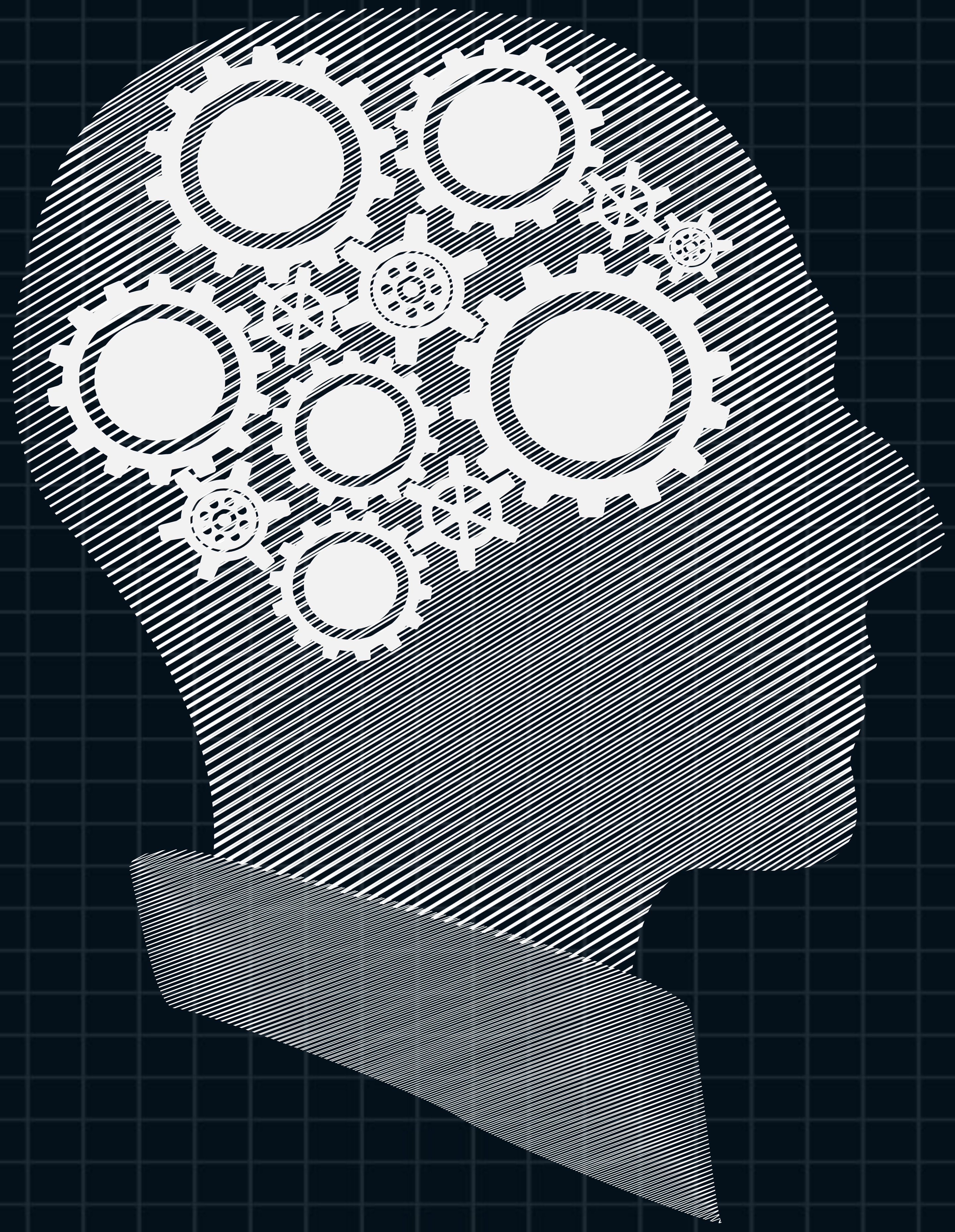


Backpropagation and Jacobian Matrices

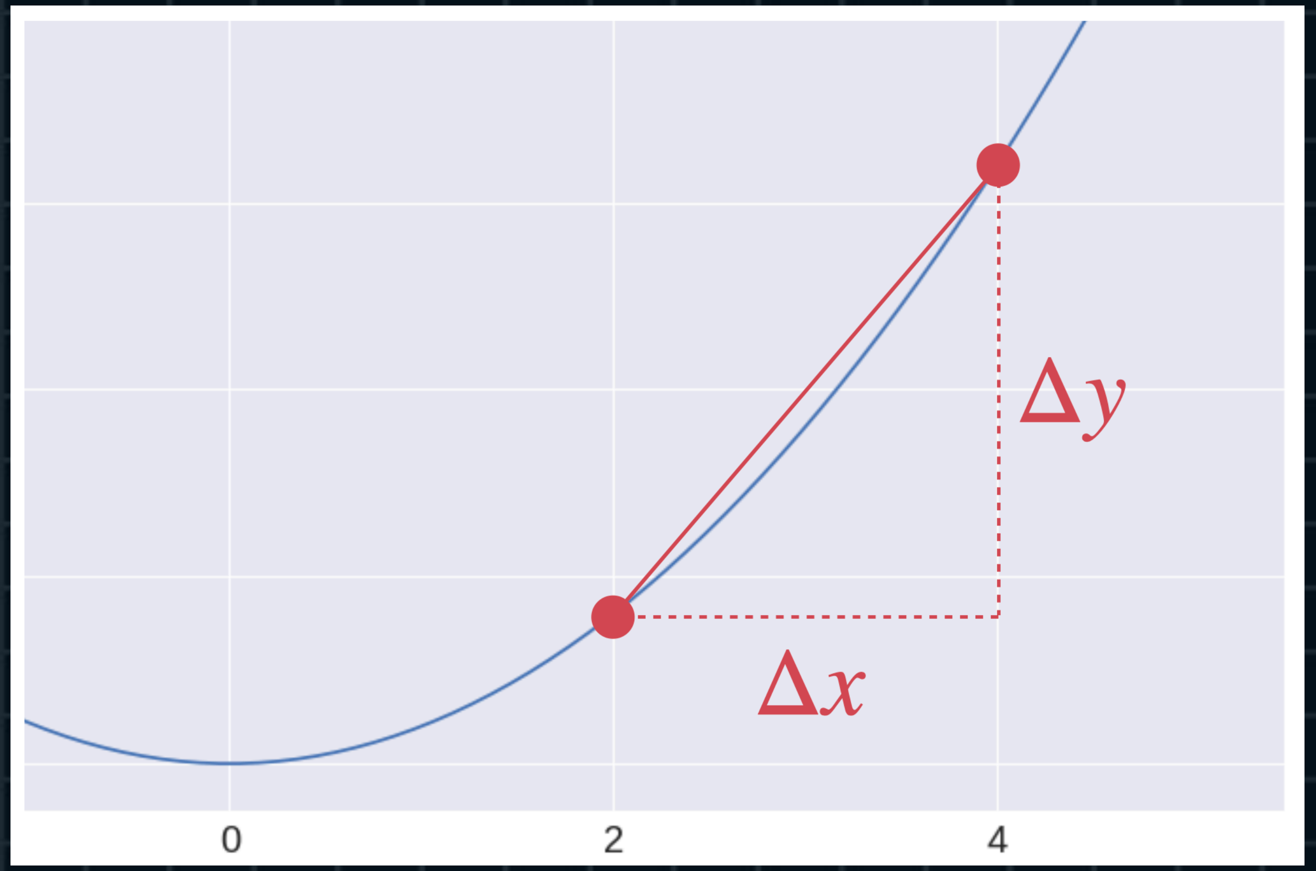
Lecture.2
Basic Differentiation

Lecture.2
Basic Differentiation

- Rate of Changes

Average Rate of Change

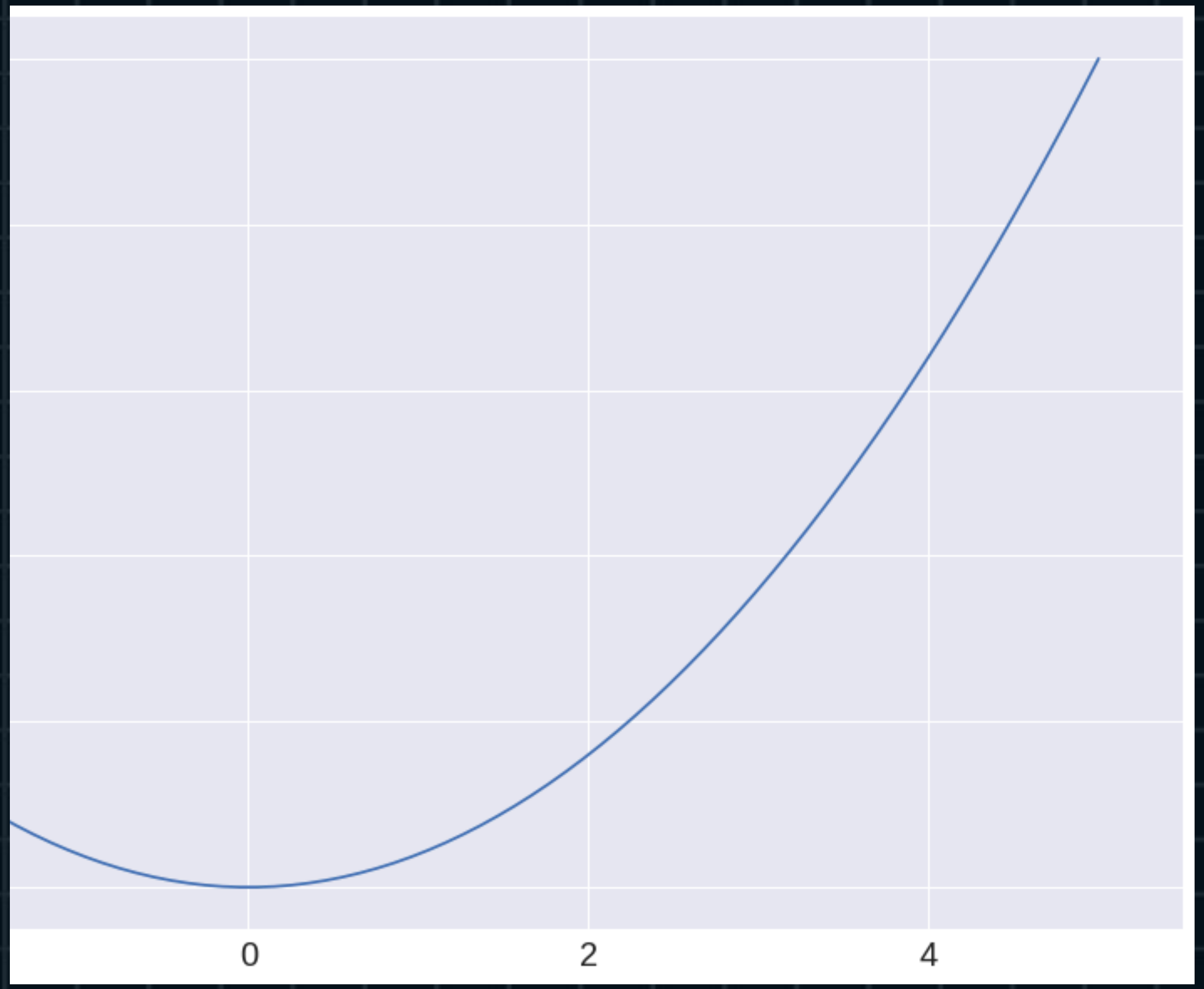
$$y = x^2$$



$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

$$\frac{\Delta y}{\Delta x} = \frac{f(4) - f(2)}{4 - 2}$$
$$= \frac{16 - 4}{4 - 2} = 6$$

Instantaneous Rate of Change



$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

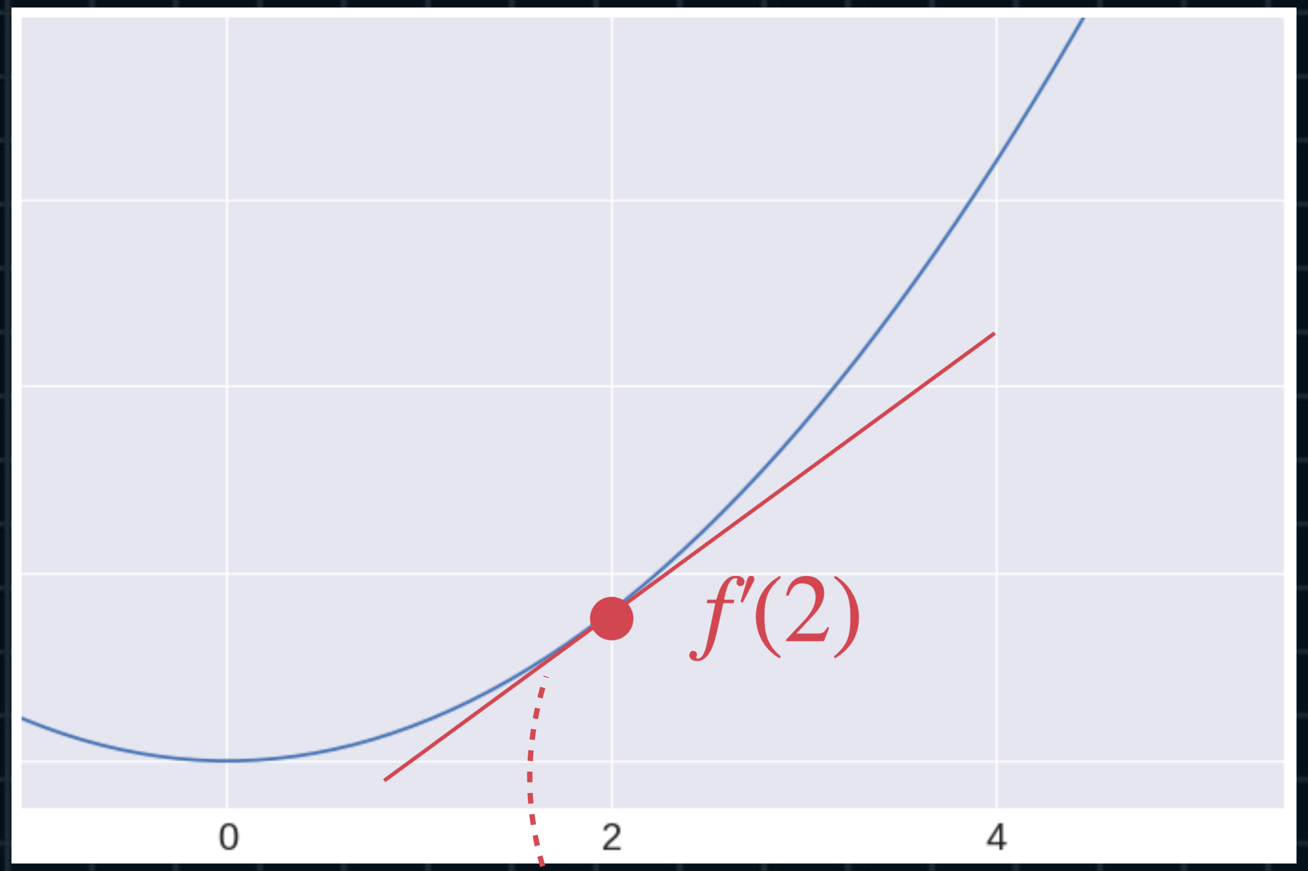
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(2) = \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h}$$

$$= \lim_{h \to 0} \left[4 + h\right] = 4$$

- Rate of Changes

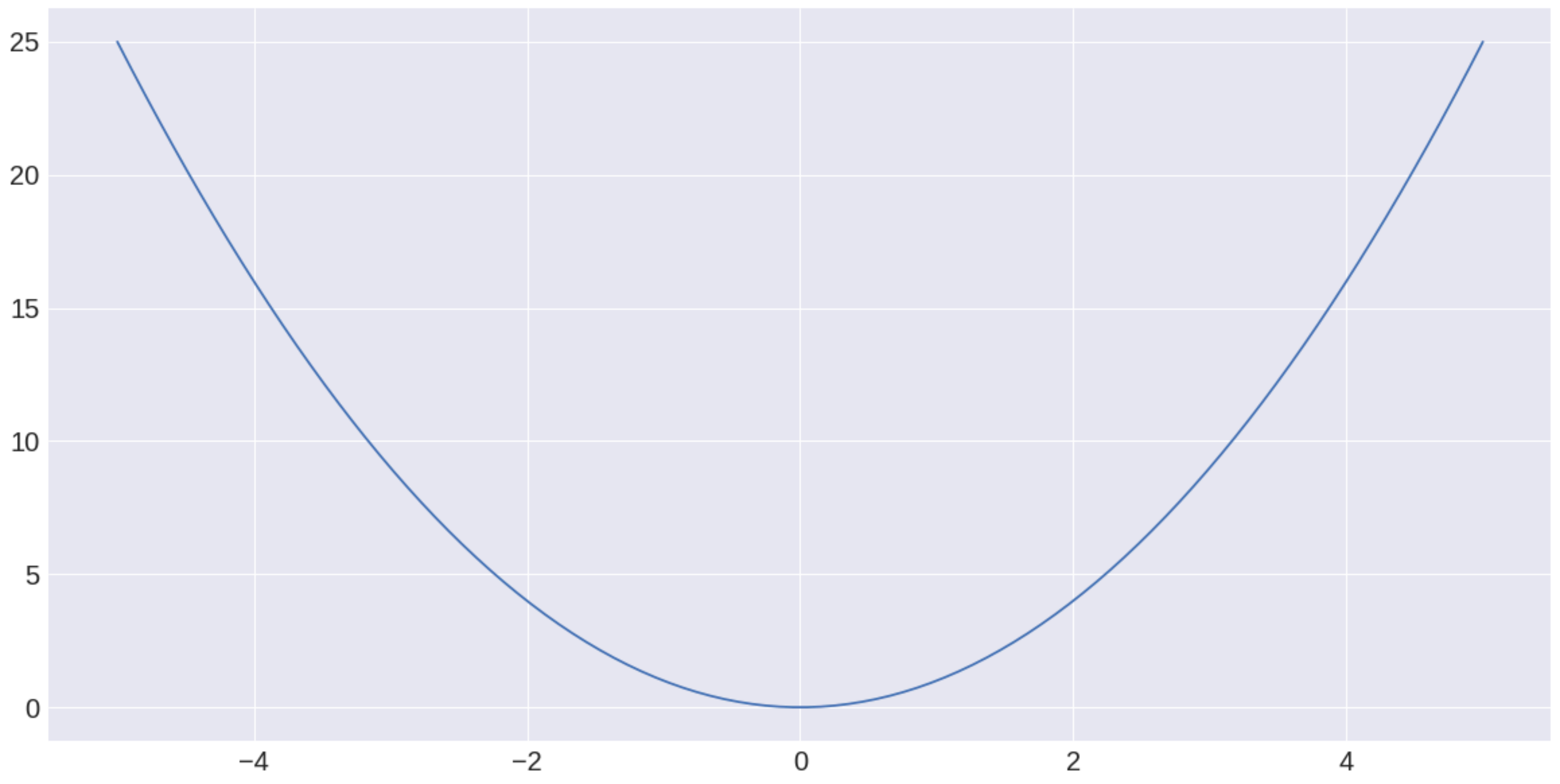
Instantaneous Rate of Change



$$y - f(2) = f'(2)(x - 2)$$

Derivatives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



$$y = x^2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

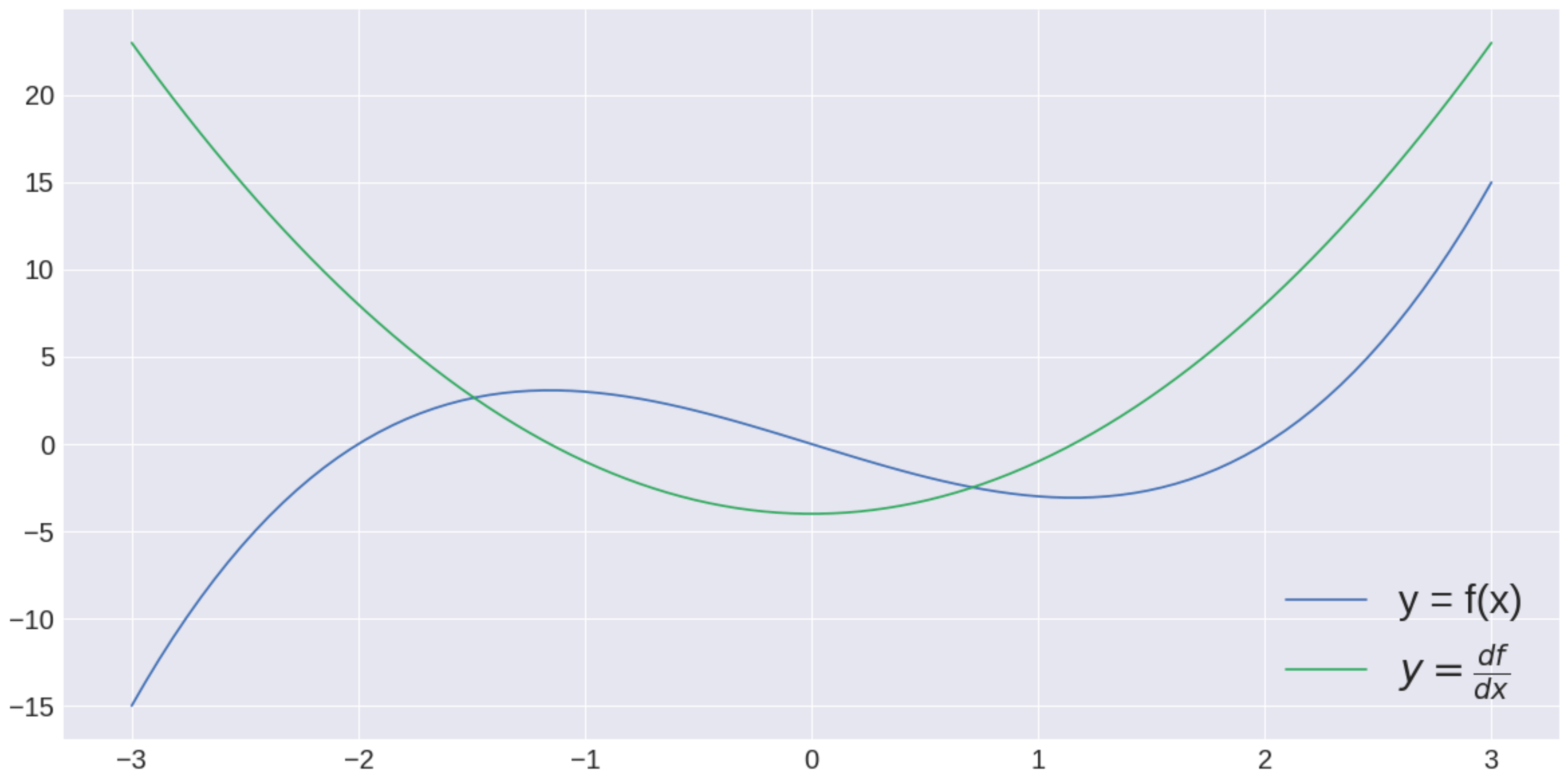
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \left[2x + h\right] = 2x$$

$$f'(-2) = 2 \cdot (-2) = -4$$

$$f'(0) = 2 \cdot 0 = 0$$

$$f'(2) = 2 \cdot 2 = 4$$

Derivatives



$$f(x) = x(x+2)(x-2)$$

Constant Functions

$$f(x) = c$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

$$f(x) = c \quad \Longrightarrow \quad f'(x) = 0$$

$f(x) = 100$

$f(x) = e^2 - ln(30)$

Power Functions

$$f(x) = x^c$$

$f(x) = x^2$

$f(x) = x^{10}$

$f(x) = \dfrac{1}{x} \dashrightarrow f(x) = x^{-1}$

$f(x) = \dfrac{1}{x^2} \dashrightarrow f(x) = x^{-2}$

$f(x) = \sqrt{x} \dashrightarrow f(x) = x^{\frac{1}{2}}$

$f(x) = \dfrac{1}{x \cdot \sqrt[3]{x}} \dashrightarrow f(x) = x^{-\frac{4}{3}}$

- Diff. of Basic Functions

Power Functions

$$f(x) = x^c \implies f'(x) = c \cdot x^{c-1}$$

$$f(x) = x$$

$$f(x) = x^2$$

$$f(x) = \frac{1}{x}$$

$$f(x) = \frac{-2}{x^3}$$

$$f(x) = \sqrt{x}$$

$$f(x) = \frac{1}{\sqrt[3]{x}}$$

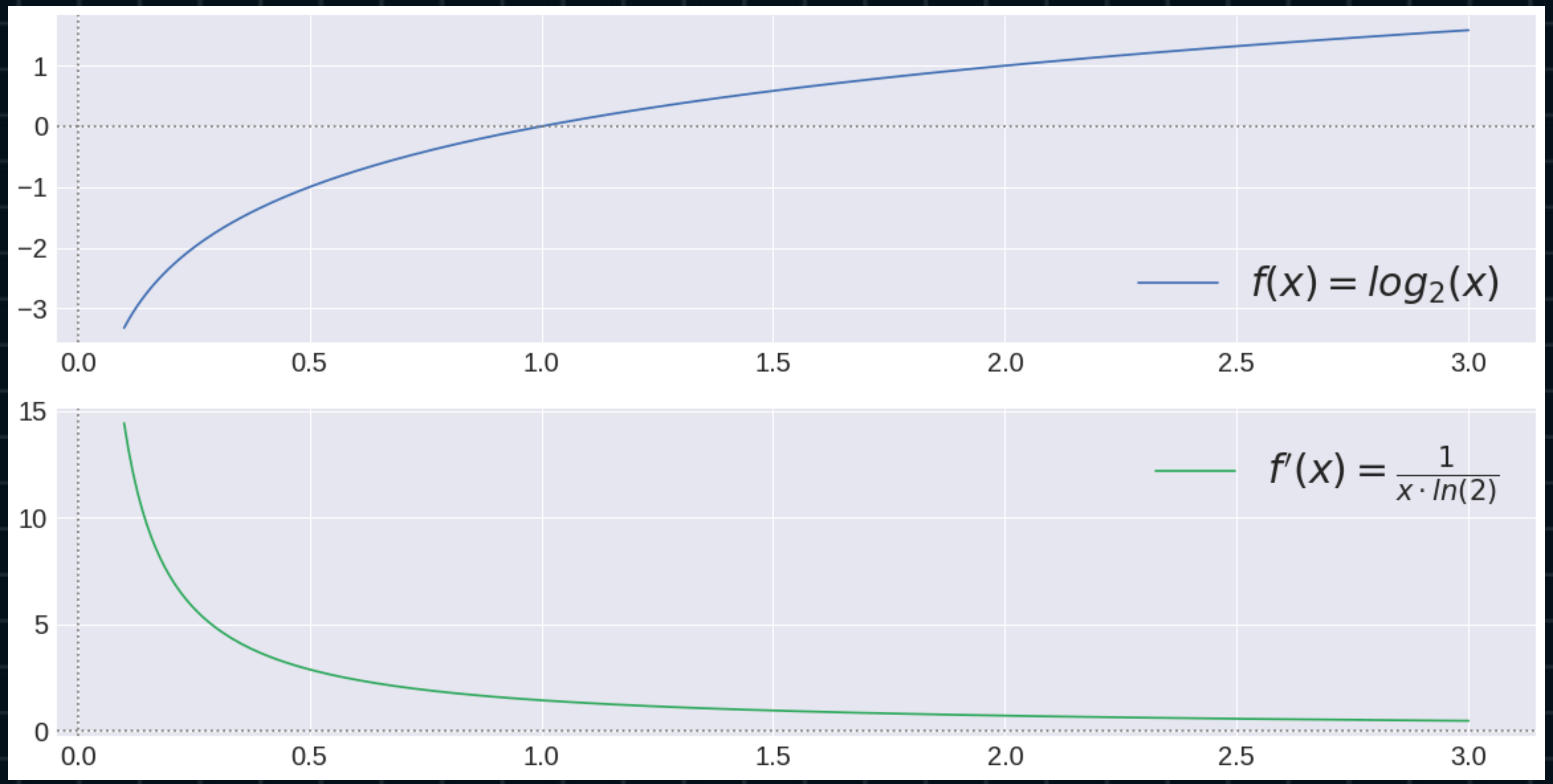
Logarithmic Functions

$$f(x) = log_a(x) \implies f'(x) = \frac{1}{x \cdot ln(a)}$$

$$f(x) = log_2(x)$$

$$f(x) = log_e(x) = ln(x)$$

Logarithmic Functions



Monotonically Increasing Functions

$$x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$$

Inverse Function of Exponential

$$f(x) = a^x$$

$$log(y) = log(a^x) = x \cdot log(a)$$

$$x = \frac{log(y)}{log(a)} = log_a(y)$$

$$f(x) = log_a(x)$$

- Diff. of Basic Functions

Exponential Functions

$$f(x) = a^x \implies f'(x) = ln(a) \cdot a^x$$

$f(x) = 2^x$

$f(x) = ln^x(2)$

$f(x) = e^x$

Trigonometric Equalities

$$\lim_{x \to 0} \frac{sin(x)}{x} = 1,\ \lim_{x \to 0} \frac{tan(x)}{x} = 1$$

$$sin(A) + sin(B) = 2sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)$$

$$sin(A) - sin(B) = 2cos\left(\frac{A+B}{2}\right)sin\left(\frac{A-B}{2}\right)$$

$$cos(A) + cos(B) = 2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)$$

$$cos(A) - cos(B) = -2sin\left(\frac{A+B}{2}\right)sin\left(\frac{A-B}{2}\right)$$

Diff. of Sin Function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{sin(x+h) - sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{2cos\left(x + \frac{h}{2}\right) sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{cos\left(x + \frac{h}{2}\right) sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = cos(x)$$

- Diff. of Basic Functions

Trigonometric Functions

$$f(x) = sin(x) \implies f'(x) = cos(x)$$

$$f(x) = cos(x) \implies f'(x) = -\,sin(x)$$

$$f(x) = tan(x) \implies f'(x) = sec^2(x) = \frac{1}{cos^2(x)}$$

- Diff. of Basic Functions

Piecewise-defined Functions

$$f(x) = \begin{cases} f_1(x), & x \geq \alpha \\ f_2(x), & x < \alpha \end{cases} \implies f'(x) = \begin{cases} f_1'(x), & x \geq \alpha \\ f_2'(x), & x < \alpha \end{cases}$$

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$f(x) = ReLU(x) = max(0, x)$$

Constant Multiple Rule

$$f(x) = x \implies 2 \cdot f(x) = 2x$$

$$f(x) = e^x \implies -3 \cdot f(x) = -3e^x$$

$$f(x) = sin(x) \implies e \cdot f(x) = e \cdot sin(x)$$

$$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]$$

$$f(x) \longrightarrow \boxed{\times C} \longrightarrow \boxed{\frac{d}{dx}} \longrightarrow \frac{d}{dx}\left[c \cdot f(x)\right] \iff f(x) \longrightarrow \boxed{\frac{d}{dx}} \longrightarrow \boxed{\times C} \longrightarrow c \cdot \frac{d}{dx}\left[f(x)\right]$$

Constant Multiple Rule

$$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]$$

$f(x) = 2x$

$f(x) = -3e^x$

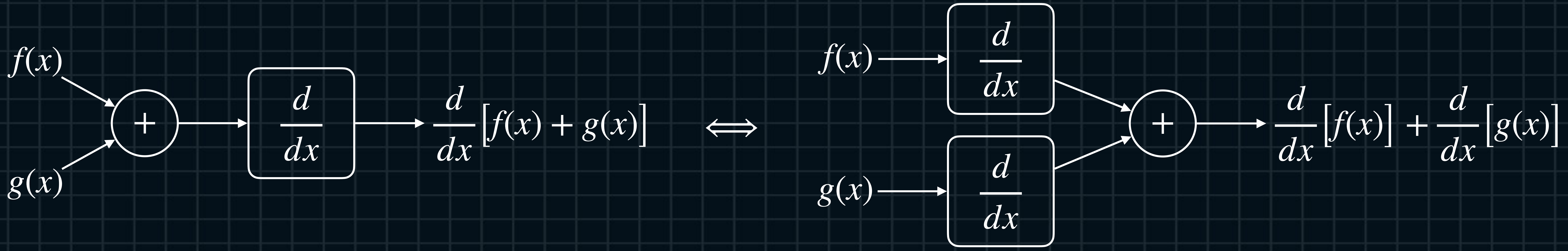
Lecture.2 Basic Differentiation

- Differentiation Rules

Sum Rule

$$f(x) = 3x^3 - 2x^2 + 10x - 20$$

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{df}{dx} + \frac{dg}{dx}$$



Lecture.2 Basic Differentiation

- Differentiation Rules

Sum Rule

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{df}{dx} + \frac{dg}{dx}$$

$$f(x) = 2x^2 - x + 7$$

$$f(x) = sin(x) - \frac{1}{\sqrt{x}}$$

$$f(x) = sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$f(x) = cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$sinh(x) \underset{\frac{d}{dx}}{\rightleftarrows} cosh(x)$$

Lecture.2 Basic Differentiation

- Differentiation Rules

Linearity of Diff.

$$Sys\{\alpha \cdot f(t)\} = \alpha \cdot Sys\{f(t)\}$$ Homogeneity

$$Sys\{f(t) + g(t)\} = Sys\{f(t)\} + Sys\{g(t)\}$$ Additivity

$$Sys\{\alpha \cdot f(t) + \beta \cdot g(t)\} = \alpha \cdot Sys\{f(t)\} + \beta \cdot Sys\{g(t)\}$$

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$$ Constant Multiple Rule

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$ Sum Rule

$$\frac{d}{dx}[\alpha \cdot f(x) + \beta \cdot g(x)] = \alpha \cdot \frac{d}{dx}[f(x)] + \beta \cdot \frac{d}{dx}[g(x)]$$

- Differentiation Rules

Time-invariance of Diff.

$$Sys\{f(t)\} = f'(t) \quad \Longrightarrow \quad Sys\{f(t-\tau)\} = f'(t-\tau)$$

$$\frac{df(t)}{dt} = f'(t) \quad \Longrightarrow \quad \frac{df(t-\tau)}{dt} = f'(t-\tau)$$

- Differentiation Rules

LTI Systems and Diff.

$$Sys\{\alpha \cdot f(t) + \beta \cdot g(t)\} = \alpha \cdot Sys\{f(t)\} + \beta \cdot Sys\{g(t)\}$$

$$Sys\{x(t)\} = y(t) \quad \Rightarrow \quad Sys\{x(t-\tau)\} = y(t-\tau)$$

$$\frac{d}{dx}\left[\alpha \cdot f(t-\tau) + \beta \cdot f(t-\tau)\right] = \alpha \cdot f'(t-\tau) + \beta \cdot g'(t-\tau)$$

Product Rule

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \frac{d}{dx}\left[f(x)\right]\cdot g(x) + f(x)\cdot\frac{d}{dx}\left[g(x)\right]$$

$f(x) = x^3$

$f(x) = e^x ln(x)$

Lecture.2
Basic Differentiation

- Differentiation Rules

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}\left[f(x)\right] \cdot g(x) - f(x) \cdot \frac{d}{dx}\left[g(x)\right]}{\left(g(x)\right)^2}$$

$$f(x) = \frac{1}{x}$$

$$f(x) = \frac{x^2}{e^x}$$

$$f(x) = tanh(x) = \frac{sinh(x)}{cosh(x)}$$

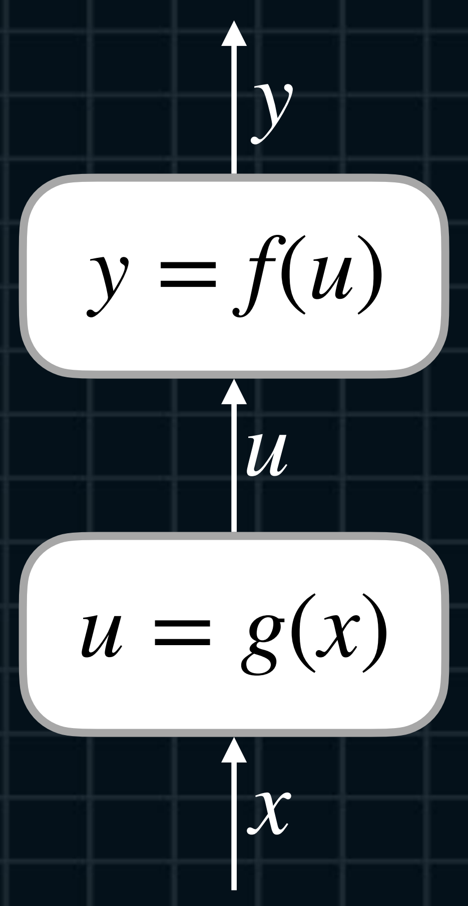
Lecture.2 Basic Differentiation - Composite Functions and Chain Rule

Composite Functions

$$y = f(g(x))$$

$$u = g(x)$$

$$y = f(u)$$



- Composite Functions and Chain Rule

Chain Rule

$$y = f(g(x))$$

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

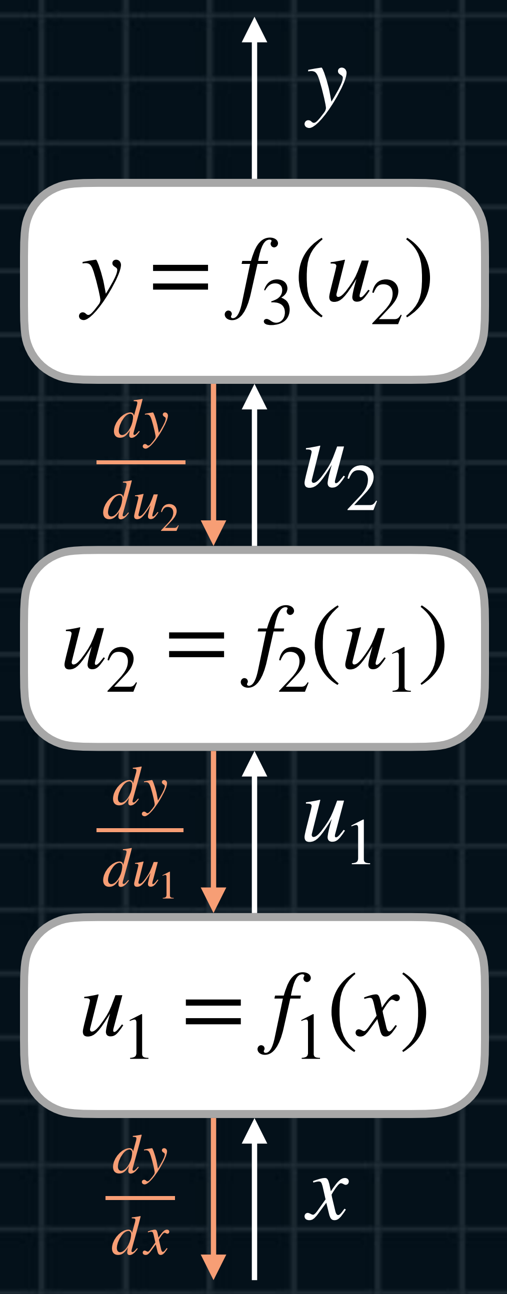
Chain Rule

$$y = f_1(f_2(f_3(x)))$$

$$\begin{tikzcd}[row sep=small]
{} \\
\boxed{y = f_3(u_2)} \arrow[u, "y"'] & \dfrac{du_1}{dx} & \dfrac{dy}{dx} = \dfrac{dy}{du_1}\dfrac{du_1}{dx} \\
\boxed{u_2 = f_2(u_1)} \arrow[u, "u_2"'] & \dfrac{du_2}{du_1} & \dfrac{dy}{du_1} = \dfrac{dy}{du_2}\dfrac{du_2}{du_1} \\
\boxed{u_1 = f_1(x)} \arrow[u, "u_1"'] & \dfrac{dy}{du_2} \\
{} \arrow[u, "x"']
\end{tikzcd}$$

Chain Rule

$$y = f_1(f_2(f_3(x)))$$



- Composite Functions and Chain Rule

Forward/Backward Calculations

$$y = f_4(f_3(f_2(f_1(x))))$$

Lecture.2 Basic Differentiation - Composite Functions and Chain Rule

Exercises

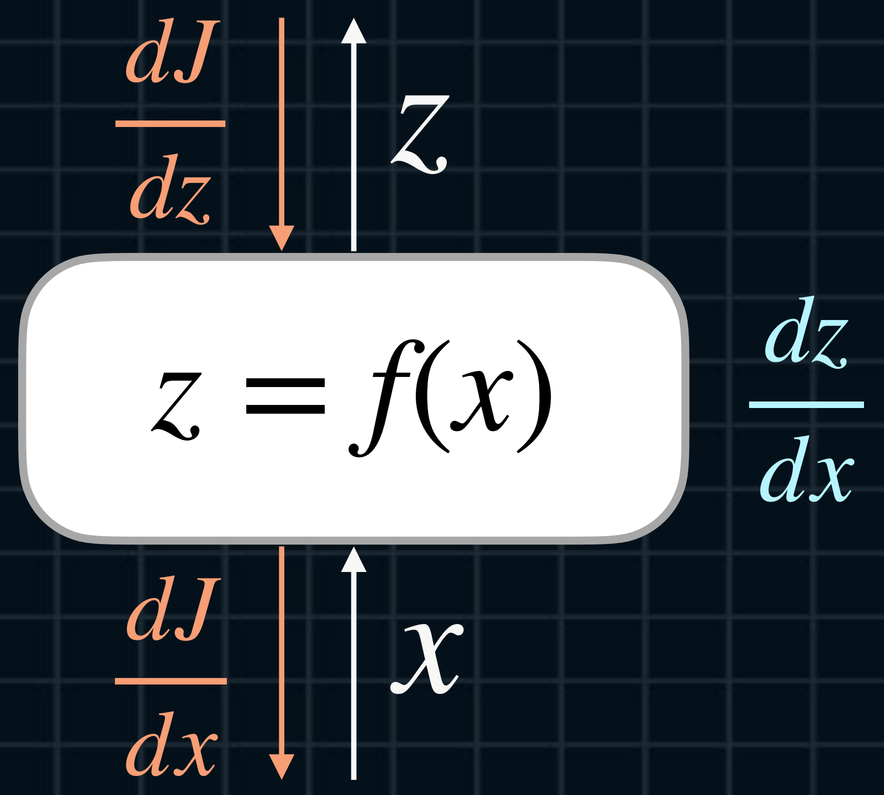
$$f(x) = cos(x^3)$$

$$f(t) = sin(t - \tau)$$

$$f(x) = (a - x)^2$$

$$f(x) = \frac{1}{1 + e^{-x}}$$

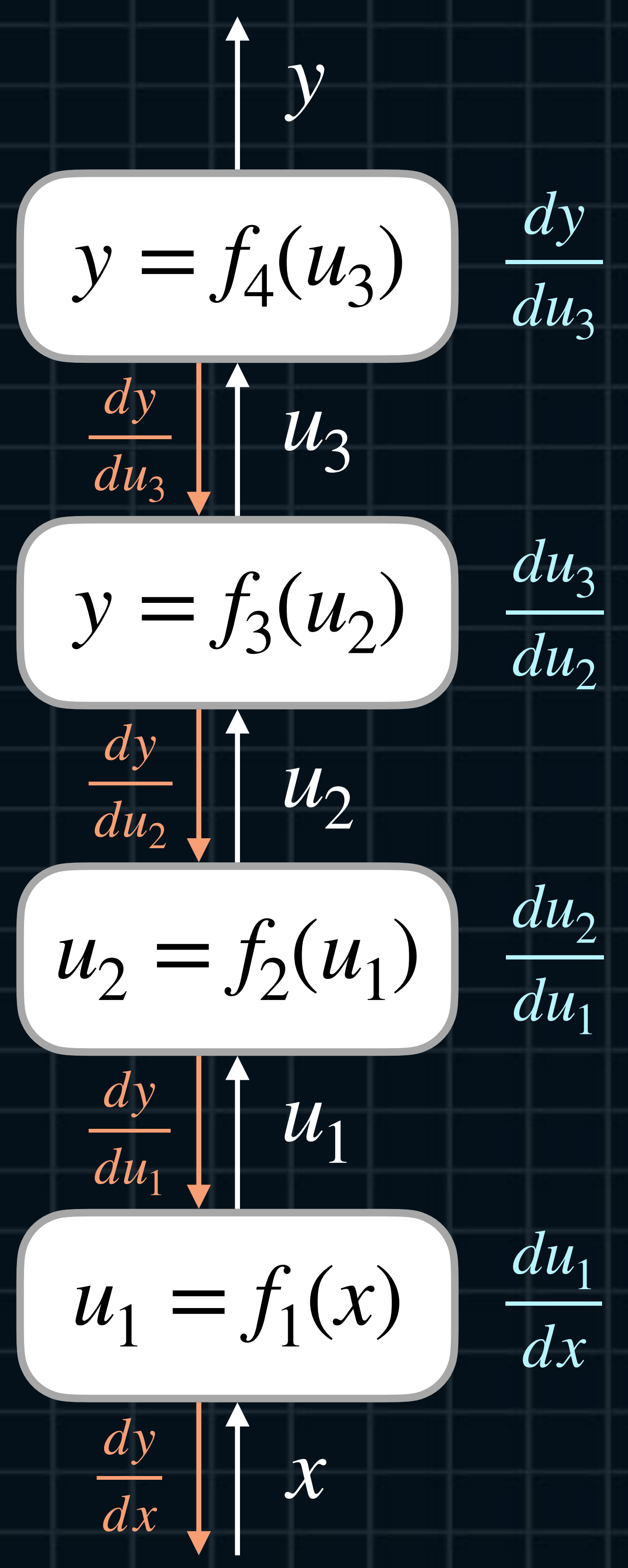
Modules of Backpropagation

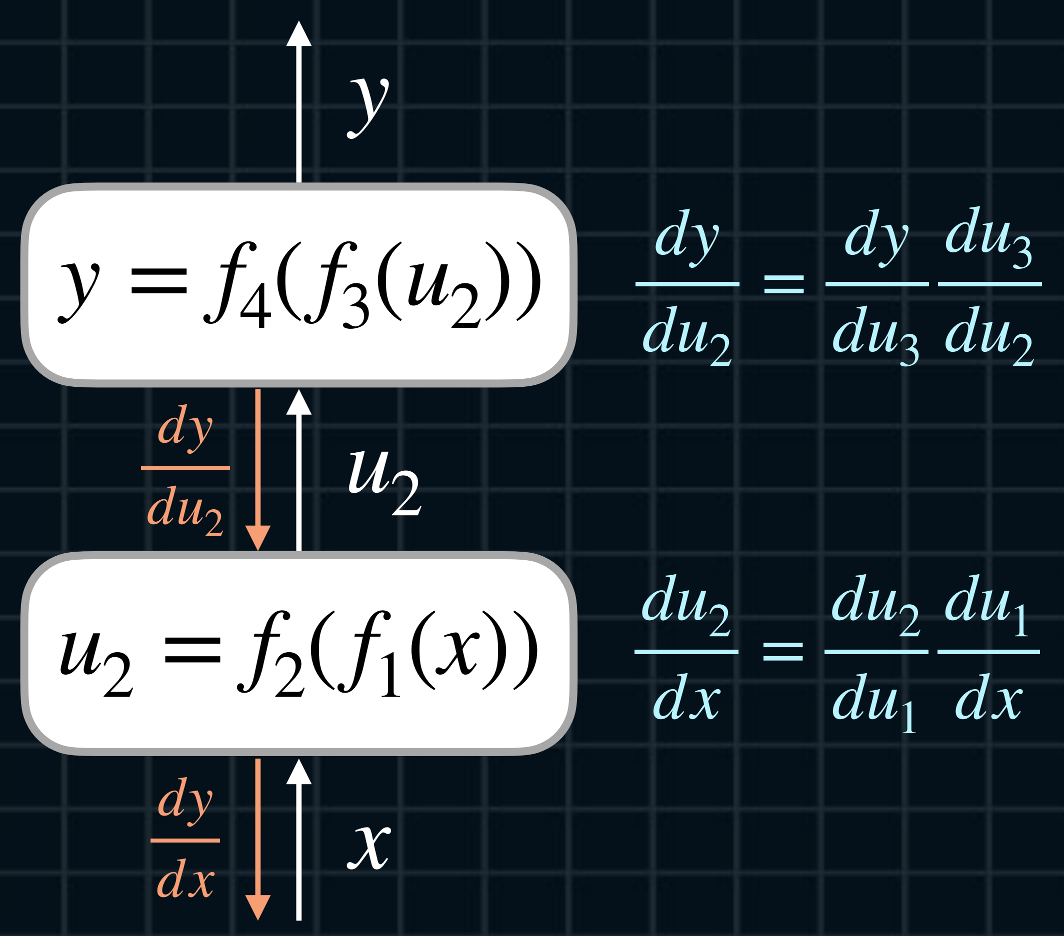


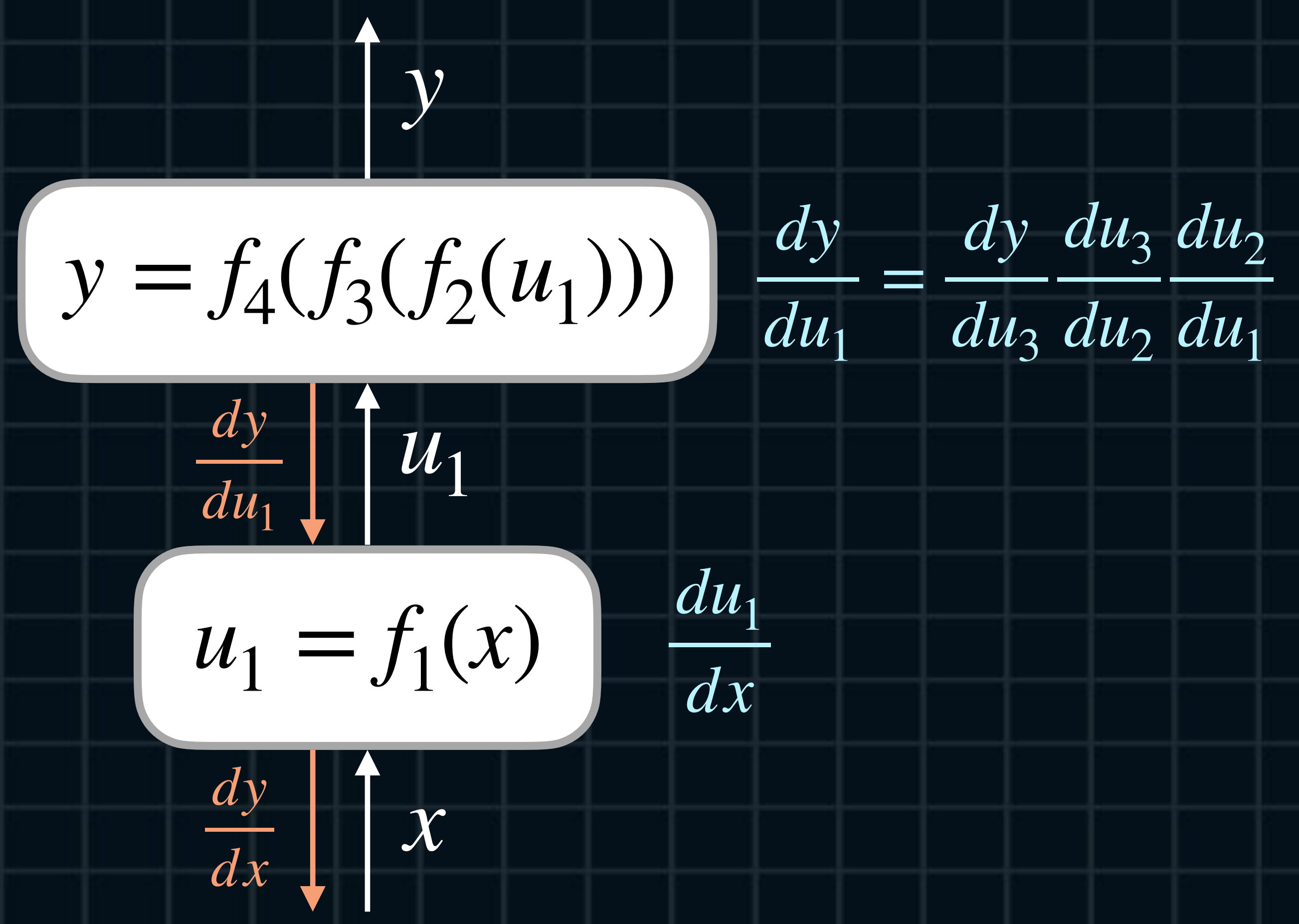
$$\frac{dJ}{dz}\frac{dz}{dx} = \frac{dJ}{dx}$$

Lecture.2 Basic Differentiation - Composite Functions and Chain Rule

Merging Modules







- Composite Functions and Chain Rule

Merging Modules

y

$$y = f_4(f_3(f_2(f_1(u_1))))$$

$\frac{dy}{dx}$ x

$$\frac{dy}{dx} = \frac{dy}{du_3}\frac{du_3}{du_2}\frac{du_2}{du_1}\frac{du_1}{dx}$$

Merging Modules

$$\frac{dy}{du_3} = -\frac{1}{(u_3)^2} = -\frac{1}{(1+e^{-x})^2}$$

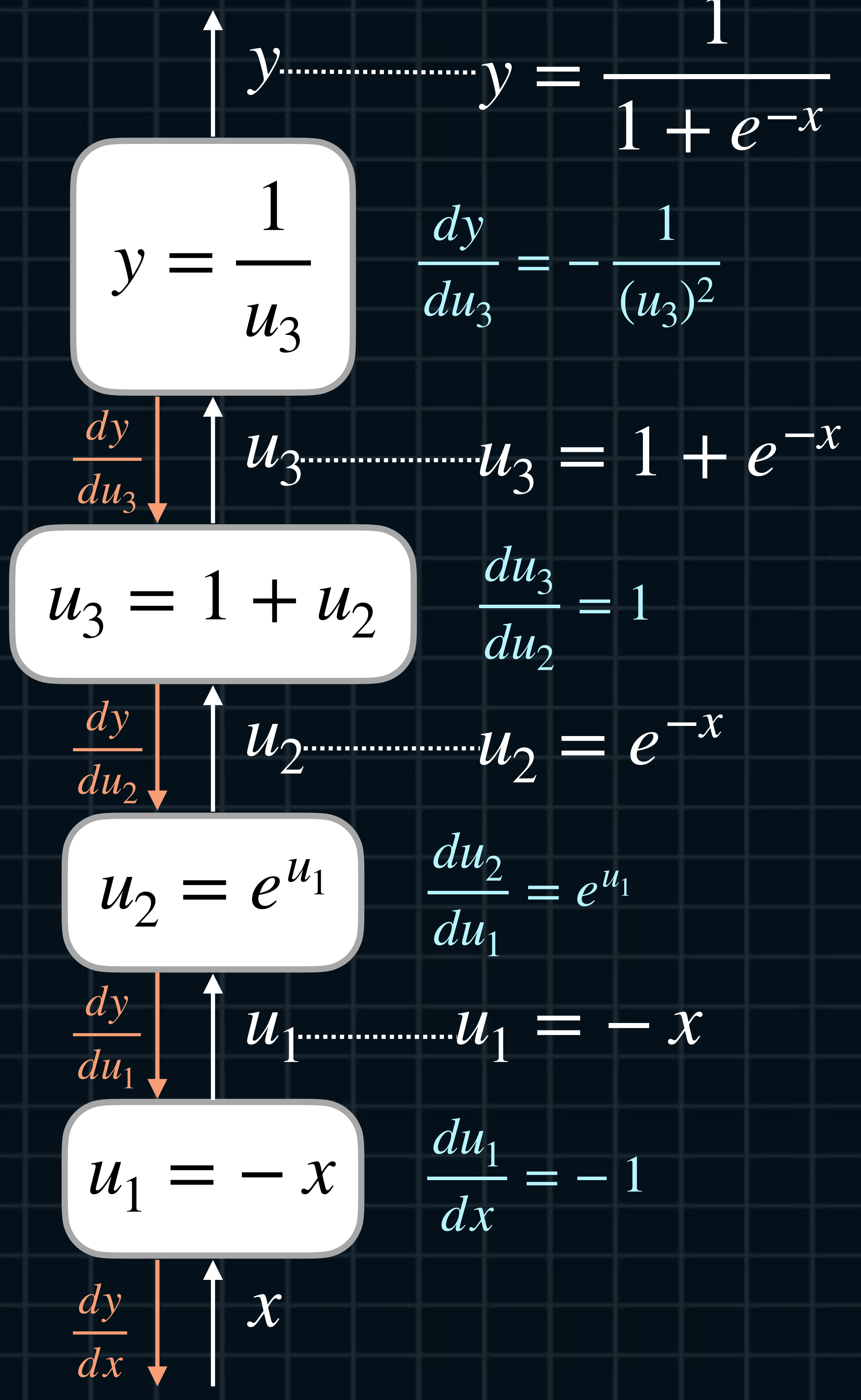
$$\frac{dy}{du_2} = \frac{dy}{du_3}\frac{du_3}{du_2} = -\frac{1}{(1+e^{-x})^2}$$

$$\frac{dy}{du_1} = \frac{dy}{du_2}\frac{du_2}{du_1} = -\frac{2}{(1+e^{-x})^2}\cdot e^{u_1}$$

$$= -\frac{e^{-x}}{(1+e^{-x})^2}$$

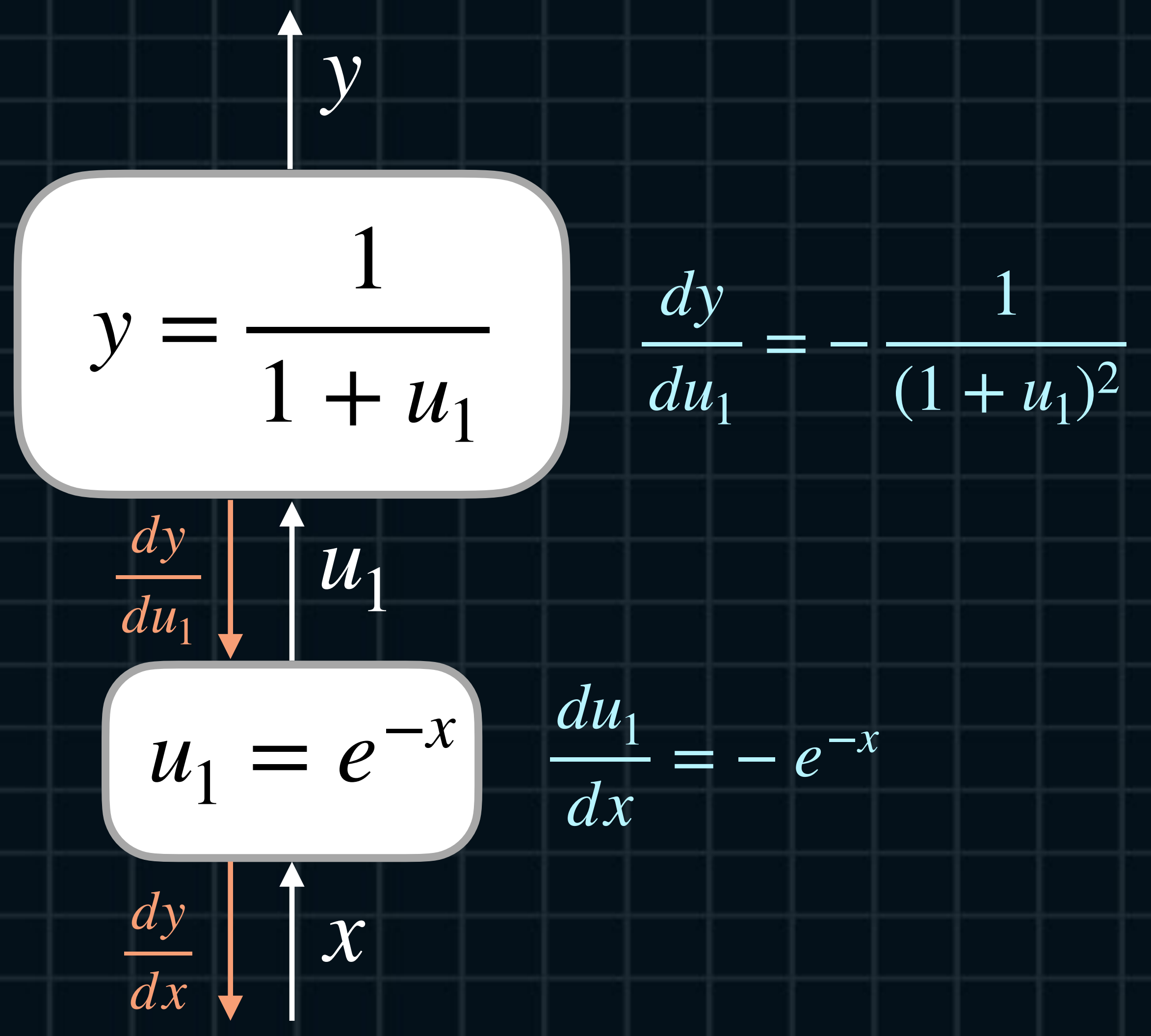
$$\frac{dy}{dx} = \frac{dy}{du_1}\frac{du_1}{dx} = \frac{e^{-x}}{(1+e^{-x})^2}$$

$$= y(1-y)$$



$$\frac{dy}{du_1} = -\frac{1}{(1+u_1)^2} = -\frac{1}{(1+e^{-x})^2}$$

$$\frac{dy}{du_1} = \frac{e^{-x}}{(1+e^{-x})^2} = y(1-y)$$



Merging Modules

$$\frac{dy}{du_1} = -\frac{1}{(1+u_1)^2} = -\frac{1}{(1+e^{-x})^2}$$

$$\frac{dy}{du_1} = \frac{e^{-x}}{(1+e^{-x})^2} = y(1-y)$$

$$y = \frac{1}{1+u_1} \qquad \frac{dy}{du_1} = -\frac{1}{(1+u_1)^2}$$

$$\frac{dy}{du_1} \qquad u_1$$

$$u_1 = e^{-x} \qquad \frac{du_1}{dx} = -e^{-x}$$

$$\frac{dy}{dx} \qquad x$$

$$y = \frac{1}{1+e^{-x}} \qquad \frac{dy}{dx} = -y(1-y)$$

$$\frac{dy}{dx} \qquad x$$

Backpropagation and Jacobian Matrices

Lecture.2
Basic Differentiation