 ⇒ 그래서 세상이 멸망할거다 라고 말함

Compare this with

the age of the earth $\approx 4.5 \times 10^9$ years

" the universe $\approx 1.38 \times 10^{10}$ ".

## §7.2. Solving Recurrence Relations

Ex) Let $a_0, a_1, \ldots$ be the sequence defined by
$a_0 = 0$, $a_1 = 1$, and

$$a_n = 3a_{n-1} - 2a_{n-2} \text{ for } n \geq 2.$$

Find a formula for $a_n$.

Idea: If we can find a sequence $b_0, b_1, \ldots$ that satisfies the same initial conditions and the same recurrence relation then we have $a_n = b_n$. Let's try to find a sequence of the form $b_n = r^n$.

If $b_n = r^n$, then

$$b_n = 3b_{n-1} - 2b_{n-2} \iff r^n = 3r^{n-1} - 2r^{n-2}$$

$$\iff r^n - 3r^{n-1} + 2r^{n-2} = 0$$

$$\iff r^2 - 3r + 2 = 0$$

$$\iff r = 1 \text{ or } 2.$$

This means if $r = 1$ or $2$, then $b_n = r^n$ satisfies the rec. rel.

Observation: If two sequences $b_n, c_n$ satisfy the rec. rel, then the new sequence

$$d_n = \alpha \cdot b_n + \beta \cdot c_n \quad (\alpha, \beta : \text{constants})$$

also satisfies the rec. rel.

$$\begin{aligned} \because d_n &= \alpha \cdot b_n + \beta \cdot c_n \\ &= \alpha(3b_{n-1} - 2b_{n-2}) + \beta(3c_{n-1} - 2c_{n-2}) \\ &= 3(\alpha b_{n-1} + \beta c_{n-1}) - 2(\alpha b_{n-2} + \beta c_{n-2}) \\ &= 3d_{n-1} - 2d_{n-2}. \end{aligned}$$

Hence, $b_n = \alpha \cdot 1^n + \beta \cdot 2^n$ then $b_n$ satisfies r.r. 점화식

We choose $\alpha, \beta$ so that $b_0 = 0$, $b_1 = 1$. ← 점화식을 만족하게

$$\left.\begin{aligned} b_0 &= \alpha + \beta = 0 \\ b_1 &= \alpha + 2\beta = 1 \end{aligned}\right) \Rightarrow \alpha = -1, \ \beta = 1.$$

$$a_n = b_n = 2^n - 1.$$

선형 상계 동차 점화식

Def) A linear homogeneous recurrence relation is

$$(*)\quad a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} \quad (n \geq r)$$

where $c_1, \dots, c_r$ are constants.

특성다항식 (점화식을 푸는데 중요)

The characteristic polynomial of the rec. rel. is

$$x^r - c_1 x^{r-1} - \cdots - c_{r-1} x - c_r.$$

Thm. Suppose that $a_0, a_1, \dots$ is a seq. satisfying $(*)$. If the char. poly. has r distinct roots $s_1, \dots, s_r$ (해) then

서로 다른 r개의 해

$$a_n = \alpha_1 s_1^n + \cdots + \alpha_r s_r^n$$

for some constants $\alpha_1, \dots, \alpha_r$.

ex). Recall the Fibonacci sequence $f_0, f_1, \dots$

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2.$$

$$f_0 = 0, \quad f_1 = 1.$$

Find a formula for $f_n$.

sol). The char poly : $x^2 - x - 1 = 0$.

$\Rightarrow$ roots $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$.

$$f_n = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

$$f_0 = \alpha + \beta = 0$$

$$f_1 = \frac{1+\sqrt{5}}{2}\alpha + \frac{1-\sqrt{5}}{2}\beta = 1.$$

$$\Rightarrow \alpha = \frac{1}{\sqrt{5}}, \quad \beta = -\frac{1}{\sqrt{5}}.$$

$$\Rightarrow f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

점화식을 푸는 과정

중복해 (중근)?

Q : What if there are multiple roots?

ex) Suppose $h_n = 4h_{n-1} - 4h_{n-2}$, $h_0 = 0$, $h_1 = 1$.

Char poly : $x^2 - 4x + 4 = 0$.

has one double root 2.

If we try the same method,

$h_n = \alpha 2^n + \beta 2^n = (\alpha+\beta) 2^n$

$(\alpha+\beta) = 0$
$(\alpha+\beta) \times 2 = 1$
만족하는 $\alpha, \beta$는 없음

We cannot find $\alpha, \beta$ satisfying init. conds.

We need a solution different from $2^n$.

$2^n$ 하나의 항으로는 점화식을 만족시킬 수 없다

Thm Suppose $a_0, a_1, \ldots$ satisfies (*).

If the char. poly. has roots with multiplicities

$$\underbrace{\alpha_1, \ldots, \alpha_1}_{m_1}, \underbrace{\alpha_2, \ldots, \alpha_2}_{m_2}, \ldots, \underbrace{\alpha_k, \ldots, \alpha_k}_{m_k}$$

then,

이런 항들이 필요함

$$a_n = c_{11}\alpha_1^n + c_{12} n\alpha_1^n + \cdots + c_{1m_1} n^{m_1-1}\alpha_1^n$$
$$+ c_{21}\alpha_2^n + c_{22} n\alpha_2^n + \cdots + c_{2m_2} n^{m_2-1}\alpha_2^n$$
$$+ \cdots$$
$$+ c_{k1}\alpha_k^n + c_{k2} n\alpha_k^n + \cdots + c_{km_k} n^{m_k-1}\alpha_k^n,$$

where $c_{ij}$ are constants. $(1 \le j \le k,\ 1 \le i \le m_j)$.

In other words, if the char poly has a root $\alpha$ of multiplicity $m$ then

$\alpha^n, n\alpha^n, n^2\alpha^n, \ldots, n^{m-1}\alpha^n$

satisfy the rec. rel. So we have $m$ distinct solutions to the rec. rel.

m개의 다른 솔루션을 갖지 않으면 항들이 $(\alpha+\beta)2^n$ 꼴처럼 뭉치는 항들로 인해 점화식을 만족시켜 항을 놓을 수 있다

초기조건 ⇐

For example, if $a_n = r a_{n-1} + s a_{n-2}$ has char. poly with double root $\alpha$, then

$a_n = c_1\alpha^n + c_2 n\alpha^n$.

ex continued)

Since $x^2 - 4x + 4 = (x-2)^2 = 0$ has double root 2,

$h_n = c_1 2^n + c_2 n 2^n$.

$h_0 = c_1 = 0$

$h_1 = 2c_1 + 2c_2 = 1$

$\Rightarrow c_1 = 0,\ c_2 = 1/2$.

$\Rightarrow h_n = n 2^{n-1}$.