metric convention: $(+,-,-,-)$

1 Scalar Loop

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \partial_\mu\chi^\dagger\partial^\mu\chi - m^2\chi^\dagger\chi - g\phi\chi^\dagger\chi.$$

One-loop self energy for ϕ:

$$\Sigma(p^2) = -ig^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2-m^2)[(q-p)^2-m^2]} - (Ap^2+BM^2).$$

Loop calculation with DimReg ($d=4-2\epsilon$):

$$\begin{aligned}
\Sigma &= -ig^2\int\frac{d^dq}{(2\pi)^d}\int_0^1 dx\left((1-x)(q^2-m^2)+x[(q-p)^2-m^2]\right)^{-2} - (Ap^2+BM^2)\\
&= -ig^2\int\frac{d^dq}{(2\pi)^d}\int_0^1 dx\,(q^2-2xq\cdot p+xp^2-m^2)^{-2} - (Ap^2+BM^2)\\
&= -ig^2\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}(l^2-\Delta)^{-2} - (Ap^2+BM^2)\\
&= \frac{g^2}{16\pi^2}\int_0^1 dx\left(\frac{4\pi}{\Delta}\right)^\epsilon\Gamma(\epsilon) - (Ap^2+BM^2)\\
&= \frac{g^2}{16\pi^2}\int_0^1 dx\left(\epsilon^{-1}-\gamma+\ln\frac{4\pi}{\Delta}\right) - (Ap^2+BM^2)\\
&= \frac{g^2}{16\pi^2}\left(\epsilon^{-1}-\gamma+\int_0^1 dx\,\ln\frac{4\pi}{\Delta}\right) - (Ap^2+BM^2),
\end{aligned}$$

where $l := q-xp$ and $\Delta := -x(1-x)p^2+m^2$.

For on-shell renorm, we calculate $\Sigma(M^2)$ and $\Sigma'(M^2)$:

$$\begin{aligned}
\Sigma(M^2) &= \frac{g^2}{16\pi^2}\left(\epsilon^{-1}-\gamma+\int_0^1 dx\,\ln\frac{4\pi}{-x(1-x)M^2+m^2}\right) - (A+B)M^2,\\
\Sigma'(M^2) &= \frac{g^2}{16\pi^2}\int_0^1 dx\,\frac{x(1-x)}{-x(1-x)M^2+m^2} - A.
\end{aligned}$$

Hence, putting $\operatorname{Re}\Sigma(M^2) = \operatorname{Re}\Sigma'(M^2) = 0$ gives

$$A = \frac{g^2}{16\pi^2}C_1, \quad B = \frac{g^2}{16\pi^2M^2}\left(\epsilon^{-1}-\gamma+\ln(4\pi)-M^2C_1+C_2\right)$$

for

$$C_1 := \operatorname{Re}\int_0^1 dx\,\frac{x(1-x)}{-x(1-x)M^2+m^2}, \quad C_2 := \operatorname{Re}\int_0^1 dx\,\ln\frac{1}{-x(1-x)M^2+m^2}.$$

The renormalised self-energy is

$$\Sigma = \frac{g^2}{16\pi^2}\left(\int_0^1 dx\,\ln\frac{1}{-x(1-x)p^2+m^2} - (p^2-M^2)C_1 - C_2\right).$$

The imaginary part is

$$\operatorname{Im}\Sigma = \frac{g^2}{16\pi}\int_0^1 dx\,\Theta(x(1-x)p^2-m^2) = \begin{cases}\frac{g^2}{16\pi}\sqrt{1-\frac{4m^2}{p^2}} & (p^2>4m^2)\\ 0 & (p^2\le 4m^2)\end{cases}.$$

2 Fermion Loop

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - y\phi\bar{\psi}\psi.$$

One-loop self energy for ϕ:

$$\Sigma(p^2) = iy^2\int\frac{d^4q}{(2\pi)^4}\frac{\mathrm{tr}[(q\!\!\!/ + m)(q\!\!\!/ - p\!\!\!/ + m)]}{(q^2-m^2)[(q-p)^2-m^2]} - (Ap^2 + BM^2).$$

Loop calculation with DimReg ($d = 4 - 2\epsilon$):

$$\begin{aligned}
\Sigma &= 4iy^2\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\frac{q^2 - q\cdot p + m^2}{[(1-x)(q^2-m^2) + x\{(q-p)^2-m^2\}]^2} - (Ap^2+BM^2)\\
&= 4iy^2\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{l^2 + (2x-1)l\cdot p - x(1-x)p^2 + m^2}{(l^2-\Delta)^2} - (Ap^2+BM^2)\\
&= 4iy^2\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{l^2+\Delta}{(l^2-\Delta)^2} - (Ap^2+BM^2)\\
&= -\frac{y^2}{4\pi^2}\int_0^1 dx\,\Delta\left(\frac{4\pi}{\Delta}\right)^\epsilon(3+\epsilon)\Gamma(\epsilon) - (Ap^2+BM^2)\\
&= -\frac{y^2}{4\pi^2}\int_0^1 dx\,\Delta\left(3\epsilon^{-1} + 1 - 3\gamma + 3\ln\frac{4\pi}{\Delta}\right) - (Ap^2+BM^2)\\
&= -\frac{y^2}{4\pi^2}\left((m^2 - p^2/6)(3\epsilon^{-1} + 1 - 3\gamma) + 3\int_0^1 dx\,\Delta\ln\frac{4\pi}{\Delta}\right) - (Ap^2+BM^2)
\end{aligned}$$

where $l := q - xp$ and $\Delta := -x(1-x)p^2 + m^2$.

The on-shell renormalisation gives

$$\begin{aligned}
A &= \frac{y^2}{4\pi^2}\left(\frac{1}{6}(3\epsilon^{-1} + 4 - 3\gamma + 3\ln 4\pi) + 3C_3\right),\\
B &= \frac{y^2}{4\pi^2}\left(-\frac{m^2}{M^2}(3\epsilon^{-1} + 1 - 3\gamma + 3\ln 4\pi) - \frac{1}{2} - \frac{3m^2}{M^2}C_2 + 6C_3\right)
\end{aligned}$$

for

$$C_3 := \mathrm{Re}\int_0^1 dx\,x(1-x)\ln\frac{1}{-x(1-x)M^2 + m^2}.$$

Then the renormalised self-energy is

$$\Sigma = -\frac{y^2}{4\pi^2}\left(\frac{p^2 - M^2}{2} + 3\int_0^1 dx\,\Delta\ln\frac{1}{\Delta} - 3m^2C_2 + (3p^2 + 6M^2)C_3\right).$$

The imaginary part is

$$\mathrm{Im}\,\Sigma = -\frac{3y^2}{4\pi}\int_0^1 dx\,\Delta\Theta(-\Delta).$$

Again, $\mathrm{Im}\,\Sigma = 0$ if $p^2 \le 4m^2$. For $p^2 > 4m^2$, we have

$$\mathrm{Im}\,\Sigma = -\frac{3y^2}{4\pi}\int_{\frac{1}{2}(1-\sqrt{1-4m^2/p^2})}^{\frac{1}{2}(1+\sqrt{1-4m^2/p^2})}\Delta\,dx = \frac{y^2}{16\pi}(p^2 - 4m^2)\sqrt{1 - \frac{4m^2}{p^2}}.$$

3 Scalar and Photon Loop

No explicit Lagrangian but let's assume that there is a $\gamma\phi\phi'$ vertex for complex scalars ϕ and ϕ' with masses M and m, respectively.

One-loop self-energy of ϕ:

$$\Sigma(p^2) = ie^2 \int \frac{d^4q}{(2\pi)^4} \frac{(p+q)^2}{(q^2-m^2)(q-p)^2} - (Ap^2 + BM^2).$$

Loop calculation with DimReg ($d = 4 - 2\epsilon$):

$$\begin{aligned}
\Sigma &= ie^2 \int \frac{d^dq}{(2\pi)^d} \int_0^1 dx \, \frac{q^2 + 2q\cdot p + p^2}{[(1-x)(q^2-m^2) + x(q-p)^2]^2} - (Ap^2 + BM^2) \\
&= ie^2 \int \frac{d^dl}{(2\pi)^d} \int_0^1 dx \, \frac{l^2 + 2(1+x)l\cdot p + (1+x)^2p^2}{(l^2-\Delta)^2} - (Ap^2 + BM^2) \\
&= ie^2 \int \frac{d^dl}{(2\pi)^d} \int_0^1 dx \, \frac{l^2 + (1+x)^2p^2}{(l^2-\Delta)^2} - (Ap^2 + BM^2) \\
&= -\frac{e^2}{16\pi^2} \int_0^1 dx \left(\frac{4\pi}{\Delta}\right)^\epsilon \left[(2+\epsilon)\Delta + (1+x)^2p^2\right]\Gamma(\epsilon) - (Ap^2 + BM^2) \\
&= -\frac{e^2}{16\pi^2} \int_0^1 dx \left((\epsilon^{-1} - \gamma)[(1+3x^2)p^2 + (2-2x)m^2] + \Delta + [(1+3x^2)p^2 + (2-2x)m^2] \ln\frac{4\pi}{\Delta} \right) \\
&\quad - (Ap^2 + BM^2) \\
&= -\frac{e^2}{16\pi^2} \left[(\epsilon^{-1} - \gamma)(2p^2 + m^2) - \frac{1}{6}p^2 + \frac{1}{2}m^2 + \int_0^1 dx \, [(1+3x^2)p^2 + (2-2x)m^2] \ln\frac{4\pi}{\Delta} \right] \\
&\quad - (Ap^2 + BM^2),
\end{aligned}$$

where $l := q - xp$ and $\Delta := -x(1-x)p^2 + (1-x)m^2$.

The imaginary part is

$$\operatorname{Im}\Sigma = -\frac{e^2}{16\pi} \int_0^1 dx \, [(1+3x^2)p^2 + (2-2x)m^2]\Theta(-\Delta).$$

If $p^2 \le m^2$ then $\operatorname{Im}\Sigma = 0$. Otherwise,

$$\operatorname{Im}\Sigma = -\frac{e^2}{16\pi} \int_{m^2/p^2}^1 dx \, [(1+3x^2)p^2 + (2-2x)m^2] = \frac{e^2}{8\pi}\left(\frac{m^4}{p^2} - p^2\right).$$