Main reference: Peskin & Schroeder QFT and Srednicki QFT

1 Formalism

Spectral representation

$$\int d^4x\, e^{ip\cdot x}\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \frac{iZ_r}{p^2-m^2+i\epsilon}+\int_{4m^2}^{\infty}\frac{dM^2}{2\pi}\rho(M^2)\frac{\rho(M^2)}{p^2-M^2+i\epsilon}.$$

We may rescale the field so that $\phi = Z_\phi^{1/2}\phi_r$. If we had started with the bare Lagrangian

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}m_0^2\phi^2-g_0\phi\mathcal{O}+\ldots,$$

where $\mathcal{O}$ is an operator which does not depend on ϕ, then

$$\begin{aligned}\mathcal{L}&=\frac{1}{2}Z_\phi(\partial_\mu\phi_r)^2-\frac{1}{2}Z_\phi m_0^2\phi_r^2-g_0Z_\phi^{1/2}\phi_r\mathcal{O}+\ldots\\&=\frac{1}{2}(\partial_\mu\phi_r)^2-\frac{1}{2}m^2\phi_r^2-g\phi_r\mathcal{O}+\frac{1}{2}\delta_Z(\partial_\mu\phi_r)^2-\frac{1}{2}\delta_m\phi_r^2-\delta_g\phi_r\mathcal{O}+\ldots,\end{aligned}$$

for

$$\delta_Z=Z_\phi-1,\quad \delta_m=Z_\phi m_0^2-m^2,\quad \delta_g=Z_\phi^{1/2}g_0-g.$$

We fix the counterterms by requiring that the exact propagator has a pole $iZ_r/(p^2-m^2)$:

$$\Sigma(m^2)=0,\quad \Sigma'(m^2)=Z_r^{-1}-1.$$

From LSZ, we have

$$\langle p|S|p\rangle = Z_r\times(\text{Diagram}).$$

2 Specific Example: Scalar Loop

The scalar loop self-energy is

$$\begin{aligned}\Sigma(p^2)&=\frac{\tilde{g}^2}{16\pi^2}\left(\epsilon^{-1}-\gamma+\int_0^1 dx\,\ln\frac{4\pi}{\Delta}\right)-(Ap^2+BM^2)+O(g^4)\\&=\frac{g^2}{16\pi^2}\left(\epsilon^{-1}+\int_0^1 dx\,\ln\frac{\mu^2}{\Delta}\right)-(Ap^2+BM^2)+O(g^4),\end{aligned}$$

where $\Delta=-x(1-x)p^2+m^2$. In the $\overline{\text{MS}}$ scheme, we just cancel the ϵ^{-1} divergence, so that

$$\Sigma(p^2)=\frac{g^2}{16\pi^2}\int_0^1 dx\,\ln\frac{\mu^2}{\Delta}+O(g^4).$$

For now, we take $2m>M$, so that the external particle is stable. The physical mass M_p is defined by $\Sigma(M_p^2)=M^2-M_p^2$, thus

$$\begin{aligned}M_p^2&=M^2-\Sigma(M_p^2)\\&=M^2-\Sigma(M^2)+O(g^4)\end{aligned}$$

$$= M^2 - \frac{g^2}{16\pi^2}\int_0^1 dx\,\ln\frac{\mu^2}{\Delta_M} + O(g^4).$$

For the residue Z_r, we have

$$\begin{aligned} Z_r^{-1} &= 1 + \Sigma'(M_p^2) \\ &= 1 + \Sigma'(M^2) + O(g^4) \\ &= 1 + \frac{g^2}{16\pi^2}\int_0^1 dx\,\frac{x(1-x)}{\Delta_M} + O(g^4). \end{aligned}$$

The physical quantity (somewhat associated with the 1-to-1 scattering amplitude?) is

$$\begin{aligned} Z_r(p^2 - M^2 + \Sigma(p^2)) &= \left[1 - \frac{g^2}{16\pi^2}\int_0^1 dx\,\frac{x(1-x)}{\Delta_M} + O(g^4)\right]\left[p^2 - m^2 + \frac{g^2}{16\pi^2}\int_0^1 dx\,\ln\frac{\mu^2}{\Delta} + O(g^4)\right] \\ &= p^2 - M^2 + \frac{g^2}{16\pi^2}\left[-(p^2 - M^2)\int_0^1 dx\,\frac{x(1-x)}{\Delta_M} + \int_0^1 dx\,\ln\frac{\mu^2}{\Delta}\right] + O(g^4) \\ &= p^2 - M_p^2 + \frac{g^2}{16\pi^2}\left[-(p^2 - M^2)\int_0^1 dx\,\frac{x(1-x)}{\Delta_M} + \int_0^1 dx\,\ln\frac{\Delta_M}{\Delta}\right] + O(g^4). \end{aligned}$$

How about the on-shell scheme? In OS, we put $M = M_p$ and $Z_r = 1$. So we get

$$\Sigma(p^2) = \frac{g^2}{16\pi^2}\left[-(p^2 - M_p^2)\int_0^1 dx\,\frac{x(1-x)}{\Delta_{M_p}} + \int_0^1 dx\,\ln\frac{\Delta_{M_p}}{\Delta}\right] + O(g^4).$$

The physical quantity is

$$p^2 - m^2 + \Sigma(p^2) = p^2 - M_p^2 + \frac{g^2}{16\pi^2}\left[-(p^2 - M_p^2)\int_0^1 dx\,\frac{x(1-x)}{\Delta_{M_p}} + \int_0^1 dx\,\ln\frac{\Delta_{M_p}}{\Delta}\right] + O(g^4).$$

We can see that the result is almost the same for the $\overline{\mathrm{MS}}$ and the OS scheme. Only difference is the mass parameter that is used inside the bracket, but its effect is $O(g^2)$ so the overall quantity is the same up to $O(g^4)$.

3 Unstable Particle

For unstable particles, the pole mass M_p defined by $M_p^2 + \Sigma(M_p^2) = M^2$ is complex. Hence, the real physical mass would be $M_R = \mathrm{Re}\,M_p$.

One of the convention in on-shell scheme of unstable particle, we choose $M = M_{\mathrm{OS}}$ so that $\mathrm{Re}\,\Sigma_{\mathrm{OS}}(M_{\mathrm{OS}}^2) = 0$. This gives

$$\Sigma_{\mathrm{OS}}(p^2) = \frac{g^2}{16\pi^2}\left[-(p^2 - M_{\mathrm{OS}}^2)\mathrm{Re}\int_0^1 dx\,\frac{x(1-x)}{\Delta_{M_{\mathrm{OS}}}} + \mathrm{Re}\int_0^1 dx\,\ln\frac{\Delta_{M_{\mathrm{OS}}}}{\Delta} + \pi i\int_0^1 dx\,\Theta(-\Delta)\right] + O(g^4).$$

Therefore, the pole mass can be determined with

$$\begin{aligned} M_p^2 &= M_{\mathrm{OS}}^2 - \Sigma_{\mathrm{OS}}(M_p^2) = M_{\mathrm{OS}}^2 - \Sigma_{\mathrm{OS}}(M_{\mathrm{OS}}^2) + O(g^4) \\ &= M_{\mathrm{OS}}^2 - \frac{ig^2}{16\pi}\int_0^1 dx\,\Theta(-\Delta_{M_{\mathrm{OS}}}) + O(g^4). \end{aligned}$$

In this sense, $M_{\rm OS} = M_R + O(g^4)$. We also had the relation

$$M_p^2 = M_{\overline{\rm MS}}^2 - \frac{g^2}{16\pi^2}\int_0^1 dx\,\ln\frac{\mu^2}{\Delta_{M_{\overline{\rm MS}}}} + O(g^4),$$

so

$$M_{\overline{\rm MS}}^2 = M_R^2 + \frac{g^2}{16\pi^2}{\rm Re}\int_0^1 dx\,\ln\frac{\mu^2}{\Delta_{M_R}} + O(g^4).$$

Finally, the on-shell pole residue on $p^2 = M_p^2$ is

$$\begin{aligned}
Z_r^{-1} &= 1 + \Sigma'_{\rm OS}(M_p^2) \\
&= 1 + \frac{g^2}{16\pi^2}\left[-{\rm Re}\int_0^1 dx\,\frac{x(1-x)}{\Delta_{M_{\rm OS}}} + \int_0^1 dx\,\frac{x(1-x)}{\Delta_{M_p}}\right] + O(g^4) \\
&= 1 + \frac{ig^2}{16\pi^2}{\rm Im}\int_0^1 dx\,\frac{x(1-x)}{\Delta_{M_R}} + O(g^4).
\end{aligned}$$

3.1 Explicit Decay Width

Consider the self-energy with an explicit constant decay term:

$$\Sigma(p^2) = iM\Gamma + \frac{g^2}{16\pi^2}\left(\epsilon^{-1} + \int_0^1 dx\,\ln\frac{\mu^2}{\Delta}\right) - (Ap^2 + BM^2) + O(g^4).$$

We can think of the following three renormalisation schemes.

1. $\overline{\rm MS}$:

$$\Sigma_{\overline{\rm MS}}(p^2) = iM_{\overline{\rm MS}}\Gamma_{\overline{\rm MS}} + \frac{g^2}{16\pi^2}\int_0^1 dx\,\ln\frac{\mu^2}{\Delta} + O(g^4).$$

2. Real on-shell:

$$\begin{aligned}
\Sigma_{\rm OS}(p^2) = iM_{\rm OS}\Gamma_{\rm OS} + \frac{g^2}{16\pi^2}\Bigg[&-(p^2 - M_{\rm OS}^2){\rm Re}\int_0^1 dx\,\frac{x(1-x)}{\Delta_{\sqrt{M_{\rm OS}^2 - iM_{\rm OS}\Gamma_{\rm OS}}}} \\
&+ {\rm Re}\int_0^1 dx\,\ln\frac{\Delta_{\sqrt{M_{\rm OS}^2 - iM_{\rm OS}\Gamma_{\rm OS}}}}{\Delta} + i{\rm Im}\int_0^1 dx\,\ln\frac{1}{\Delta}\Bigg] + O(g^4).
\end{aligned}$$

3. Complex on-shell:

$$\Sigma_p(p^2) = \frac{g^2}{16\pi^2}\left[-(p^2 - M_p^2)\int_0^1 dx\,\frac{x(1-x)}{\Delta_{M_p}} + \int_0^1 dx\,\ln\frac{\Delta_{M_p}}{\Delta}\right] + O(g^4).$$

We must match the (complex) pole mass:

$$\begin{aligned}
M_p^2 &= M_{\overline{\rm MS}}^2 - iM_{\overline{\rm MS}}\Gamma_{\overline{\rm MS}} - \frac{g^2}{16\pi^2}\int_0^1 dx\,\ln\frac{\mu^2}{\Delta_{\sqrt{M_{\overline{\rm MS}}^2 - iM_{\overline{\rm MS}}\Gamma_{\overline{\rm MS}}}}} + O(g^4) \\
&= M_{\rm OS}^2 - iM_{\rm OS}\Gamma_{\rm OS} - \frac{ig^2}{16\pi^2}\Bigg[M_{\rm OS}\Gamma_{\rm OS}{\rm Re}\int_0^1 dx\,\frac{x(1-x)}{\Delta_{\sqrt{M_{\rm OS}^2 - iM_{\rm OS}\Gamma_{\rm OS}}}} \\
&\qquad + {\rm Im}\int_0^1 dx\,\ln\frac{1}{\Delta_{\sqrt{M_{\rm OS}^2 - iM_{\rm OS}\Gamma_{\rm OS}}}}\Bigg] + O(g^4).
\end{aligned}$$

Hence,

$$M_{\overline{\rm MS}}^2 = M_{\rm OS}^2 + \frac{g^2}{16\pi^2}{\rm Re}\int_0^1 dx\,\ln\frac{\mu^2}{\Delta_{\sqrt{M_{\rm OS}^2-iM_{\rm OS}\Gamma_{\rm OS}}}} + O(g^4)$$

and

$$\Gamma_{\overline{\rm MS}} = \frac{M_{\rm OS}}{M_{\overline{\rm MS}}}\Gamma_{\rm OS}\left(1+\frac{g^2}{16\pi^2}{\rm Re}\int_0^1 dx\frac{x(1-x)}{\Delta_{\sqrt{M_{\rm OS}^2-iM_{\rm OS}\Gamma_{\rm OS}}}}\right) + O(g^4).$$

Then we have to evaluate pole residue:

$$\begin{aligned}
Z_{\overline{\rm MS}} &= 1-\frac{g^2}{16\pi^2}\int_0^1 dx\frac{x(1-x)}{\Delta_{\sqrt{M_{\overline{\rm MS}}^2-iM_{\overline{\rm MS}}\Gamma_{\overline{\rm MS}}}}} + O(g^4),\\
Z_{\rm OS} &= 1-\frac{ig^2}{16\pi^2}{\rm Im}\int_0^1 dx\frac{x(1-x)}{\Delta_{\sqrt{M_{\rm OS}^2-iM_{\rm OS}\Gamma_{\rm OS}}}} + O(g^4),\\
Z_p &= 1+O(g^4).
\end{aligned}$$

Note that the g^2 term in the real on-shell residue is purely imaginary again.

4 General Formula

Let

$$\Sigma_{\overline{\rm MS}}(p^2) = iM_{\overline{\rm MS}}\Gamma_{\overline{\rm MS}} + \alpha f(p^2) + O(\alpha^2)$$

for the loop factor α. The real on-shell self-energy is

$$\Sigma_{\rm OS}(p^2) = iM_{\rm OS}\Gamma_{\rm OS}+\alpha\left[f(p^2)-{\rm Re}\,f(M_{\rm OS}^2-iM_{\rm OS}\Gamma_{\rm OS})-(p^2-M_{\rm OS}^2){\rm Re}\,f'(M_{\rm OS}^2-iM_{\rm OS}\Gamma_{\rm OS})\right]+O(\alpha^2),$$

and the complex on-shell self-energy is

$$\Sigma_p(p^2) = \alpha\left[f(p^2)-f(M_p^2)-(p^2-M_p^2)f'(M_p^2)\right]+O(\alpha^2).$$

The pole mass can be found by

$$\begin{aligned}
M_p^2 &= M_{\overline{\rm MS}}^2 - iM_{\overline{\rm MS}}\Gamma_{\overline{\rm MS}} - \alpha f(M_{\overline{\rm MS}}^2 - iM_{\overline{\rm MS}}\Gamma_{\overline{\rm MS}}) + O(\alpha^2)\\
&= M_{\rm OS}^2 - iM_{\rm OS}\Gamma_{\rm OS} - i\alpha\left[{\rm Im}\,f(M_{\rm OS}^2-iM_{\rm OS}\Gamma_{\rm OS}) + M_{\rm OS}\Gamma_{\rm OS}{\rm Re}\,f'(M_{\rm OS}^2-iM_{\rm OS}\Gamma_{\rm OS})\right]+O(\alpha^2),
\end{aligned}$$

so

$$M_{\overline{\rm MS}}^2 = M_{\rm OS}^2 + \alpha{\rm Re}\,f(M_{\rm OS}^2-iM_{\rm OS}\Gamma_{\rm OS}) + O(\alpha^2)$$

and

$$\Gamma_{\overline{\rm MS}} = \frac{M_{\rm OS}}{M_{\overline{\rm MS}}}\Gamma_{\rm OS}\left[1+\alpha{\rm Re}\,f'(M_{\rm OS}^2-iM_{\rm OS}\Gamma_{\rm OS})\right]+O(\alpha^2).$$

Finally, the pole residue are

$$\begin{aligned}
Z_{\overline{\rm MS}} &= 1-\alpha f'(M_{\overline{\rm MS}}^2 - iM_{\overline{\rm MS}}\Gamma_{\overline{\rm MS}}) + O(\alpha^2),\\
Z_{\rm OS} &= 1-i\alpha{\rm Im}\,f'(M_{\rm OS}^2-iM_{\rm OS}\Gamma_{\rm OS}) + O(\alpha^2).
\end{aligned}$$