Linear Algebra in Voting Theory

Mathematics usually takes part in every scientific fields so that it's known as mother of sciences.
In this essay, we'll consider only the participation of Linear Algebra in Voting Theory, and won't understand which method is the best to announce the winner of an election.

We'll consider an election between three candidates at first, and then, will check some methods related to
Linear Algebra for any arbitrary number of candidates 'n'.

Profile

Ask n judges to fully rank A1 , ..., Am, from most preferred to least preferred, and encode the resulting data as a profile $p \in \mathbb{R}$^(m!)

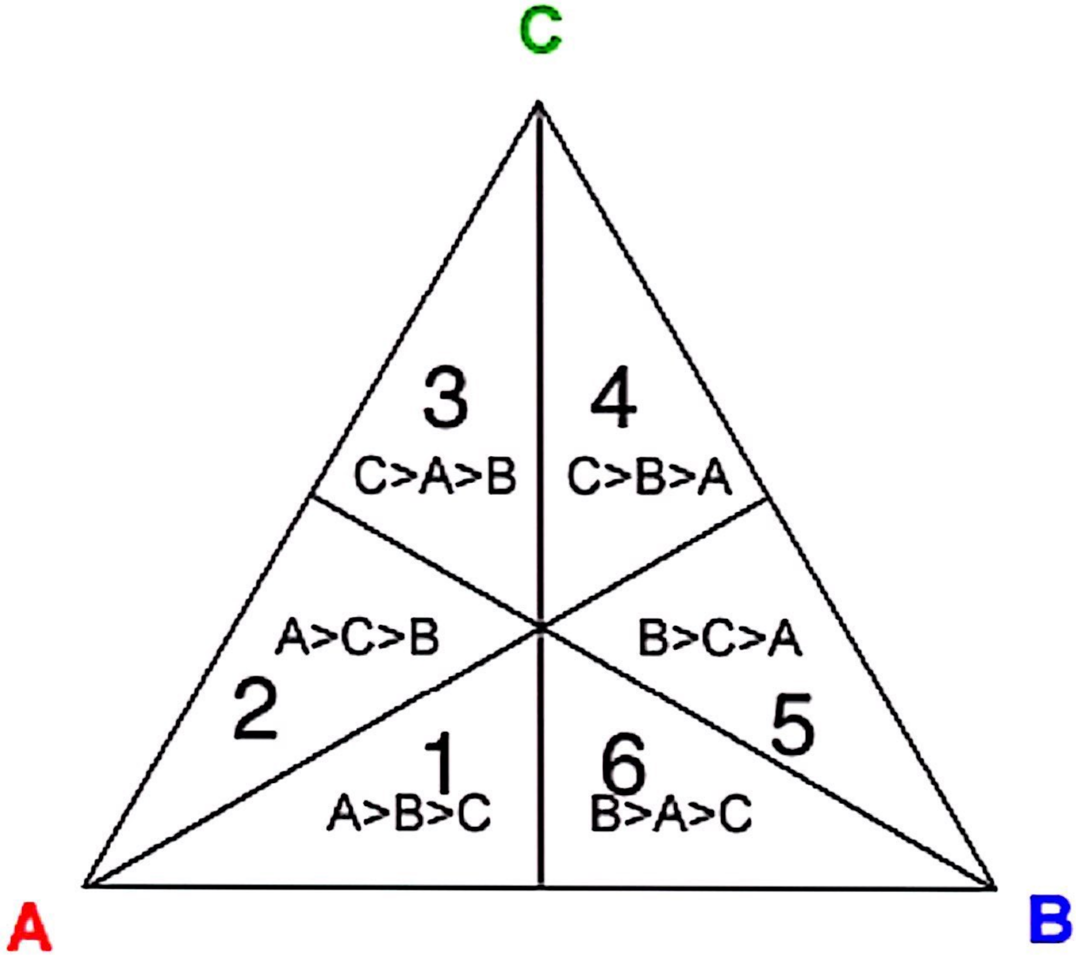
For example for m = 3, and the alternatives are ordered lexicographically, then the profile

$p = [10, 15, 2, 7, 9, 21]t \in \mathbb{R}$^6

encodes the situation where 10 judges chose the ranking A>B>C, 15 chose A>C>B, 2 chose B>A>C, and so on.

Triangular linear method in three-candidates election

Assume we have an election between candidates A, B, and C with 11 voters. So we have a triangle above:



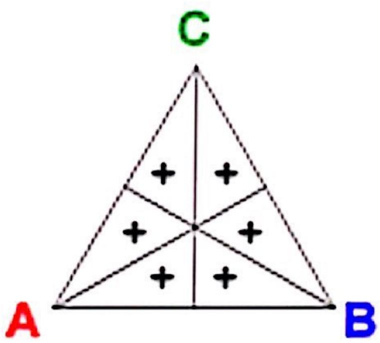
Since each preference have a number, we have the following chart:

Preference	Number of votes
A > B > C	2
A > C > B	3
C > A > B	0
C > B > A	2
B > C > A	4
B > A > C	0

We have some operator vectors which may change the number of votes of a preference:

Kernel Vector

The *kernel vector* is $\boldsymbol{K} = (1, 1, 1, 1, 1, 1)$. Adding $\boldsymbol{K}$ to a profile adds one voter to each of the six preference orders.



If we add it in mentioned election, the chart will be updated:

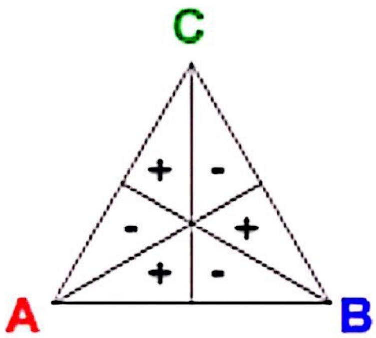
Preference	Number of votes
A > B > C	3
A > C > B	4
C > A > B	1
C > B > A	3
B > C > A	5
B > A > C	1

Condorcet Vector

The *Condorcet vector* is

C = (1, −1, 1, −1, 1, −1).

Adding ***C*** to a profile adds a voter to preferences A > B > C, B > C > A, and C > A > B, and deducts a voter from the reverse preferences C > B > A, A > C > B, and B > A > C.



If we add it in mentioned election, the chart will be updated:

Preference	Number of votes
A > B > C	3
A > C > B	2
C > A > B	1
C > B > A	1
B > C > A	5
B > A > C	0 <~ -1

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Reversal Vectors

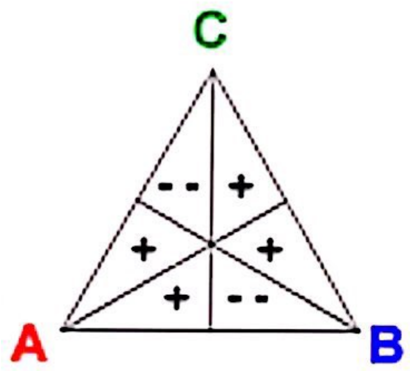
The vectors *Reversal A*, *Reversal B*, and *Reversal C* are defined by

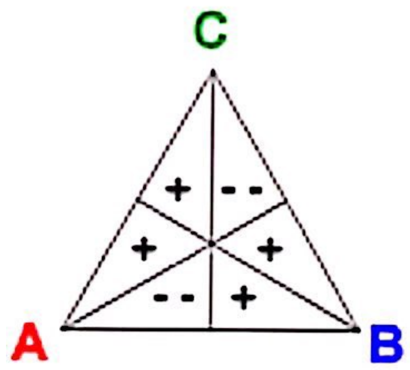
RA = (1, 1, −2, 1, 1, −2)

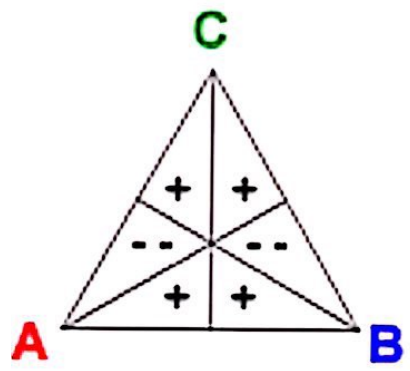
RB = (−2, 1, 1, −2, 1, 1)

RC = (1, −2, 1, 1, −2, 1).

If we add whole reversal vectors, we won't have any change in the result.







Basic Vectors

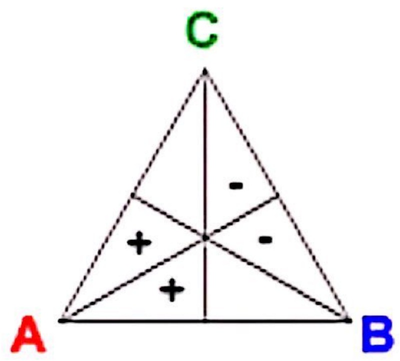
The vectors *Basic A*, *Basic B*, and *Basic C* are defined by

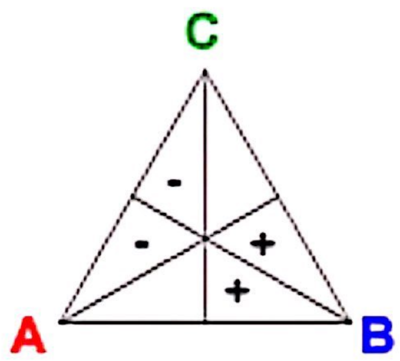
BA = (1, 1, 0, −1, −1, 0)

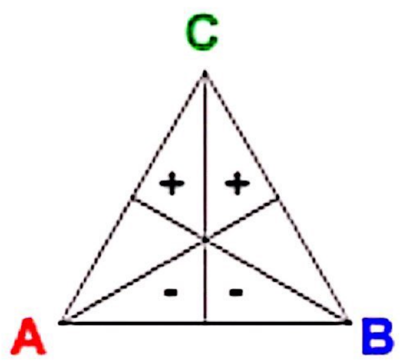
BB = (0, −1, −1, 0, 1, 1)

BC = (−1, 0, 1, 1, 0, −1).

As reversal vectors, if we add whole reversal vectors, we won't have any change in the result.







All profiles can be expressed as

$p = pK + pB + pC + pR$,

where the profile differentials on the right-hand side come from, respectively, the Kernel, the Basic, the Condorcet, and the Reversal subspaces.

The four subspaces are mutually orthogonal.

Equivalency in profiles

Let $X, Y \in C$.

We say two profiles p, q are

XY –equivalent ($p \sim q$) iff If X defeats Y in a

head–to–head race in p,

then X does so in q as well.

There's a vector Rxy such that p~q iff (p-q)Rxy = 0 means that Rxy rotates p to q.

Diagonal matrix as a chart

According to plurality method, we can define a diagonal matrix diag(a1, a2, ..., an) for n candidates in which ai is equal to i-th candidate's total votes.

Positional Voting

We start this subject with an example.
Let $Q = \{A, B, C, D, E, F\}$ be the set of candidates and $\lambda=(2, 3 ,1)$.
If $w=(5, 3, 1)$ be a weight vector means we consider 5, 3, and 1 point for the ballots respectively, we would calculate a criterion for each candidate.

$$P_\lambda = \mathrm{sp}_{\mathbb{R}} \left\{ \cdot \begin{array}{lll} C1 & C2 & \\ C3 & C4 & C5 \\ C6 & & \end{array} : C_i \in \mathbf{Q} \right\}$$

$$\mathbf{p} = 2 \begin{array}{lll} C & D & \\ A & B & E \\ F & & \end{array} + 4 \begin{array}{lll} A & B & \\ D & C & F \\ E & & \end{array} + 7 \begin{array}{lll} A & B & \\ F & D & C \\ E & & \end{array}$$

Positional point totals are then ...

A	$2 \cdot 3 + 4 \cdot 5 + 7 \cdot 5 = 61.$
B	$2 \cdot 3 + 4 \cdot 5 + 7 \cdot 5 = 61.$
C	$2 \cdot 5 + 4 \cdot 3 + 7 \cdot 3 = 43.$
D	$2 \cdot 5 + 4 \cdot 3 + 7 \cdot 3 = 43.$
E	$2 \cdot 3 + 4 \cdot 1 + 7 \cdot 1 = 17.$
F	$2 \cdot 1 + 4 \cdot 3 + 7 \cdot 3 = 35.$

Tally Matrices

In general, we have a weighting vector w = [w1, w2, ..., wn] ∊ R^n and n ≥ 2 for C={C1, C2, ..., Cn} which means candidate Ci's weigh is wi.

We call linear transformation

Tw: R^(n!) ———-> R^n a Tally matrix iff for any profile p ∊ R^(n!), Tw(p) = p.w

Hence we have N(Tw) ⊕ N(Tw) ⊥ = R^(n!).

For example for the following profile and weighting vector w=[1, s, 0], we have:

$$\mathbf{p} = \begin{array}{l} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 4 \\ 0 \\ 2 \end{bmatrix} \end{array} \begin{array}{l} \text{ABC} \\ \text{ACB} \\ \text{BAC} \\ \text{BCA} \\ \text{CAB} \\ \text{CBA} \end{array}$$

$$T_{\mathbf{w}}(\mathbf{p}) = \begin{bmatrix} 1 & 1 & s & 0 & s & 0 \\ s & 0 & 1 & 1 & 0 & s \\ 0 & s & 0 & s & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 4 \\ 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ s \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ s \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ s \end{bmatrix} + 2 \begin{bmatrix} 0 \\ s \\ 1 \end{bmatrix}$$

$$= (2e + 3(23) + 4(123) + 2(13)) \cdot \begin{bmatrix} 1 \\ s \\ 0 \end{bmatrix} = \mathbf{p} \cdot \mathbf{w}.$$

Equivalency in Weighting Vectors

Two nonzero weighting vectors x, y ∈ R^n are equivalent (x~y) iff there exist α, β ∈ R such that α > 0 and x = αy+ β1.

Let n ≥ 2, and let w and x be nonzero weighting vectors in R.

The ordinal rankings of Ty(p) and Tx(p) will be the same for all p∈R^(n!) iff x~y.

There're many methods to find out the winner of an election.

They may be assisted by other sciences such as Sociology, Data Science, etc.

In general, whole methods will be related to mathematics.

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