sign convention: $(+,-,-,-)$

1 Field Space Manifold

1.1 Basic Concepts

Consider a theory with N real scalar fields, ϕ^i for $i=1,\dots,N$. The fields can be regarded as a coordinate chart on the field-space manifold $\mathcal{M}$. That is, $\phi : U \to V$ is an invertible map from an open subset $U \subset \mathcal{M}$ to an open subset $V \subset \mathbb{R}^n$. A Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi)\partial_\mu \phi^i \partial^\mu \phi^j - V(\phi).$$

Under the field redefinition $\psi(\phi)$, the derivative transforms as

$$\partial_\mu \psi^i = \frac{\partial \psi^i}{\partial \phi^j}\partial_\mu \phi^j,$$

so

$$g'_{ij}(\psi) = \frac{\partial \phi^k}{\partial \psi^i}\frac{\partial \phi^l}{\partial \psi^j} g_{kl}(\phi).$$

Hence g transforms as a symmetric tensor with two lower indices, so we identify it as a metric on $\mathcal{M}$.

1.2 Tree-Level Amplitudes

Putting vacuum at the origin, we may expand

$$\mathcal{L} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{n!}\bar{g}_{ij,k_1\dots k_n}(\partial_\mu \phi^i)(\partial^\mu \phi^j)\phi^{k_1}\cdots\phi^{k_n} - \sum_{n=0}^{\infty}\frac{1}{n!}\bar{V}_{,k_1\dots k_n}\phi^{k_1}\cdots\phi^{k_n},$$

where bar denotes quantities evaluated at the origin. We can always assume that (with suitable field redefinition) $\bar{g}_{ij} = \delta_{ij}$, then we can further diagonalise $V_{,ij}$ so that the masses of the fields are given by $V_{,ij} = m_i \delta_{ij}$.

Vierbein e_{ai}: $g_{ij} = e_{ai}e_{bj}\delta^{ab}$.

Hence, the propagator is

$$\underset{i \qquad\quad j}{\overset{p}{\overline{}}} = \frac{i\delta_{ij}}{p^2 - m_i^2}.$$

The n-point vertex is given by

$$iV_n = -i\bar{V}_{,k_1\dots k_n} - i\sum_{i<j} p_i^\mu p_{j\mu}\bar{g}_{k_i k_j, k_1\dots\hat{k}_i\dots\hat{k}_j\dots k_n}.$$