수학으로부터 인류를 자유롭게 하라

Free Humankind from Mathematics

Basic Algebra

Chap.12 Composite Functions

12.1 Arithmetic Operations of Functions

Arithmetic Operations

$$(f+g)(x)=f(x)+g(x) \qquad (f\times g)(x)=f(x)\times g(x)$$
$$(f-g)(x)=f(x)-g(x) \qquad (f\div g)(x)=f(x)\div g(x)$$

f, g는 서로 다른 항 f, g는 같은 항

ex.1) 다음 함수 f, g에 대해 $f+g, f-g, f\times g, f\div g$를 구하세요.

(1) $f(x)=x^2,\ g(x)=2x$

$$\longrightarrow (f+g)(x)=f(x)+g(x)=x^2+2x$$
$$(f-g)(x)=f(x)-g(x)=x^2-2x$$
$$(f\times g)(x)=f(x)\times g(x)=2x^3$$
$$(f\div g)(x)=f(x)\div g(x)=\frac{1}{2}x$$

(2) $f(x)=sin(x),\ g(x)=e^x$

$$\longrightarrow (f+g)(x)=f(x)+g(x)=sin(x)+e^x$$
$$(f-g)(x)=f(x)-g(x)=sin(x)-e^x$$
$$(f\times g)(x)=f(x)\times g(x)=e^x sin(x)$$
$$(f\div g)(x)=f(x)\div g(x)=e^{-x}sin(x)$$

Terms in Functions

$$Sys\left\{\alpha x(t)+\beta y(t)\right\} = \alpha Sys\left\{x(t)\right\} + \beta Sys\left\{y(t)\right\}$$

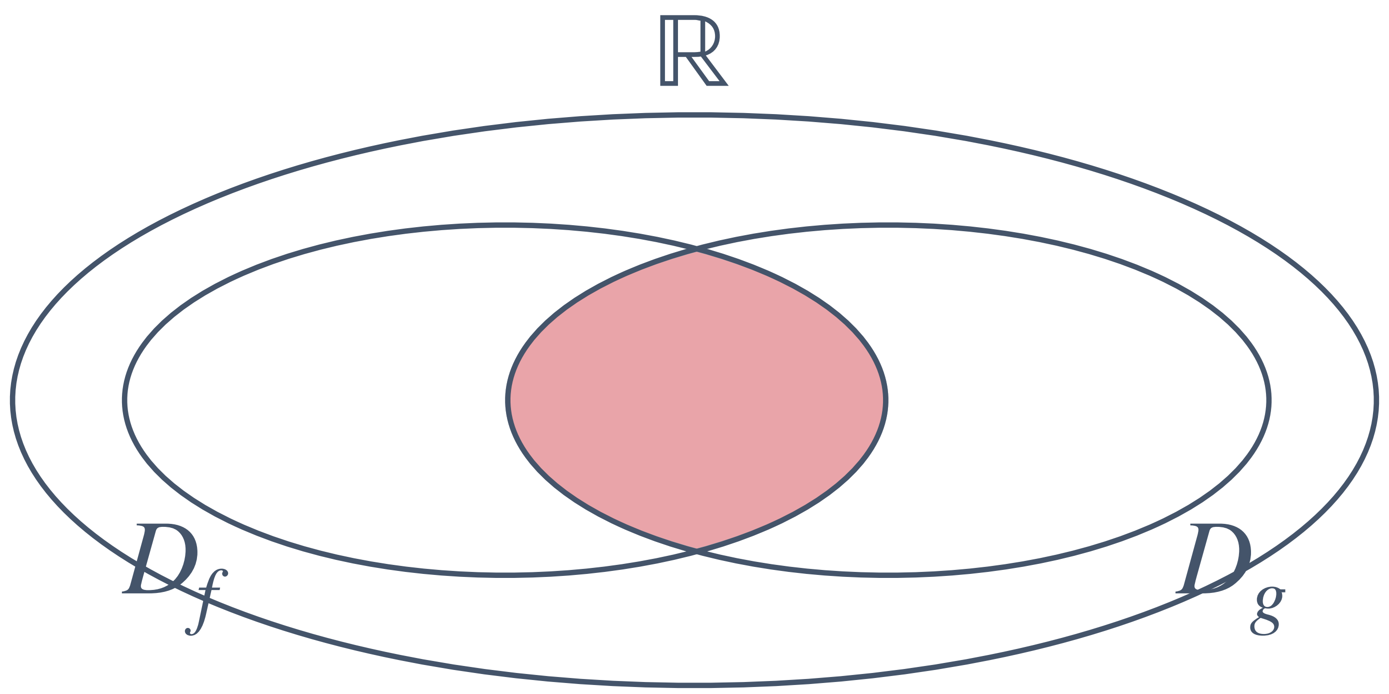
$$\frac{d}{dx}\left[\alpha f(x)+\beta g(x)\right] = \alpha\frac{d}{dx}\left[f(x)\right] + \beta\frac{d}{dx}\left[g(x)\right]$$

$$\int\left[\alpha f(x)+\beta g(x)\right]dx = \alpha\int f(x)\ dx + \beta\int g(x)\ dx$$

$$\mathscr{F}\left\{\alpha f(t)+\beta g(t)\right\} = \alpha\mathscr{F}\left\{f(t)\right\} + \beta\mathscr{F}\left\{g(t)\right\}$$

$$\int_{-\infty}^{\infty}\left[\alpha f(t)+\beta g(t)\right]\cdot e^{-j\omega t}\ dt = \alpha\int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}\ dt + \beta\int_{-\infty}^{\infty} g(t)\cdot e^{-j\omega t}\ dt$$

Domains



$(f+g)(x)$

$(f-g)(x)$

$(f\times g)(x)$

$(f\div g)(x)$

$$D = D_f \cap D_g$$

Domains

Examples

ex.1) $f(x) = \dfrac{1}{x},\ g(x) = ln(x) \quad \longrightarrow \quad f(x) + g(x) = \dfrac{1}{x} + ln(x)$

$$\longrightarrow \begin{array}{l} D_f = (-\infty,0) \cup (0,\infty) \\ D_g = (0,\infty) \end{array}$$

$$D = D_f \cap D_g = \left[(-\infty,0) \cup (0,\infty)\right] \cap (0,\infty)$$

$$\longrightarrow \quad = \left[(-\infty,0) \cap (0,\infty)\right] \cup \left[(0,\infty) \cap (0,\infty)\right]$$

$$= \varnothing \cup (0,\infty) = (0,\infty)$$

ex.2) $f(x) = x,\ g(x) = sin(x)$

$$\longrightarrow D_f = D_g = \mathbb{R}$$

$$\longrightarrow D = D_f \cap D_g = \mathbb{R}$$

ex.3) $f(x) = x^2,\ g(x) = \dfrac{1}{x(x-1)}$

$$\longrightarrow D_f = \mathbb{R},\ \ D_g = \mathbb{R} - \{0,\ 1\}$$

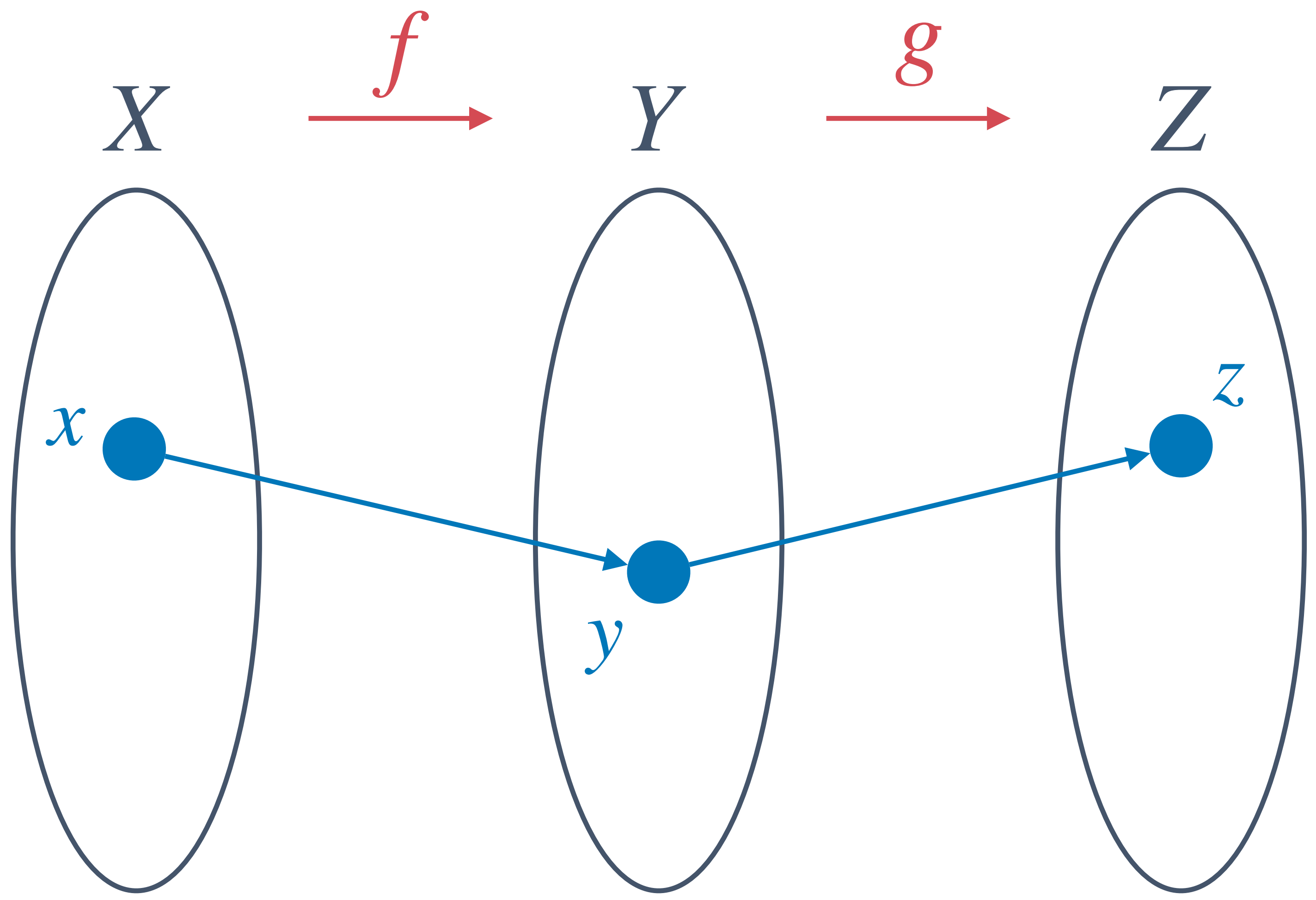
$$\longrightarrow D = \mathbb{R} - \{0,\ 1\}$$

ex.4) $f(x) = ln(x-1),\ g(x) = ln(2-x)$

$$\longrightarrow D_f = (1,\infty),\ \ D_g = (-\infty,2)$$

$$\longrightarrow D = (-\infty,2) \cap (1,\infty) = (1,2)$$

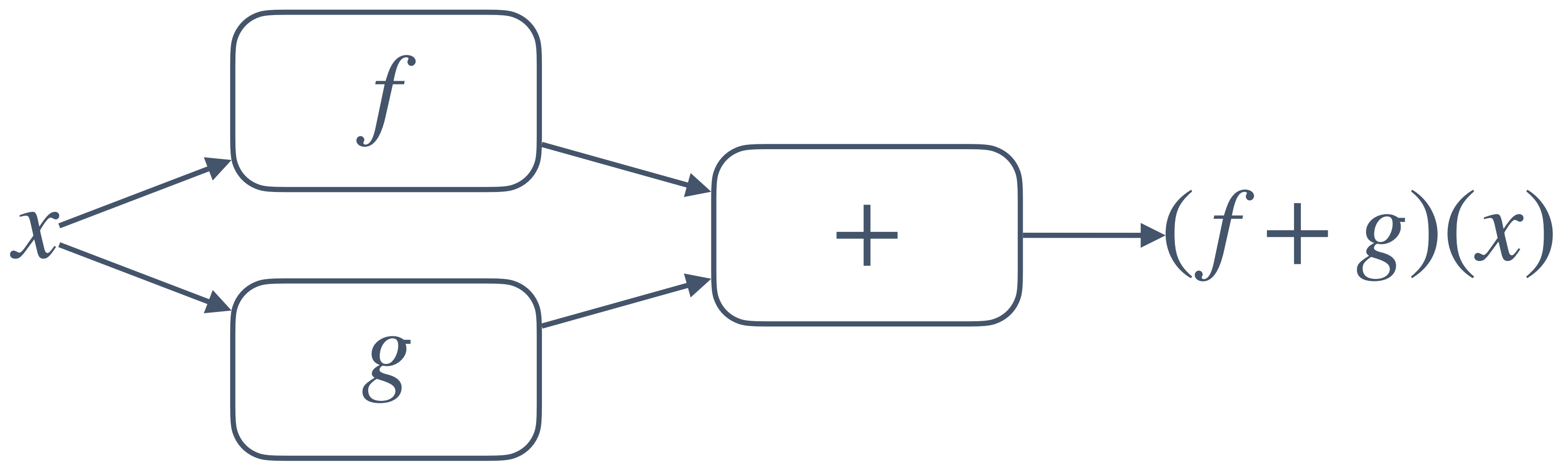
Sets within Composite Functions



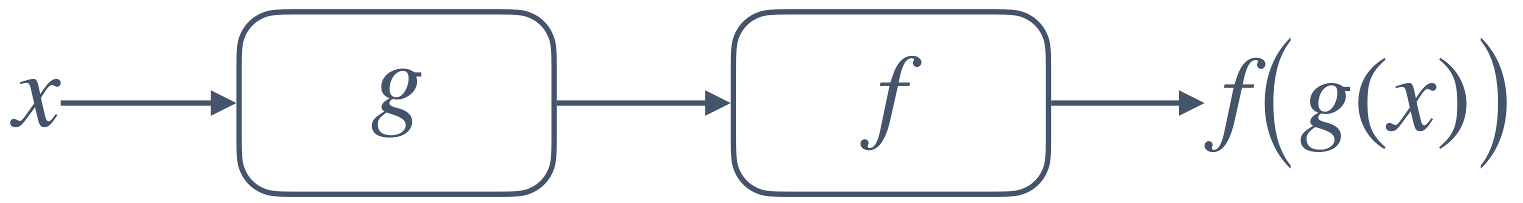
$$(g \circ f)(x) = g\big(f(x)\big)$$

Arithmetic Operations vs Compositions

$$(f+g)(x) = f(x) + g(x)$$



$$(f+g)(x) = f\big(g(x)\big)$$



$$f(x) = x^2, \quad g(x) = log_2(x)$$

$$\longrightarrow \begin{array}{l} f(x) + g(x) = x^2 + log_2(x) \\ g\big(f(x)\big) = log_2\big(f(x)\big) = log_2(x^2) \end{array}$$

12.2 Composite Functions

Examples

ex.1) $f(x) = \frac{1}{x}, \quad g(x) = e^x, \quad h(x) = sin(x)$

(1) $(f \circ f)(x) \quad \longrightarrow f\big(f(x)\big) = \frac{1}{f(x)} = x$

(2) $(g \circ g)(x) \quad \longrightarrow g\big(g(x)\big) = e^{g(x)} = e^{e^x}$

(3) $(h \circ h)(x) \quad \longrightarrow h\big(h(x)\big) = sin\big(h(x)\big) = sin\big(sin(x)\big)$

Examples

ex.1) $f(x) = \frac{1}{x}, \quad g(x) = e^x, \quad h(x) = sin(x)$

(4) $(f \circ g)(x) \quad \longrightarrow f\left(g(x)\right) = \frac{1}{g(x)} = \frac{1}{e^x} = e^{-x}$

(5) $(f \circ h)(x) \quad \longrightarrow f\left(h(x)\right) = \frac{1}{h(x)} = \frac{1}{sin(x)} = csc(x)$

(6) $(g \circ h)(x) \quad \longrightarrow g\left(h(x)\right) = e^{h(x)} = e^{sin(x)}$

(7) $(g \circ f)(x) \quad \longrightarrow g\left(f(x)\right) = e^{f(x)} = e^{\frac{1}{x}}$

(8) $(h \circ f)(x) \quad \longrightarrow h\left(f(x)\right) = sin\left(f(x)\right) = sin\left(\frac{1}{x}\right)$

(9) $(h \circ g)(x) \quad \longrightarrow h\left(g(x)\right) = sin\left(g(x)\right) = sin(e^x)$

Examples

ex.1) $f(x) = \frac{1}{x}, \quad g(x) = e^{x}, \quad h(x) = sin(x)$

(10) $(f \circ g \circ h)(x) \quad \longrightarrow f(g(h(x))) = \frac{1}{g(h(x))} = \frac{1}{e^{h(x)}} = \frac{1}{e^{sin(x)}} = e^{-sin(x)}$

(11) $(g \circ h \circ f)(x) \quad \longrightarrow g(h(f(x))) = e^{h(f(x))} = e^{sin(f(x))} = e^{sin(\frac{1}{x})}$

(12) $(h \circ f \circ g)(x) \quad \longrightarrow h(f(g(x))) = sin(f(g(x))) = sin\left(\frac{1}{g(x)}\right) = sin\left(\frac{1}{e^{x}}\right) = sin\left(e^{-x}\right)$

Internal Variables

$$z = g\big(f(x)\big) = g(y)$$

$$x \longrightarrow \boxed{f} \xrightarrow{y} \boxed{g} \longrightarrow z$$

$$w = h(g(f(x))) = h(g(y)) = h(z)$$

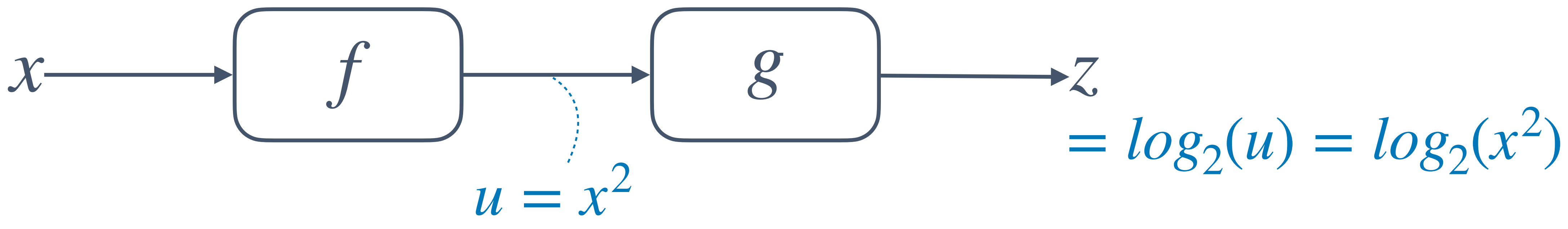
$$x \longrightarrow \boxed{f} \xrightarrow{y} \boxed{g} \xrightarrow{z} \boxed{h} \longrightarrow w$$

Decomposing Composite Functions

Examples

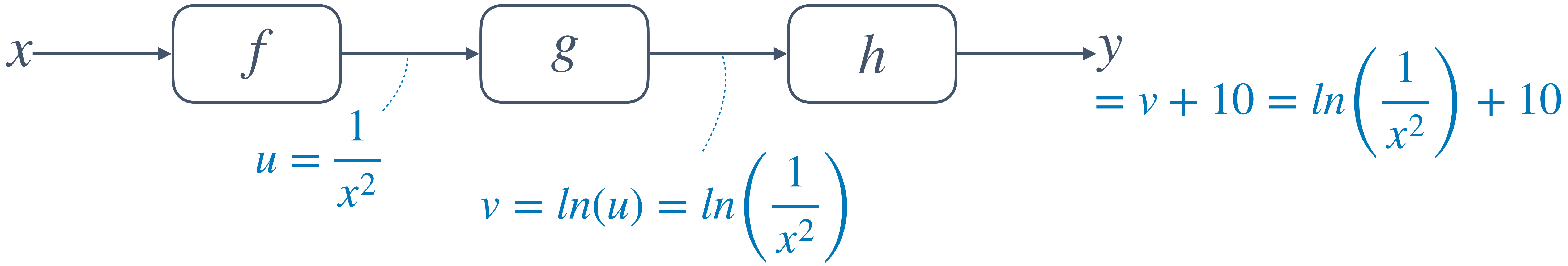
ex.1) $y = log_2\left(x^2\right)$

$$u = f(x) = x^2$$
$$g(u) = log_2(u)$$



ex.2) $y = ln\left(\frac{1}{x^2}\right) + 10$

$$u = f(x) = \frac{1}{x^2}$$
$$v = g(u) = ln(u)$$
$$y = h(v) = v + 10$$

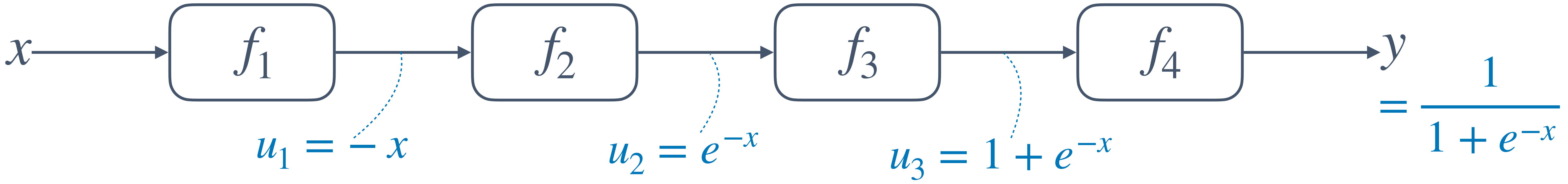


Decomposing Composite Functions

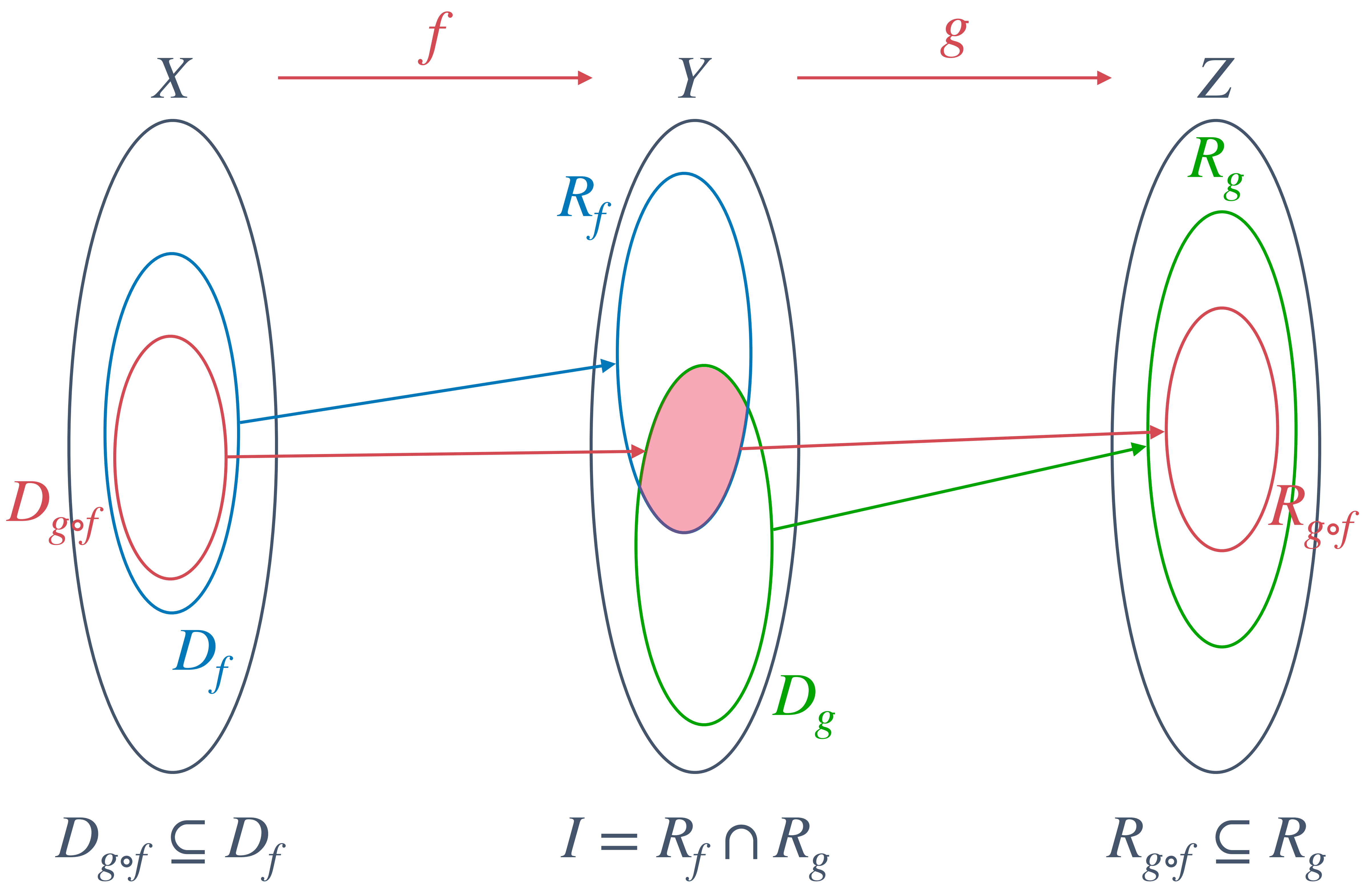
Examples

ex.3) $y = \sigma(x) = \dfrac{1}{1+e^{-x}}$

$$u_1 = f_1(x) = -x$$
$$u_2 = f_2(u_1) = e^{u_1} = e^{-x}$$
$$u_3 = f_3(u_2) = 1 + u_2 = 1 + e^{-x}$$
$$y = f_4(u_3) = \frac{1}{u_3} = \frac{1}{1+e^{-x}}$$



Domains of Composite Functions



step.1) D_f, R_f, D_g, R_g

step.2) $I = R_f \cap D_g$

step.3) $D_{g\circ f}$

step.4) $R_{g\circ f}$

Examples

ex.1) $y = ln\left(x^2+4\right)$

step.1) $u = f(x) = x^2 + 4$ $\quad y = g(u) = ln(u)$

$D_f = (-\infty, \infty)$ $\quad D_g = (0, \infty)$

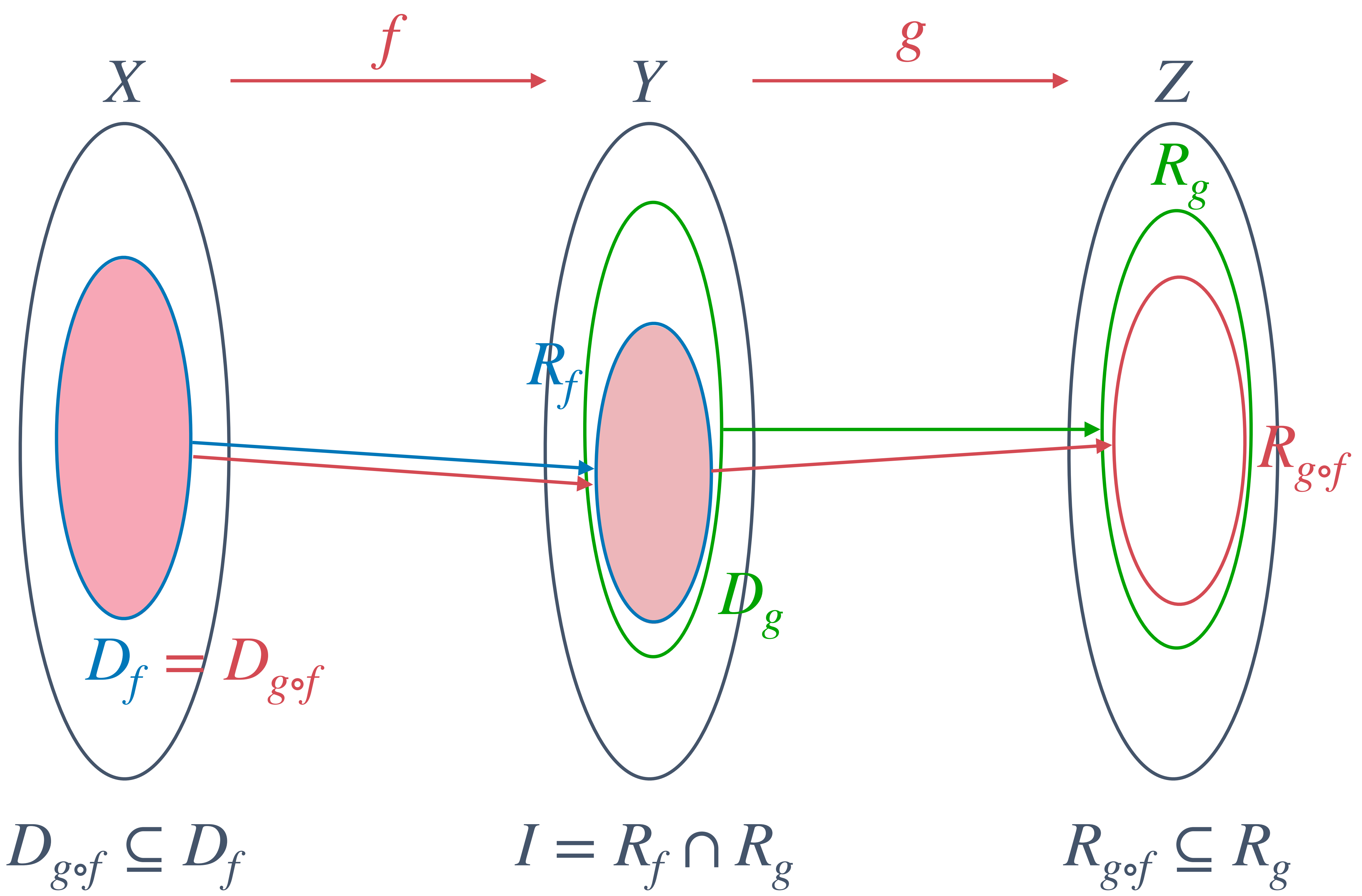
$R_f = [4, \infty)$ $\quad R_g = (-\infty, \infty)$

step.2) $I = R_f \cap D_g = [4, \infty)$

step.3) $4 \le x^2 + 4 \longrightarrow D_{g \circ f} = (-\infty, \infty)$

step.4) $4 \le u \longrightarrow ln(4) \le ln(u)$

$\longrightarrow R_{g \circ f} = [ln(4), \infty)$



Examples

ex.2) $y = ln\left(x^2 - 4\right)$

step.1) $u = f(x) = x^2 - 4$ $y = g(u) = ln(u)$

$D_f = (-\infty, \infty)$ $D_g = (0, \infty)$

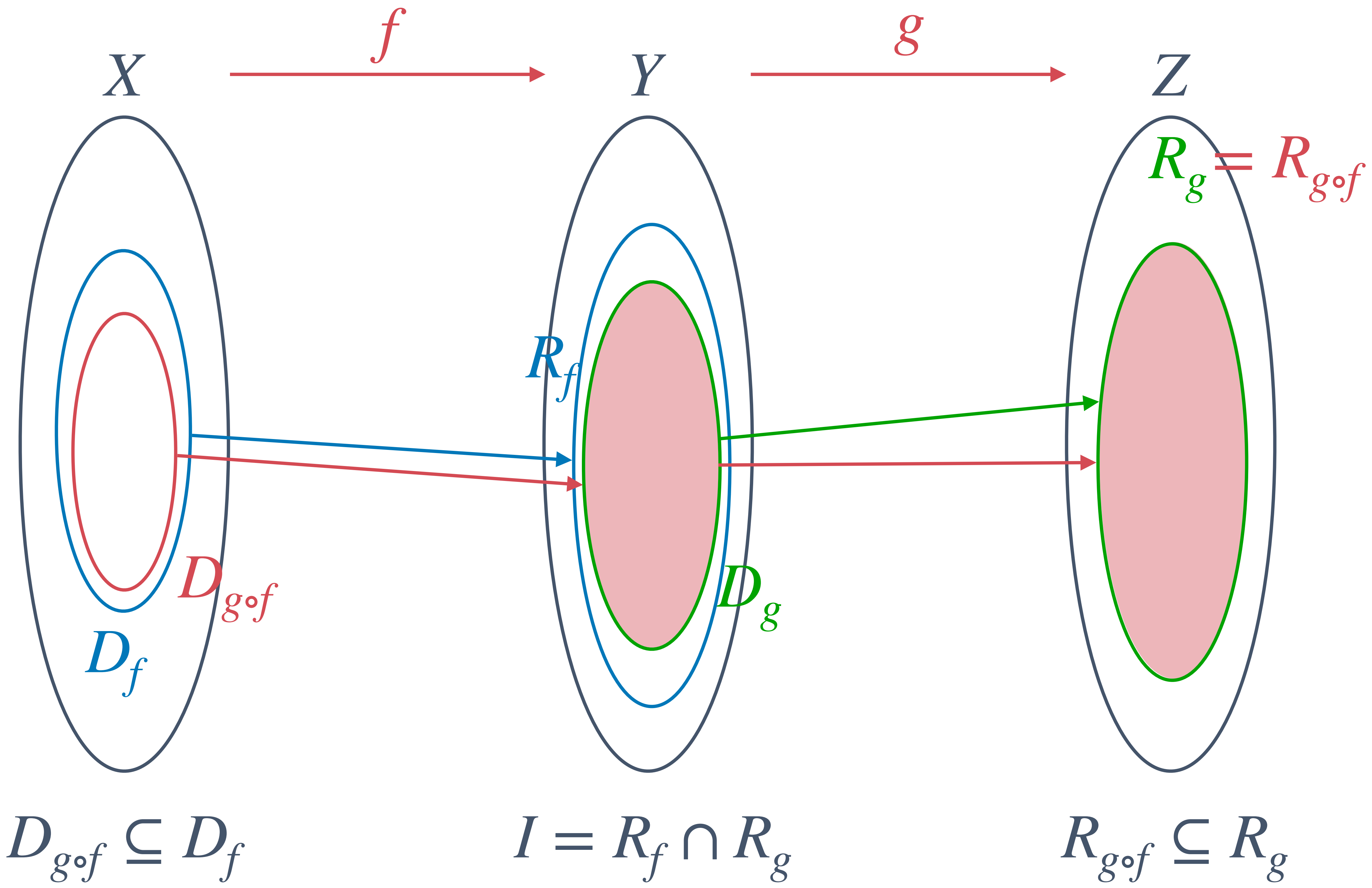
$R_f = [-4, \infty)$ $R_g = (-\infty, \infty)$

step.2) $I = R_f \cap D_g = (0, \infty)$

step.3) $x^2 - 4 > 0 \longrightarrow D_{g\circ f} = (-\infty, -2) \cup (2,\infty)$

step.4) $u > 0 \longrightarrow ln(u) > -\infty$

$\longrightarrow R_{g\circ f} = (-\infty, \infty)$



Examples

ex.3) $y = ln\left(\frac{2}{2x^2+1} - 1\right)$

step.1) $u = f(x) = \frac{2}{2x^2+1}$ $\quad y = g(u) = ln(u-1)$

$D_f = (-\infty, \infty)$ $\quad D_g = (1, \infty)$

$R_f = (0, 2]$ $\quad R_g = (-\infty, \infty)$

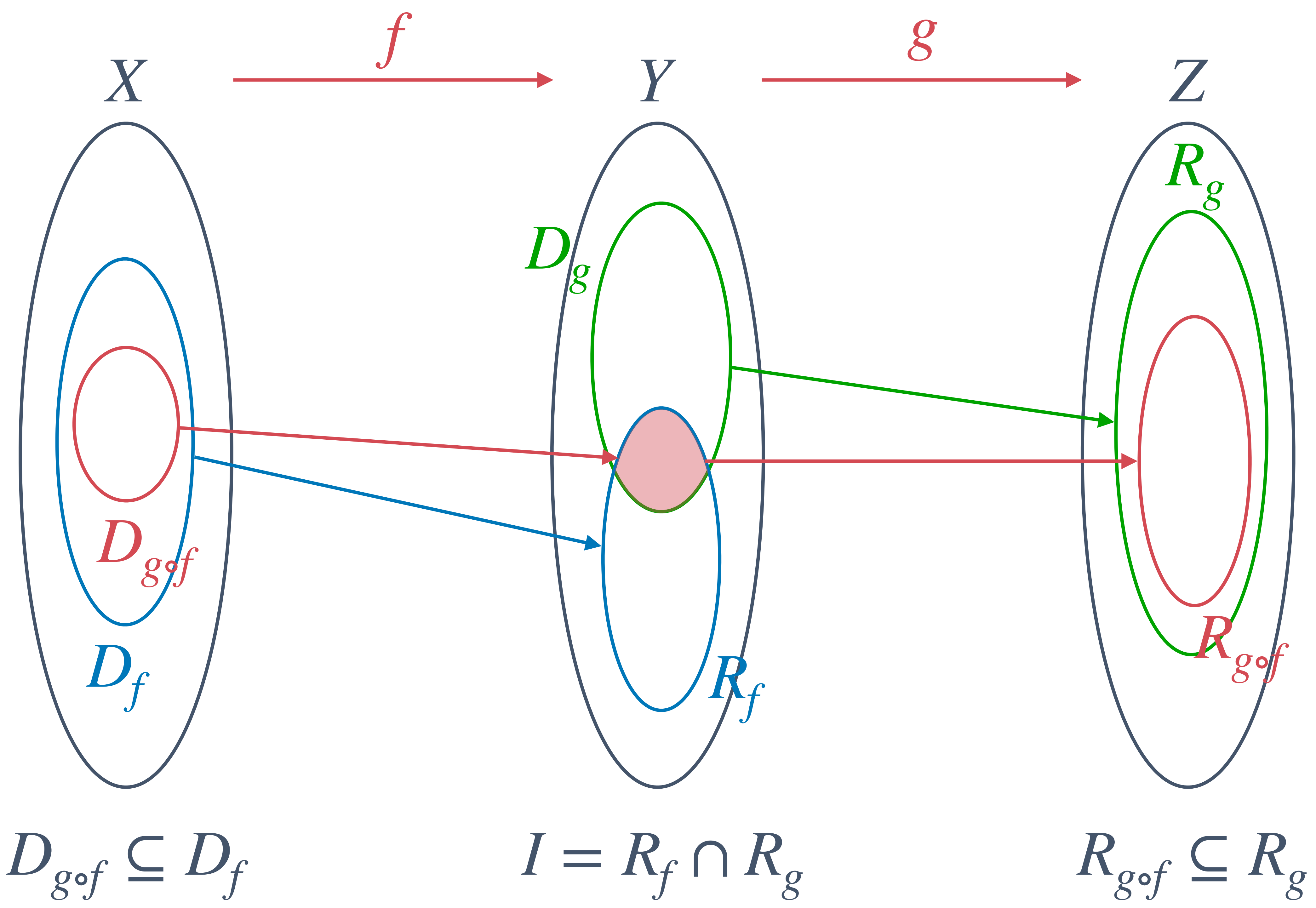
step.2) $I = R_f \cap D_g = (1, 2]$

step.3) $1 < \frac{2}{2x^2+1} \leq 2 \longrightarrow -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

$\longrightarrow D_{g \circ f} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

step.4) $1 < u \leq 2 \longrightarrow -\infty < ln(u-1) \leq 0$

$\longrightarrow R_{g \circ f} = (-\infty, 0]$



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Basic Algebra

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