Discrete Mathematics Midterm Exam (GEDB007-46, Fall 2021)

You may write your solutions in English or Korean (or both). Calculators are not allowed.

Problem 1. Prove or disprove each statement.

(1) $\forall x \exists y (x^2 = y)$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(2) $\forall y \exists x (x^2 = y)$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(3) $\forall x \forall y ((x^2 = y^2) \to (x = y))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(4) $\forall x \exists y ((x^2 = y^2) \to (x = y))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

(5) $\exists x \forall y ((x^2 = y^2) \to (x = y))$, the domain of discourse is $\mathbb{R} \times \mathbb{R}$.

Problem 2. Prove or disprove each statement. Here $\mathcal{P}(A)$ is the set of all subsets of A.

(1) For all sets A and B, we have $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

(2) For some sets A and B, we have $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

(3) For some sets A and B, we have $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

(4) For all sets A and B, we have $\mathcal{P}(A - B) - \mathcal{P}(\emptyset) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

Problem 3. Let $X = \{1, 2, \ldots, 10\}$. Define a relation R on X by $(a, b) \in R$ if and only if $\mathrm{lcm}(a, b) \leq 10$. Prove or disprove each statement.

(1) R is reflexive.

(2) R is symmetric.

(3) R is antisymmetric.

(4) R is transitive.

Problem 4. Let $X = \{1, 2, \ldots, 1000\}$ and define a relation R on X by $(a, b) \in R$ if and only if the binary expressions of a and b have the same number of digits. For example, $(6, 7) \in R$ because both $6 = 110_2$ and $7 = 111_2$ have 3 digits in their binary expressions.

(1) Show that R is an equivalence relation.

(2) Find the equivalent class containing 278.

Problem 5. For a sequence $s = (s_1, \ldots, s_n)$ of integers, the *diameter* of s is defined to be the largest difference $|s_i - s_j|$ of two elements in this sequence. For example, the diameter of $(3, 8, 2, 3, 5)$ is $|8 - 2| = 6$. Write an algorithm that receives (s, n), where $s = (s_1, \ldots, s_n)$ is a sequence of n integers ($n \geq 2$), and returns the diameter of s.

Problem 6. Prove or disprove:

$$\log_2 1 + \log_2 2 + \cdots + \log_2(2n) = \Theta(n \log_2 n).$$

Problem 7. For each positive integer i, let p_i be the ith smallest prime. For example, $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11$. Prove that for every integer n with $n \geq 2$, at least one of the following is true:

- $p_2 p_3 \cdots p_n + 1 = 2^k$ for some integer k,
- $p_2 p_3 \cdots p_n + 1 \geq 2p_{n+1}$.

Problem 8. (1) Find the inverse of 44 mod 213.

(2) Find the inverse of 213 mod 44.