One-Loop Scheme Independence

What does complex-on-shell scheme really mean?

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1 Basics

1.1 Self energy

Consider a one-loop self energy calculation:

$$\Pi(p^2) = \frac{\alpha_b}{4\pi} F_{\text{loop}}(p^2) + \delta_Z p^2 - \delta_M,$$

where F_{loop} is the result of a divergent loop integration and α_b is calculated by a bare coupling between the mother particle and loop particles. In a renormalised QFT, we choose the counterterms δ_Z and δ_M so that they can eliminate the infinities ($\sim \frac{1}{\epsilon}$ in dimreg) generated by loop integration. There is no canonical choice for those counterterms; We must cancel the divergent terms, however, for the finite parts, it is merely a matter of choice. We may separate counterterms in two parts. The first part only cancels the $1/\epsilon$ divergences, and the second part is remaining finite part of the counterterms:

$$\delta_Z = \frac{\alpha_b}{4\pi} i_Z + f_Z, \quad \delta_M = \frac{\alpha_b}{4\pi} i_M + f_M.$$

Now, $F_{\text{fin}} := F_{\text{loop}} + i_Z p^2 - i_M$ has no $1/\epsilon$ term, so is finite unless there are other divergences such as IR divergence. With UV divergence canceled, we may rewrite the expression for the one-loop self energy:

$$\Pi(p^2) = \frac{\alpha_b}{4\pi} F_{\text{fin}}(p^2) + f_Z p^2 - f_M.$$

In the MS scheme, we only cancel the divergences and nothing more, hence $f_Z = f_M = 0$. But in the other schemes, they are generally not zero.

1.2 Propagator and pole mass

The one-loop corrected propagator is

$$\frac{Z}{p^2 - M_b^2 + \Pi(p^2)},$$

where M_b is a bare mass parameter of the mother particle. Generally M_b is not a physical mass of the particle, so it is desirable to mention which renormalisation scheme we are working on. So I will notate the scheme on the mass parameter (e.g. M_{MS}) if we work in the specific scheme.

Until now, every parameters were not physical. The only physical mass is the pole mass μ, which is defined by the position of pole of the propagator:

$$\mu^2 - M_b^2 + \Pi(\mu^2) = 0.$$

Using this equation, we can find μ if we fixed the scheme and given the model parameters such as M_b and α_b. If μ is complex then we may write $\mu = M - \frac{i}{2}\Gamma$, where M is a real physical mass of the mother particle and Γ is a decay width. (citation needed)

2 Complex-on-shell scheme

2.1 Bridge between different schemes

Suppose that two researchers, Alice and Bob, are examining the same process. However, they are using different scheme, thus they have two different expressions for the propagator:

$$\Delta_A(p^2) = \frac{Z_A}{p^2 - M_A^2 + \Pi_A(p^2)}, \quad \Delta_B(p^2) = \frac{Z_B}{p^2 - M_B^2 + \Pi_B(p^2)},$$

where the self-energies are

$$\Pi_A(p^2) = \frac{\alpha_A}{4\pi} F_{\text{fin}}(p^2) + f_{Z;A} p^2 - f_{M;A}, \quad \Pi_B(p^2) = \frac{\alpha_B}{4\pi} F_{\text{fin}}(p^2) + f_{Z;B} p^2 - f_{M;B}.$$

Since they are describing the same physical process, the pole mass has to be the same:

$$\mu^2 - M_A^2 + \Pi_A(\mu^2) = \mu^2 - M_B^2 + \Pi_B(\mu^2) = 0,$$

so $M_A^2 = \mu^2 + \Pi_A(\mu^2)$ and $M_B^2 = \mu^2 + \Pi_B(\mu^2)$. Expanding $\Pi(p^2)$, we get

$$\begin{aligned} 0 &= \mu^2 - M_A^2 + \Pi_A(\mu^2) \\ &= (1 + f_{Z;A})\mu^2 - M_A^2 + \frac{\alpha_A}{4\pi} F_{\text{fin}}(\mu^2) - f_{M;A}, \end{aligned}$$

and the similar equation holds for B. If we assume that the (scheme-dependent) model parameters α_b, M_b, f_Z, and f_M are real, then we can take imaginary part on the RHS and get

$$(1 + f_{Z;A})\text{Im}[\mu^2] + \frac{\alpha_A}{4\pi}\text{Im}[F_{\text{fin}}(\mu^2)] = (1 + f_{Z;B})\text{Im}[\mu^2] + \frac{\alpha_B}{4\pi}\text{Im}[F_{\text{fin}}(\mu^2)] = 0.$$

Therefore,

$$\begin{aligned} p^2 + \Pi_A(p^2) - M_A^2 &= p^2 + \Pi_A(p^2) - \mu^2 - \Pi_A(\mu^2) \\ &= (1 + f_{Z;A})(p^2 - \mu^2) + \frac{\alpha_A}{4\pi}[F_{\text{fin}}(p^2) - F_{\text{fin}}(\mu^2)] \\ &= \frac{\alpha_A}{4\pi}\left[-\frac{\text{Im}[F_{\text{fin}}(\mu^2)]}{\text{Im}[\mu^2]}(p^2 - \mu^2) + F_{\text{fin}}(p^2) - F_{\text{fin}}(\mu^2)\right]. \end{aligned}$$

This clearly shows that

$$\frac{\alpha_A}{4\pi Z_A}\Delta_A(p^2) = \frac{\alpha_B}{4\pi Z_B}\Delta_B(p^2) = \frac{1}{-\frac{\text{Im}[F_{\text{fin}}(\mu^2)]}{\text{Im}[\mu^2]}(p^2 - \mu^2) + F_{\text{fin}}(p^2) - F_{\text{fin}}(\mu^2)}.$$

Indeed, although their model parameters are scheme-dependent, their propagators are the same up to multiplicative factor.

2.2 Complex-on-shell scheme

Our third researcher, Chris, is trying to fix the form of a propagator only given the pole mass μ. He focuses to the result of the previous chapter,

$$\frac{\alpha_A}{4\pi Z_A}\Delta_A(p^2) = \frac{\alpha_B}{4\pi Z_B}\Delta_B(p^2) = \frac{1}{-\frac{\mathrm{Im}[F_{\mathrm{fin}}(\mu^2)]}{\mathrm{Im}[\mu^2]}(p^2-\mu^2) + F_{\mathrm{fin}}(p^2) - F_{\mathrm{fin}}(\mu^2)}.$$

The RHS is totally determined by μ and F_{fin}, and we showed that it doesn't contain any scheme-dependent parameters. This is a good news, but we have to determine the total multiplicative factor to define $\Delta_C(p^2)$. Chris wants to compare his result with the Breit-Wigner distribution,

$$D_{BW}(p^2) = \left|\frac{1}{p^2-\mu^2}\right|^2.$$

If Chris' expression for the propagator is

$$\Delta_C(p^2) = \frac{k}{-\frac{\mathrm{Im}[F_{\mathrm{fin}}(\mu^2)]}{\mathrm{Im}[\mu^2]}(p^2-\mu^2) + F_{\mathrm{fin}}(p^2) - F_{\mathrm{fin}}(\mu^2)}$$

for a real k then near $p^2 = \mu^2$,

$$|\Delta_C(p^2)|^2 \approx k^2 \left|\frac{1}{p^2-\mu^2}\right|^2 \left|\frac{1}{-\frac{\mathrm{Im}[F_{\mathrm{fin}}(\mu^2)]}{\mathrm{Im}[\mu^2]} + F'_{\mathrm{fin}}(\mu^2)}\right|^2.$$

Hence, if we require that $|\Delta_C(p^2)|^2 \approx D_{BW}(p^2)$ near the pole then

$$k = \left|-\frac{\mathrm{Im}[F_{\mathrm{fin}}(\mu^2)]}{\mathrm{Im}[\mu^2]} + F'_{\mathrm{fin}}(\mu^2)\right|.$$

Therefore, Chris finally obtains

$$\Delta_C(p^2) = \frac{\left|-\frac{\mathrm{Im}[F_{\mathrm{fin}}(\mu^2)]}{\mathrm{Im}[\mu^2]} + F'_{\mathrm{fin}}(\mu^2)\right|}{-\frac{\mathrm{Im}[F_{\mathrm{fin}}(\mu^2)]}{\mathrm{Im}[\mu^2]}(p^2-\mu^2) + F_{\mathrm{fin}}(p^2) - F_{\mathrm{fin}}(\mu^2)} = \frac{\left|1 - \frac{\mathrm{Im}[\mu^2]}{\mathrm{Im}[F_{\mathrm{fin}}(\mu^2)]}F'_{\mathrm{fin}}(\mu^2)\right|}{p^2-\mu^2 - \frac{\mathrm{Im}[\mu^2]}{\mathrm{Im}[F_{\mathrm{fin}}(\mu^2)]}[F_{\mathrm{fin}}(p^2) - F_{\mathrm{fin}}(\mu^2)]}.$$

Looking at the RHS, it seems like we are using the complex pole mass μ in place of M_b. Also, the factor $-\frac{\mathrm{Im}[\mu^2]}{\mathrm{Im}[F_{\mathrm{fin}}(\mu^2)]}$ is taking a role of coupling α_b and $F_{\mathrm{fin}}(\mu^2)$ is taking a role of the finite part of the counterterm. For a stable particle, the on-shell scheme uses the real pole mass M as a mass parameter, so $M_{\mathrm{OS}} = M$. For this reason, we may call Chris' scheme the **complex-on-shell scheme** (COS).

2.3 Additional justifications

One may question, why aren't the coupling α showing in the COS scheme? That is because μ and α is not completely independent. Let a be the mother particle and b is a two-particles state which appears in the loop, so we are dealing with $a \to b \to a$ loop process. If we try to measure the $a - b$ coupling constant α, then we may observe the decay process $a \to b$ and measure the decay width $\Gamma(a \to b)$ to determine α. However, that decay width $\Gamma(a \to b)$ is precisely the width Γ showing in the complex pole $\mu = M - \frac{i}{2}\Gamma$. Therefore, we expect the relation $\alpha \propto \Gamma = -\frac{1}{m}\mathrm{Im}[\mu^2]$. (For stable particles, there is no decay width to be fixed so we do not have this issue.)

To put it another way, in the MS or $\overline{\mathrm{MS}}$ scheme, there are two degrees of freedom (M_b and α_b) for the model parameters. If we fix μ, then it already has two degrees of freedom (M and Γ), so there should be no additional free parameters.

3 Example: scalar loop

We take a massive real scalar particle as a mother particle and a complex real scalar with mass $m = 0.45\,\text{GeV}$ as a loop particle. One-loop calculation gives

$$\Pi_{\overline{\text{MS}}}(p^2) = \frac{g^2}{16\pi^2} \int_0^1 dx\, \ln \frac{\nu^2}{-x(1-x)p^2 + m^2}$$

for a fixed $\overline{\text{MS}}$ reference scale ν. We will investigate three different scheme describing the same physics (same pole mass). Especially, we fix $\mu = 1\,\text{GeV} - \frac{i}{2} 0.2\,\text{GeV}$.

Our first scheme is $\overline{\text{MS}}$ scheme. Taking $\nu = 1\,\text{GeV}$, we must have $M_{\overline{\text{MS}}} = 1.172\,\text{GeV}$ and $g_{\overline{\text{MS}}} = 4.066\,\text{GeV}$ so that the pole mass μ can be the value we fixed.

The next scheme is real-on-shell scheme, where we require $\text{Re}[\Pi_{\text{OS}}(M_{\text{OS}}^2)] = 0$ and $\text{Re}[\Pi'_{\text{OS}}(M_{\text{OS}}^2)] = 0$. Hence, the self-energy takes the form of

$$\Pi_{\text{OS}}(p^2) = \frac{g^2}{16\pi^2} \Bigg(\int_0^1 dx\, \left[\ln \frac{1}{-x(1-x)p^2 + m^2} - \ln \left| \frac{1}{-x(1-x)M_{\text{OS}}^2 + m^2} \right| \right] - (p^2 - M_{\text{OS}}^2) \int_0^1 dx\, \frac{x(1-x)}{x(1-x)M_{\text{OS}}^2 - m^2} \Bigg).$$

Also the on-shell mass and couplings are $M_{\text{OS}} = 1.024\,\text{GeV}$ and $g_{\text{OS}} = 4.500\,\text{GeV}$ to match the location of complex pole.

The last scheme is COS scheme. Here we take

$$F_{\text{fin}} = \int_0^1 dx\, \ln \frac{\nu^2}{-x(1-x)p^2 + m^2}.$$

It seems like the arbitrary parameter ν is presenting, but it actually cancels out in the final expression.

We plot the resonance shapes for three schemes in Figure 1. For the COS scheme, we plotted $|\Delta_C(p^2)|^2$, and for the rest, we plotted $|p^2 - M^2 + \Pi(p^2)|^{-2}$. We can notice that the shape looks similar. Indeed, if we rescale the height of the peaks so that each peak's height is equal to that of the complex-on-shell scheme (Figure 2), then the graphs overlap perfectly. To compare, we also plotted Breit-Wigner distribution on top of them, also matching the peak height.

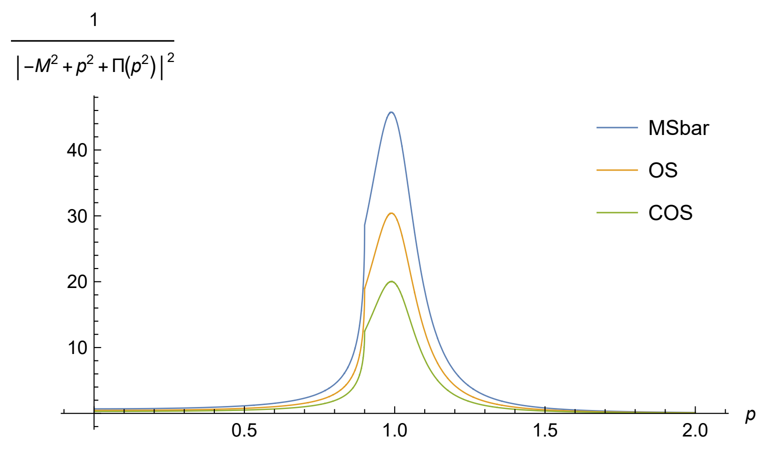

Figure 1: resonance shapes without field strength renormalisation.

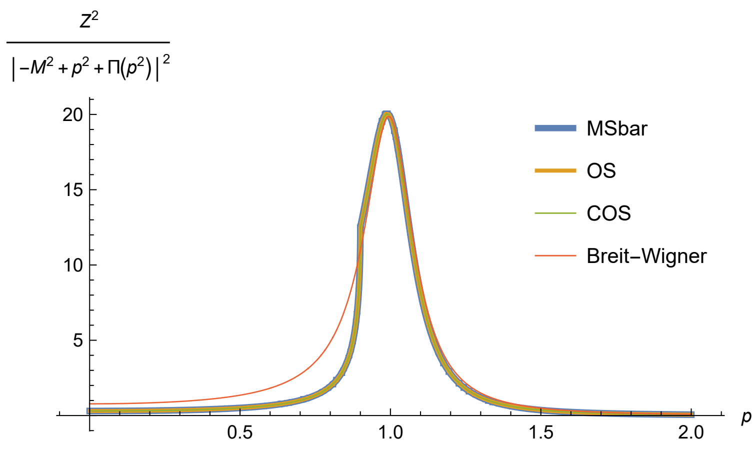

Figure 2: resonance shapes with field strength renormalisation.

3.1 A faulty naive complex-on-shell scheme

Following the usual procedure for the on-shell renormalisation, one may be tempted to use the following 'naive complex-on-shell' scheme (NCOS):

$$\Pi_{\text{NCOS}}(\mu^2) = 0, \quad \Pi'_{\text{NCOS}}(\mu^2) = 0$$

while choosing μ and α_{NOS} freely. Then the self-energy is

$$\Pi_{\text{NCOS}}(p^2) = \frac{g^2}{16\pi^2}\left(\int_0^1 dx \left[\ln \frac{1}{-x(1-x)p^2+m^2} - \ln \frac{1}{-x(1-x)\mu^2+m^2}\right]\right.$$
$$\left. - (p^2-\mu^2)\int_0^1 dx \, \frac{x(1-x)}{x(1-x)\mu^2-m^2}\right).$$

This shouldn't be right, since we introduced fake degrees of freedom. In Figure 3, we can see that the NCOS scheme produces different resonance shape even when we match the peak height.

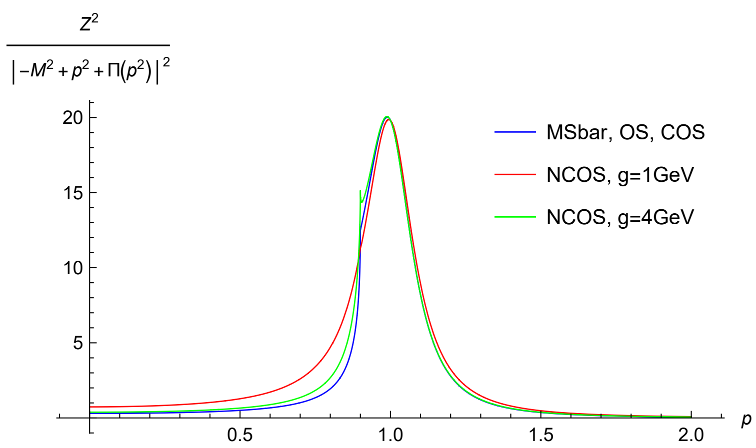

Figure 3: Naive complex-on-shell scheme cannot represent the resonance shape correctly.

4 Many decay channels and partial width

In general, there can be many decay channels for the mother particle A. That is, we have several two-particle states $B_1, \ldots, B_N$ so that A can decay into B_i. This automatically means that there are loop diagrams $A \to B_i \to A$. In this case, the pole mass μ alone is not enough to parametrise the whole process; We need to fix either coupling constants or partial width for each processes.

Again, we start with the self-energy

$$\Pi_b(p^2) = f_Z p^2 - f_M + \sum_{i=1}^{N} \frac{\alpha_{b,i}}{4\pi} F_{\mathrm{fin},i}(p^2).$$

Then taking imaginary part of the pole equation $\mu^2 - M_b^2 + \Pi(\mu^2) = 0$ gives

$$(1 + f_Z)\mathrm{Im}[\mu^2] + \sum_{i=1}^{N} \frac{\alpha_{b,i}}{4\pi} \mathrm{Im}[F_{\mathrm{fin},i}(\mu^2)] = 0.$$

Since $\mathrm{Im}[\mu^2] = -M\Gamma$, we rewrite the equation:

$$\Gamma = \sum_{i=1}^{N} \frac{\alpha_{b,i}}{4\pi M(1 + f_Z)} \mathrm{Im}[F_{\mathrm{fin},i}(\mu^2)].$$

Then it is natural to associate each summand in the RHS to partial width Γ_i:

$$\Gamma_i := \frac{\alpha_{b,i}}{4\pi M(1 + f_Z)} \mathrm{Im}[F_{\mathrm{fin},i}(\mu^2)].$$

Then we have

$$\begin{aligned} p^2 - M_b^2 + \Pi(p^2) &= p^2 - \mu^2 + \Pi(p^2) - \Pi(\mu^2) \\ &= (1 + f_Z)(p^2 - \mu^2) + \sum_{i=1}^{N} \frac{\alpha_{b,i}}{4\pi} [F_{\mathrm{fin},i}(p^2) - F_{\mathrm{fin},i}(\mu^2)] \\ &= (1 + f_Z)(p^2 - \mu^2) + \frac{(1 + f_Z)M}{\mathrm{Im}[F_{\mathrm{fin},i}(\mu^2)]} \sum_{i=1}^{N} \Gamma_i [F_{\mathrm{fin},i}(p^2) - F_{\mathrm{fin},i}(\mu^2)]. \end{aligned}$$

Again, if we require that $|\Delta_C(p^2)|^2 \approx D_{BW}(p^2)$ near the pole, then we can fix the overall scaling of the propagator and get

$$\Delta_C(p^2) = \frac{\left| 1 + \dfrac{M}{\mathrm{Im}[F_{\mathrm{fin},i}(\mu^2)]} \displaystyle\sum_{i=1}^{N} \Gamma_i [F'_{\mathrm{fin},i}(\mu^2)] \right|}{p^2 - \mu^2 + \dfrac{M}{\mathrm{Im}[F_{\mathrm{fin},i}(\mu^2)]} \displaystyle\sum_{i=1}^{N} \Gamma_i [F_{\mathrm{fin},i}(p^2) - F_{\mathrm{fin},i}(\mu^2)]}.$$

We have one subtle point: The partial width Γ_i is nonzero even if two-particle state B_i is heavier than the real part of the pole mass M. That is because since μ^2 is complex, $\mathrm{Im}[F_{\mathrm{fin},i}(\mu^2)]$ is nonzero regardless of the mass of the loop particles. One intuitive explanation for this would be that the unstable mother particle is never on-shell, so its (real) mass is always ambiguous. Thus, A can decay into B_i on the tail of its resonance.

To sidestep this subtlety, we can 'trust the process' and use other schemes ($\overline{\mathrm{MS}}$, OS, etc., but not NCOS) instead. Then we can explicitly fix every α_i, but matching μ (so the total width) is not easy.