수학으로부터 인류를 자유롭게 하라

Free Humankind from Mathematics

Basic Algebra

Chap.11 Systems of Equations

11.1 Systems of Equations

Multivariate Equations

$$E(x_1, x_2, \ldots, x_n) = 0 \longrightarrow n \text{ variables(unknowns)}$$

ex.1) $2x_1 + 5x_2 = 0$

ex.2) $(x_1)^2 - 5ln(x_2) - 4 = 0$

ex.3) $x_1 + 2x_2 + \ldots(n-1)x_{n-1} + nx_n = 0$

$$E(x_1, x_2, \ldots, x_n, y) = 0$$

ex.1) $3x - 2y = 0$

ex.2) $x^2 + y^2 = 4$

ex.3) $(x_1)^2 - 2x_2 + y = 0$

Systems of Equations

$$
\begin{aligned}
E_1(x_1, x_2, \ldots, x_n) &= 0 \\
E_2(x_1, x_2, \ldots, x_n) &= 0 \\
&\vdots \\
E_m(x_1, x_2, \ldots, x_n) &= 0
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
&m \text{ equations} \\
&n \text{ unknowns}
\end{aligned}
$$

Systems of Equations

Solutions

m개의 equation을 모두 만족시키는 $(x_1, x_2, \ldots, x_n)$

Solution Sets

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| \, E_i(x_1, x_2, \ldots, x_n) = 0,\ 1 \le i \le m \right\}$$

Solving Systems of Equations

Problem Setting

$$E_1(x_1, x_2, x_3) = 0 \quad x_1 - 2x_2 + x_3 = 0$$
$$E_2(x_1, x_2, x_3) = 0 \quad 2^{4x_1+x_2} - 4^{x_3} = 0$$
$$E_3(x_1, x_2, x_3) = 0 \quad x_1 + 2(x_2)^2 - (x_3)^2 = 0$$

11.1 Systems of Equations

Solving Systems of Equations

Step.1) $E_1(x_1, x_2, x_3) = 0$에서 하나의 변수(x_i)로 식을 정리($x_3 = E_1'(x_1, x_2)$로 변환)

$$E_1(x_1, x_2, x_3) = 0 \qquad x_1 - 2x_2 + x_3 = 0 \qquad\qquad x_3 = E_1'(x_1, x_2) \qquad x_3 = -x_1 + 2x_2$$
$$E_2(x_1, x_2, x_3) = 0 \qquad 2^{4x_1+x_2} - 4^{x_3} = 0 \qquad\qquad E_2(x_1, x_2, x_3) = 0 \qquad 2^{4x_1+x_2} - 4^{x_3} = 0$$
$$E_3(x_1, x_2, x_3) = 0 \qquad x_1 + 2(x_2)^2 - (x_3)^2 = 0 \qquad\qquad E_3(x_1, x_2, x_3) = 0 \qquad x_1 + 2(x_2)^2 - (x_3)^2 = 0$$

Step.2) 나머지 모든 식에 $x_3 = E_1'(x_1, x_2)$를 대입

$$x_3 = E_1'(x_1, x_2) \qquad x_3 = -x_1 + 2x_2 \qquad\qquad x_3 = E_1'(x_1, x_2) \qquad x_3 = -x_1 + 2x_2$$
$$E_2(x_1, x_2, x_3) = 0 \qquad 2^{4x_1+x_2} - 4^{x_3} = 0 \qquad\qquad E_2'(x_1, x_2) = 0 \qquad 2^{4x_1+x_2} - 4^{-x_1+2x_2} = 0 \longrightarrow 6x_1 - 3x_2 = 0$$
$$E_3(x_1, x_2, x_3) = 0 \qquad x_1 + 2(x_2)^2 - (x_3)^2 = 0 \qquad\qquad E_3'(x_1, x_2) = 0 \qquad x_1 + 2(x_2)^2 - (-x_1 + 2x_2)^2 = 0$$

Solving Systems of Equations

Step.3) $E_2'(x_1, x_2) = 0$에서 하나의 변수(x_j)로 식을 정리($x_2 = E_2''(x_1)$)

$$x_3 = E_1'(x_1, x_2) \qquad x_3 = -x_1 + 2x_2$$
$$E_2'(x_1, x_2) = 0 \qquad 6x_1 - 3x_2 = 0$$
$$E_3'(x_1, x_2) = 0 \qquad x_1 + 2(x_2)^2 - (-x_1 + 2x_2)^2 = 0$$

$$x_3 = E_1'(x_1, x_2) \qquad x_3 = -x_1 + 2x_2$$
$$x_2 = E_2''(x_1) \qquad x_2 = 2x_1$$
$$E_3'(x_1, x_2) = 0 \qquad x_1 + 2(x_2)^2 - (-x_1 + 2x_2)^2 = 0$$

Step.4) 마지막 식에 $x_2 = E_2''(x_1)$를 대입

$$x_3 = E_1'(x_1, x_2) \qquad x_3 = -x_1 + 2x_2$$
$$x_2 = E_2''(x_1) \qquad x_2 = 2x_1$$
$$E_3'(x_1, x_2) = 0 \qquad x_1 + 2(x_2)^2 - (-x_1 + 2x_2)^2 = 0$$

$$x_3 = E_1'(x_1, x_2) \qquad x_3 = -x_1 + 2x_2$$
$$x_2 = E_2''(x_1) \qquad x_2 = 2x_1$$
$$E_3'(x_1, x_2) = 0 \qquad x_1 + 2(2x_1)^2 - (-x_1 + 2 \cdot 2x_1)^2 = 0$$
$$\longrightarrow (x_1)^2 - x_1 = 0$$

Solving Systems of Equations

Step.5) $E_3''(x_1)$의 solution set

$$x_3 = E_1'(x_1, x_2) \qquad x_3 = -x_1 + 2x_2$$
$$x_2 = E_2''(x_1) \qquad x_2 = 2x_1$$
$$E_3'(x_1, x_2) = 0 \qquad (x_1)^2 - x_1 = 0 \longrightarrow S_{x_1} = \{0, 1\}$$

Step.6) 이전 관계식에 대입하여 나머지 값 구하기

$$\begin{matrix} & & x_2 = 2x_1 & & x_3 = -x_1 + 2x_2 & \\ x_1 = 0 & \longrightarrow & x_2 = 0 & \longrightarrow & x_3 = 0 & \longrightarrow \quad S = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \\ x_1 = 1 & & x_2 = 2 & & x_3 = 3 & \end{matrix}$$

Linear Equations

$$E_1(x_1, x_2, \ldots, x_n) = b \quad \longrightarrow \quad a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

(linear function) = (constant)

$n + 1$ constants, n variables

Systems of Linear Equations

General SLEs

$$
\begin{aligned}
E_1(x_1, x_2, \ldots, x_n) = b_1 \quad &\longrightarrow \quad a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n = b_1 \\
E_2(x_1, x_2, \ldots, x_n) = b_2 \quad &\longrightarrow \quad a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n = b_2 \\
&\vdots \\
E_m(x_1, x_2, \ldots, x_n) = b_m \quad &\longrightarrow \quad a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n = b_m
\end{aligned}
$$

m equations, n variables

Linear Equations

Hyperplanes

2 variables: $a_1x_1 + a_2x_2 = b$

2차원 공간에 그려짐 / 직선(1차원)

2차원 공간을 양분

3 variables: $a_1x_1 + a_2x_2 + a_3x_3 = b$

3차원 공간에 그려짐 / 평면(2차원)

3차원 공간을 양분

4 variables: $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b$

4차원 공간에 그려짐 / **공간**(3차원)

4차원 공간을 양분

$\Longrightarrow$

n variables: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$

n차원 공간에 그려짐 / **평면같은** 것($n-1$차원)

n차원 공간을 양분

$\longrightarrow$ n차원 공간의 hyperplane

Systems of Linear Equations

사과 2개, 바나나 1개 = 1,000원

사과 3개, 바나나 2개 = 1,700원

↓ 사과값(x), 바나나값(y)

$$2x + y = 1000$$
$$3x + 2y = 1700$$

Solutions

Algebraic: 두 방정식을 만족시키는 (x, y)

Geometric: 두 직선들의 교점의 좌표

Solving SLEs

(사과 2개) + (바나나 1개) = 1,000원 (사과 3개) + (바나나 2개) = 1,700원

(Step.1) x 또는 y의 계수를 맞춘다.

(사과 4개) + (바나나 2개) = 2,000원 (사과 3개) + (바나나 2개) = 1,700원

(Step.2) 식을 전체적으로 뺀다.

(사과 4개) + (바나나 2개) = 2,000원
(사과 3개) + (바나나 2개) = 1,700원

(사과 1개) = 300원

(Step.3) 나머지 변수의 값을 구한다.

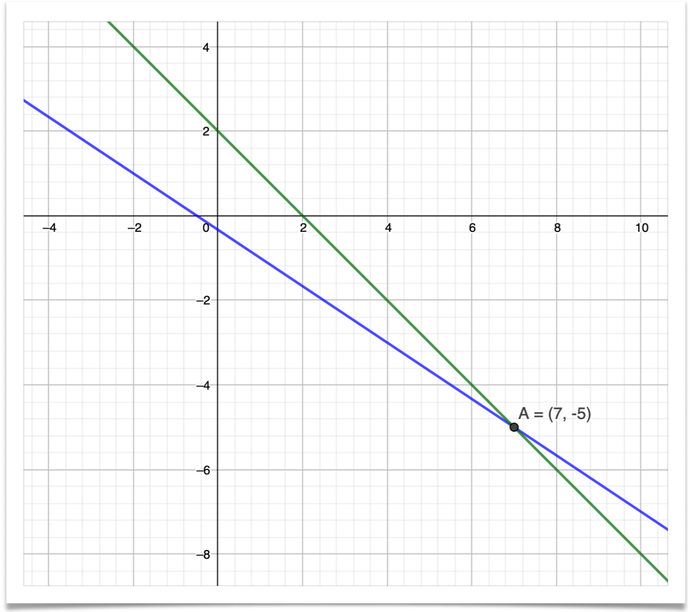
600원 + (바나나 1개) = 1,000원

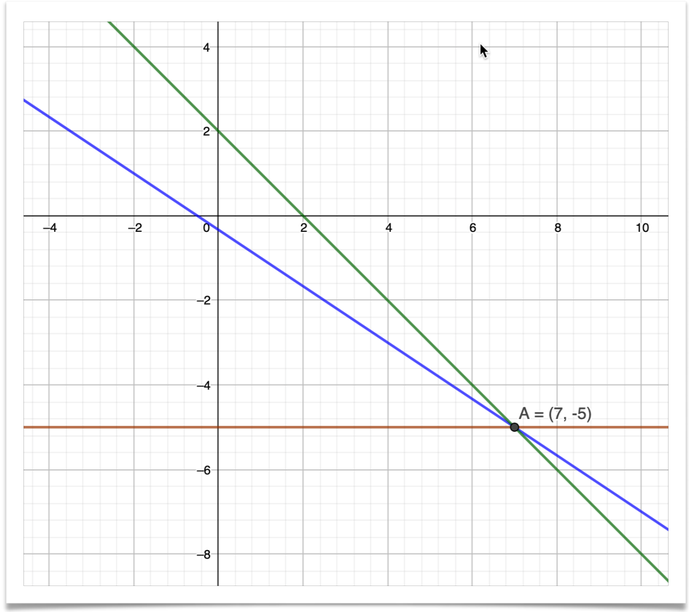
(바나나 1개) = 400원

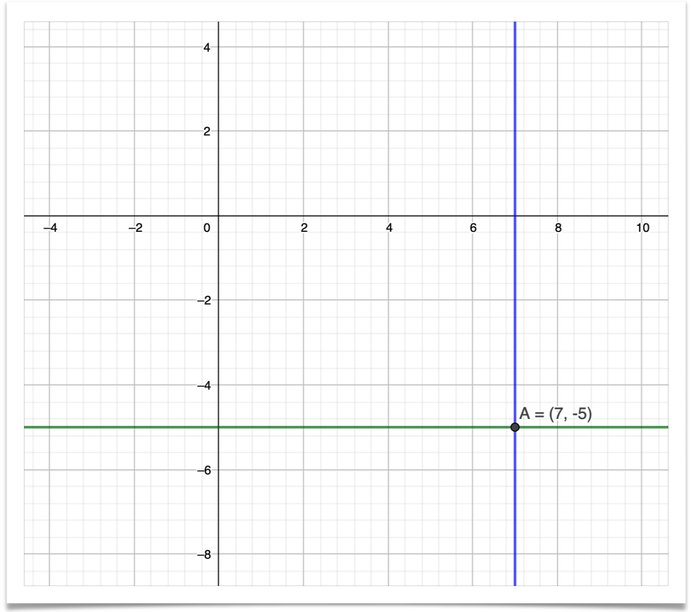
Solving SLEs

Examples

ex.1) $\begin{aligned} 2x+3y+1&=0 \\ x+y-2&=0 \end{aligned}$ $\longrightarrow$ $\begin{aligned} 2x+3y+1&=0 \\ 2x+2y-4&=0 \\ \hline y+5&=0 \end{aligned}$ $\longrightarrow$ $\begin{aligned} x&=7 \\ y&=-5 \end{aligned}$



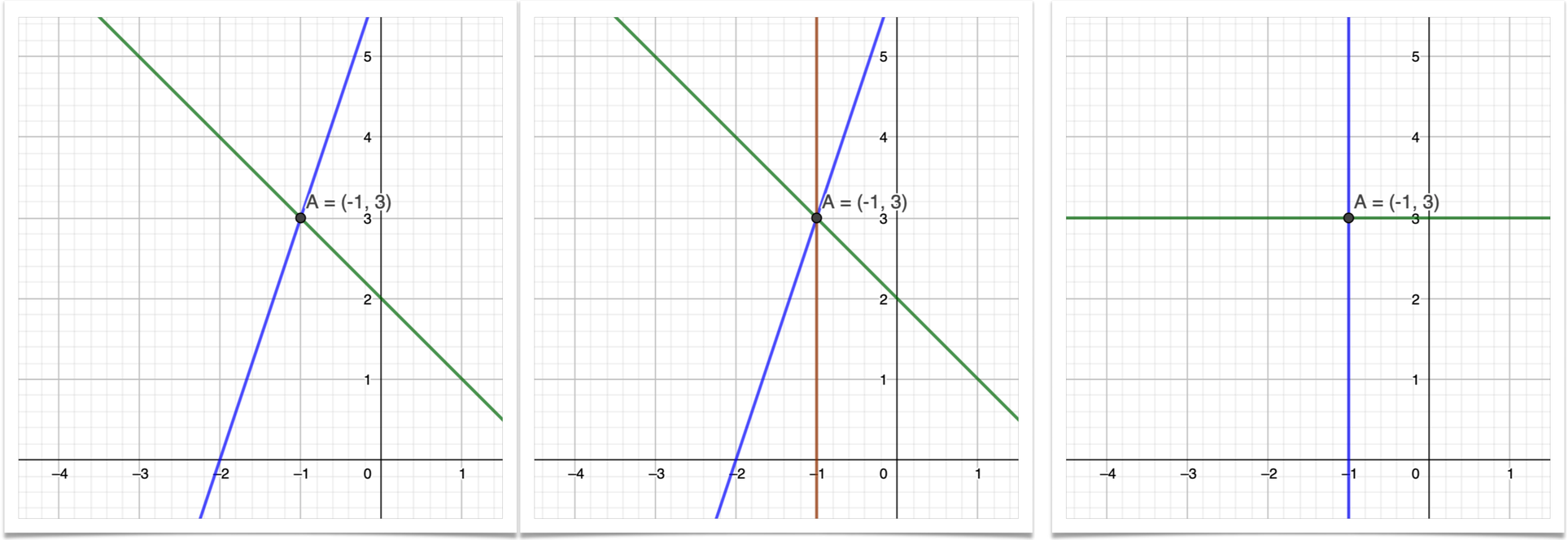




Solving SLEs

Examples

ex.2) $\begin{aligned} 3x - y + 6 &= 0 \\ x + y - 2 &= 0 \end{aligned}$ $\longrightarrow$ $\begin{aligned} 3x - y + 6 &= 0 \\ -x - y + 2 &= 0 \\ \hline 4x \quad &+ 4 = 0 \end{aligned}$ $\longrightarrow$ $\begin{aligned} x &= -1 \\ y &= 3 \end{aligned}$



Solving SLEs

Examples

ex.3)
$$\begin{array}{r} 3x - y + 2 = 0 \\ 3x - y + 3 = 0 \\ \hline 1 = 0 \end{array} \longrightarrow S = \varnothing$$

ex.4)
$$\begin{array}{r} 3x - y + 2 = 0 \\ 6x - 2y + 4 = 0 \\ \hline 0 = 0 \end{array} \longrightarrow S = \{(x, y) \mid 3x - y + 2 = 0\}$$

Solving SLEs

(Step.1) x 또는 y의 계수를 맞춘다.

(Step.2) 식을 전체적으로 뺀다.

(Step.3) 나머지 변수의 값을 구한다.

$$\begin{matrix} ax+by=c \\ a'x+b'y=c' \end{matrix} \longrightarrow \begin{matrix} x=\alpha \\ \text{or} \\ y=\beta \end{matrix}$$

Systems of Inequalities

$$
\begin{aligned}
E_1(x_1, x_2, \ldots, x_n) &\begin{matrix} > & < \\ \geq & \leq \end{matrix} 0 \\
E_2(x_1, x_2, \ldots, x_n) &\begin{matrix} > & < \\ \geq & \leq \end{matrix} 0 \\
&\vdots \\
E_m(x_1, x_2, \ldots, x_n) &\begin{matrix} > & < \\ \geq & \leq \end{matrix} 0
\end{aligned}
$$

Systems of Inequalities

Linearized Systems of Inequalities

$$E_1(x_1, x_2, \ldots, x_n) < E_2(x_1, x_2, \ldots, x_n) < E_3(x_1, x_2, \ldots, x_n)$$

inequality 1

$$E_1(x_1, x_2, \ldots, x_n) < E_2(x_1, x_2, \ldots, x_n)$$

inequality 2

$$E_2(x_1, x_2, \ldots, x_n) < E_3(x_1, x_2, \ldots, x_n)$$

Systems of Linear Inequalities

$$E_1(x_1, x_2, \ldots, x_n) \begin{matrix} > \\ < \\ \geq \\ \leq \end{matrix} b \quad \longrightarrow \quad a_1x_1 + a_2x_2 + \ldots + a_nx_n \begin{matrix} > \\ < \\ \geq \\ \leq \end{matrix} b$$

First Order Linear Inequalities

$$y \begin{matrix} > \\ \geq \end{matrix} ax + b \quad \longrightarrow \text{직선을 기준으로 } \textbf{위}$$

$$y \begin{matrix} < \\ \leq \end{matrix} ax + b \quad \longrightarrow \text{직선을 기준으로 } \textbf{아래}$$

그래프에서 $>$, $<$ 는 점선으로, $\geq$, $\leq$ 는 실선으로 표현

ex.1) $y \geq 2x + 1$ ex.2) $y > 2x + 1$

Systems of Linear Inequalities

Examples

ex.1) $\begin{cases} y \geq x \\ y \geq -x \end{cases}$

ex.2) (1) $\begin{cases} y \geq x+1 \\ y \geq -2x+4 \end{cases}$ (2) $\begin{cases} y \geq x+1 \\ y \leq -2x+4 \end{cases}$

(3) $\begin{cases} y \leq x+1 \\ y \geq -2x+4 \end{cases}$ (4) $\begin{cases} y \leq x+1 \\ y \leq -2x+4 \end{cases}$

Systems of Linear Inequalities

Maximum, Minimum Values

ex) $\begin{cases} 2x+y-5 \le 0 \\ x+2y-4 \le 0 \\ x \ge 0 \\ y \ge 0 \end{cases}$

에 대해 $x+y$의 최댓값, 최솟값을 구하세요.

Systems of Inequalities

Examples

ex.1) $2^{x^2} < 4^{x+12}$

$\longrightarrow 2^{x^2} < 2^{2x+24} \longrightarrow x^2 < 2x + 24$

$\longrightarrow x^2 - 2x - 24 < 0 \longrightarrow (x-6)(x+4) < 0 \longrightarrow -4 < x < 6$

ex.2) $\left(\frac{1}{2}\right)^{1-x} < 2^{x^2-1} \leq \left(\frac{1}{2}\right)^{x-1}$

$\longrightarrow 2^{x-1} < 2^{x^2-1} \leq 2^{1-x} \longrightarrow x - 1 < x^2 - 1 \leq 1 - x$

(1) $x - 1 < x^2 - 1 \longrightarrow x^2 - x > 0 \longrightarrow x(x-1) > 0 \longrightarrow x < 0$ or $x > 1$

(2) $x^2 - 1 \leq 1 - x \longrightarrow x^2 + x - 2 \leq 0 \longrightarrow (x+2)(x-1) \leq 0 \longrightarrow -2 \leq x \leq 1$

$\longrightarrow S = [-2, 0)$

Systems of Inequalities

Examples

ex.3) $log_2(x+3)+log_2(x-1)<0$

$$\longrightarrow log_2(x+3)(x-1)<log_2 1 \longrightarrow x^2+2x-3<1 \longrightarrow x^2+2x-4<0$$

$$\longrightarrow -1-\sqrt{5}<x<-1+\sqrt{5}$$

ex.4) $x^{log_3 x} \le \frac{3^{12}}{x}$

$$\longrightarrow log_3\left(x^{log_3 x}\right) \le log_3\left(\frac{3^{12}}{x}\right) \longrightarrow \left(log_3 x\right)^2 \le 12-log_3 x$$

$$\longrightarrow \left(log_3 x\right)^2+log_3 x-12 \le 0 \longrightarrow \left(log_3 x+4\right)\left(log_3 x-3\right) \le 0$$

$$\longrightarrow -4 \le log_3 x \le 3 \longrightarrow \frac{1}{81} \le x \le 27$$

CLOSING

Basic Algebra

Chap.11 Systems of Equations