

Finite automata and formal languages

Assignment 1

DIT323 (Finite automata and formal languages)
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1

Question: Prove using induction that, for every finite alphabet, Σ , $\forall n \in \mathbb{N}$. $|\Sigma^n| = |\Sigma|^n$.

Solution: Let $P(n) := |\Sigma^n| = |\Sigma|^n$.

Proof: By induction on n .

Basis: $P(0)$ is true since $|\Sigma^0| = 1 = |\Sigma|^0$.

Inductive step: Assume $P(n)$ is true for some arbitrary $n \in \mathbb{N}$ (i.h.). We want to show that $P(n+1)$ is true.

Let Σ be a finite alphabet. Then, $|\Sigma^{\{n+1\}}| = |\Sigma| |\Sigma^n|$ by definition of $\Sigma^{\{n+1\}}$.

By the induction hypothesis, $|\Sigma^n| = |\Sigma|^n \Rightarrow |\Sigma^{\{n+1\}}| = |\Sigma| |\Sigma^n| = |\Sigma| |\Sigma|^n = |\Sigma|^{\{n+1\}}$.

$\therefore \forall n \in \mathbb{N}. P(n)$. □

2

Define a language S containing words over the alphabet $\Sigma = \{a, b\}$ inductively in the following way:

- The empty word is in S : $\varepsilon \in S$
- If $u, v \in S$, then $auv \in S$.
- If $u, v, w \in S$, then $buavaw \in S$.

2.1

Question: Use recursion to define two functions $\#_a, \#_b \in \Sigma^* \rightarrow \mathbb{N}$ that return the number of occurrences of a and b , respectively, in their input.

Solution:

$$\#_a(uv) = \begin{cases} 0 & \text{if } uv = \varepsilon \\ 1 + \#_a(v) & \text{if } u = a \\ \#_a(v) & \text{if } u \neq a \end{cases} \quad [1]$$

$$\#_b(uv) = \begin{cases} 0 & \text{if } uv = \varepsilon \\ 1 + \#_b(v) & \text{if } u = b \\ \#_b(v) & \text{if } u \neq b \end{cases} \quad [2]$$

2.2

Question: Use induction to prove that $\forall w \in S. \#_a(w) = 2\#_b(w)$.

Hint: You might want to show that $\#_a(auavb) = 2 + \#_a(u) + \#_a(v)$. How do you prove this? This property follows from a lemma that you can perhaps prove by induction: $\forall u, v \in \Sigma^*. \#_a(uv) = \#_a(u) + \#_a(v)$

Solution:

Proof of lemma from hint

Lemma to prove: $l_1 := \forall u, v \in \Sigma^*. \#_a(uv) = \#_a(u) + \#_a(v)$

Let $P(u) := \#_a(uv) = \#_a(u) + \#_a(v)$.

Proof: By induction on u .

Basis: $P(\varepsilon)$ is true since $\#_a(\varepsilon) = 0 = \#_a(\varepsilon) + \#_a(\varepsilon)$.

Inductive step: Assume $P(u)$ is true for some arbitrary $u \in \Sigma^*$ (i.h.). We want to show that $P(au)$ is true.

Let $v \in \Sigma^*$. Then,

$$\begin{aligned} \#_a(auv) &= 1 + \#_a(uv) && \text{(by definition of } \#_a) \\ &= 1 + \#_a(u) + \#_a(v) && \text{(i.h.)} \\ &= \#_a(au) + \#_a(v) && \text{(by definition of } \#_a) \end{aligned} \quad [3]$$

$\therefore \forall u, v \in \Sigma^*. P(u)$. □

Because $\#_a$ and $\#_b$ are defined in the same way, the same proof can be applied to:

$$\forall u, v \in \Sigma^*. \#_b(uv) = \#_b(u) + \#_b(v) \quad [4]$$

Using lemma 1 to prove lemma 2 from hint

Lemma to prove: $l_2 := \#_a(auavb) = 2 + \#_a(u) + \#_a(v)$

Proof:

$$\begin{aligned} \#_a(auavb) &= 1 + \#_a(uavb) && \text{(by definition of } \#_a) \\ &= 1 + \#_a(u) + \#_a(avb) && \text{(lemma } l_1) \\ &= 1 + \#_a(u) + 1 + \#_a(vb) && \text{(by definition of } \#_a) \\ &= 2 + \#_a(u) + \#_a(v) + \#_a(b) && \text{(lemma } l_1) \\ &= 2 + \#_a(u) + \#_a(v) + \#_a(\varepsilon) && \text{(by definition of } \#_a) \\ &= 2 + \#_a(u) + \#_a(v) && \text{(by definition of } \#_a) \end{aligned} \quad [5]$$

$\therefore \#_a(auavb) = 2 + \#_a(u) + \#_a(v)$ □

Because $\#_a$ and $\#_b$ are defined in the same way, the same proof can be applied to:

$$\#_b(bubva) = 2 + \#_b(u) + \#_b(v) \quad [6]$$

Using statement to prove original question

Statement to prove: $\forall w \in S. \#_a(w) = 2\#_b(w)$

Let $P(w) := \#_a(w) = 2\#_b(w)$.

Proof: By induction on w .

Basis: $P(\varepsilon)$ is true since $\#_a(\varepsilon) = 0 = 2\#_b(\varepsilon)$.

Inductive step: Assume $P(w)$ is true for some arbitrary $w \in S$ (i.h.). We want to show that $P(auavb) \wedge P(buavaw)$ is true.

Let $u, v, w \in S$. Then,

$$\begin{aligned}
 \#_a(auavb) &= 2 + \#_a(u) + \#_a(v) && \text{(lemma } l_2) \\
 &= 2 + 2\#_b(u) + 2\#_b(v) && \text{(i.h)} \\
 &= 2(1 + \#_b(u) + \#_b(v)) && \text{(arithmetic)} \\
 &= 2\#_b(auavb) && [7]
 \end{aligned}$$

$$\begin{aligned}
 \#_a(buavaw) &= 2 + \#_a(u) + \#_a(v) && \text{(by statement)} \\
 &= 2 + 2\#_b(u) + 2\#_b(v) && \text{(i.h)} \\
 &= 2(1 + \#_b(u) + \#_b(v)) && \text{(arithmetic)} \\
 &= 2\#_b(buavaw) && \text{(by statement)} \\
 &&& [8]
 \end{aligned}$$

$\therefore \forall w \in S. P(w)$. □

3

Question: Let $\Sigma = \{0\}$ and define $f, g, h \in \Sigma^* \rightarrow \mathbb{N}$ recursively in the following way:

- $g(\varepsilon) = 1$
 $g(0w) = |w| + g(w) + -h(w)$
- $h(\varepsilon) = 0$
 $h(0w) = |w| + g(w)$
- $f(\varepsilon) = 1$
 $f(0w) = h(w) = 2g(w)$

3.1

Question: Compute the values of $f(00), g(00), h(00), f(000), g(000), h(000), f(0000), g(0000), h(0000)$

3.2

Question: Prove that $\forall n \in \mathbb{N}. f(0^n) = 1 + n$.

Hint: First prove that $\forall n \in \mathbb{N}. g(0^n) = 1 \wedge h(0^n) = n$.