수학으로부터 **인류**를 **자유**롭게 하라

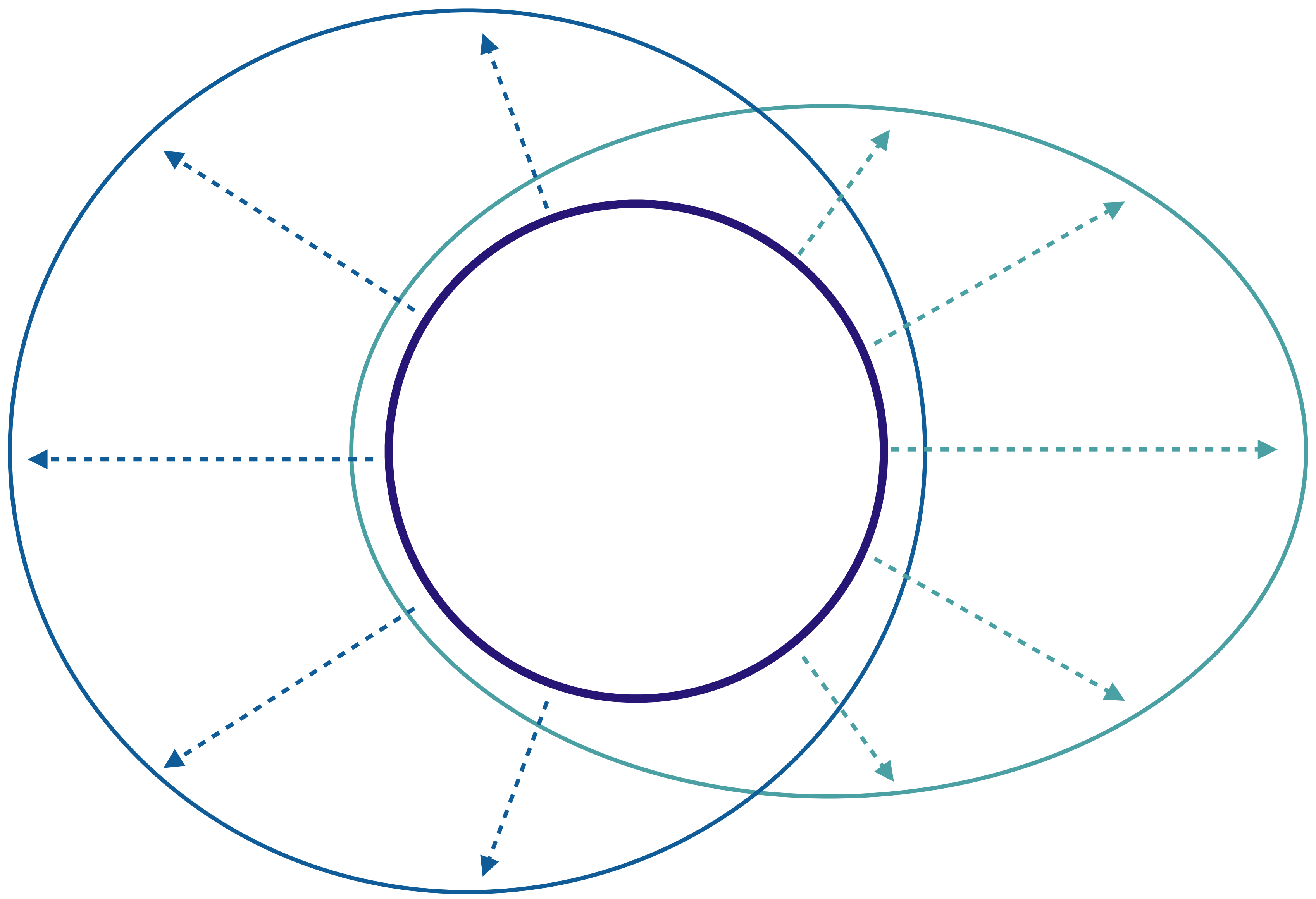
Free Humankind from Mathematics

Basic Algebra

Chap.3 Mathematical Logic

Expansion of Knowledge

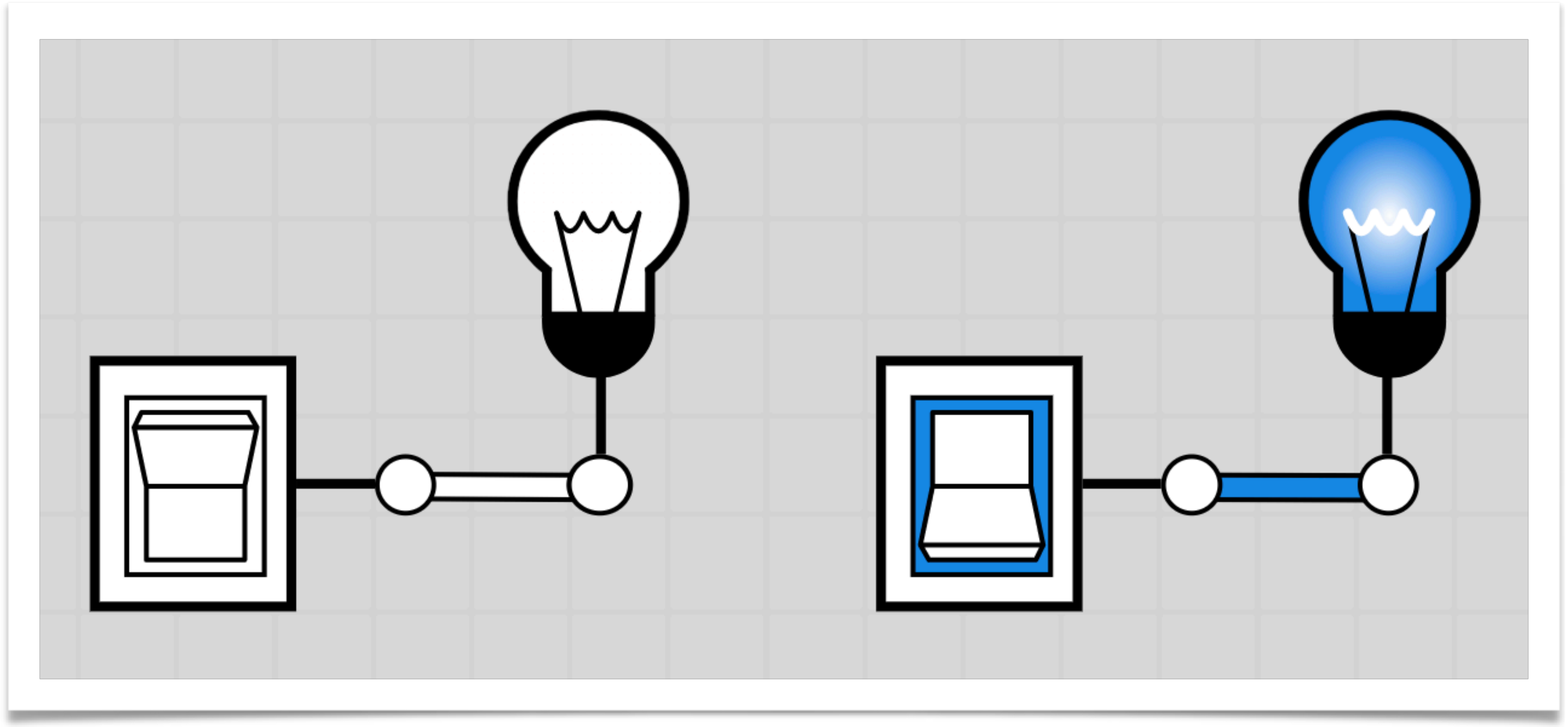
참으로 알고있는 것으로부터 **논리적인 과정을 통해** 새로운 참을 이끌어내는 과정



Boolean Values

True: 1(참)
False: 0(거짓)

Y: 0 Y: 1



Logicly

Propositions

True, False로 판단할 수 있는 문장

ex.1) p_1 :모든 사람은 죽는다. $\longrightarrow$ True(1)

ex.2) p_2 :일주일은 10일로 이루어져있다. $\longrightarrow$ False(0)

ex.3) p_3 :4는 2의 배수이다. $\longrightarrow$ True(1)

ex.4) p_4 :6은 prime number이다. $\longrightarrow$ False(0)

참일 수도, 거짓일 수도 있는 경우 확률의 개념이 도입된다.

Axioms

참으로 증명없이 받아들이는 proposition

Vector Space Definition

The set $\mathcal{V}$ is called a ***vector space over*** $\mathcal{F}$ when the vector addition and scalar multiplication operations satisfy the following properties.

(A1) $\mathbf{x}+\mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$. This is called the ***closure property for vector addition.***

(A2) $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$.

(A3) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A4) There is an element $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x}+\mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

(A5) For each $\mathbf{x} \in \mathcal{V}$, there is an element $(-\mathbf{x}) \in \mathcal{V}$ such that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$.

(M1) $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathcal{F}$ and $\mathbf{x} \in \mathcal{V}$. This is the ***closure property for scalar multiplication.***

(M2) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for all $\alpha, \beta \in \mathcal{F}$ and every $\mathbf{x} \in \mathcal{V}$.

(M3) $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$ for every $\alpha \in \mathcal{F}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(M4) $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$ for all $\alpha, \beta \in \mathcal{F}$ and every $\mathbf{x} \in \mathcal{V}$.

(M5) $1\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

[1]

Axiomatic Definition of Vector Space

$$0 \leq P(A_i)$$

$$P(S) = 1$$

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

if $A_i, A_j,\ i \neq j$가 서로 mutually exclusive event일때

Axiomatic Definition of Probability

Mathematical Proof

$$\boxed{\text{Axioms}} \xrightarrow{\text{logical process}} \boxed{\text{True Statements}}$$

or

$$\boxed{\text{Axioms}} \longrightarrow \boxed{\text{True Statements}} \longrightarrow \boxed{\text{True Statements}}$$

Conditions

변수에 따라 True, False가 달라지는 식

항상 True이거나 False이면 각각 true proposition, false proposition으로 생각한다.

*많은 경우에 위의 모든 경우를 condition으로 취급하기도 한다.

ex.1) $x^2 = 4 \longrightarrow$ True, if $x = 2$ or $x = -2$

$\longrightarrow$ False, if otherwise

ex.2) $x^2 > 4 \longrightarrow$ True, if $x \geq 2$ or $x \leq -2$

$\longrightarrow$ False, if $-2 \leq x \leq 2$

Truth Sets

어떤 condition을 만족하는 원소들의 집합

ex.1) $U = \{x \mid 1 \leq (x \in \mathbb{N}) \leq 20\}$

$p : x$는 12의 약수 $\longrightarrow A = \{x \mid p\} = \{1, 2, 3, 4, 6, 12\}$

$q : x$는 4의 배수 $\longrightarrow B = \{x \mid q\} = \{4, 8, 12, 16, 20\}$

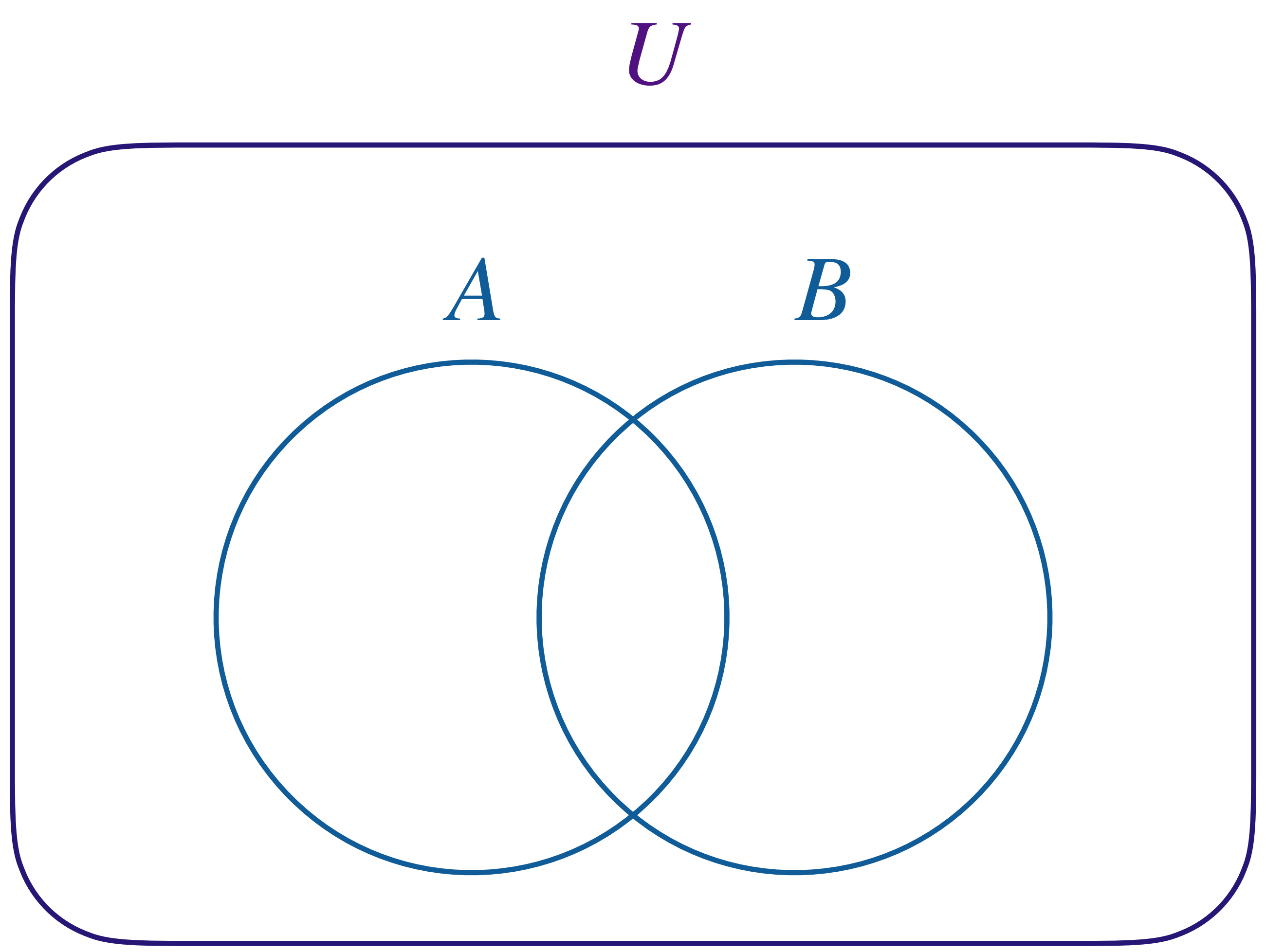
ex.2) $U = \mathbb{R}$

$p : (x+1)(x-2) = 0 \longrightarrow A = \{x \mid p\} = \{-1, 2\}$

condition이 정해지면 truth set이 생김

Truth Sets

condition을 통해 집합이 만들어진다.



집합 A는 조건 p의 truth set / 집합 B는 조건 q의 truth set

Unary/Binary Operations

Logical Operations

condition(s) $\longrightarrow$ new condition

Unary/Binary Operations

Unary Operations

$$☺ A = Y$$

- Logical Negation(¬)

Binary Operations

$$A ☺ B = Y$$

- Logical Conjunction($\land$)
- Logical Disjunction($\lor$)
- Logical Exclusive Disjunction($\oplus$)
- Logical Implication($\longrightarrow$)
- NAND, NOR, XNOR

Truth Tables

input condition(s)이 true/false인지에 따라 output condition의 true/false를 정리한 표

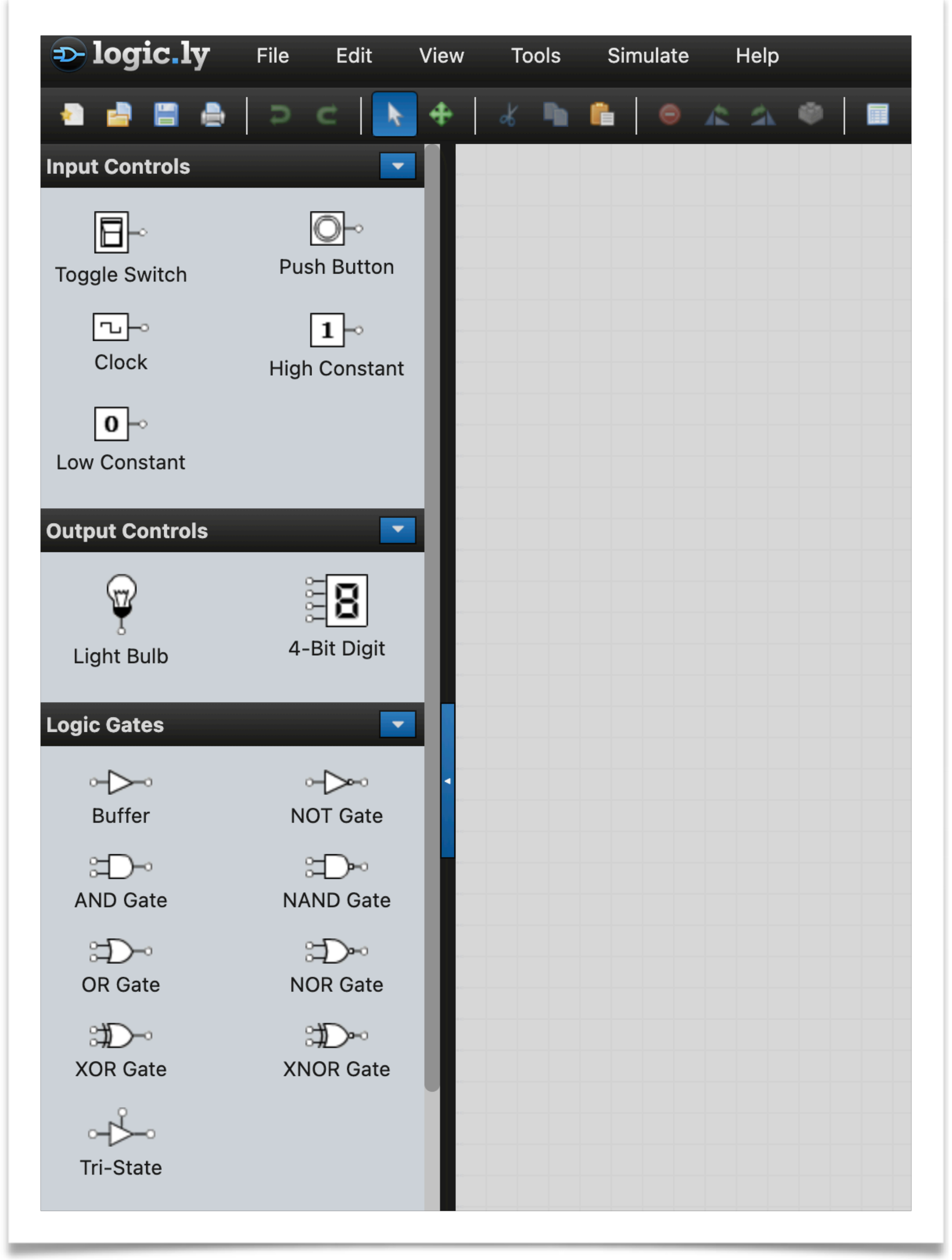
truth table for unary operation

A	Y
0	0/1
1	0/1

truth table for binary operation

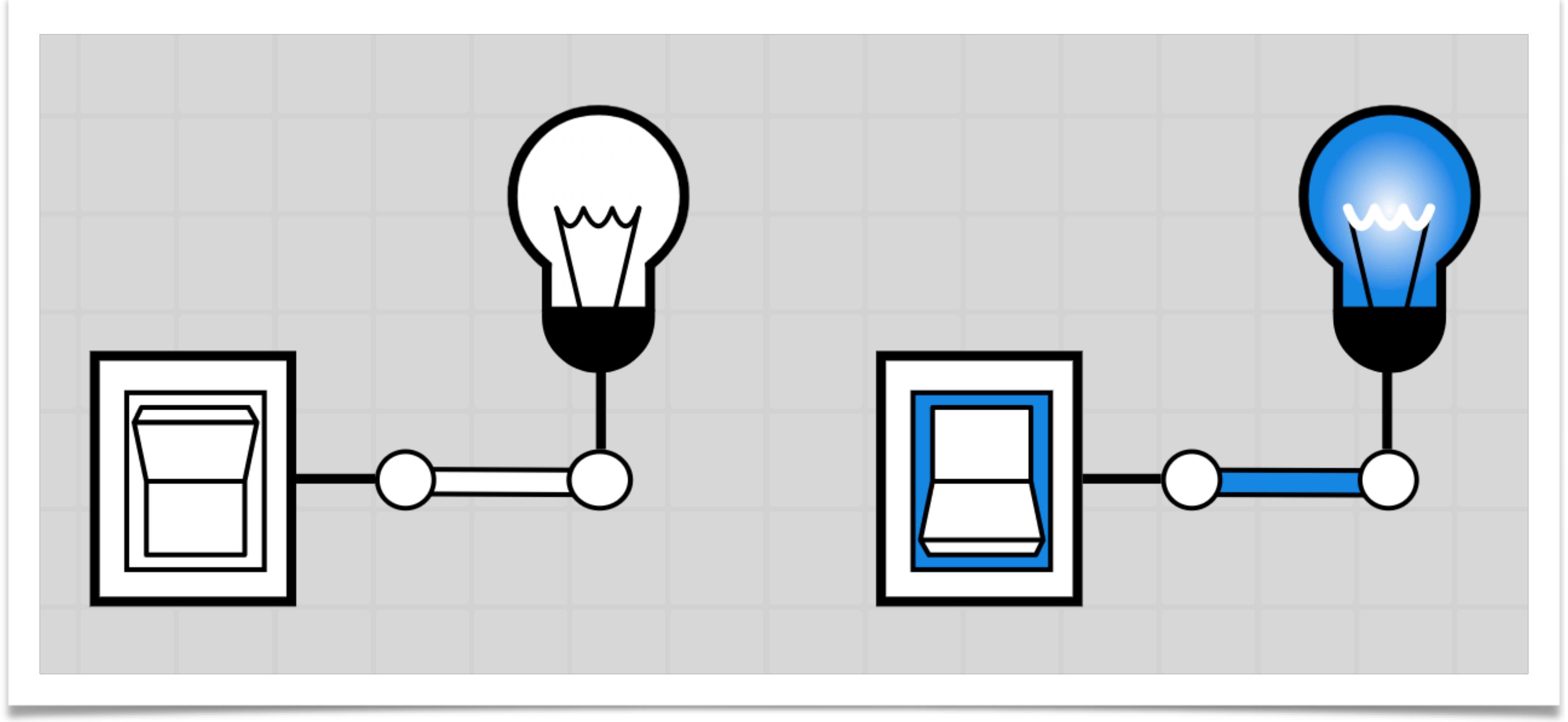
A	B	Y
0	0	0/1
0	1	0/1
1	0	0/1
1	1	0/1

Truth Tables



logic.ly

Y: 0 Y: 1



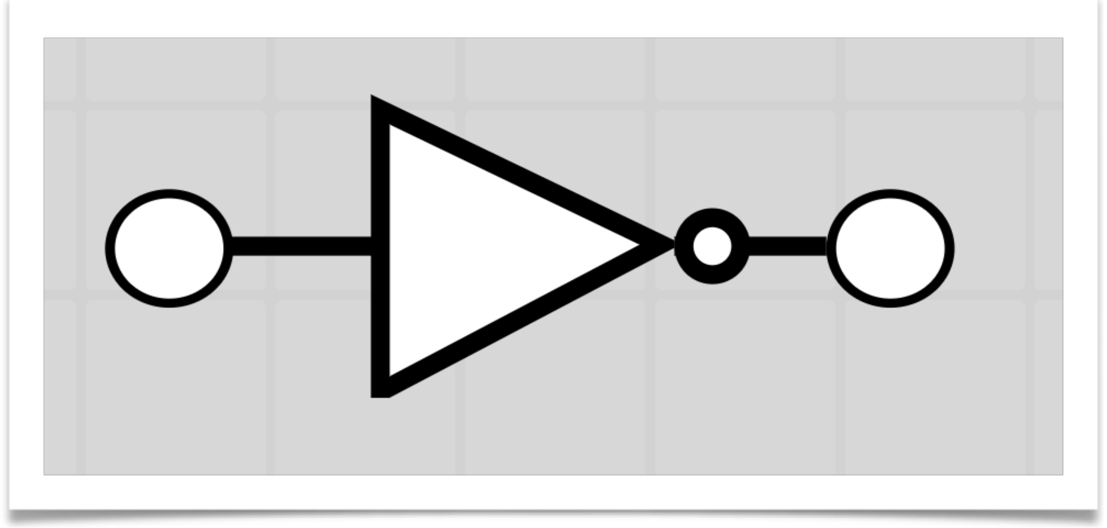
X: 0 X: 1

Logical Negation(¬)

input condition의 참과 거짓을 바꾸는 연산

$$\neg A = Y$$

A	Y
0	1
1	0



ex.1) ¬(a는 prime number이다.) = (a는 prime number가 아니다.)

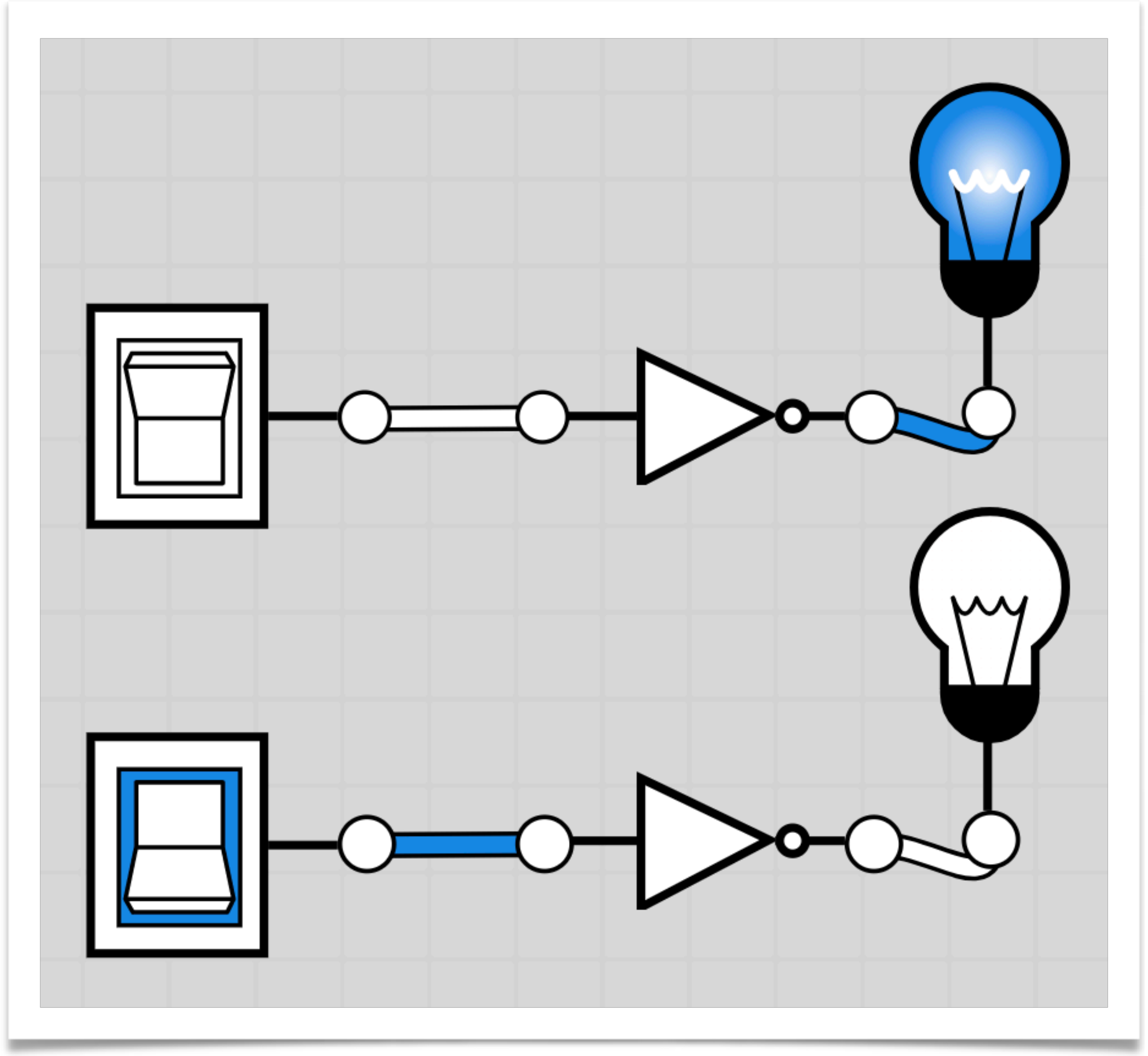
ex.2) ¬(a와 b는 서로 independent하다.) = (a와 b는 서로 dependent하다.)

Logical Negation(¬)

input condition의 참과 거짓을 바꾸는 연산

$$\neg A = Y$$

A	Y
0	1
1	0

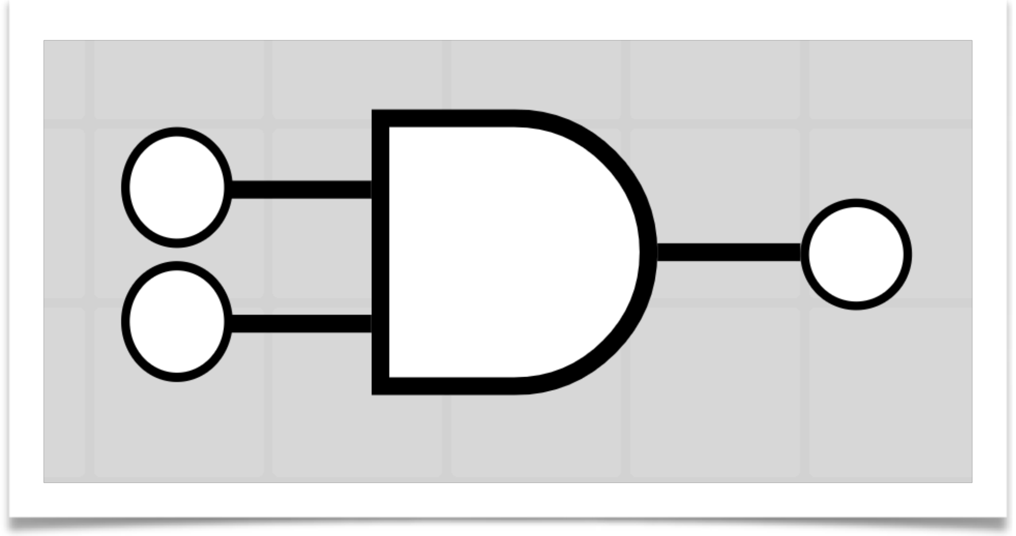


Logical Conjunction(∧)

두 input condition A, B가 모두 참일 때만 output condition이 참인 연산

$$A \wedge B = Y$$

A	B	Y
0	0	0
0	1	0
1	0	0
1	1	1



ex.1) (a는 3의 배수이다.) ∧ (a는 4의 배수이다.) = (a는 3의 배수이고, 4의 배수이다.)

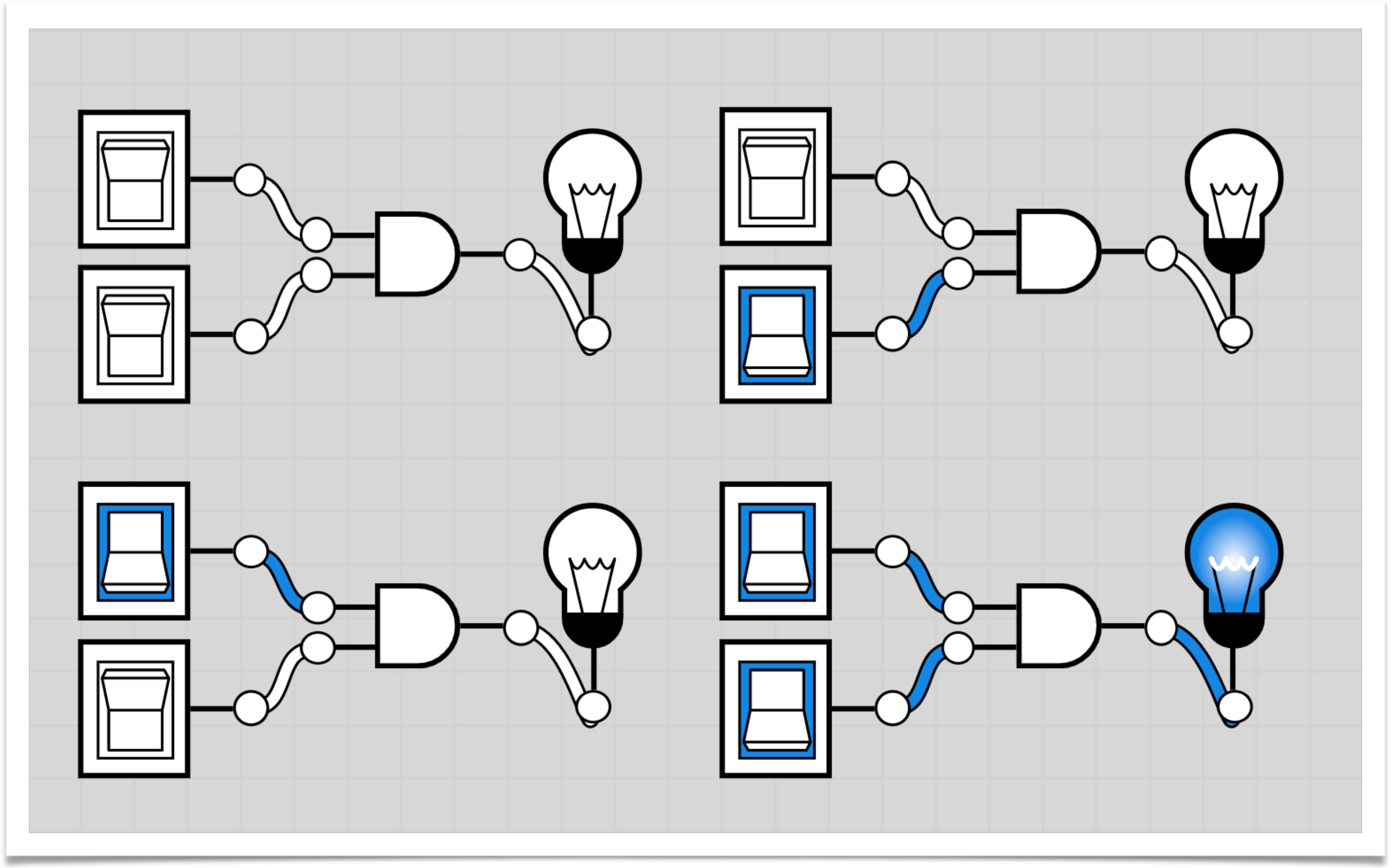
ex.2) (입력된 이미지에 강아지가 있다.) ∧ (입력된 이미지에 고양이가 있다.)
= (입력된 이미지에 강아지와 고양이가 모두 있다.)

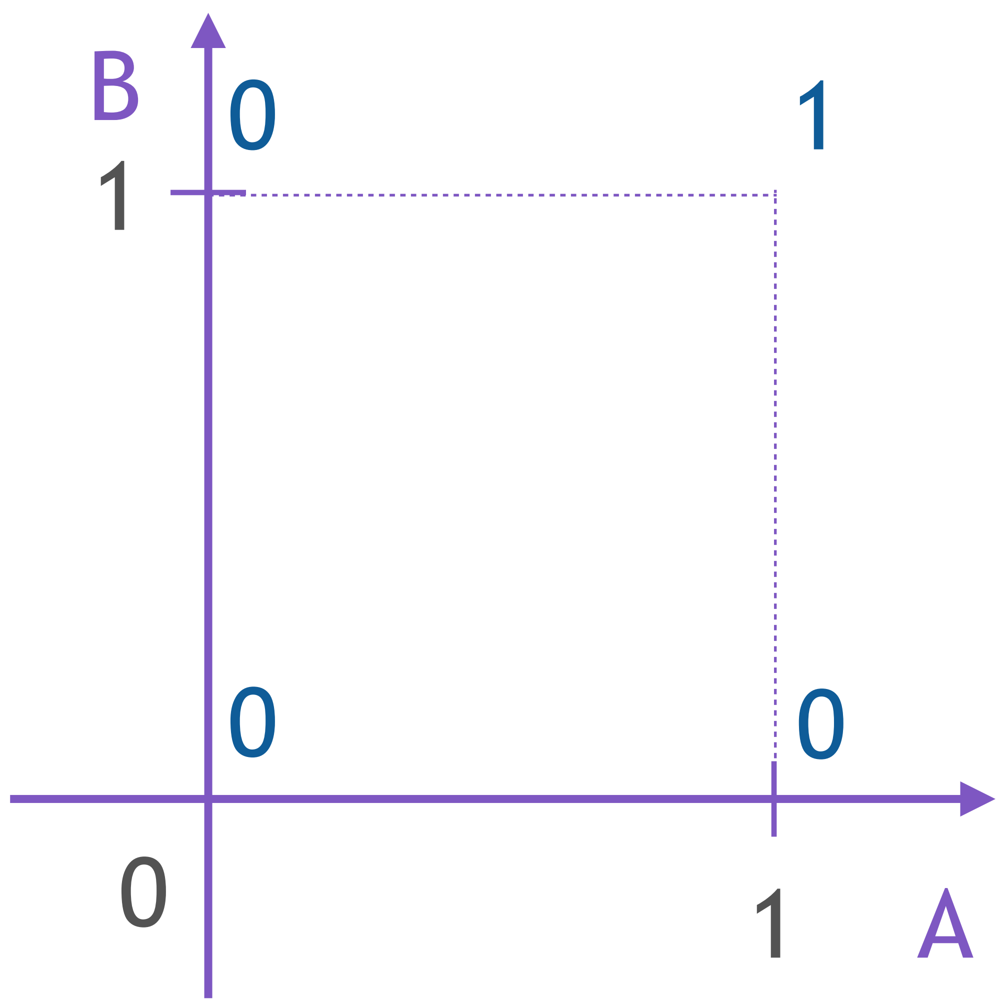
Logical Conjunction(∧)

두 input condition A, B가 모두 참일 때만 output condition이 참인 연산

$$A \wedge B = Y$$

A	B	Y
0	0	0
0	1	0
1	0	0
1	1	1



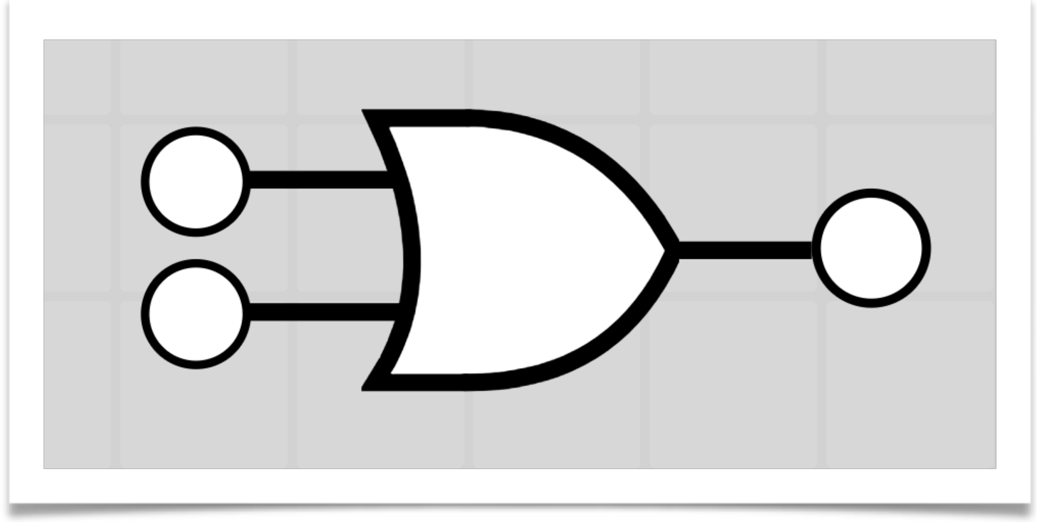


Logical Disjunction(∨)

두 input condition A, B 중 하나라도 참이면 output condition이 참인 연산

$$A \vee B = Y$$

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1



ex.1) (a는 3의 배수이다.) ∨ (a는 4의 배수이다.) = (a는 3의 배수이거나, 4의 배수이다.)

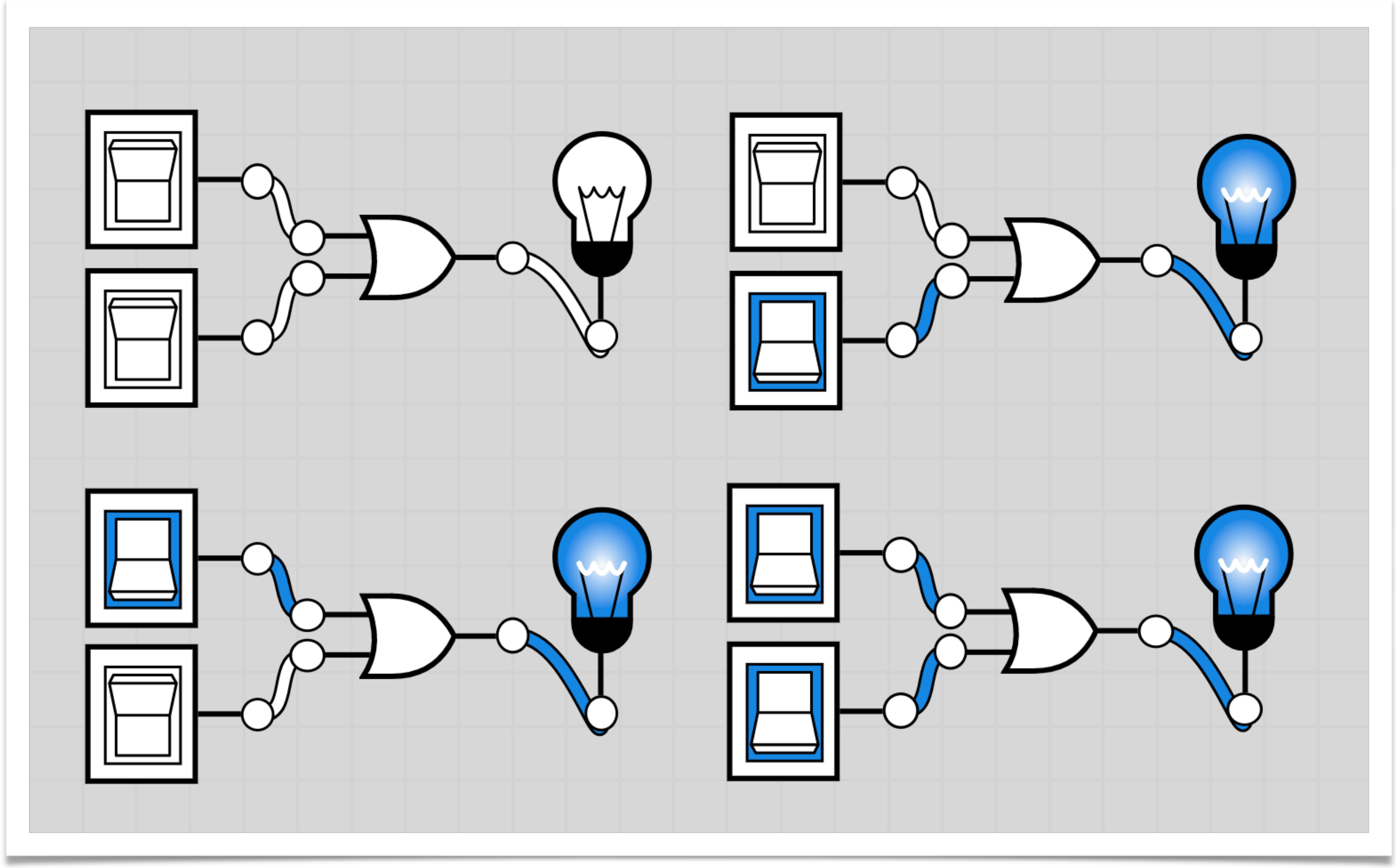
ex.2) (입력된 이미지에 강아지가 있다.) ∨ (입력된 이미지에 고양이가 있다.)
= (입력된 이미지에 강아지가 있거나 고양이가 있다.)

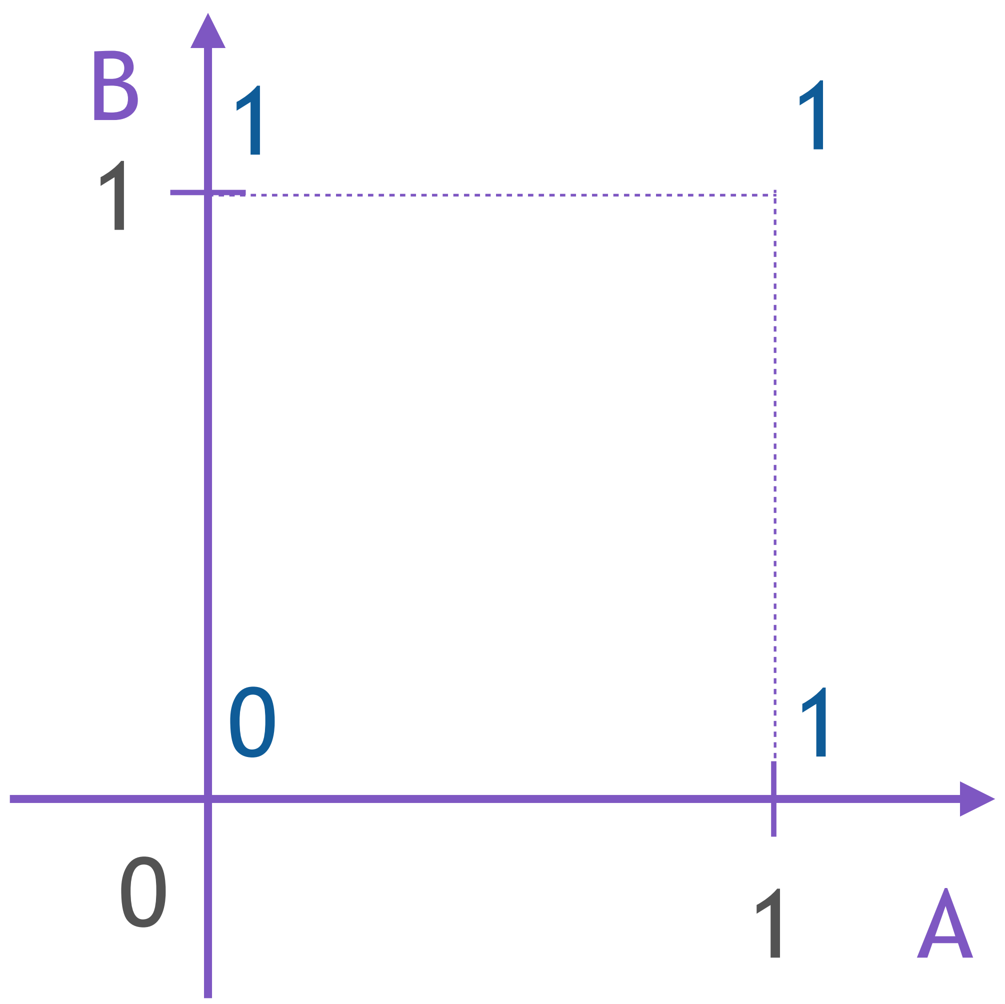
Logical Disjunction(∨)

두 input condition A, B 중 하나라도 참이면 output condition이 참인 연산

$$A \vee B = Y$$

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1



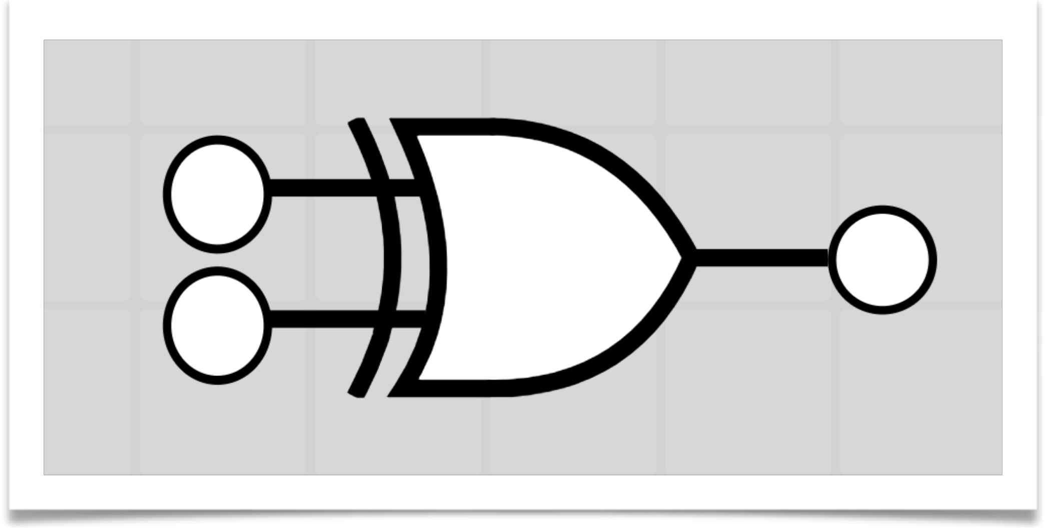


Logical Exclusive Disjunction(⊕)

두 input condition A, B 중 하나만 참일때, output condition이 참인 연산

$$A \oplus B = Y$$

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	0



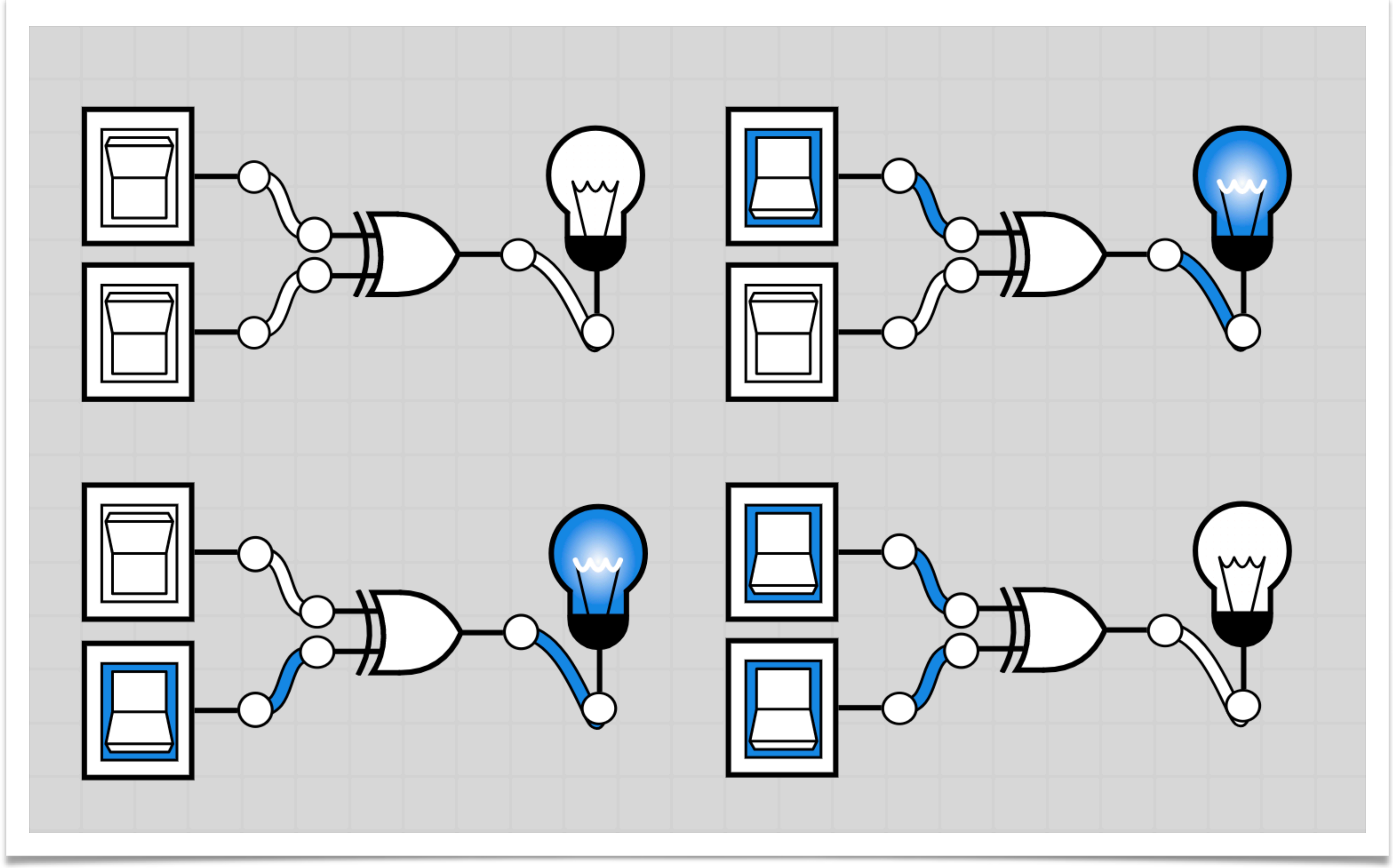
⊕는 입력이 서로 같은지, 다른지를 판별하는 연산으로도 사용된다.

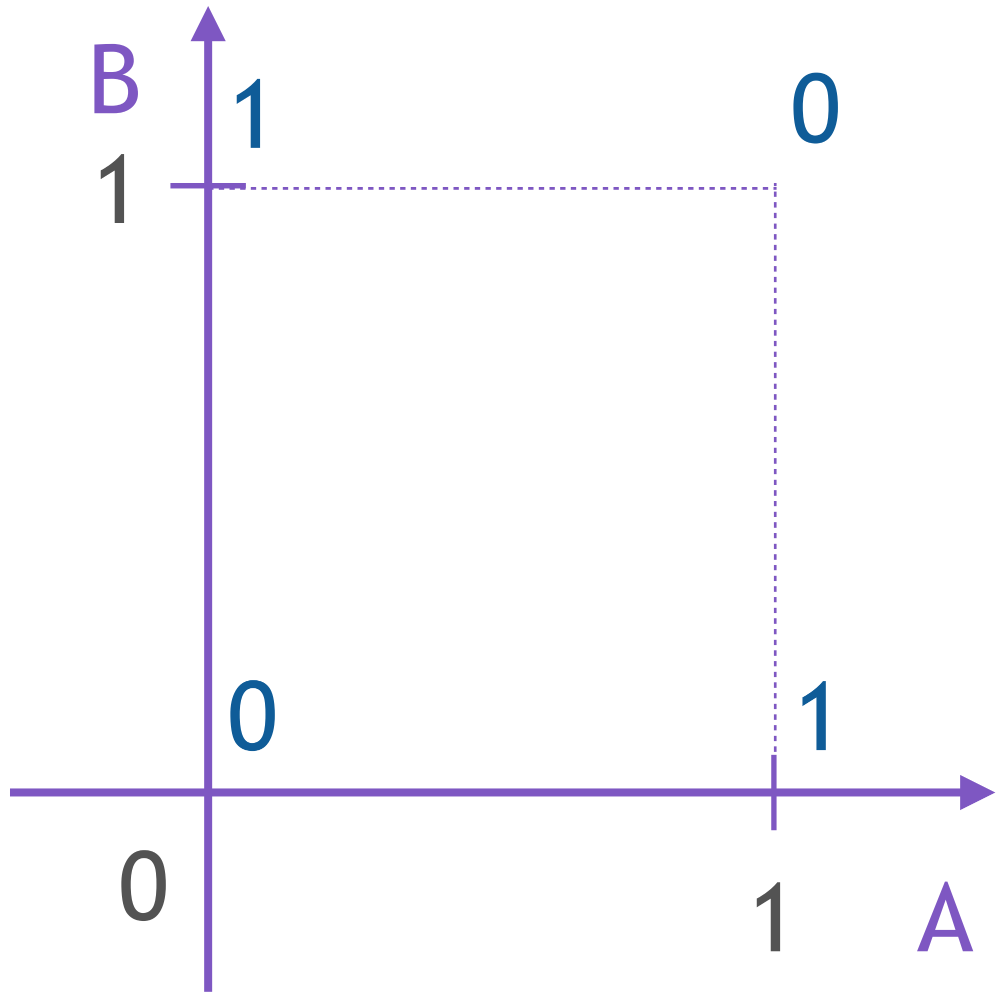
Logical Exclusive Disjunction(⊕)

두 input condition A, B 중 하나만 참일때, output condition이 참인 연산

$$A \oplus B = Y$$

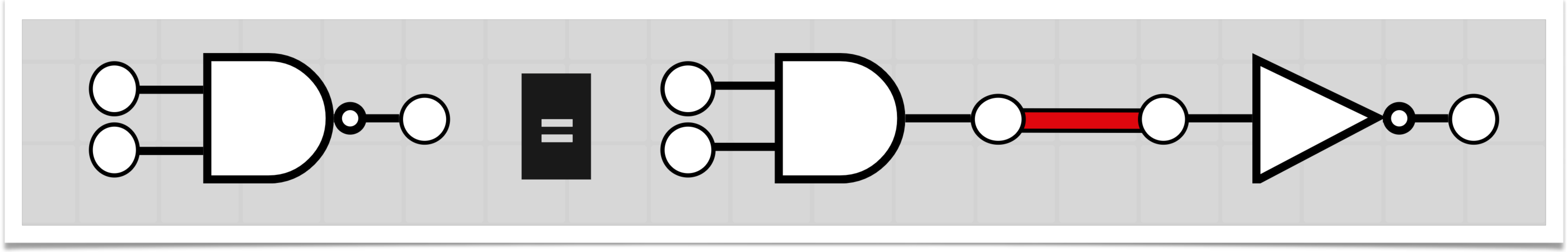
A	B	Y
0	0	0
0	1	1
1	0	1
1	1	0



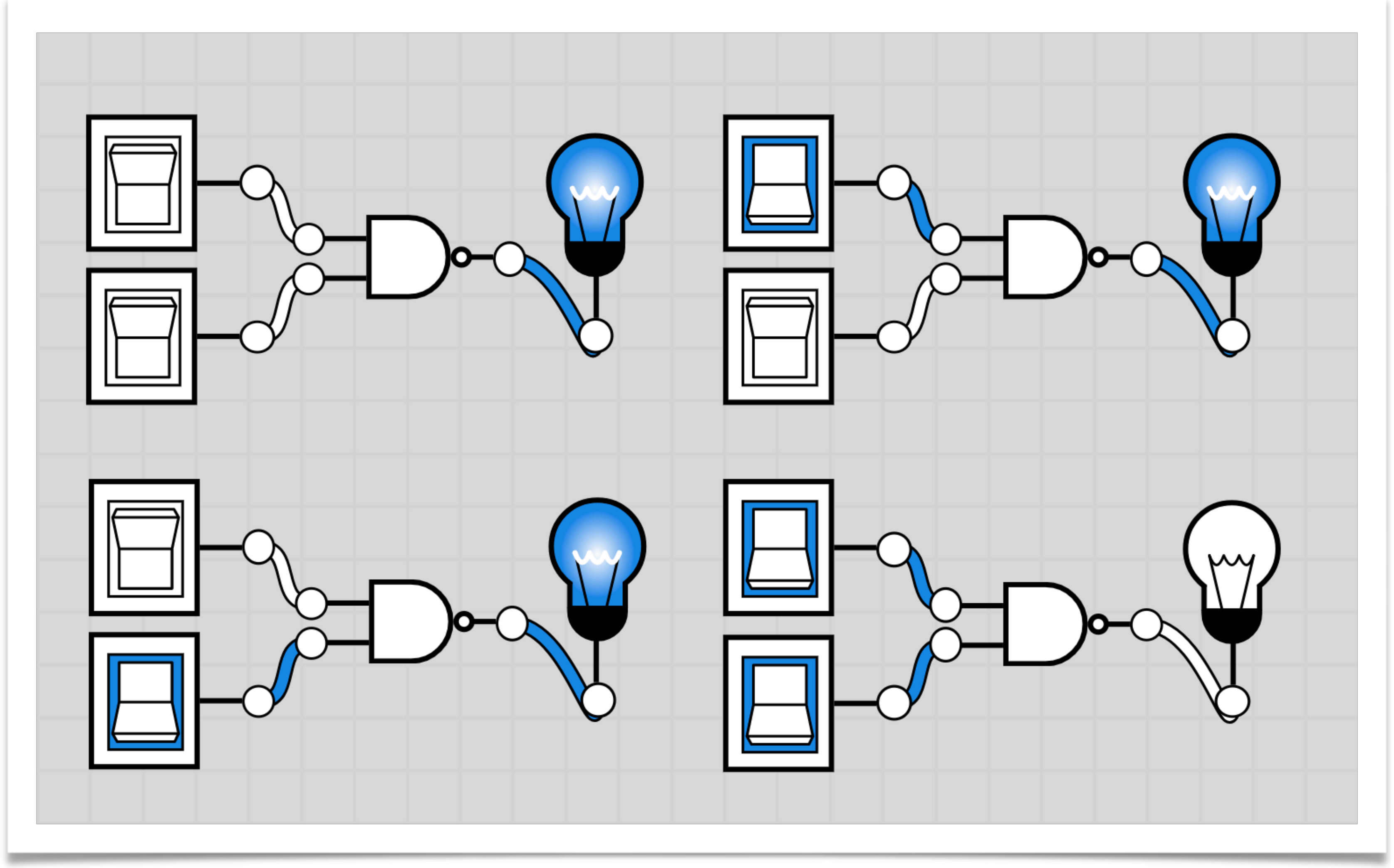


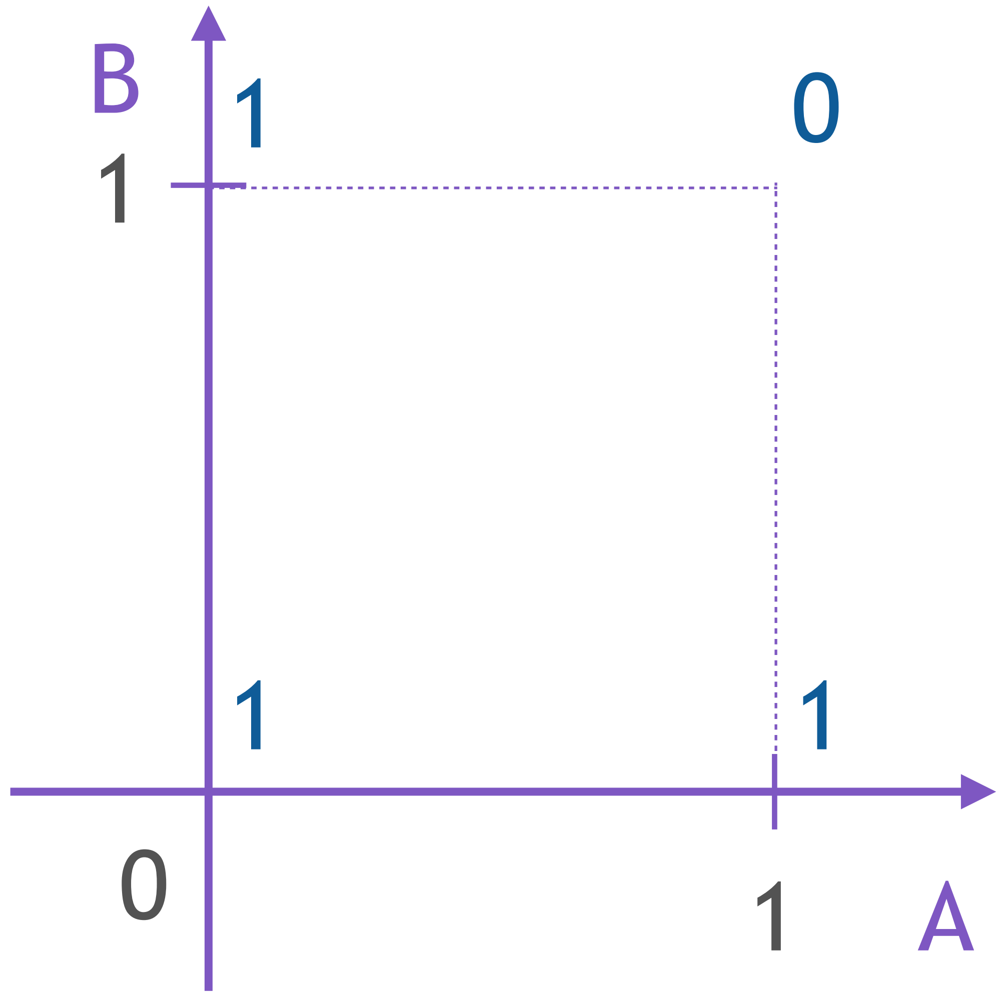
Other Digital Logic Gates

NAND Operations



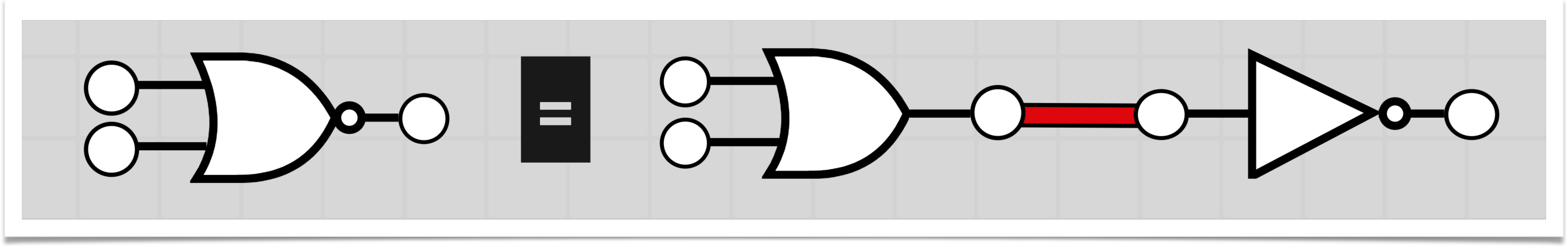
A	B	Y
0	0	1
0	1	1
1	0	1
1	1	0



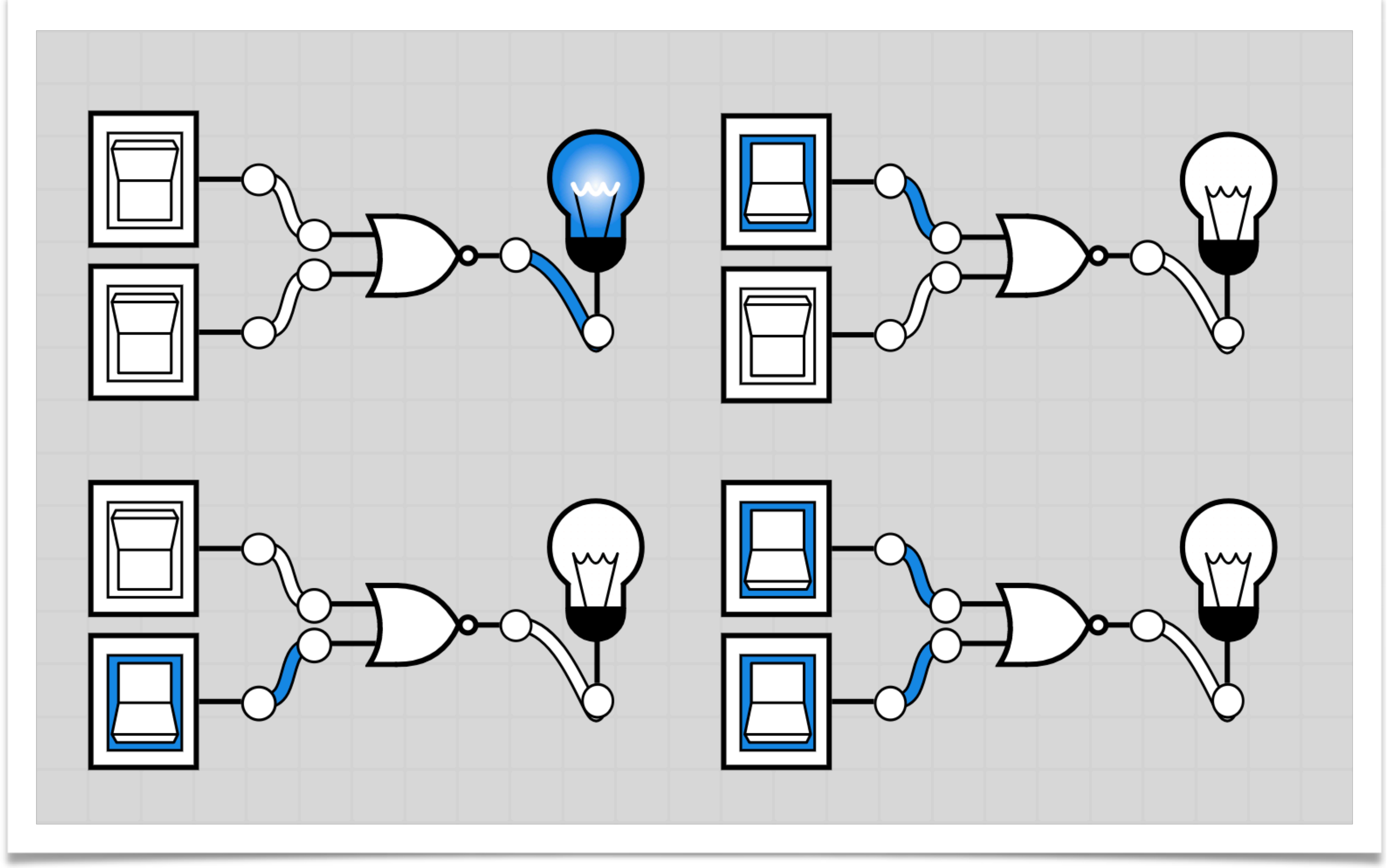


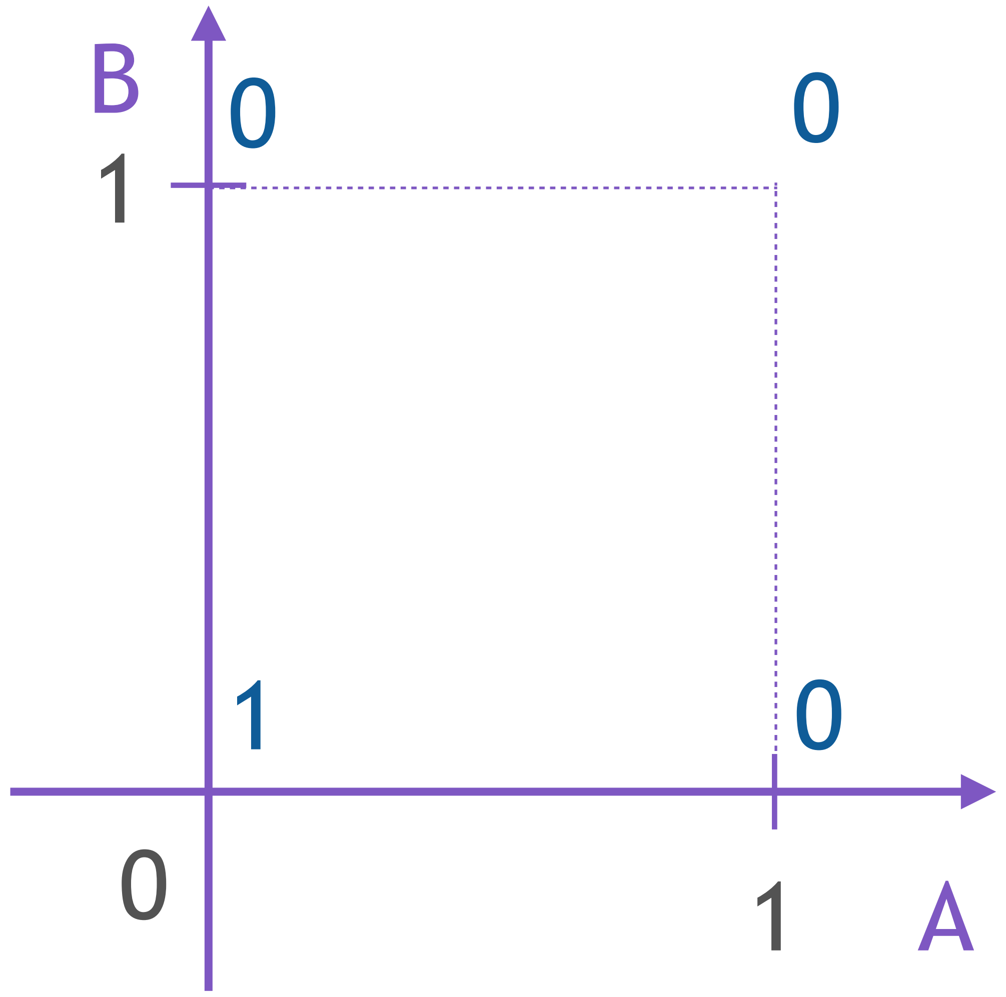
Other Digital Logic Gates

NOR Operations



A	B	Y
0	0	1
0	1	0
1	0	0
1	1	0

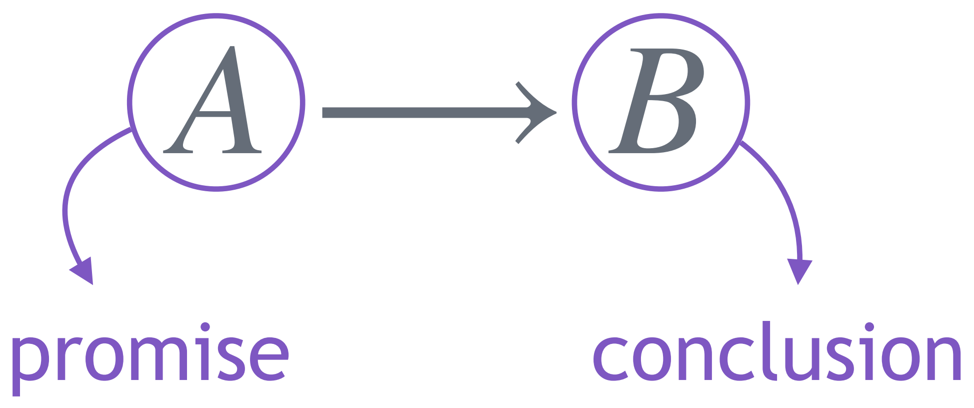




Logical Implications

조건 A가 만족될 때, B가 만족됨을 추론하는 연산

If A, then B



Innocent Until Proven Guilty

A is True, but B is False $\Longrightarrow$ 약속을 지키지 않은 경우

Logical Implications

Truth Table of Logical Implication

$$A \longrightarrow B$$

A	B	Y
0	0	1
0	1	1
1	0	0
1	1	1

Logical Implications

ex.1)

A: (수학 시험에서 100점을 맞는다.)
B: (용돈을 받는다.)

ex.2)

A: (구름이 낀 날씨)
B: (비가 온다.)

Converse, Inverse, Contrapositive

$$A \longrightarrow B$$

Converse $B \longrightarrow A$

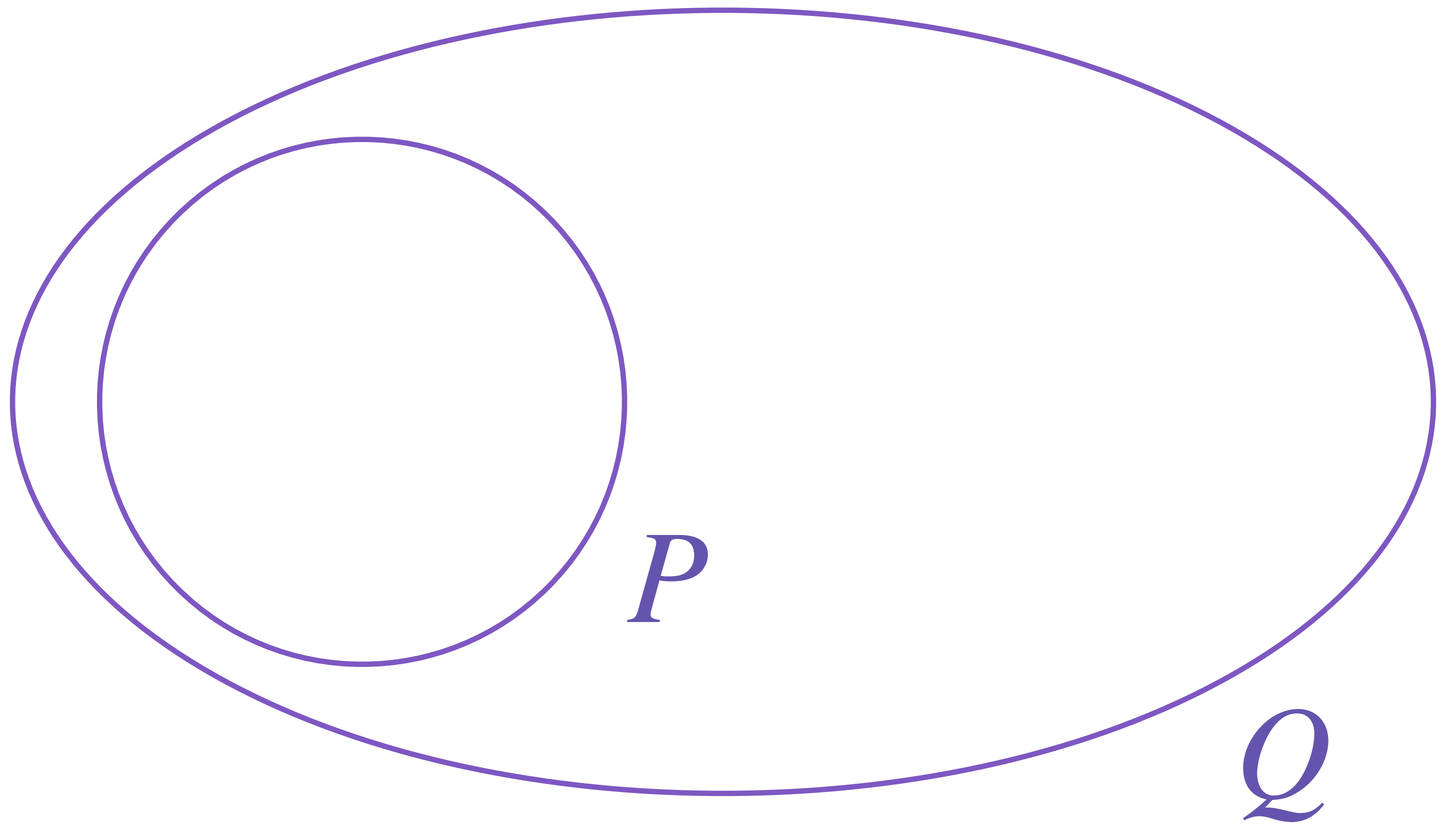
Inverse $\neg A \longrightarrow \neg B$

Contrapositive $\neg B \longrightarrow \neg A$

Sufficiency and Necessity

condition p, q와 이 조건들에 대한 각각의 truth set P, Q일때,
$p \longrightarrow q$가 참이기 위해선 $P \subseteq Q$여야 한다.

P	Q	Y
0	0	1
0	1	1
1	0	0
1	1	1



p : True, q : False인 경우가 없는 상황

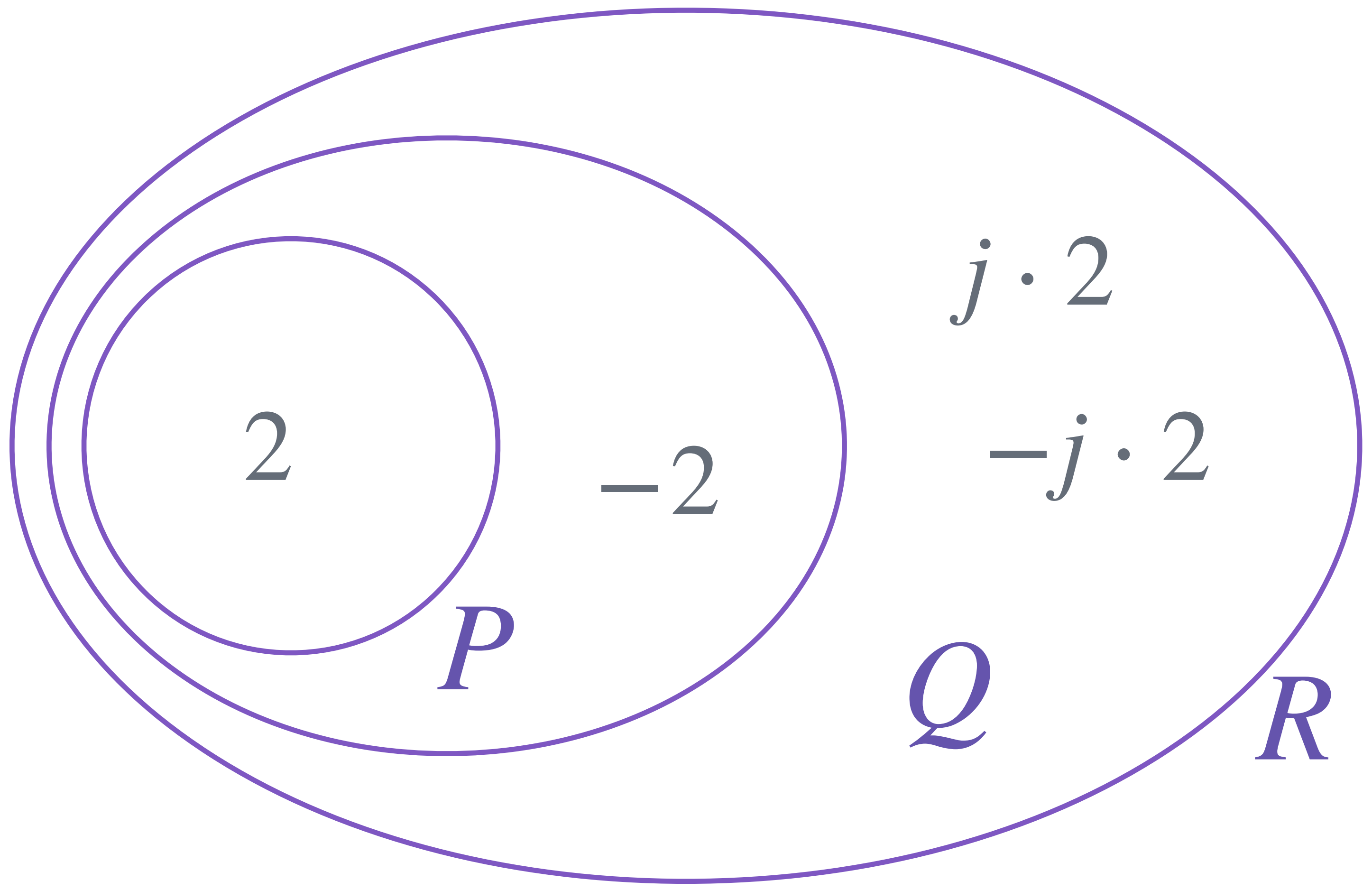
Sufficiency and Necessity

참으로 알고있는 것으로부터 **논리적인 과정을 통해** 새로운 참을 이끌어내는 과정

ex.1) $p : x = 2 \longrightarrow q : x^2 = 4 \longrightarrow r : x^4 = 16$

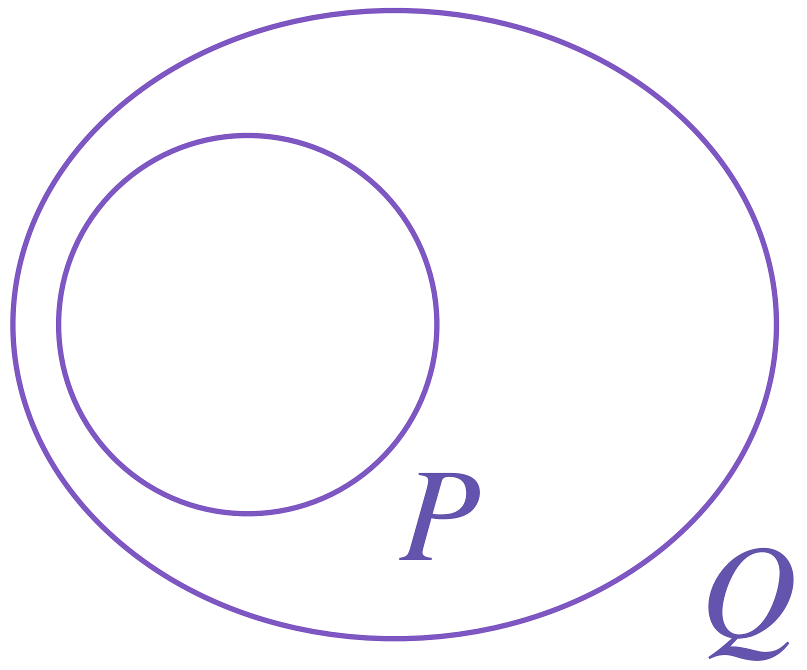
$x = \pm 2$

$x = \pm 2$ or $\pm j \cdot 2$



Sufficiency and Necessity

$$p \longrightarrow q$$



Sufficiency

p는 q이기 위한 충분조건이다.

q가 참이기 위해 p가 참임을 보여주면 충분하다.

p is sufficient for q

q가 참이기 위해 'p가 참임'만으로 충분하다.

Necessity

q는 p이기 위한 필요조건이다.

p가 참이기 위해 q가 참임이 필요하다.

q is necessary for p

p가 참이기 위해 'q가 참임'이 필요하다.

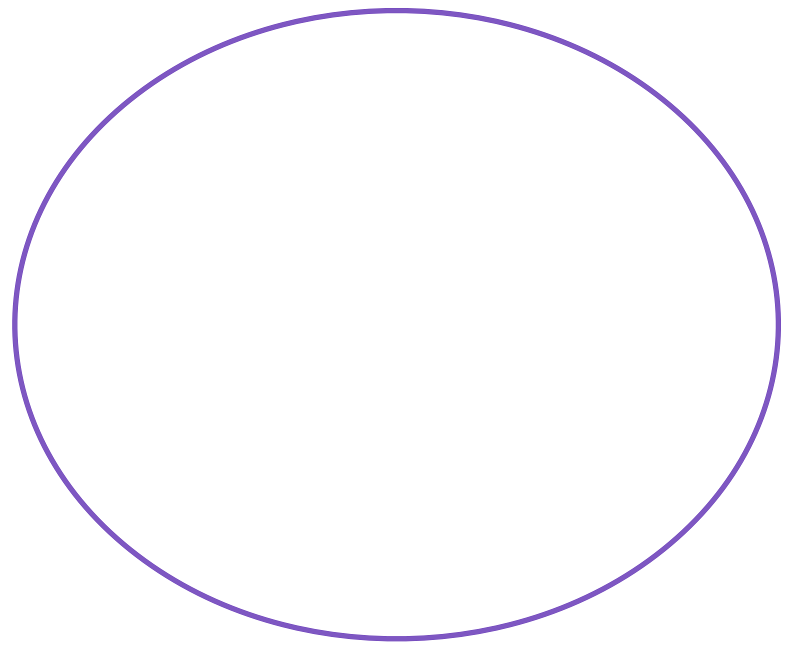
Sufficiency and Necessity

iff Condition

if and only if

두 조건 p, q가 서로 같은 truth set을 만들 때, 두 조건은 **같은 조건**이다.

ex.1) $p : x^2 = 4 \Longleftrightarrow q : x = \pm 2$



$P = Q$

For All $\forall$

$\forall x,\ \ p$

x가 될 수 있는 모든 경우에 대해 조건 p가 만족된다.

ex.1) ∀강아지는 동물이다.

ex.2) $\forall x,\ \ 0 \cdot x = 0$

ex.3) $\forall x$ except x=0, $x \cdot \dfrac{1}{x} = 1$

There Exists $\exists$

$\exists x,\ \ q$

어떤(최소한 하나 이상의) x의 값이 조건 q를 만족시킨다.

ex.1) $\exists x,\ \ x + 3 = 0$

ex.2) $\exists x,\ \ \dfrac{1}{x}$ cannot be defined

ex.3) $\exists x,\ \ x^2 = 1$

ex.4) $\nexists x \in \mathbb{R},\ \ x^2 = -5$

Identities and Inverses

Additive Identities and Inverses

$$\forall x, \exists a,\ \ x + a = x \longrightarrow a = 0$$
$$\forall x, \exists a,\ \ x + a = 0 \longrightarrow a = -x$$

Multiplicative Identities and Inverses

$$\forall x \text{ except x=0}, \exists a,\ \ x \cdot a = x \longrightarrow a = 1$$
$$\forall x \text{ except x=0}, \exists a,\ \ x \cdot a = 1 \longrightarrow a = x^{-1}$$

Basic Computations

(1) $\neg(A \wedge B)$

A	B	$A \wedge B$	$\neg(A \wedge B)$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

(2) $(\neg A) \vee (\neg B)$

A	B	$\neg A$	$\neg B$	$(\neg A) \vee (\neg B)$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

(3) $A \longrightarrow B,\ \neg A \longrightarrow \neg B,\ \neg B \longrightarrow \neg A$

A	B	$\neg A$	$\neg B$	$A \longrightarrow B$	$\neg A \longrightarrow \neg B$	$\neg A \longrightarrow \neg B$
0	0	1	1	1	1	1
0	1	1	0	1	0	1
1	0	0	1	0	1	0
1	1	0	0	1	1	1

(4) $\big[\underbrace{(A \wedge B)}_{=C} \longrightarrow B\big]_{=D} \longrightarrow \underbrace{(A \vee B)}_{=E}$

A	B	$A \wedge B$	$C \to B$	$A \vee B$	$D \to E$
0	0	0	1	0	0
0	1	0	1	1	1
1	0	0	1	1	1
1	1	1	1	1	1

Logical Equivalence

truth set A, B의 연산으로 X, Y가 나오고, 모든 경우에 대해 X, Y의 True, False가 같을 때, 두 X, Y는 logical equivalent하다.

ex.1 $(A \longrightarrow B) \longleftrightarrow (\neg B \longrightarrow \neg A)$

A	B	$A \to B$	$\neg A$	$\neg B$	$\neg B \to \neg A$
0	0	1	1	1	1
0	1	1	1	0	1
1	0	0	0	1	0
1	1	1	0	0	1

ex.2 $(A \oplus B) \longleftrightarrow \Big[\big(\underbrace{A \wedge (\neg B)}_{= C}\big) \vee \big(\underbrace{(\neg A) \wedge B}_{= D}\big)\Big]$

A	B	$A \oplus B$	$\neg A$	$\neg B$	$A \wedge (\neg B)$	$(\neg A) \wedge B$	$C \vee D$
0	0	0	1	1	0	0	0
0	1	1	1	0	0	1	1
1	0	1	0	1	1	0	1
1	1	0	0	0	0	0	0

De Morgan's Law

$$\neg(A \wedge B) \longleftrightarrow (\neg A) \vee (\neg B)$$

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$(\neg A) \vee (\neg B)$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

$$\neg(A \vee B) \longleftrightarrow (\neg A) \wedge (\neg B)$$

A	B	$A \vee B$	$\neg(A \vee B)$	$\neg A$	$\neg B$	$(\neg A) \wedge (\neg B)$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Other Equivalences

(1) $\neg(\neg A) \longleftrightarrow A$
(2) $A \wedge A \longleftrightarrow A$
(3) $A \vee A \longleftrightarrow A$

Commutativity
(4) $A \wedge B \longleftrightarrow B \wedge A$
(5) $A \vee B \longleftrightarrow B \vee A$

Associativity
(6) $(A \wedge B) \wedge C \longleftrightarrow A \wedge (B \wedge C)$
(7) $(A \vee B) \vee C \longleftrightarrow A \vee (B \vee C)$

Distributivity
(8) $A \wedge (B \vee C) \longleftrightarrow (A \wedge B) \vee (A \wedge C)$
(9) $A \vee (B \wedge C) \longleftrightarrow (A \vee B) \wedge (A \vee C)$

Tautologies and Contradictions

Tautologies

항상 결과가 참이 되는 연산

ex.1) $A \lor (\neg A)$

A	$\neg A$	$A \lor (\neg A)$
0	1	1
1	0	1

ex.2) $[(A \to B) \land (B \to C)] \longrightarrow (A \to C)$

$= P \quad = Q \quad = R \quad = S$

A	B	C	$A \to B$	$B \to C$	$P \land Q$	$A \to C$	$R \to S$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

Tautologies and Contradictions

Contradictions

항상 결과가 거짓이 되는 연산

ex.1) $A \wedge (\neg A)$

A	$\neg A$	$A \wedge (\neg A)$
0	1	0
1	0	0

ex.2) $\underbrace{(A \vee B)}_{=P} \wedge \underbrace{\left[(\neg A) \wedge (\neg B)\right]}_{=Q}$

A	B	$A \vee B$	$\neg A$	$\neg B$	$(\neg A) \wedge (\neg B)$	$P \wedge Q$
0	0	0	1	1	1	0
0	0	1	1	0	0	0
0	1	1	0	1	0	0
0	1	1	0	0	0	0

References

[1] Meyer, Carl D. Matrix analysis and applied linear algebra. Vol. 71. Siam, 2000.

CLOSING

Basic Algebra

Chap.3 Mathematical Logic