수학으로부터 인류를 자유롭게 하라

Free Humankind from Mathematics

Basic Algebra

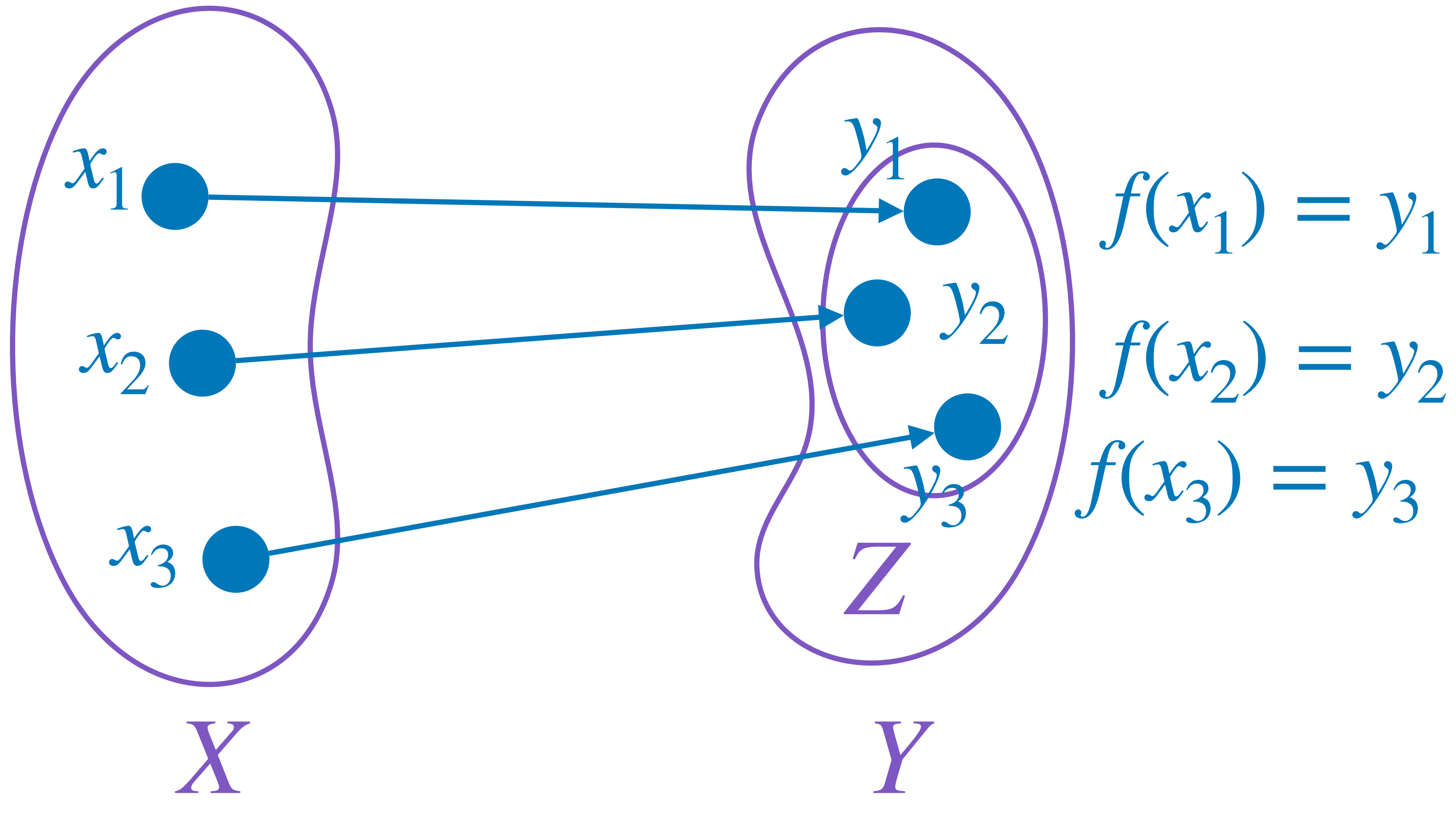
Chap.10 Multivariate Functions and Linearity

Bivariate Functions

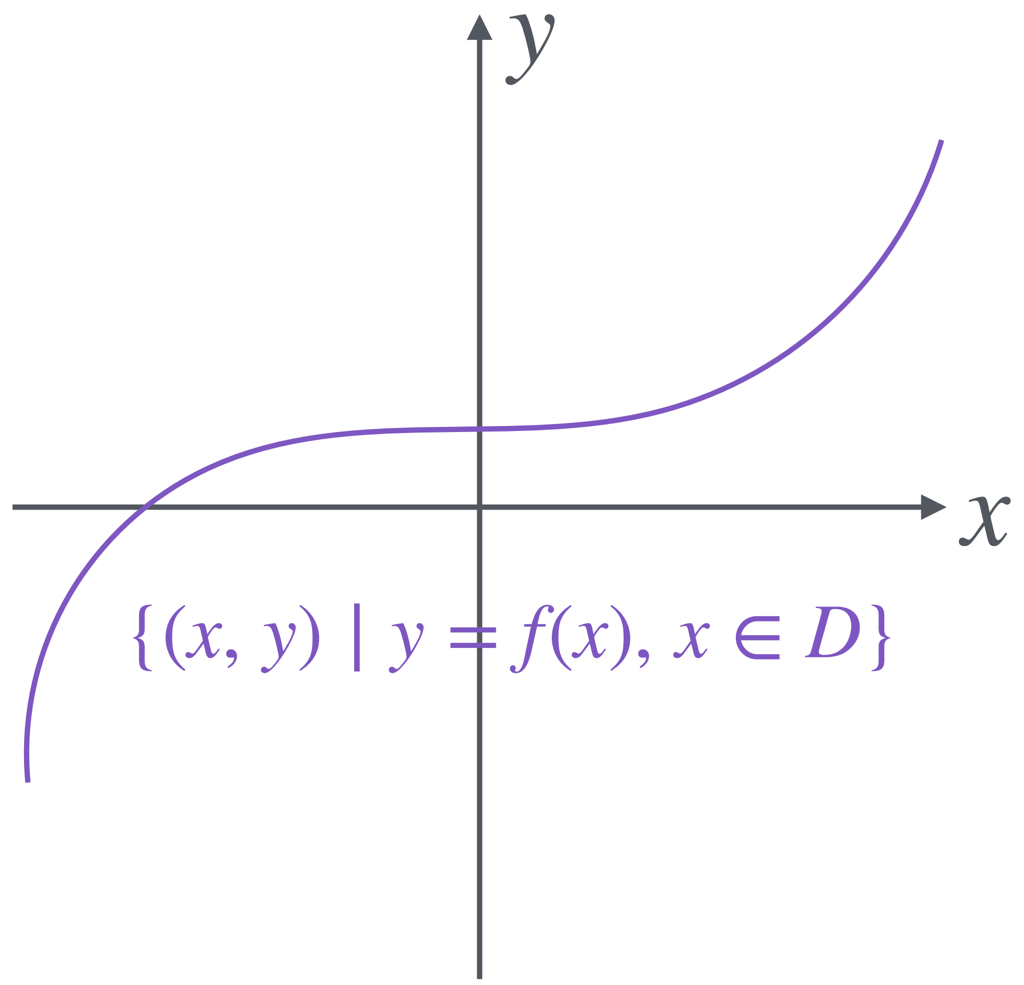
Univariate Functions: Review

$$y = f(x)$$

Equations



Sets



Graphs

Bivariate Functions

with Equations

$$z = f(x, y) \qquad\qquad y = f(x_1, x_2)$$

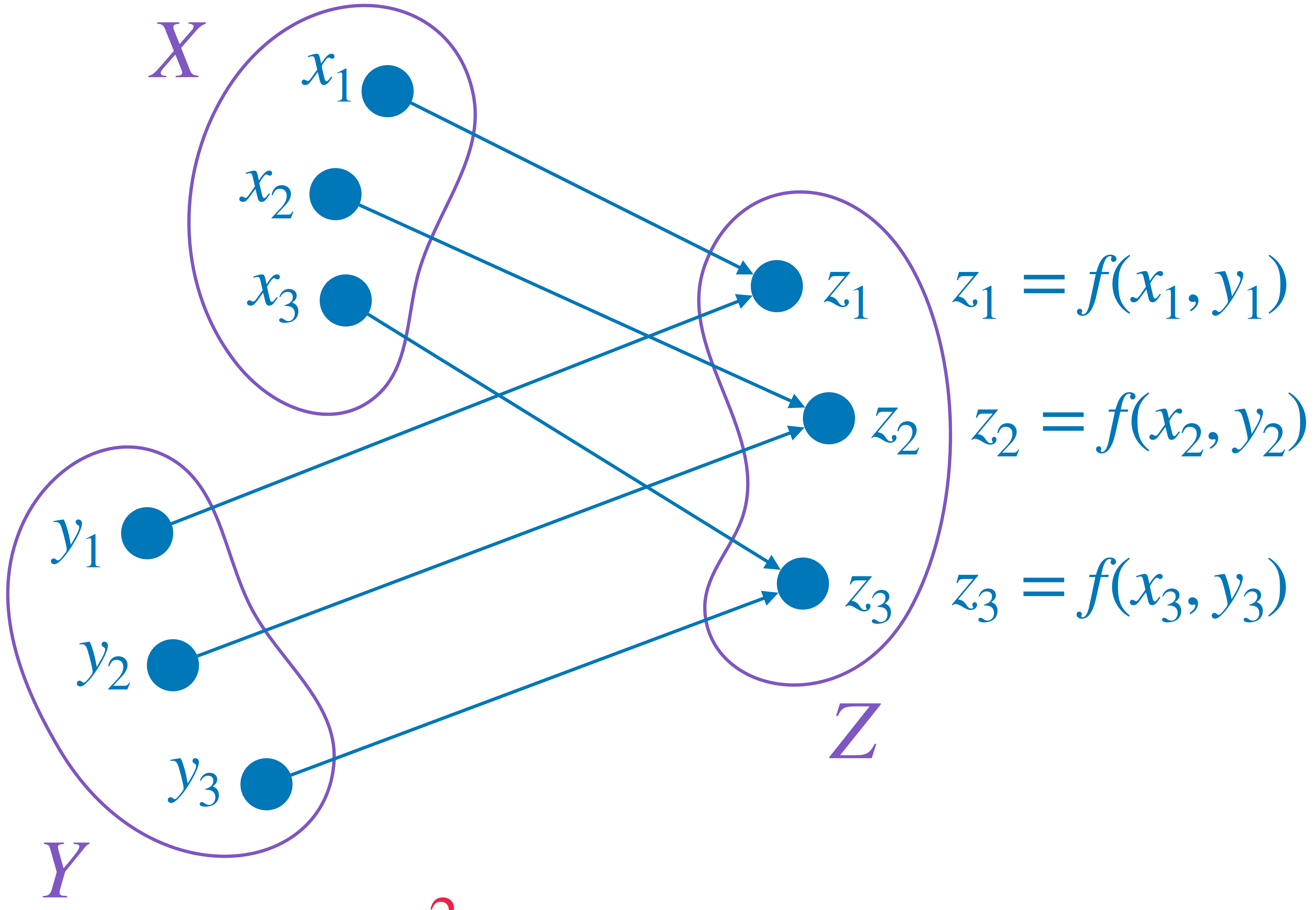
ex.1) $z = f_1(x, y) = x + y$

ex.2) $z = f_2(x, y) = e^x - ln(y)$

ex.3) $z = f_3(x, y) = \dfrac{x^2 + 2x - 2}{y + 1}$

Bivariate Functions

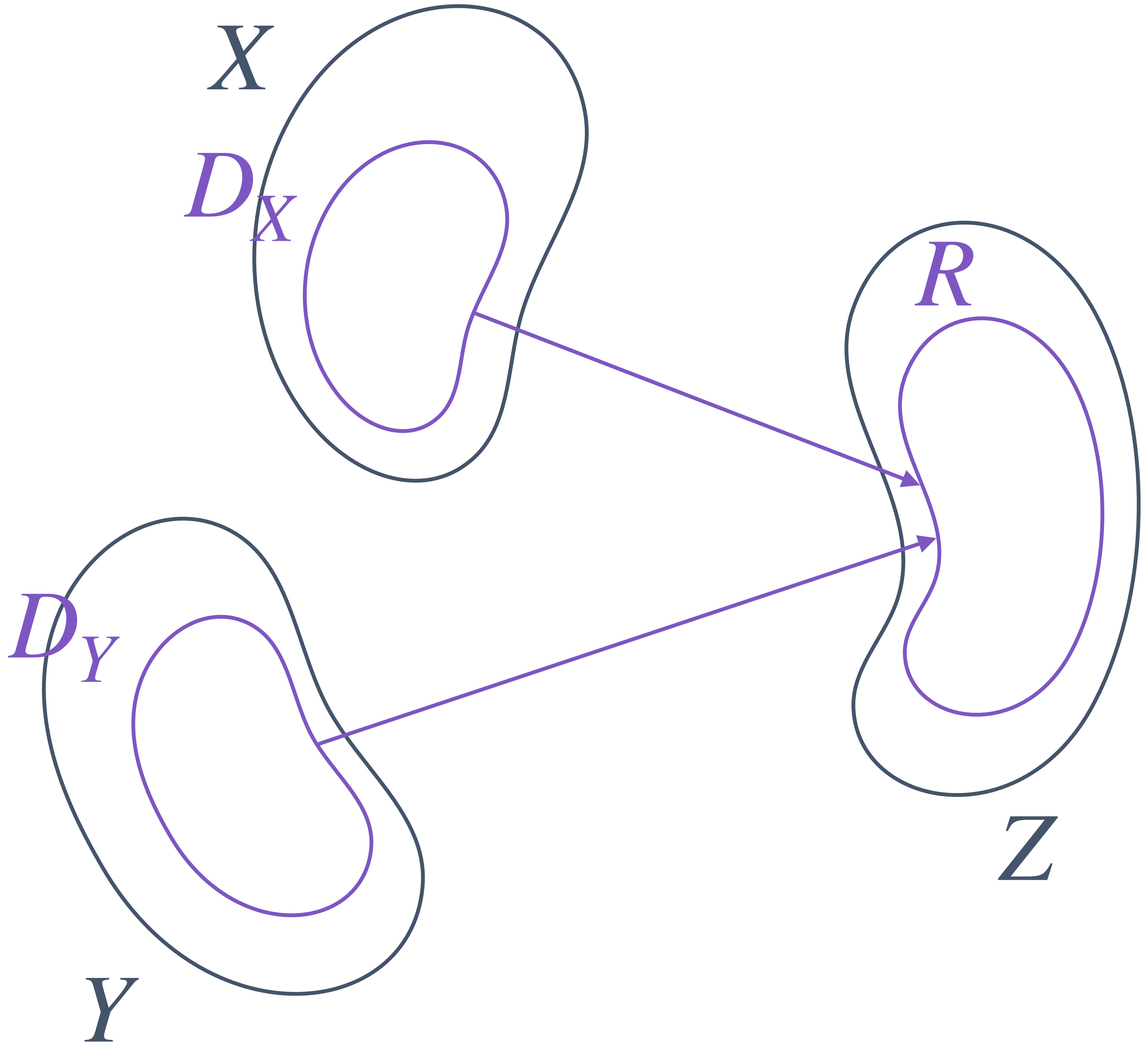
with Sets



$$\mathbb{R}^2 \longrightarrow \mathbb{R}$$

Bivariate Functions

with Sets



$$D_X \times D_Y \longrightarrow R$$

Bivariate Functions

with Sets

ex.1) $z = f(x, y) = x^2 + y^3$

$$\longrightarrow D_x = \mathbb{R}, \quad D_y = \mathbb{R}$$

ex.2) $z = f(x, y) = ln(x) + \dfrac{1}{y}$

$$\longrightarrow D_x = (0,\infty), \quad D_y = (-\infty,0) \cup (0,\infty)$$

Graphs of Bivariate Functions

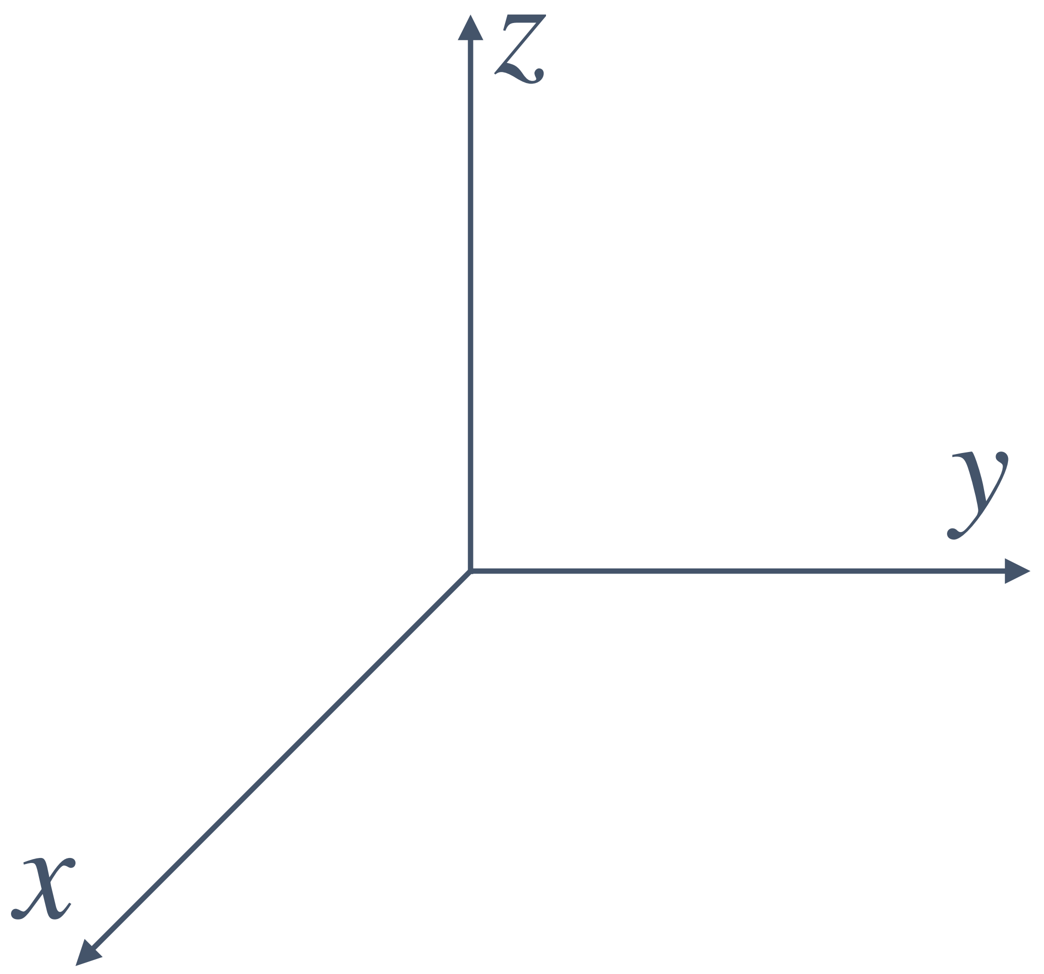
Axes

$$z = f(x, y)$$

axes for inputs → one axis for $x - axis$

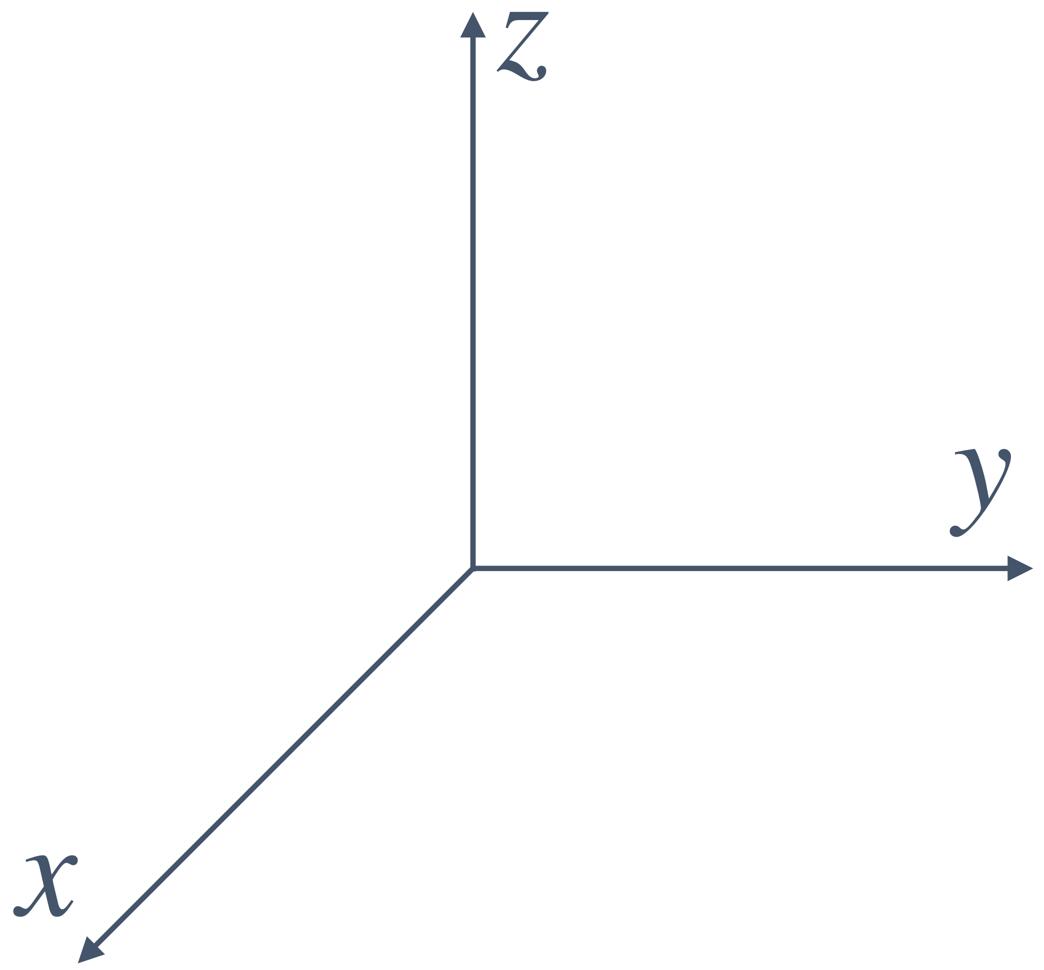
axes for inputs → one axis for $y - axis$

axis for output → one axis for $z - axis$



Graphs of Bivariate Functions

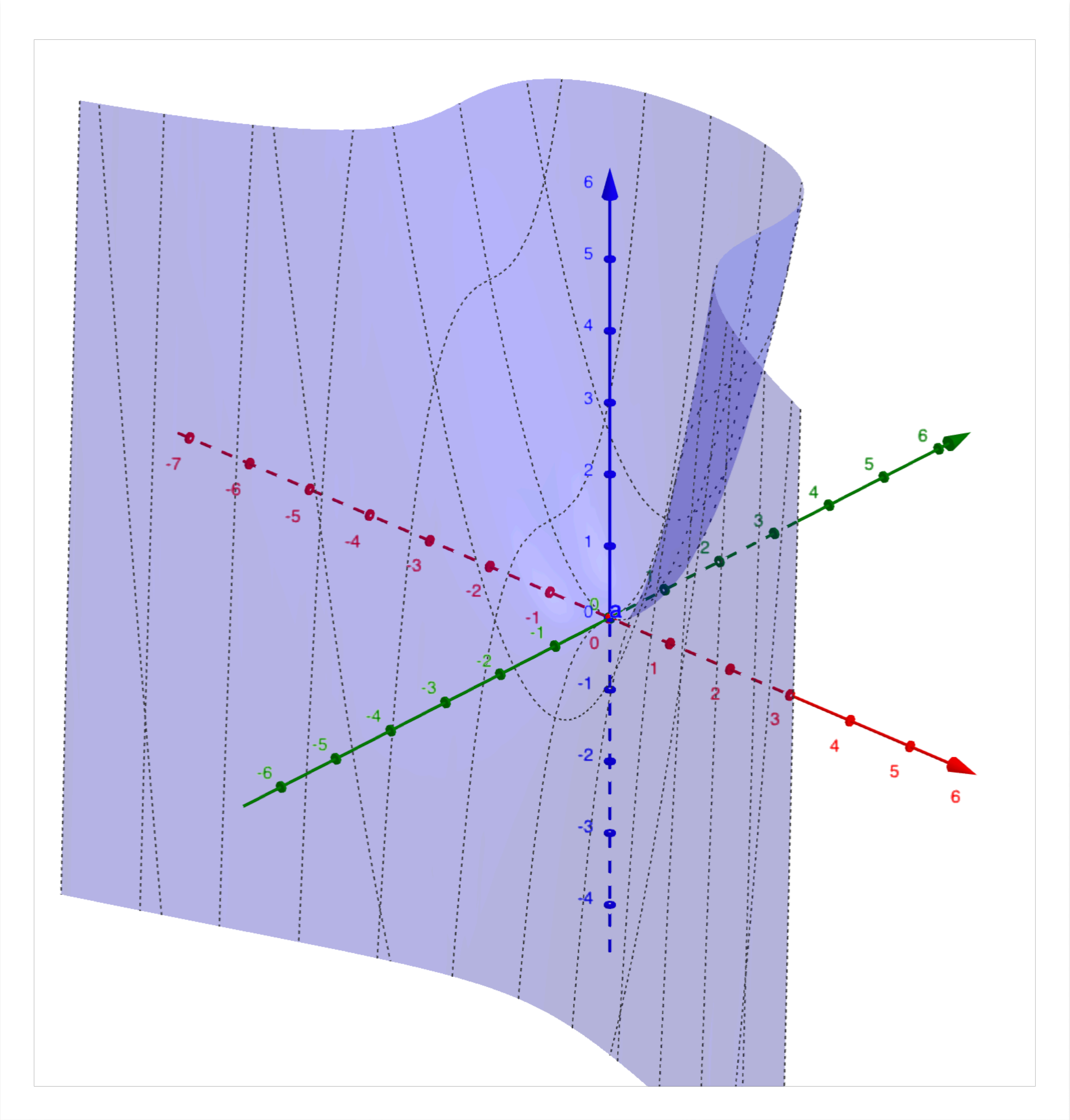
Graphs



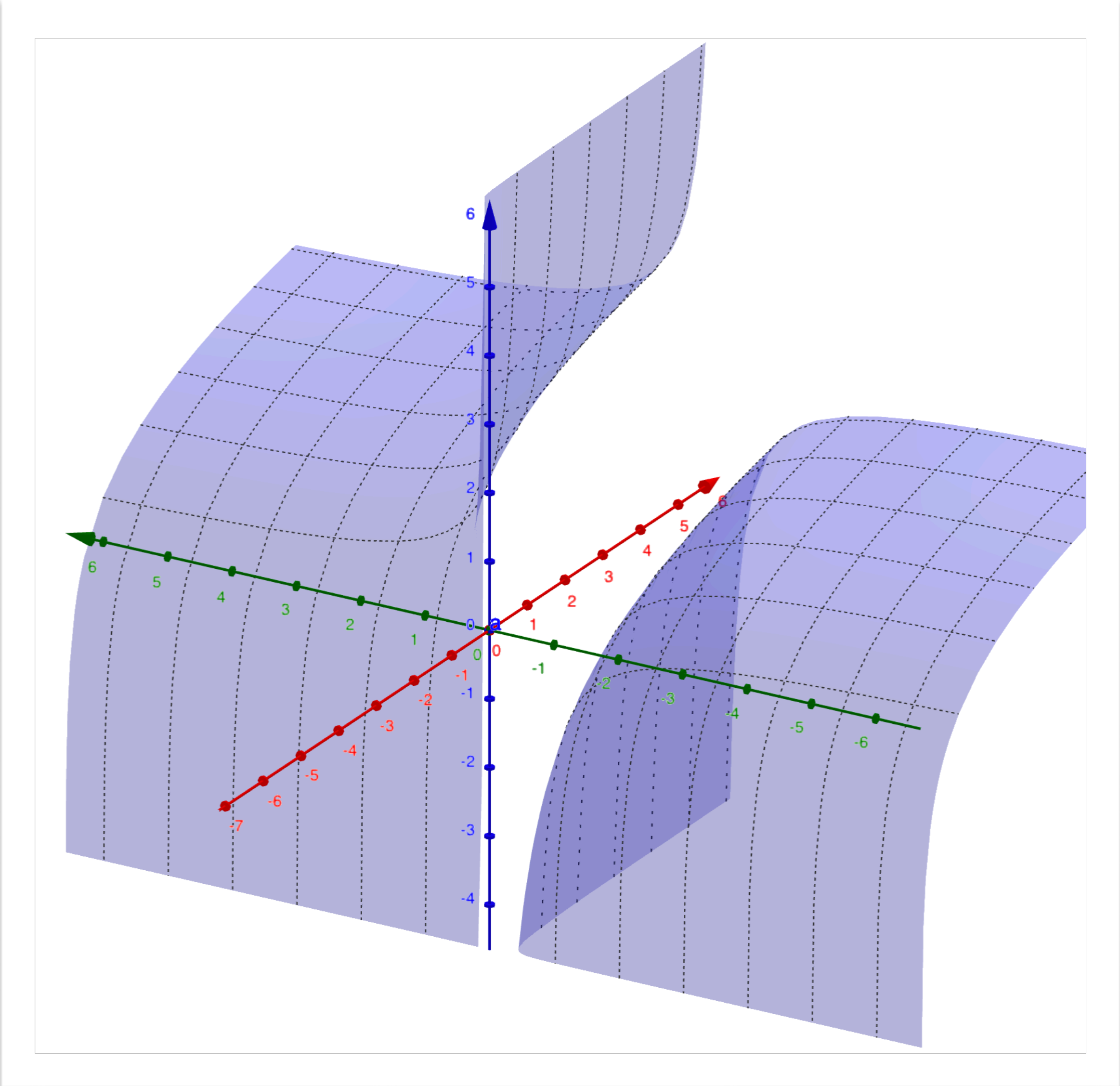
$$\{(x, y, z) \mid z = f(x, y),\ x \in D_X,\ y \in D_Y\}$$

Graphs of Bivariate Functions

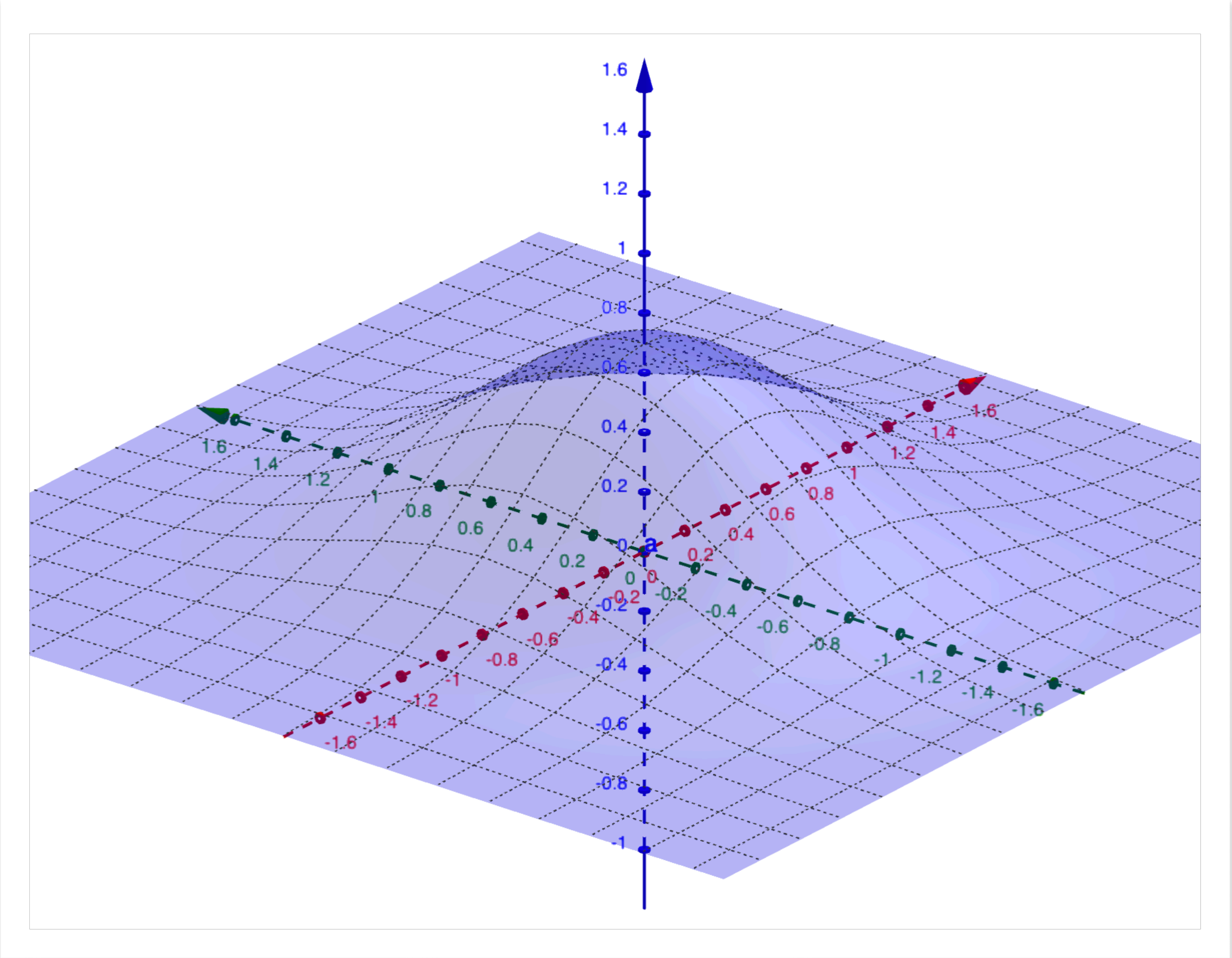
Examples



$$z = f(x, y) = x^2 + y^3$$

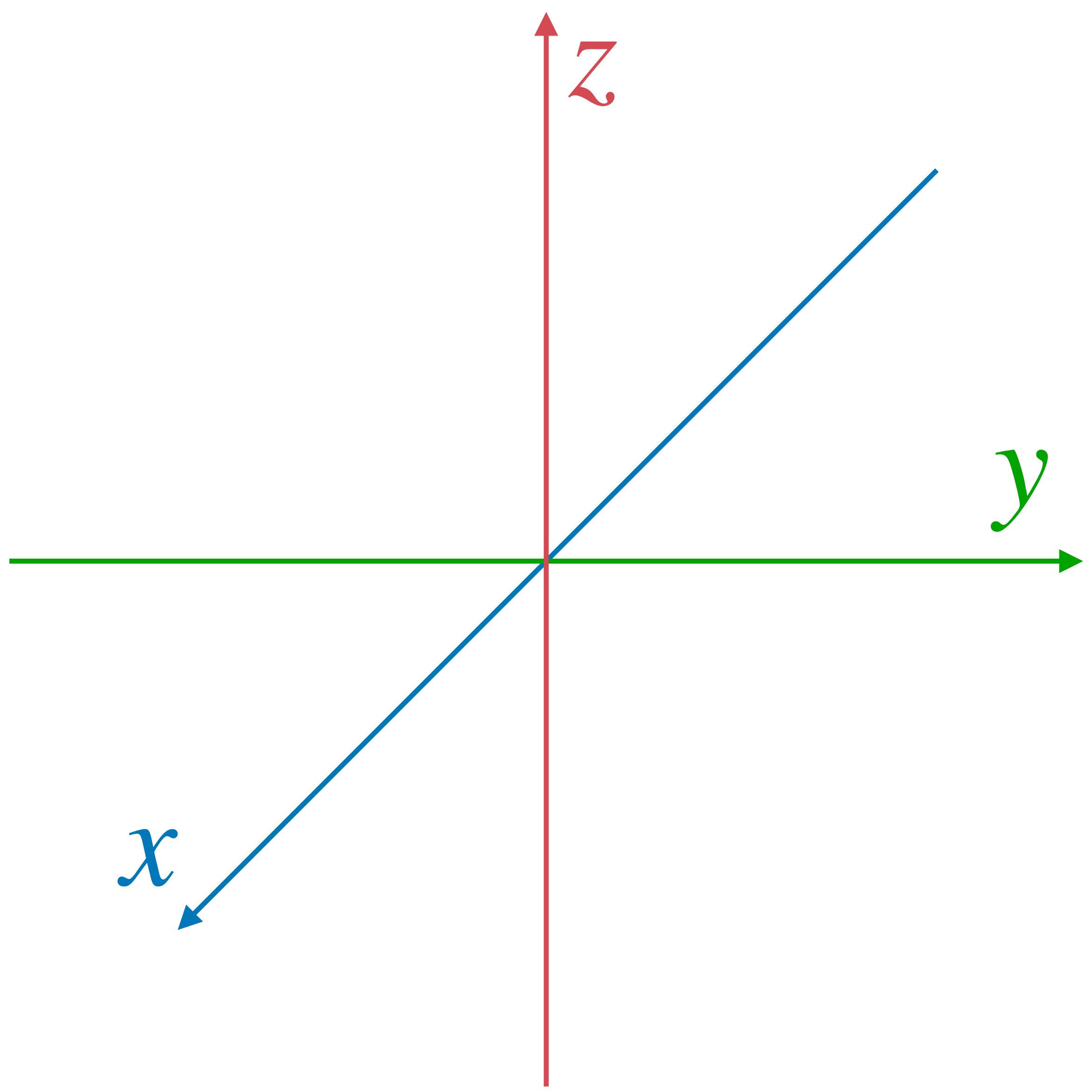


$$z = f(x, y) = ln(x) + \frac{1}{y}$$

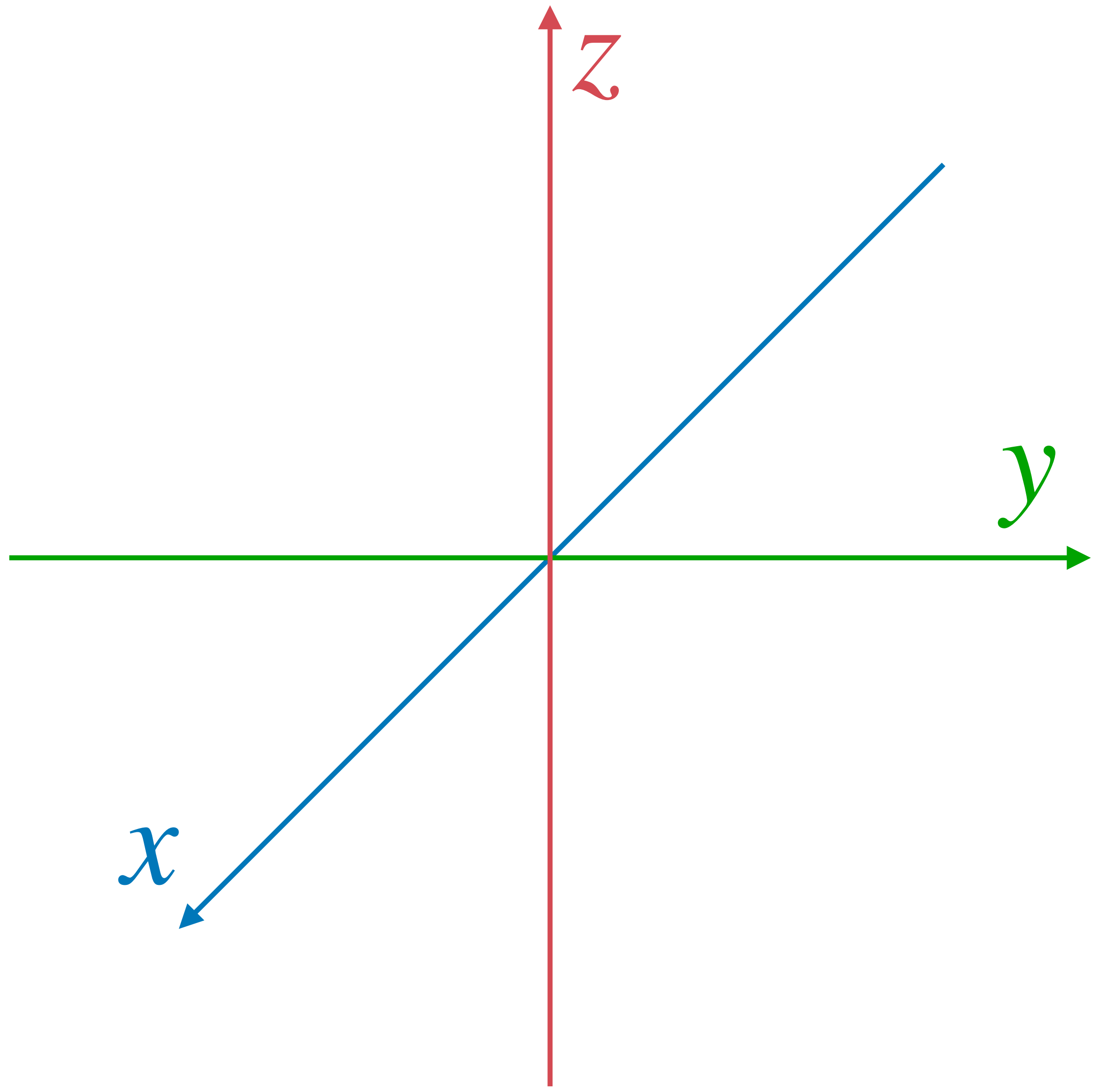


$$z = f(x, y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Three Axes in Coord. Space



Equations of Axes



x-axis

$$X = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x \in \mathbb{R},\, y = z = 0 \right\} \quad \longleftrightarrow \quad y = 0,\, z = 0$$

y-axis

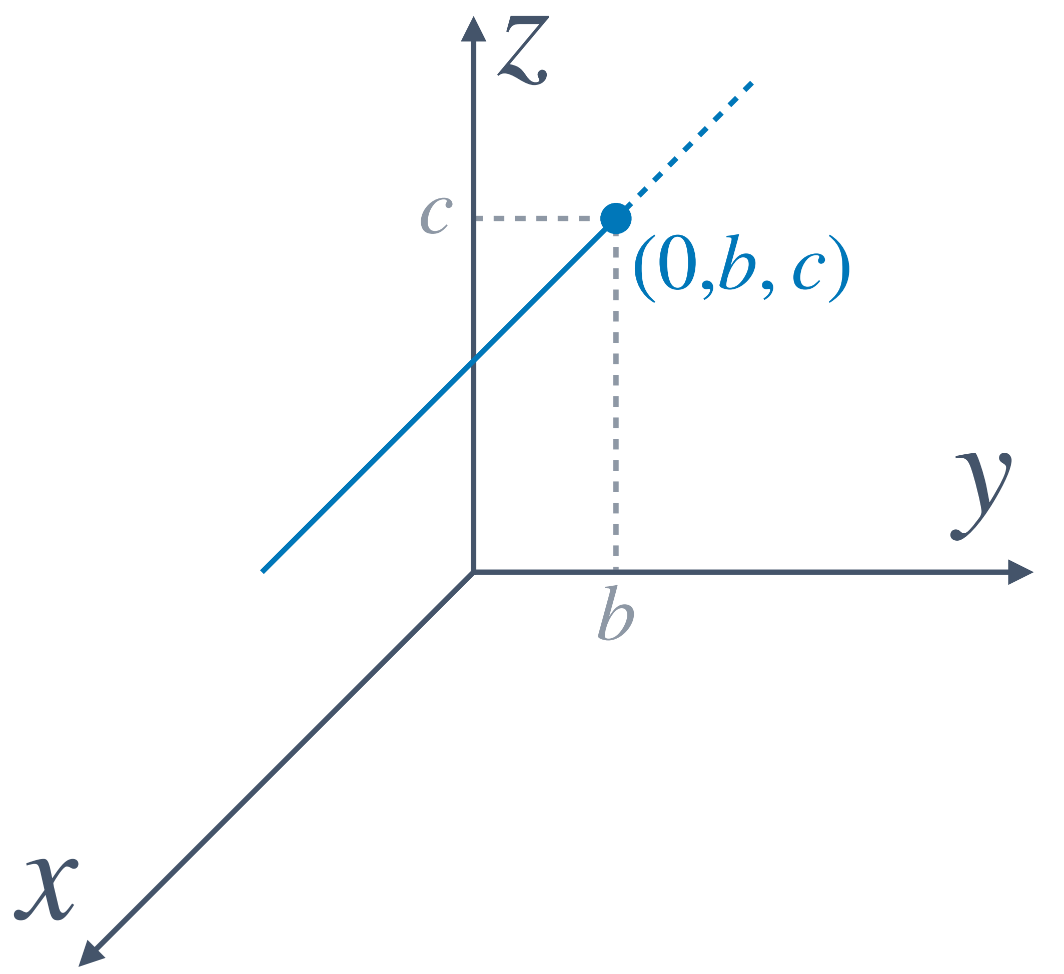
$$Y = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, y \in \mathbb{R},\, z = x = 0 \right\} \quad \longleftrightarrow \quad z = 0,\, x = 0$$

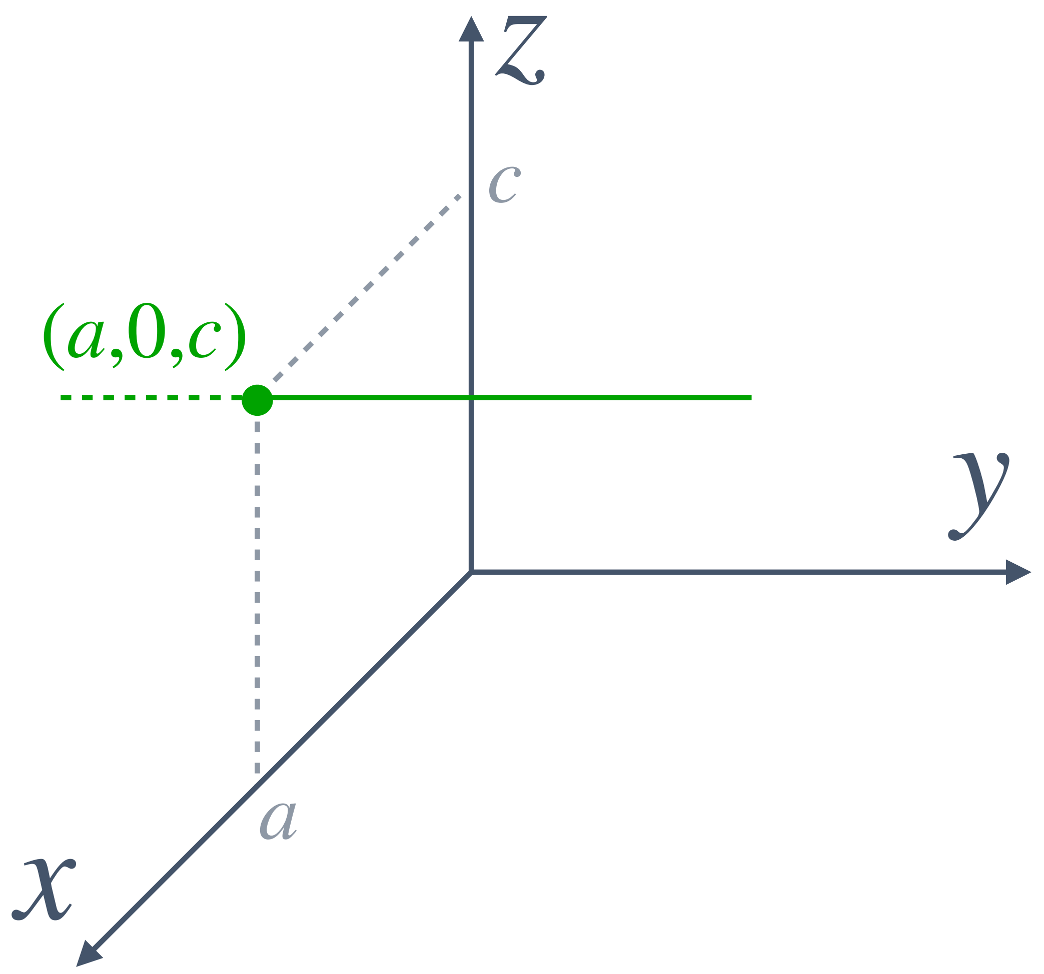
z-axis

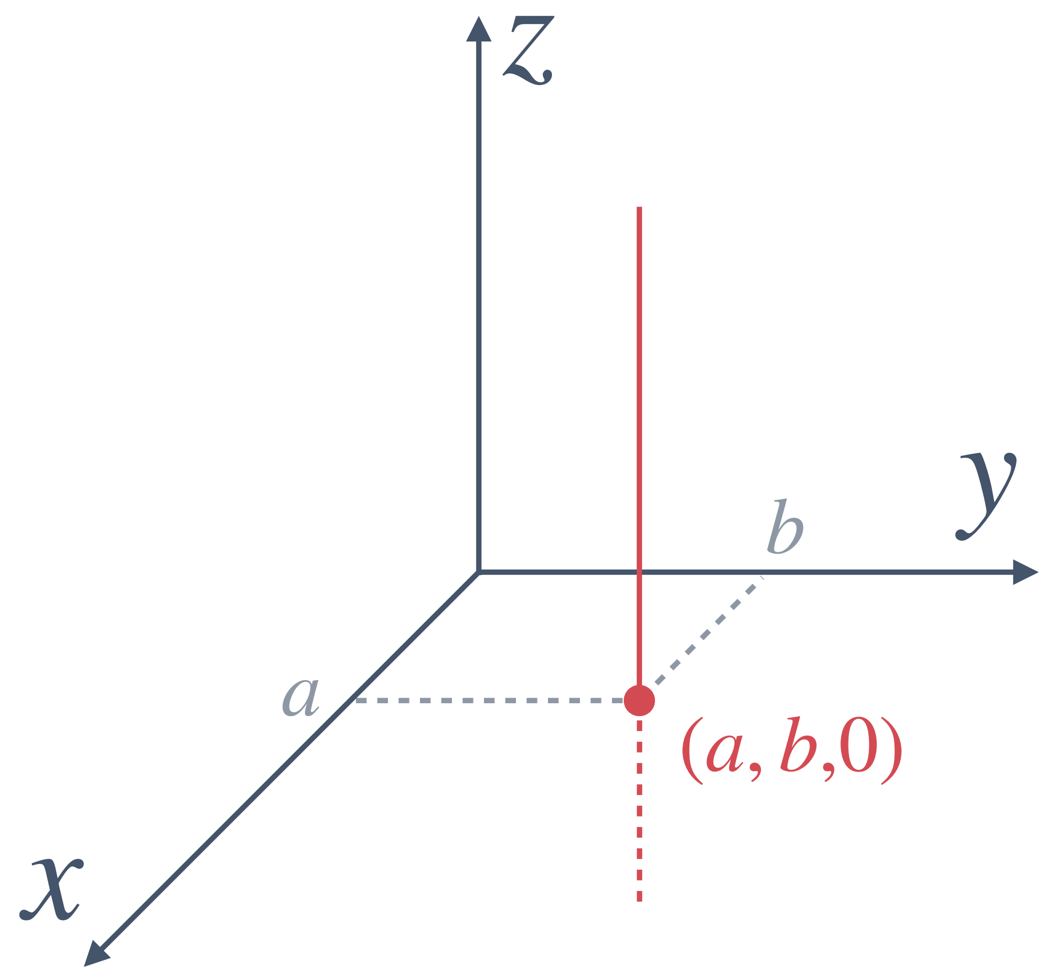
$$Z = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, z \in \mathbb{R},\, x = y = 0 \right\} \quad \longleftrightarrow \quad x = 0,\, y = 0$$

Equations of Axes

Lines Parallel to Axes







$$L_X = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x \in \mathbb{R}, \, y = b, \, z = c \right\}$$

$$L_X : y = b, z = c$$

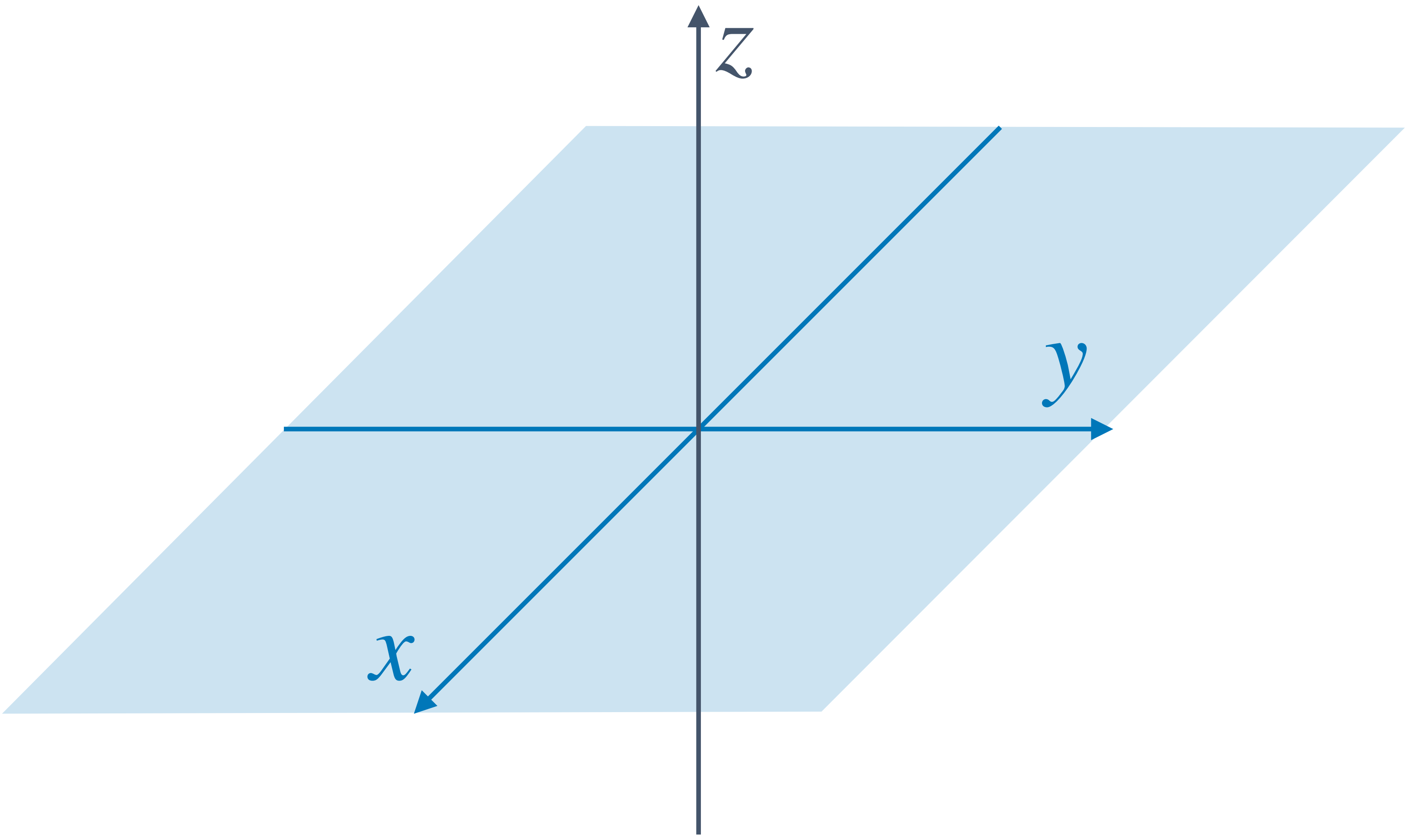
$$L_Y = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, y \in \mathbb{R}, \, z = c, \, x = a \right\}$$

$$L_Y : z = c, \, x = a$$

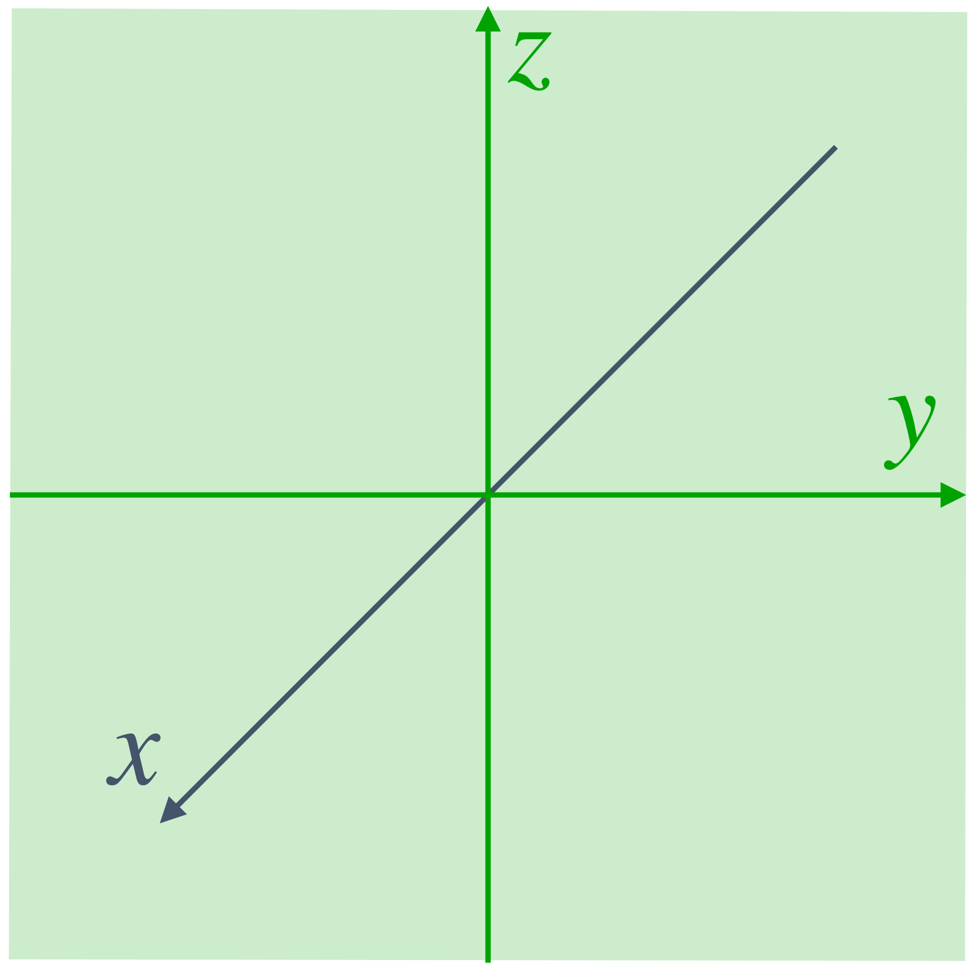
$$L_Z = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, z \in \mathbb{R}, \, x = a, \, y = b \right\}$$

$$L_Z : x = a, \, y = b$$

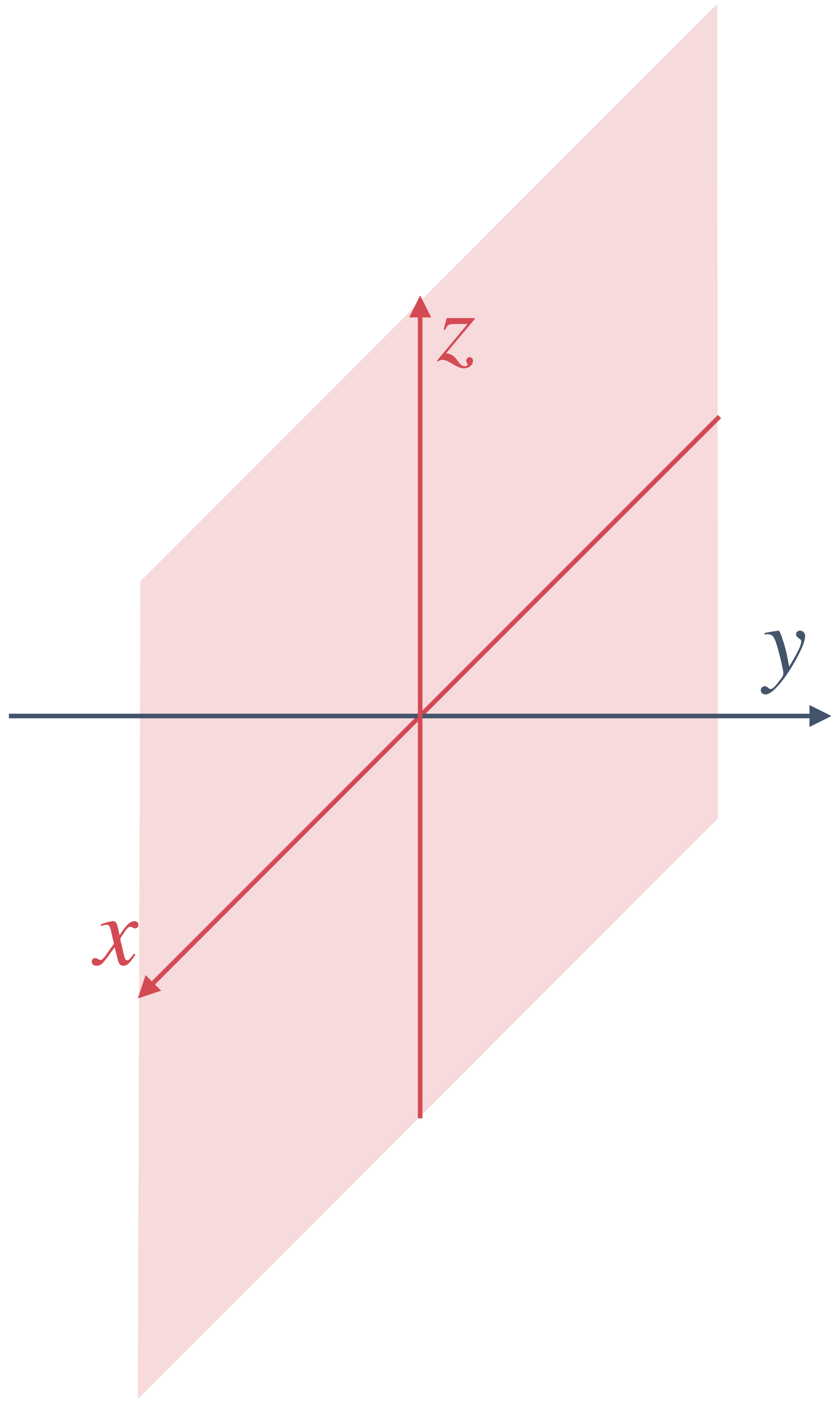
Three Planes in Coord. Space



xy-plane

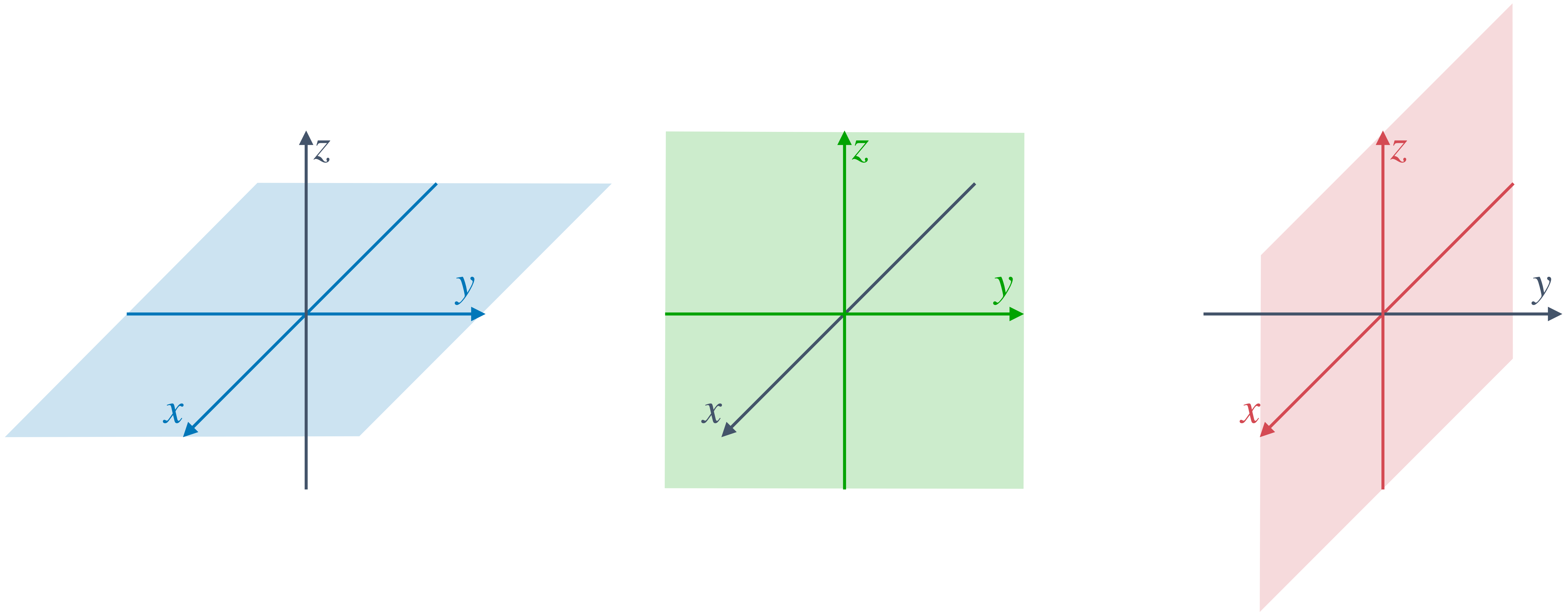


yz-plane



zx-plane

Equations of xy, yz, zx Planes



xy-plane yz-plane zx-plane

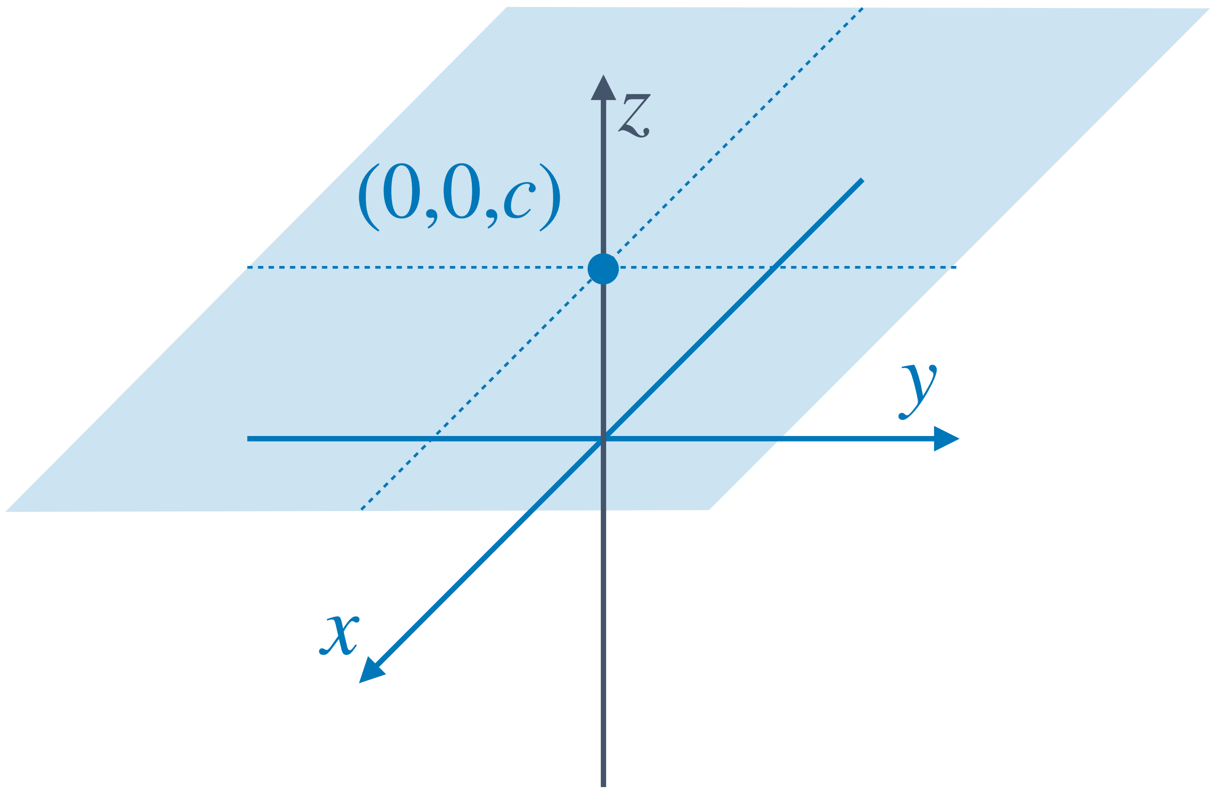
$$P_{XY} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x, y \in \mathbb{R}, \, z = 0 \right\}$$
$$P_{XY} : z = 0$$

$$P_{YZ} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, y, z \in \mathbb{R}, \, x = 0 \right\}$$
$$P_{YZ} : x = 0$$

$$P_{ZX} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, z, x \in \mathbb{R}, \, y = 0 \right\}$$
$$P_{ZX} : y = 0$$

Equations of Planes

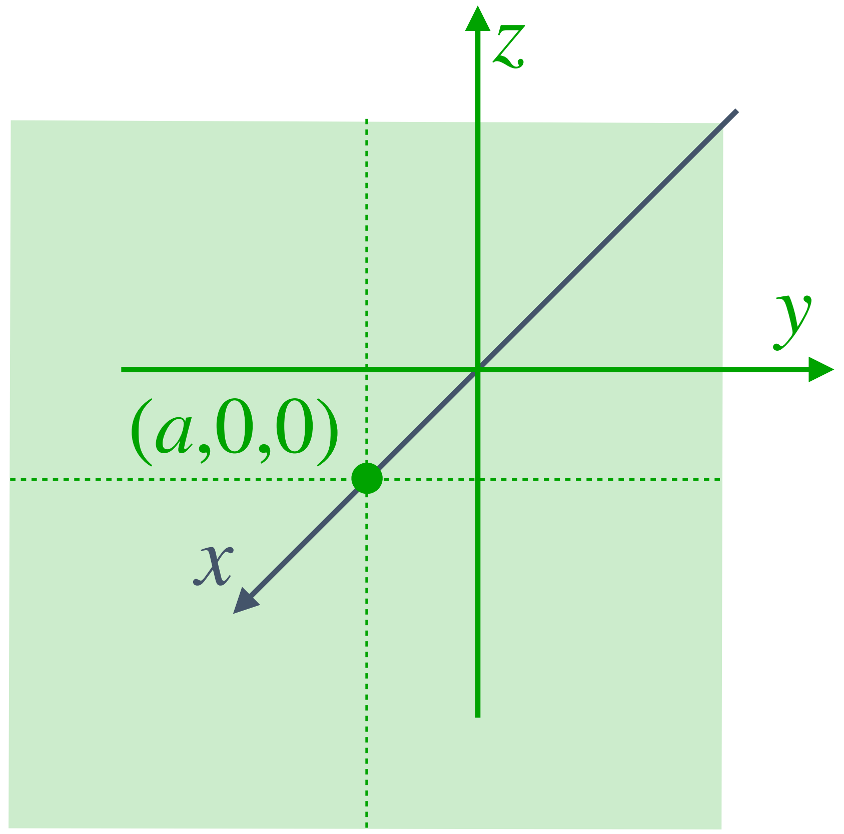
Parallel to xy, yz, zx Planes



Parallel to xy-plane

$$P_{XY} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x, y \in \mathbb{R},\ z = c \right\}$$

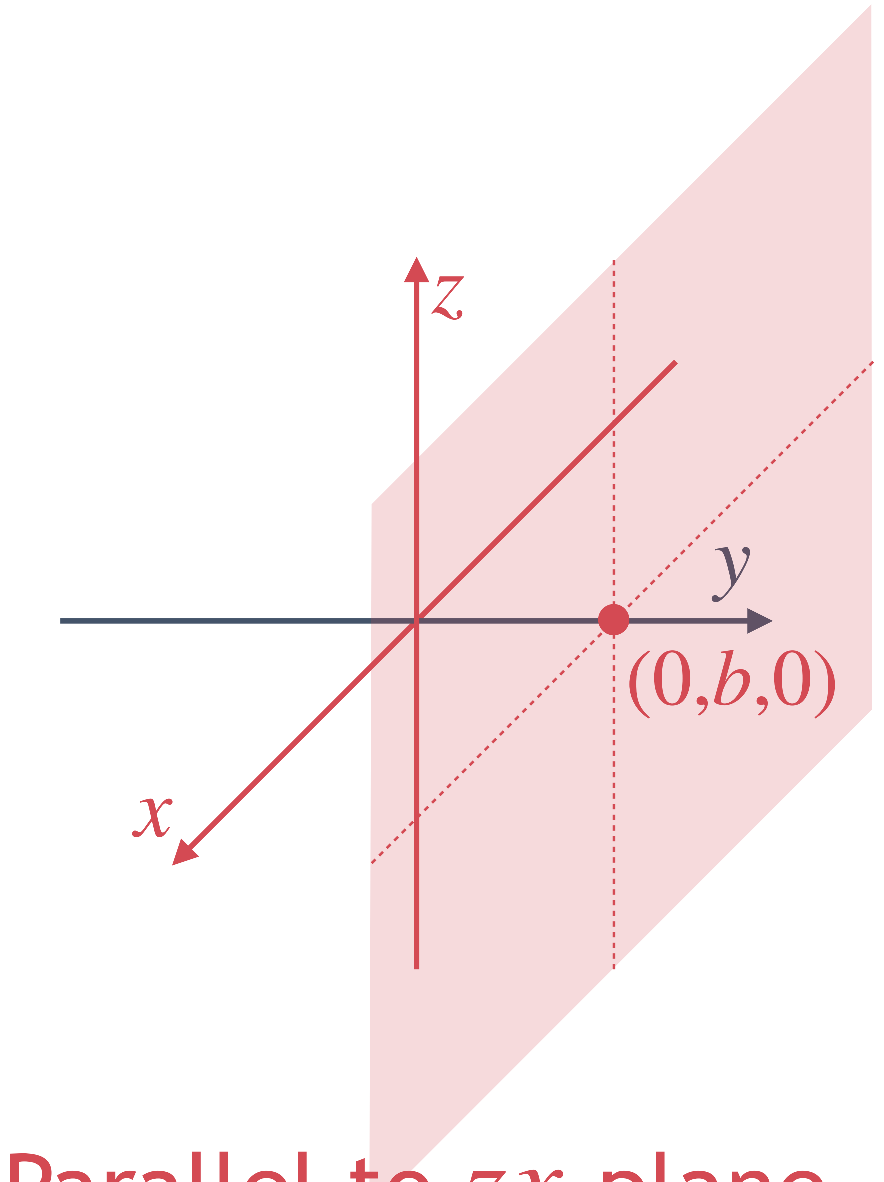
$$P_{XY} : z = c$$



Parallel to yz-plane

$$P_{YZ} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, y, z \in \mathbb{R},\ x = a \right\}$$

$$P_{YZ} : x = a$$



Parallel to zx-plane

$$P_{ZX} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, z, x \in \mathbb{R},\ y = b \right\}$$

$$P_{ZX} : y = b$$

Vertical Sections of Bivariate Functions

$$z = f(x, y)$$

$$z_x = f(a, y) \qquad\qquad z_y = f(x, b)$$

$$Z_x : \left\{ \begin{pmatrix} a \\ y \\ f(a, y) \end{pmatrix} \middle| \, y \in D_Y \right\} \qquad\qquad Z_y : \left\{ \begin{pmatrix} x \\ b \\ f(x, b) \end{pmatrix} \middle| \, x \in D_X \right\}$$

Example $z = x^2 + y^2$

$f(1,y) = 1 + y^2$ $\qquad$ $f(x,1) = x^2 + 1$

$f(2,y) = 4 + y^2$ $\qquad$ $f(x,2) = x^2 + 4$

$f(3,y) = 9 + y^2$ $\qquad$ $f(x,3) = x^2 + 9$

Vertical Sections of Bivariate Functions

ex.1) 함수 $z = f(x, y) = x^2y$와 각 평면사이의 intersection을 구하세요.

(1) $y = 0 \longrightarrow f(x, 0) = 0 \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$

$y = 1 \longrightarrow f(x, 1) = x^2 \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 1 \\ x^2 \end{pmatrix}$

$y = -1 \longrightarrow f(x, -1) = -x^2 \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -1 \\ -x^2 \end{pmatrix}$

$y = 2 \longrightarrow f(x, 2) = 2x^2 \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2 \\ 2x^2 \end{pmatrix}$

$y = -2 \longrightarrow f(x, -2) = -2x^2 \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -2 \\ -2x^2 \end{pmatrix}$

(2) $x = 0 \longrightarrow f(0, y) = 0 \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$

$x = 1 \longrightarrow f(1, y) = y \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ y \\ y \end{pmatrix}$

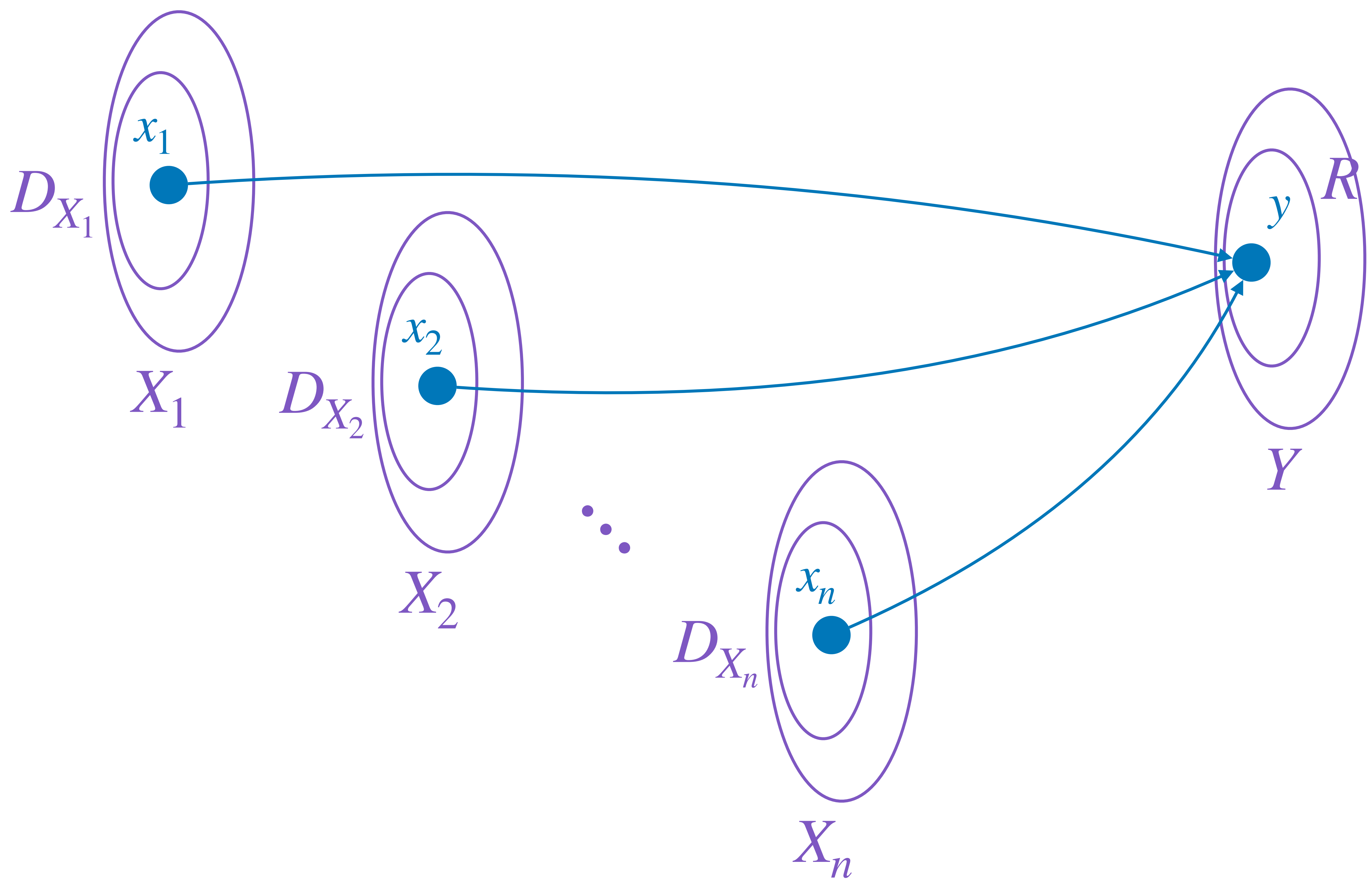
$x = -1 \longrightarrow f(-1, y) = y \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ y \\ y \end{pmatrix}$

$x = 2 \longrightarrow f(2, y) = 4y \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 4y \end{pmatrix}$

$x = -2 \longrightarrow f(-2, y) = 4y \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ y \\ 4y \end{pmatrix}$

Multivariate Functions

$$y = f(x_1, x_2, \dots\ , x_n)$$



Operations on Multivariate Functions

constant multiplications $\alpha \cdot f : (x_1, x_2, \ldots, x_n) \longmapsto \alpha \cdot f(x_1, x_2, \ldots, x_n)$

function additions $f + g : (x_1, x_2, \ldots, x_n) \longmapsto f(x_1, x_2, \ldots, x_n) + g(x_1, x_2, \ldots, x_n)$

Examples of Multivariate Functions

Vector Norm

$$f(x_1, x_2, \ldots, x_n) = \sqrt{(x_1)^2 + (x_2)^2 + \ldots + (x_n)^2}$$

Weighted Sum

$$f(x_1, x_2, \ldots, x_n) = w_1 x_1 + w_2 x_2 + \ldots + w_n x_n$$

Linearity

Linearity = Homogeneity + Additivity

Linearity

Homogeneity

$$f(\alpha \cdot x) = \alpha f(x)$$

Additivity

$$f(x + y) = f(x) + f(y)$$

Linearity

$$f(\alpha \cdot x + \beta \cdot y) = \alpha f(x) + \beta f(y)$$

Multivariate Linear Functions

Homogeneity

$$f(\alpha \cdot x_1, \alpha \cdot x_2, \ldots\ , \alpha \cdot x_n) = \alpha f(x_1, x_2, \ldots, x_n)$$

Additivity

$$f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = f(x_1, x_2, \ldots, x_n) + f(y_1, y_2, \ldots, y_n)$$

Linearity

$$f(\alpha \cdot x_1 + \beta \cdot y_1, \alpha \cdot x_2 + \beta \cdot y_2,\ \ldots\ , \alpha \cdot x_n + \beta \cdot y_n) = \alpha f(x_1, x_2, \ldots, x_n) + \beta f(y_1, y_2, \ldots, y_n)$$

Examples

ex.1) 다음 함수가 linear한지 확인하세요.

(1) $f(x) = ax$

(2) $f(x) = ax + b$

(3) $f(x) = x^2$

(4) $f(x_1, x_2, \ldots, x_n) = w_1x_1 + w_2x_2 + \ldots + w_nx_n$

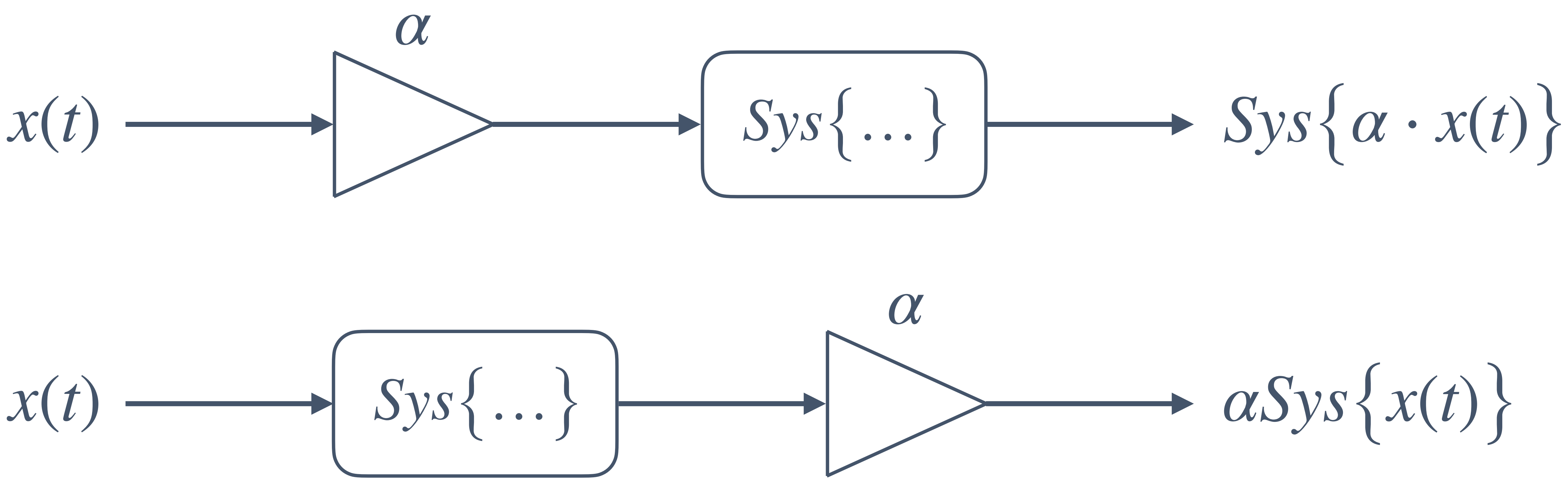
Systems

Function, Combinations of Functions
Operation, Combinations of Operations

Linearity

Homogeneity

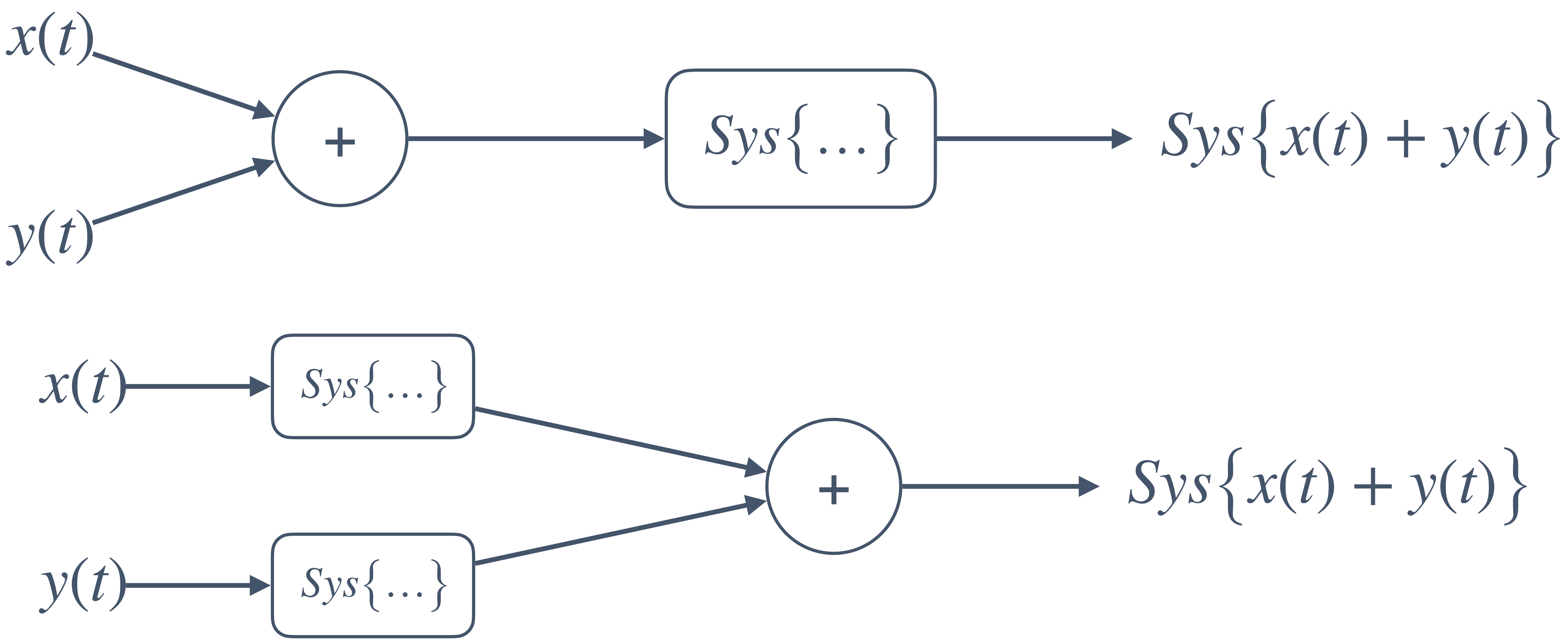
$$Sys\{\alpha \cdot x(t)\} = \alpha Sys\{x(t)\}$$



Linearity

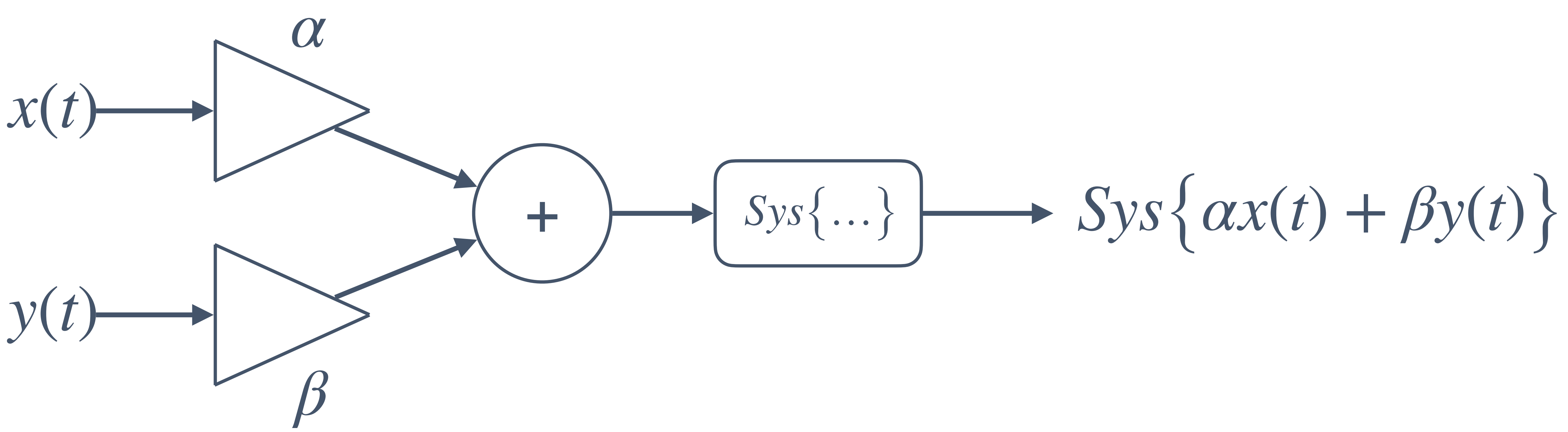
Additivity

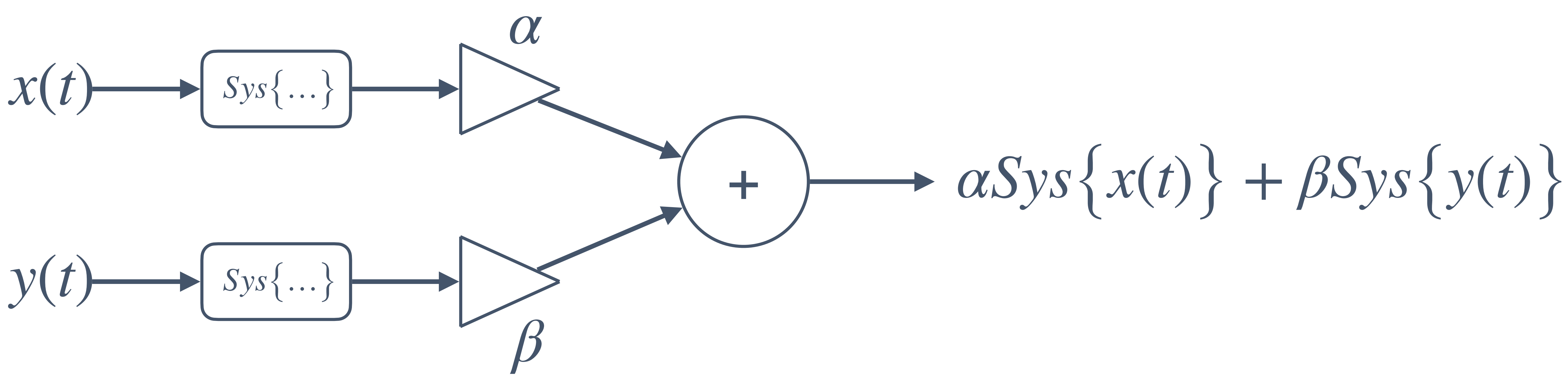
$$Sys\left\{x(t)+y(t)\right\} = Sys\left\{x(t)\right\} + Sys\left\{y(t)\right\}$$



Linear Systems

$$Sys\{\alpha x(t) + \beta y(t)\} = \alpha Sys\{x(t)\} + \beta Sys\{y(t)\}$$





Examples

ex.1) Differentiation $\dfrac{d}{dx}[\ldots]$

Homogeneity $\dfrac{d}{dx}\left[\alpha f(x)\right] = \alpha \dfrac{d}{dx}\left[f(x)\right]$

Additivity $\dfrac{d}{dx}\left[f(x) + g(x)\right] = \dfrac{d}{dx}\left[f(x)\right] + \dfrac{d}{dx}\left[g(x)\right]$

$\Longrightarrow$ Linearity $\dfrac{d}{dx}\left[\alpha f(x) + \beta g(x)\right] = \alpha \dfrac{d}{dx}\left[f(x)\right] + \beta \dfrac{d}{dx}\left[g(x)\right]$

(1) $\dfrac{d}{dx}\left[3x^2 - 2x\right] = 3 \cdot \dfrac{d}{dx}\left[x^2\right] - 2 \cdot \dfrac{d}{dx}\left[x\right]$

(2) $\dfrac{d}{dx}\left[2sin(x) - 5ln(x)\right] = 2 \cdot \dfrac{d}{dx}\left[sin(x)\right] - 5 \cdot \dfrac{d}{dx}\left[ln(x)\right]$

Examples

ex.2) Integraion $\int [\dots]\, dx$

Homogeneity $\int [\alpha f(x)]\, dx = \alpha \int f(x)\, dx$

Additivity $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

$\Longrightarrow$ Linearity $\int [\alpha f(x) + \beta g(x)]\, dx = \alpha \int f(x)\, dx + \beta \int g(x)\, dx$

(1) $\int [3x^2 - 2x]\, dx = 3 \int x^2\, dx - 2 \int x\, dx$

(2) $\int [2sin(x) - 5ln(x)]\, dx = 2 \int sin(x)\, dx - 5 \int ln(x)\, dx$

Examples

ex.3) Fourier Transform $\mathscr{F}\left\{x(t)\right\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$

Homogeneity $\mathscr{F}\left\{\alpha f(t)\right\} = \alpha \mathscr{F}\left\{f(t)\right\}$

$$\int_{-\infty}^{\infty} \left[\alpha f(t)\right] \cdot e^{-j\omega t}\, dt = \alpha \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t}\, dt$$

Additivity $\mathscr{F}\left\{f(t) + g(t)\right\} = \mathscr{F}\left\{f(t)\right\} + \mathscr{F}\left\{g(t)\right\}$

$$\int_{-\infty}^{\infty} \left[f(t) + g(t)\right] \cdot e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} g(t) \cdot e^{-j\omega t}\, dt$$

$\Longrightarrow$ Linearity $\mathscr{F}\left\{\alpha f(t) + \beta g(t)\right\} = \alpha \mathscr{F}\left\{f(t)\right\} + \beta \mathscr{F}\left\{g(t)\right\}$

$$\int_{-\infty}^{\infty} \left[\alpha f(t) + \beta g(t)\right] \cdot e^{-j\omega t}\, dt = \alpha \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t}\, dt + \beta \int_{-\infty}^{\infty} g(t) \cdot e^{-j\omega t}\, dt$$

(1) $\mathscr{F}\left\{3x^2 - 2x\right\} = 3\mathscr{F}\left\{x^2\right\} - 2\mathscr{F}\left\{x\right\}$

$$= 3\int_{-\infty}^{\infty} x^2 \cdot e^{-j\omega t}\, dt - 2\int_{-\infty}^{\infty} x \cdot e^{-j\omega t}\, dt$$

(2) $\mathscr{F}\left\{2sin(x) - 5ln(x)\right\} = 2\mathscr{F}\left\{sin(x)\right\} - 5\mathscr{F}\left\{ln(x)\right\}$

$$= 2\int_{-\infty}^{\infty} sin(x) \cdot e^{-j\omega t}\, dt - 5\int_{-\infty}^{\infty} ln(x) \cdot e^{-j\omega t}\, dt$$

CLOSING

Basic Algebra

Chap.10 Multivariate Functions and Linearity