Counterexamples of Probability

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Contents

1 Prerequisites

This is a *sample* book written in **Markdown**. You can use anything that Pandoc's Markdown supports, e.g., a math equation $a^2 + b^2 = c^2$.

The **bookdown** package can be installed from CRAN or Github:

```
install.packages("bookdown")
# or the development version
# devtools::install_github("rstudio/bookdown")
```

Remember each Rmd file contains one and only one chapter, and a chapter is defined by the first-level heading `#`.

To compile this example to PDF, you need XeLaTeX. You are recommended to install TinyTeX (which includes XeLaTeX): https://yihui.name/tinytex/.

2 Introduction

You can label chapter and section titles using `{#label}` after them, e.g., we can reference Chapter 2. If you do not manually label them, there will be automatic labels anyway, e.g., Chapter 4.

Figures and tables with captions will be placed in `figure` and `table` environments, respectively.

```
par(mar = c(4, 4, .1, .1))
plot(pressure, type = 'b', pch = 19)
```

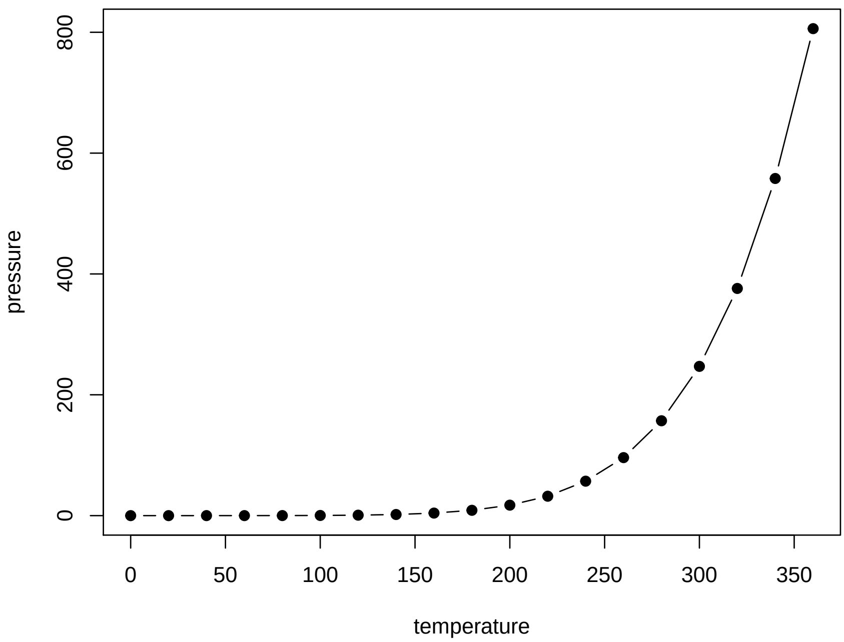

Figure 2.1: Here is a nice figure!

Reference a figure by its code chunk label with the `fig:` prefix, e.g., see Figure 2.1. Similarly, you can reference tables generated from `knitr::kable()`, e.g., see Table 2.1.

```
knitr::kable(
  head(iris, 20), caption = 'Here is a nice table!',
  booktabs = TRUE
)
```

Table 2.1: Here is a nice table!

Sepal.Length	Sepal.Width	Petal.Length	Petal.Width	Species
5.1	3.5	1.4	0.2	setosa
4.9	3.0	1.4	0.2	setosa
4.7	3.2	1.3	0.2	setosa
4.6	3.1	1.5	0.2	setosa
5.0	3.6	1.4	0.2	setosa
5.4	3.9	1.7	0.4	setosa
4.6	3.4	1.4	0.3	setosa
5.0	3.4	1.5	0.2	setosa
4.4	2.9	1.4	0.2	setosa
4.9	3.1	1.5	0.1	setosa
5.4	3.7	1.5	0.2	setosa
4.8	3.4	1.6	0.2	setosa
4.8	3.0	1.4	0.1	setosa
4.3	3.0	1.1	0.1	setosa
5.8	4.0	1.2	0.2	setosa
5.7	4.4	1.5	0.4	setosa
5.4	3.9	1.3	0.4	setosa
5.1	3.5	1.4	0.3	setosa
5.7	3.8	1.7	0.3	setosa
5.1	3.8	1.5	0.3	setosa

You can write citations, too. For example, we are using the **bookdown** package (Xie, 2018) in this sample book, which was built on top of R Markdown and **knitr** (**?**).

3 Classes of Random Events

3.1 Sample space and Event

Let Ω (the **sample space** of all possible outcomes of an experiments) be an arbitrary non-empty set. Its elements, denoted by ω, will be interpreted as **outcomes (results)** of some experiment. These outcomes sometimes called **simple events** or **elements** becaus they cannot be decomposed further. Other events (subsets of Ω) are **compound events** because they are composed of two or more simple events. An **event** A occurs if and only if the randomly selected ω belongs to A. As usual, we use $A \cup B$ and $A \cap B$ to represent the union and the intersection of any two subsets A and B of Ω, respectively. Also, A^c is the complement of $A \subset \Omega$. In particular, $\Omega^c = \theta$, where θ is the empty set.
Example 3.1 (an example of sample space and event)**.** When drawing randomly from $[0, 1]$ the sample space is $\Omega = [0, 1]$, each $\omega \in [0, 1]$ is a sample event, and the interval $A = (\frac{1}{2}, 1]$ is an example of a compound event.

3.2 Field and sigma-field

Definition 3.1 (Field)**.** The class $\mathcal{A}$ of subsets of Ω is called a **field** if it contains Ω and is closed under the formation of complements and finite unions, that is if:

1. $\Omega \in \mathcal{A}$;
2. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$;
3. $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cup A_2 \in \mathcal{A}$.

Proposition 3.1 (Definition of a field)**.** *The condition 3 of the definition of a field can be replaced by*

3. $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cap A_2 \in \mathcal{A}$.

That means, $\mathcal{A}$ is closed under finite intersections.

Proof. By de Morgan laws, $(A_1 \cap A_2)^C = A_1^C \cup A_2^C$ and $(A_1 \cup A_2)^c = A_1^C \cap A_2^C$. By the condition 2 of the definition of a field, we can get the desired result. □

Definition 3.2 (Semi-field)**.** Let Ω be an arbitrary set. A non-empty class $\mathcal{J}$ of subsets of Ω is called a **semi-field** if $\Omega = \mathcal{J}, \emptyset \in \mathcal{J}$, $\mathcal{J}$ is closed under the formation of finite intersections, and the complement of any set in $\mathcal{J}$ is a finite sum of disjoint sets of $\mathcal{J}$.

It is easy to see that any field of subsets of Ω is also a semi-field. However, the following simple examples show that the converse is not true.

Example 3.2 (A class of events which is a semi-field but not a field)**.** 1. Let $\Omega = [-\infty, \infty)$ and $\mathcal{J}_1$, contain Ω, $\{\infty\}$ and all intervals of the type $[a, b)$ where $-\infty < a \leq b \leq \infty$. Then, $\emptyset \in \mathcal{J}_1$, $\Omega \in \mathcal{J}_1$, $[a_1, b_1) \cap [a_2, b_2) = [a_1 \vee a_2, b_1 \wedge b_2) \in \mathcal{J}_1$, and $[a, b)^C = [-\infty, a) \cup [b, \infty)$. So, $\mathcal{J}_1$, is a semi-field. Obviously $\mathcal{J}_1$ is not a field (Check 3.3).

2. Take $\Omega = \mathbb{R}^1$ and denote by $\mathcal{J}_2$ the class of all subsets of the form $A \cap B$ where A is closed and B is an open set in Ω. Then again, $\mathcal{J}_2$ is a semi-field but not a field.

Definition 3.3 (Sigma-field)**.** The class $\mathcal{F}$ of subsets of Ω is called a **σ-field** if it is a field and closed under the formation of countable unions, that is if:

4. $A_1, A_2, \ldots, \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Proposition 3.2 (Definition of a sigma-field)**.** *The condition 4 of the definition of a sigma-field can be replaced by*

4. $A_1, A_2, \ldots, \in \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

That means, σ-field $\mathcal{F}$ is closed under countable intersections because $\bigcap_{n=1}^{\infty} A_n = (\bigcup_{n=1}^{\infty} A_n^C)^C$ *and each A_n^C belongs to $\mathcal{F}$.*

Proof. By de Morgan lawsand condition 2 of the definition of a field, $(\bigcap_{n=1}^{\infty} A_n)^C = \bigcup_{n=1}^{\infty} A_n^C \in \mathcal{F}$ and $(\bigcup_{n=1}^{\infty} A_n)^C = \bigcap_{n=1}^{\infty} A_n^C \in \mathcal{F}$. So, we can change the condition 4. □

Usually, $\mathcal{F}$ is the set of events (subsets of Ω) we are allowed to consider.

Definition 3.4 (Random events (events))**.** The elements of any field of σ-field are called **random events** (or simply, **events**).

Example 3.3 (A class of events which is a field but not a sigma-field)**.** Let $\Omega = [0, \infty)$ and $\mathcal{F}_1$ be the class of all intervals of the type $[a, b)$ or $[a, \infty)$, where $0 \leq a < b < \infty$. Let $\mathcal{F}_2$ be the class of all finite sums of intervals of $\mathcal{F}_1$. Then $\mathcal{F}_1$ is not a field, and $\mathcal{F}_2$ is a field but not a σ-field.

Take arbitrary numbers a and b, $0 < a < b < \infty$. Then $A = [a, b) \in \mathcal{F}_1$. However, $A^C = [0, a) \cup [b, \infty) \neq \mathcal{F}_1$ and thus $\mathcal{F}_1$ is not a field.

It is easy to see that

1. the finite union of finite sums of intervals (of $\mathcal{F}_1$) is again a sum of intervals;

2. the complement of a finite sum of intervals is also a sum of intervals. This means that $\mathcal{F}_2$ is a field.

However, $\mathcal{F}_2$ is not a σ-field because, for example, the set $A_n = [0, \frac{1}{n}) \in \mathcal{F}_1$ for each $n = 1, 2, \dots$, and the intersection $\bigcap_{n=1}^{\infty} A_n = \{0\}$ does not belong to $\mathcal{F}_1$.

Let us look at two additional cases.

1. Let $\Omega = \mathbb{R}^1$ and $\mathcal{F}$ be the class of all finite sums of intervals of the type \$(-∞, a], $(b, c]$ and (d, ∞). Then $\mathcal{F}$ is a field. But the intersection $\bigcap_{n=1}^{\infty}(b - \frac{1}{n}, c]$ is equal to $[b, c]$ which does not belong to $\mathcal{F}$. Hence, the field $\mathcal{F}$ is not a σ-field.

2. Let Ω be any infinite set and $\mathcal{A}$ the collection of all subsets $A \in \Omega$ such that either A or A^c is finite. Then it is easy to see that $\mathcal{A}$ is a field but not a σ-field.

Example 3.4 (A class of events can be closed under finite unions and finite intersections but not under complements)**.** Let $\Omega = \mathbb{R}$ and the class $\mathcal{A}$ consist of intervals of the type (x, ∞), $x \in \Omega$. Then using the notations $u = x \wedge y := \min\{x, y\}$ and $v = x \vee y := \max\{x, y\}$ we have:

$$(x, \infty) \cup (y, \infty) = (u, \infty) \in \mathcal{A}$$

$$(x, \infty) \cap (y, \infty) = (v, \infty) \in \mathcal{A}.$$

However, $(x, \infty)^C = (-\infty, x] \notin \mathcal{A}$.
Remark. Because any field or σ-field $\mathcal{F}$ is nonempty, it contains a set A, and therefore A^C. Consequently, any field or σ-field contains $\Omega = A \cup A^C$ and $\emptyset = \Omega^C$.

A σ-field is more restrictive than a field. Because we want to be able to consider countable unions, we require the allowable sets $\mathcal{F}$ to be a σ-field. Therefore, it is understood in probability theory and throughout the remainder we are allowed to consider only the events belonging to the σ-field $\mathcal{F}$. We will see that fields also play an important role in probability because a common technique is first to define a probability measure on a field and then extend it to a σ-field.
Definition 3.5 (D-system)**.** A system $\mathcal{D}$ of subsets of a given set Ω is called a **D-system** (Dynkin system) in Ω if the following three conditions hold:

1. $\Omega \in \mathcal{D}$;

2. $A, B \in \mathcal{D}$ and $A \subset B \Rightarrow B \backslash A \in \mathcal{D}$;

3. $A_n \in \mathcal{D}, n = 1, 2, \dots$. and $A_1 \subset A_2 \subset \dots \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.

It is obvious that every σ-field is a D-system, but the converse may not be true.
Example 3.5 (Every sigma-field of events is a D-system, but the converse does not always hold)**.** Take $\Omega = \{\omega_1, \omega_2, \dots, \omega_{2n}\}, n \in \mathbb{N}$. Denote by $\mathcal{D}_e$ the collection of all subsets $D \in \Omega$ consisting of an even number of elements. Conditions 1, 2, and 3 above are satisfied, and hence $\mathcal{D}_e$ is a D-system. However, if $n < 1$ and we take $A = \{\omega_1, \omega_2\}$ and $B = \{\omega_2, \omega_3\}$, we see that $A \in \mathcal{D}_e, B \in \mathcal{D}_e$ and $A \cap B = \{\omega_2\} \notin \mathcal{D}_e$. Thus $\mathcal{D}_e$ is not even a field and hence not a σ-field.

Note that a D-system $\mathcal{D}$ is a σ-field if and only if the intersection of any two sets in $\mathcal{D}$ is again in $\mathcal{D}$.

Example 3.6 (Trivial and total sigma-fields)**.** For any sample space Ω, the smallest and largest σ-fields are the **trivial σ-field** $2^\Omega = \{\emptyset, \Omega\}$ and the **total σ-field** (also called the power σ-field) $(\Omega) = \{\text{all subsets of } \Omega\}$, respectively.

Remember that we are allowed to determine only whether or not event A occured for $A \in \mathcal{F}$, not for sets outside $\mathcal{F}$. The trivial σ-field is not very useful because the only events that we are allowed to consider are the entire sample space and the empty set. In other words, we can determine only whether something or nothing happened. With the total σ-field we can determine, for each subset $A \subset \Omega$, whether or not event A occurred. The trouble with 2^Ω is that it may be too large to ensure that the properties we desire for a probability measure hold for all $A \in \mathcal{F}$.

Therefore, we want $\mathcal{F}$ to be large, but not too large.
Example 3.7 (For countable sample sapce, the sigma-field should be the total sigma-field)**.** Suppose Ω is a countable set like $\{0, 1, 2, \dots, \}$. We certainly want $\mathcal{F}$ to contain each singleton. But any σ-field that contains each singleton ω must contain all subsets of Ω. This follows from the fact that any subset A of Ω is a countable union $\bigcup_{n=1}^{\infty} \{\omega_i\}$ of singletons, and σ-fields are closed under countable unions. Therefore, when the sample space is countable, we should always use the total σ-field.
Example 3.8 (The union of a sequence of sigma-fields need not be a sigma-field)**.** Let $\mathcal{F}_1, \mathcal{F}_2, \dots$ be a sequence of σ-fields of subsets of the set Ω. Then their intersection $\bigcap_{n=1}^{\infty} \mathcal{F}_n$ is always a σ-field and it is natural to ask whether the union $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a σ-field. We shall now show that the answer to this question is negative.

Consider the set $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and the following two classes of its subsets: $\mathcal{F}_1 = \{\emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega\}$, $\mathcal{F}_2 = \{\emptyset, \{\omega_2\}, \{\omega_1, \omega_3\}, \Omega\}$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are fields and hence σ-fields. Obviously the intersection $\mathcal{F}_1 \cap \mathcal{F}_2 = \{\emptyset, \Omega\}$, the trivial σ-field. However, the union

$$
\begin{aligned}
\mathcal{F} &= \mathcal{F}_1 \cup \mathcal{F}_2 \\
&= \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_2, \omega_3\}, \{\omega_1, \omega_3\}, \Omega\}
\end{aligned}
$$

is not a field, and hence not a σ-field because the element $\{\omega_1\} \cup \{\omega_2\} = \{\omega_1, \omega_2\} \notin \mathcal{F}$.

3.3 Sigma-fields in real line

We next consider σ-fields when Ω is the uncountable set $\mathbb{R}$ (or an interval). Again, any useful σ-field must contain each singleton ω.

4 Methods

We describe our methods in this chapter.

5 Applications

Some *significant* applications are demonstrated in this chapter.

5.1 Example one

5.2 Example two

6 Final Words

We have finished a nice book.

Bibliography

Xie, Y. (2018). *bookdown: Authoring Books and Technical Documents with R Markdown*. R package version 0.9.