A Biggner's Guide to Probability and Extremes

Seoncheol Park

2025-08-26

Table of contents

Preface

. .

.

This is a Quarto book.

To learn more about Quarto books visit https://quarto.org/docs/books.

```
1 + 1
```

```
[1] 2
```

Part I

Intro

Part II

Probability

1 Measure and Integration

1.1 Limit of sets

σ-field .

Definition 1.1.

- S: a collection of subsets of Ω. We say that S is a

1. π-**system**: if $A, B \in S \Longrightarrow A \cap B \in S$
2. λ-**system**: if
    - $\Omega \in S$
    - $A, B \in S$ and $A \subseteq B \Longrightarrow B \backslash A \in S$
    - $A_n \uparrow A$ and $A_n \in S \Longrightarrow A \in S$
3. **Algebra** (field) if
    - $\emptyset, \Omega \in S$
    - $A \in S \Longrightarrow A^c \in S$
    - $A, B \in S \Longrightarrow A \cup B \in S$
4. **Monotone class** if
    - $A_n \in S$ and $A_n \uparrow A \Longrightarrow A \in S$
    - $A_n \in S$ and $A_n \downarrow A \Longrightarrow A \in S$
    - Recall that $A_n \uparrow A$ means that $A_1 \subseteq A_2 \subseteq ...$ and $\cup_n A_n = A$ and $A_n \downarrow A$ means that $A_1 \supseteq A_2 \supseteq ...$ and $\cap_n A_n = A$
5. σ-**algebra** (field) if
    - $\emptyset, \Omega \in S$
    - $A \in S \Longrightarrow A^c \in S$
    - $A_n \in S \Longrightarrow \cup A_n \in S$

Remark

- σ-algebra π-system, λ-system, monotone class algebra .
- Arbitrary intersections of σ-algebra, algebra, π-system, λ-system σ-algebra, algebra, π-system, λ-system .

PROBABILITY THEORY - PART 1 MEASURE THEORETICAL FRAMEWORK
.

Ω	S(π-system)	$\mathcal{A}(S)$ (algebra generated by S)	$\sigma(S)$
$(0,1]$	$\{(a,b] : 0 < a \le b \le 1\}$	$\{\cup_{k=1}^N I_k : I_k \in S$ are pairwise disjoint$\}$	$\mathcal{B}(0,1]$
$[0,1]$	$\{(a,b] \cap [0,1] : a \le b\}$	$\{\cup_{k=1}^N R_k : R_k \in S$ are pairwise disjoint$\}$	$\mathcal{B}[0,1]$
$\mathbb{R}^d$	$\{\prod_{i=1}^d (a_i, b_i] : a_i \le b_i\}$	$\{\cup_{k=1}^N R_k : R_k \in S$ are pairwise disjoint$\}$	$\mathcal{B}_{\mathbb{R}^d}$
$\{0,1\}^{\mathbb{N}}$	Collection of all cylinder sets	Finite disjoint unions of cylinders	$\mathcal{B}(\{0,1\}^{\mathbb{N}})$

, π-system π-system algebra , σ-algebra . Borel set countable number of operations on intervals .

σ-algebra . .

Lemma 1.1 ($\pi - \lambda$ theorem)**.**

- *Let Ω be a set and $\mathcal{F}$ be a set of subsets of Ω.*

1. *$\mathcal{F}$ is a σ-algebra $\iff$ it is a π-system as well as a λ-system.*
2. *If S is a π-system, then $\lambda(S) = \sigma(S)$. $\lambda(S)$ S λ-system intersection .*

Proof

1. . $\mathcal{F}$ π-system λ-system . $\Omega \in \mathcal{F}$ $A \in \mathcal{F}$, $A^c = \Omega \backslash A \in \mathcal{F}$. $A_n \in \mathcal{F}$, finite unions $B_n := \cup_{k=1}^n A_k = (\cap_{k=1}^n A_k^c)^c \in \mathcal{F}$. (intersection $\mathcal{F}$ π-system) countable union $\cup A_n$ B_n increasing limit $\mathcal{F}$ λ-property
2. $\lambda(S)$ S λ-system . $\lambda(S) = \sigma(S)$ $\mathcal{F} := \lambda(S)$ π-system . $A, B \in \mathcal{F}$ $A \cap B \in \mathcal{F}$. $A \in S$ $\mathcal{F}_A := \{B \in \mathcal{F} : B \cap A \in \mathcal{F}\}$. S

π-system $\mathcal{F}_A \supset S$. $\mathcal{F}_A$ λ-system . $\Omega \in \mathcal{F}_A$. $B, C \in \mathcal{F}_A$ $B \subseteq C$, $\mathcal{F}$ $C \cap A$ $B \cap A$ λ-system $(C \backslash B) \cap A = (C \cap A) \backslash (B \cap A) \in \mathcal{F}$. $(C \backslash B) \in \mathcal{F}_A$. $B_n \in \mathcal{F}_A$ $B_n \uparrow B$, $B_n \cap A \in \mathcal{F}_A$ $B_n \cap A \uparrow B \cap A$. $B \in \mathcal{F}_A$. $\mathcal{F}_A$ S λ-system $\mathcal{F}_A \supset \mathcal{F}$. , $A \in S$ $B \in \mathcal{F}$ $A \cap B \in \mathcal{F}$. $A \in \mathcal{F}$. $\mathcal{F}_A := \{B \in \mathcal{F} : B \cap A \in \mathcal{F}\}$. $\mathcal{F}_A \supset S$. $\mathcal{F}_A$ λ-system $A \in \mathcal{F}$ $\mathcal{F}_A = \mathcal{F}$. $\mathcal{F}$ π-system .

Lemma 1.2 (Monotone class theorem)**.**

- *Let Ω be a set and let S be a collection of Ω. If S is an algebra, then the monotone class generated by S is a σ-algebra. That is, $\mathcal{M}(S) = \sigma(S)$.*

Q. (Unique extension of measures) $(\mathbb{R}, \mathcal{B})$, ?

.

Example 1.1 ($\pi - \lambda$ theorem)**.**

- $\Omega = \{1, 2, 3, 4\}$ $S = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. $\sigma(S) = 2^\Omega$ (power set) .
- Ω μ, ν
  - $\mu_i = \frac{1}{4}, \forall i$
  - $\nu_1 = \nu_3 = \frac{1}{2}, \nu_2 = \nu_4 = 0$. $\mu(A) = \frac{1}{2} = \nu(A)$ for all $A \in S$, $\mu \neq \nu$ on $\sigma(S)$.

.

Lemma 1.3. S Ω *π-system* $\mathcal{F} = \sigma(S)$. $\mathbb{P} = \mathbb{Q}$ $\mathcal{F}$ $A \in S$ $\mathbb{P}(A) = \mathbb{Q}(A)$. $A \in \mathcal{F}$ $\mathbb{P}(A) = \mathbb{Q}(A)$.

Proof

- $\mathcal{G} = \{A \in \mathcal{F} : \mathbb{P}(A) = \mathbb{Q}(A)\}$. $\mathcal{G} \supseteq S$.
- $\mathcal{G}$ λ-system . $\Omega \in \mathcal{G}$; . $A, B \in \mathcal{G}$ $A \supseteq B$, $\mathbb{P}(A \backslash B) = \mathbb{P}(A) - \mathbb{P}(B) = \mathbb{Q}(A) - \mathbb{Q}(B) = \mathbb{Q}(A \backslash B)$. $A \backslash B \in \mathcal{G}$. $A_n \in \mathcal{G}$ $A_n \uparrow A$, $\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \mathbb{Q}(A_n) = \mathbb{Q}(A)$. (measure countable additivity)
- $\mathcal{G} \supseteq \lambda(S)$ Lemma **??** $\sigma(S)$. $\mathcal{F}$ $\mathbb{P} = \mathbb{Q}$.

Example **??** , S π-system .

> Remark
>
> - σ-algebra , Borel interval , σ-algebra , collection , collection σ-algebra . λ-system monotone class (π-system algebra) .

1.2 Measures

Definition 1.2 (σ-additive)**.**

- $\mathcal{A}$: a collection of subsets of Ω containing the empty set $\emptyset$
- A **set function** on $\mathcal{A}$: $\mu : \mathcal{A} \to [0, \infty]$ with $\mu(\emptyset) = 0$
- We say that μ is **countably additive**, or σ**-additive**, if for all sequences (A_n) of disjoint sets in $\mathcal{A}$ with $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

> Remark
>
> - Recall that a **measurable space** is a pair $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is a σ-algebra on Ω.

Definition 1.3 (Measure space)**.**

- A **measure space** is a triple $(\Omega, \mathcal{F}, \mu)$, where
  - Ω: set
  - $\mathcal{F}$: σ-algebra on Ω
  - μ: $\mathcal{F} \to [0, \infty]$ is a countably additive set function.
  - Then μ is a measure on $(\Omega, \mathcal{F})$.

power set . Ω subset measure . " " collection of sets . Ω smaller class of subsets σ-algebra generate . .

1. (Construction) measure whole σ-algebra ? (**Caratheodory's Extension Theorem**)
2. (Uniqueness) σ-algebra ? ($\pi - \lambda$ **systems Lemma**)

1.2.1 Lebesgue measure

Definition 1.4 (Existence and uniqueness of Lebesgue measure)**.**

- There exists a unique Borel measure λ on $[0,1]$ such that $\lambda(I) = |I|$ for any interval I.

Proof

- $S = \{(a,b] \cap [0,1]\}$ $\mathcal{B}$ π-system . Lemma **??** uniqueness .
- Existence , . PROBABILITY THEORY - PART 1 MEASURE THEORETICAL FRAMEWORK . $\Omega = [0,1]$.

1. **Outer measure λ_*** .

   $$\lambda_*(A) = \inf\{\sum |I_k| : \text{ each } I_k \text{ is an open interval and } \{I_k\} \text{ a countable cover for } A\}.$$

   outer measure .

   - subset $A \subseteq \Omega$ $0 \le \lambda_*(A) \le 1$
   - $\lambda_*(A \cup B) \le \lambda_*(A) + \lambda_*(B)$, for any $A, B \subseteq \Omega$
   - $\lambda_*(\Omega) = 1$

2. λ_* σ-field . λ_* a set Ω outer measure . Caratheodary

   $$\mathcal{F} := \{A \subseteq \Omega : \lambda_*(E) = \lambda_*(A \cap E) + \lambda_*(A^c \cap E) \text{ for any } E\}.$$

   $\mathcal{F}$ σ-algebra λ_* restricted to $\mathcal{F}$.

3. $A = (a,b] \subseteq [0,1]$ $A \in \mathcal{F}$ $\mathcal{F}$. $\mathcal{F}$.

1.2.2 Construction of Lebesgue measure

Lebesgue class .

1. algebra $\mathcal{A}$ ($(a,b]$) $\mathcal{A}$ countably additive probability measure P on $\mathcal{A}$, outer measure P^* $\mathcal{A}$ countable cover . , E

   $$P^*(E) = \inf\{\sum_{i=1}^{\infty} P(A_i) : E \supseteq \cup_{i=1}^{\infty} A_i, A_i \in \mathcal{A}\}.$$

2. $\mathcal{F}$, $\mathcal{F} \supset \mathcal{A}$ σ-algebra P^* $\mathcal{A}$.
3. $P^* = P$ on $\mathcal{A}$.

1.2.3 Push-forward (image) measure

- Measure
  - Caratheodory Extension Theorem
  - Transportation between spaces via functions.

Definition 1.5 (Push-forward measure)**.**

- $(\Omega, \mathcal{F}, \mathbb{P})$, X $(\Omega, \mathcal{F})$ $(E, \mathcal{E})$.

$$\mathbb{Q}(A) = \mathbb{P}(X^{-1}(A)), \quad A \in \mathcal{E}$$

  $(E, \mathcal{E})$ (**push-forward measure**).

- $\mathbb{Q} = \mathbb{P} \circ X^{-1}$ **law of the distribution** of X .

> Remark
>
> - E X^{-1} Ω $\mathbb{P}$.
> - Push-forward measure change-of-variables formula
> - $\mathbb{Q}$ A_n pairwise disjoint $X^{-1}(A_n)$ pairwise disjoint . B_n Ω pairwise disjoint , $X(B_n)$ disjoint . **pull-back measure** . X X^{-1} pull-back .

1.3 Integration

1.3.1 Integration notations

> Remark
>
> notation

- $(E, \mathcal{E}, \mu)$: measure space, $f : E \to \mathbb{R}$: a real-valued transformation
-
  - $\int_E f(x) d\mu(x)$
  - $\int_E f d\mu$ ( )
  - $\int_E f(x) \mu(dx)$

1.3.2 -

.

$$E(X) = \int x dF(x).$$

x f(x) .

$$\int_a^b f(x)dx.$$

x $g(x)$ $f(x)$.

$$\int_{x=a}^b f(x)dg(x).$$

$dg(x)$ $g(x)$ (differential) , $g(x)$ x . **- (Riemann-Stieltjes Integral)** . - .

$$\int_{x=a}^b f(x)dg(x) = \lim_{N\to\infty} \sum_{n=0}^{N-1} f(t_n)[g(x_{n+1}) - g(x_n)].$$

x_n $[a,b]$, t_n $[x_n, x_{n+1}]$.

Example 1.2 (- **).**

- X: random variable with support $R_X = [0,1]$ and distribution function

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{2}x, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1. \end{cases}$$

-

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x dF_X(x) \\ &= \int_0^1 x dF_X(x) + 1 \cdot \Big[F_X(1) - \lim_{x\to 1, x<1} F_X(x)\Big] \\ &= \int_0^1 x \frac{d}{dx}\Big(\frac{1}{2}x\Big)dx + 1 \cdot \Big[1 - \frac{1}{2}\Big] \\ &= \Big[\frac{1}{4}x^2\Big]_0^1 + \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}. \end{aligned}$$

1.4

Confused when changing from Lebesgue Integral to Riemann Integral .

.

$(\Omega, \mathcal{F}, P)$ X .

$$P(X < x) = 1 - e^{-\lambda x}.$$

, X .

$$E[X] = \int_{\{\omega | X(\omega) \geq 0\}} X(\omega) dP(\omega).$$

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x} dx$$

.

.

By definition, given $X : \Omega \to \mathbb{R}$ a random variable, $E[X] = \int_\Omega X$. X defines a measure $\tilde{m}$ in $\mathbb{R}$, called the **push-forward**, by $\tilde{m}(A) = P(X^{-1}(A))$. By definition, this measure is invariant under X, and hence

$$\int_{\mathbb{R}} f d\tilde{m} = \int_\Omega f \circ X dP.$$

The equality follows from the usual arguments (prove for characteristics, simple functions, then use convergence. Recall that $1_A \circ X = 1_{X^{-1}(A)}$).

Let h be the density of X. We then have, by definition of density, that $\tilde{m}(A) = P(X^{-1}(A)) = \int_A hdm$ for any $A \in \mathcal{B}(\mathbb{R})$, where m is the Lebesgue measure. By **change of variables**, we have

$$\int_{\mathbb{R}} f d\tilde{m} = \int_{\mathbb{R}} f \cdot hdm.$$

Combining these equations,

$$\int_{\mathbb{R}} f \cdot hdm = \int_\Omega f \circ X dP.$$

Taking $f = \mathrm{Id}$ yields

$$\int_{\mathbb{R}} xh(x)dx = \int_\Omega XdP = E[X].$$

Taking $f = \mathrm{Id} \cdot \mathbf{1}_I$, where I is some interval (for example, $(0, +\infty)$ as in your case), we have

$$\int_I xh(x)dx = \int_{X^{-1}} XdP,$$

recalling again that $\mathbf{1}_A \circ X = \mathbf{1}_{X^{-1}(A)}$. Since $P(X < 0)$ in your case is 0, this last integral is actually equal to the integral over the whole space, and hence to $E[X]$, which gives your equality.

2 Probability

2.1 Probability Triples

.

Q. probability triple ? Single double triple ?

- **Sample space** Ω ( ): any non-empty set . uniform distribution $\Omega = [0, 1]$.
- $\mathcal{F}$: σ-algebra σ-field: Ω subset collection $\emptyset, \Omega$.
- **Probability** P: a mapping from $\mathcal{F}$ to $[0, 1]$ with
  - $P(\emptyset) = 0$
  - $P(\Omega) = 1$
  - P is countably additive, $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$
- Recall that the elements of any field or σ-field (Definition **??**) are called **random events** (or simply **events**).

2.2 Probabilities

Definition 2.1 (Probability)**.**

- Let Ω be any set and $\mathcal{A}$ be a field of its subsets. We say that P is a **probability** on the measurable space $(\Omega, \mathcal{A})$ if P is defined for all events $A \in \mathcal{A}$ and satisfies the following axioms.

1. $P(A) \geq 0$ for each $A \in \mathcal{A}$; $P(\Omega) = 1$
2. P is **finitely additive**. That is, for any finite number of pairwise disjoint events $A_1, \ldots, A_n \in \mathcal{A}$ we have
$$P\Big(\cup_{i=1}^{n} A_i \Big) = \sum_{i=1}^{n} P(A_i).$$

3. P is continuous at $\emptyset$. That is, for any events $A_1, A_2, \ldots, \mathcal{A}$ such that $A_{n+1} \subset A_n$ and $\cap_{n=1}^{\infty} A_n = \emptyset$, it is true that
$$\lim_{n \to \infty} P(A_n) = 0.$$

Note that conditions 2 and 3 are equivalent to the next one 4.

4. P is σ-additive (countably additive), that is
$$P\Big(\cup_{n=1}^{\infty} A_n \Big) = \sum_{n=1}^{\infty} P(A_n)$$
for any events $A_1, A_2, \ldots \in \mathcal{A}$ which are pairwise disjoint.

Example 2.1 (A probability measure which is additive but not σ-additive)**.** Let Ω be the set of all rational numbers r of the unit interval $[0, 1]$ and $\mathcal{F}_1$ the class of the subsets of Ω of the form $[a, b]$, $(a, b]$, (a, b) or $[a, b)$ where a and b are rational numbers. Denote by $\mathcal{F}_2$ the class of all finite sums of disjoint sets of $\mathcal{F}_1$. Then $\mathcal{F}_2$ is a field. Let us define the probability measure P as follows:
$$\begin{aligned} &P(A) = b - a, \quad \text{if } A \in \mathcal{F}_1, \\ &P(B) = \sum_{i=1}^{n} P(A_i), \quad \text{if } B \in \mathcal{F}_2, \text{ that is, } B = \sum_{i=1}^{n} A_i, A_i \in \mathcal{F}_1. \end{aligned}$$

Consider two disjoint sets of $\mathcal{F}_2$ say
$$B = \sum_{i=1}^{n} A_i \quad \text{ and } B' = \sum_{j=1}^{m} A'_j,$$
where $A_i, A'_j \in \mathcal{F}_1$ and all A_i, A'_j are disjoint. Then $B + B' = \sum_{k=1}^{m+n} C_k$ where either $C_k = A_i$ for some $i = 1, \ldots, n$, or $C_k = A'_j$ for some $j = 1, \ldots, m$. Moreover,
$$\begin{aligned} P(B + B') = P\Big(\sum_k C_k \Big) &= \sum_k P(C_k) = \sum_{i,j} (P(A_i) + P(A'_j)) \\ &= P(A_i) + \sum_j P(A'_j) = P(B) + P(B'). \end{aligned}$$
and hence P is an additive measure.

Obviously every one-point set $\{r\} \in \mathcal{F}_2$ and $P(\{r\}) = 0$. Since Ω is a countable set and $\Omega = \sum_{i=1}^{\infty} \{r_i\}$, we get
$$P(\Omega) = 1 \neq 0 = \sum_{i=1}^{\infty} P(\{r_i\}).$$
This contradiction shows that P is not σ-additive.

3 Random Variables

3.1 Random Variables

Definition 3.1 (( )).

- : $(\Omega, \mathcal{F}, P)$, $f : \Omega \to \mathbb{R}$

$$B \in \mathcal{B}(\mathbb{R}) \implies f^{-1}(B) \in \mathcal{F}$$

f **(measurable function)**

Example 3.1 ( ).

- $\Omega = \{1, 2, 3\}$, $\mathcal{F} = \{\Omega, \emptyset, \{1, 2\}, \{3\}\}$
- $f(1) = 1, f(2) = 2, f(3) = 3$ $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$
- $\{1\} \in \mathcal{B}(\mathbb{R})$ $f^{-1}(\{1\}) = \{1\} \notin \mathcal{F}$ f ,

Remark

-

Definition 3.2 (Random Variables)**.** Given a probability triple $(\Omega, \mathcal{F}, P)$, a **random variable** is a function X from Ω to $\mathbb{R}$, such that

$$\{\omega \in \Omega; X(\omega) \leq x\} \in \mathcal{F}, \quad x \in \mathbb{R}.$$

Q. Random variable inverse image ?

Commonly a probability measure P is added to $(\Omega, \mathcal{F})$. Then sets like $\{X \in A\} := \{\omega \in \Omega | X(\omega) \in A\}$ can $= X^{-1}(A)$ be **measured** if they belong to $\mathcal{F}$. $X : \Omega \to \mathbb{R}$ $X < 1$ $X^{-1}(-\infty, 1)$.

Example 3.2 (inverse image)**.** Proschan (2016) 4.2 .

- $(\Omega, \mathcal{F}, P) = ((0,1), \mathbb{B}_{(0,1)}, \mu_L)$

$$X(\omega) = \frac{1}{\omega(1-\omega)}$$

.

- Borel set B $\{(6.25, \infty) \cup \{4\}\}$
- : $X^{-1}(B) = \{(0, 0.2) \cup (0.8, 1) \cup \{0.5\}\} \in \mathcal{B}_{(0,1)}$

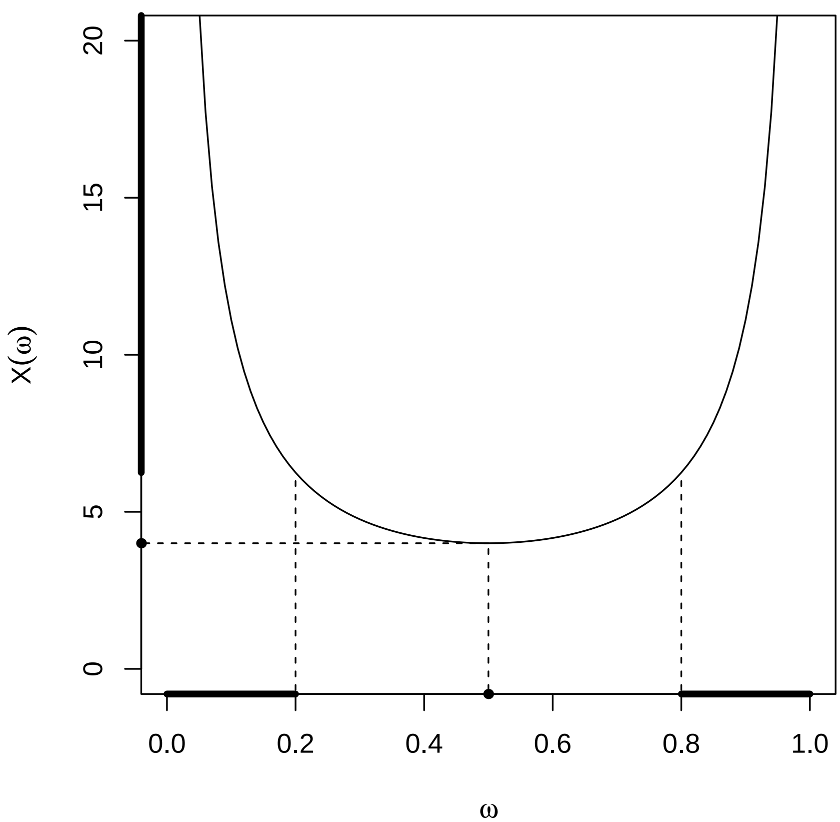

Figure 3.1: Figure: X .

3.2 Radon-nikodym derivative

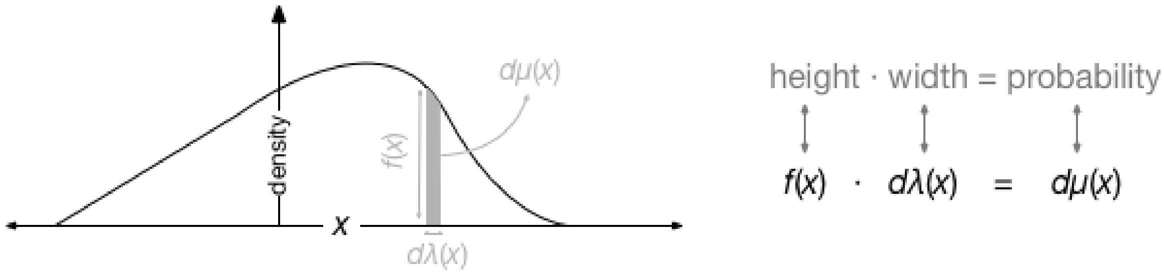

Figure 3.2: Change of measures.

volume element .

- $\mu(x)$: probability measure, interval set of points area/volume ( ) .
- $\lambda(x)$: reference measure. We often take $\lambda(x)$ as the Lebesgue measure which is essentially just a uniform function over the sample space.

The reference measure $\lambda(x)$ is essentially just a meter-stick that allows us to express the probability measure as a simple function $f(x)$. That is, we represent the probability measure $\mu(x)$ as $f(x)$ by comparing the probability measure to some specified reference measure $\lambda(x)$. This is essentially the intuition that is given by the Radon-Nikodym derivative

$$f(x) = \frac{d\mu(x)}{d\lambda(x)}$$

or equivalently

$$\text{height} = \text{area} \ / \ \text{width}.$$

Note that we can also represent the same idea by

$$\mu(A) = \int_{A \in X} f(x) d\lambda(x),$$

where $\mu(A)$ is the sum of the probability of events in the set A which is itself a subset of the entire sample space X. Note that when $A = X$ then the integral must equal 1 by definition of probability.

- .

Definition 3.3 (Integrable Random Variable)**.** Gut (2014) 53 , $E|X| < \infty$ random variable X integrable .

Example 3.3. Given a probability measure P and sample space Ω, it is true that

$$\int_\Omega dP = 1.$$

Because

$$\int_\Omega dP = P(\Omega) = 1.$$

More generally

$$\int_A dP = \int_\Omega 1_A dP = P(A), \quad A \in \mathcal{F}.$$

Definition 3.4 ($\mathcal{L}^p$)**.** $(\Omega, \mathcal{F}, P)$. $p > 1$, X $E|X|^p < \infty$ $X \in \mathcal{L}^p$ $\|X_p\| = (E|X|^p)^{\frac{1}{p}}$.

3.3 Distribution

- measurable subset of possible outcomes (points, bounded/unbounded intervals) measure ( )
- Semi-infinite interval $(-\infty, x]$ measurable subset $\mathbb{R}$ CDF
- PDF CDF CDF PDF

3.4 Expectation

- **Expectation**: integral with respect to the probability measure

Definition 3.5 (Expectation)**.**

- $(\Omega, \mathcal{A}, \mathbb{P})$: Probability space
  - Ω: set (sample space)
  - $\mathcal{A}$: σ-algebra on Ω
  - $\mathbb{P}$: Probability measure
- $X : \Omega \to \mathbb{R}$: Random variable (a measurable fct)
- **Expectation**: The concept of integral of X w.r.t. $\mathbb{P}$

$$E[X] \triangleq \int_\Omega X(\omega) d\mathbb{P}[\omega]$$