Finite automata and formal languages
Assignment 1

DIT323 (Finite automata and formal languages)
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1

Question: Prove using induction that, for every finite alphabet, Σ, $\forall n \in \mathbb{N}.\ |\Sigma^n| = |\Sigma|^n$.

Solution: Let $P(n) \coloneqq |\Sigma^n| = |\Sigma|^n$.

Proof: By induction on n.

Basis: $P(0)$ is true since $|\Sigma^0| = 1 = |\Sigma|^0$.

Inductive step: Assume $P(n)$ is true for some arbitrary $n \in \mathbb{N}$ *(i.h.)*. We want to show that $P(n+1)$ is true.

Let Σ be a finite alphabet. Then, $|\Sigma^{\{n+1\}}| = |\Sigma|\ |\Sigma^n|$ by definition of $\Sigma^{\{n+1\}}$.

By the induction hypothesis, $|\Sigma^n| = |\Sigma|^n \Rightarrow |\Sigma^{\{n+1\}}| = |\Sigma|\ |\Sigma^n| = |\Sigma|\ |\Sigma|^n = |\Sigma|^{\{n+1\}}$.

$\therefore \forall n \in \mathbb{N}.P(n)$. $\square$

2

Define a language S containing words over the alphabet $\Sigma = \{a, b\}$ inductively in the following way:

- The empty word is in S: $\varepsilon \in S$
- If $u, v \in S$, then *auavb* $\in S$.
- If $u, v, w \in S$, then *buavaw* $\in S$.

2.1

Question: Use recursion to define two functions $\#_a, \#_b \in \Sigma^* \to \mathbb{N}$ that return the number of occurrences of a and b, respectively, in their input.

Solution:

$$\#_a(uv) = \begin{cases} 0 & \text{if } uv = \varepsilon \\ 1 + \#_a(v) & \text{if } u = a \\ \#_a(v) & \text{if } u \neq a \end{cases} \qquad [1]$$

$$\#_b(uv) = \begin{cases} 0 & \text{if } uv = \varepsilon \\ 1 + \#_b(v) & \text{if } u = b \\ \#_b(v) & \text{if } u \neq b \end{cases} \qquad [2]$$

2.2

Question: Use induction to prove that $\forall w \in S . \#_a(w) = 2\#_b(w)$.

Hint: You might want to show that $\#_a(auavb) = 2 + \#_a(u) + \#_a(v)$. How do you prove this? This property follows from a lemma that you can perhaps prove by induction: $\forall u, v \in \Sigma^* . \#_a(uv) = \#_a(u) + \#_a(v)$

Solution:

Proof of lemma from hint

Lemma to prove: $l_1 \coloneqq \forall u, v \in \Sigma^* . \#_a(uv) = \#_a(u) + \#_a(v)$

Let $P(u) \coloneqq \#_a(uv) = \#_a(u) + \#_a(v)$.

Proof: By induction on u.

Basis: $P(\varepsilon)$ is true since $\#_a(\varepsilon) = 0 = \#_a(\varepsilon) + \#_a(\varepsilon)$.

Inductive step: Assume $P(u)$ is true for some arbitrary $u \in \Sigma^*$ *(i.h.)*. We want to show that $P(au)$ is true.

Let $v \in \Sigma^*$. Then,

$$\begin{aligned} \#_a(auv) &= 1 + \#_a(uv) && \text{(by definition of } \#_a\text{)} \\ &= 1 + \#_a(u) + \#_a(v) && \text{(i.h)} \\ &= \#_a(au) + \#_a(v) && \text{(by definition of } \#_a\text{)} \end{aligned} \tag{3}$$

$\therefore \forall u, v \in \Sigma^* . P(u)$. $\square$

Because $\#_a$ and $\#_b$ are defined in the same way, the same proof can be applied to:

$$\forall u, v \in \Sigma^* . \#_b(uv) = \#_b(u) + \#_b(v) \tag{4}$$

Using lemma 1 to prove lemma 2 from hint

Lemma to prove: $l_2 \coloneqq \#_a(auavb) = 2 + \#_a(u) + \#_a(v)$

Proof:

$$\begin{aligned} \#_a(auavb) &= 1 + \#_a(uavb) && \text{(by definition of } \#_a\text{)} \\ &= 1 + \#_a(u) + \#_a(avb) && \text{(lemma } l_1\text{)} \\ &= 1 + \#_a(u) + 1 + \#_a(vb) && \text{(by definition of } \#_a\text{)} \\ &= 2 + \#_a(u) + \#_a(v) + \#_a(b) && \text{(lemma } l_1\text{)} \\ &= 2 + \#_a(u) + \#_a(v) + \#_a(\varepsilon) && \text{(by definition of } \#_a\text{)} \\ &= 2 + \#_a(u) + \#_a(v) && \text{(by definition of } \#_a\text{)} \end{aligned} \tag{5}$$

$\therefore \#_a(auavb) = 2 + \#_a(u) + \#_a(v)$ $\square$

Because $\#_a$ and $\#_b$ are defined in the same way, the same proof can be applied to:

$$\#_b(bubva) = 2 + \#_b(u) + \#_b(v) \tag{6}$$

Using statement to prove original question

Statement to prove: $\forall w \in S.\#_a(w) = 2\#_b(w)$

Let $P(w) := \#_a(w) = 2\#_b(w)$.

Proof: By induction on w.

Basis: $P(\varepsilon)$ is true since $\#_a(\varepsilon) = 0 = 2\#_b(\varepsilon)$.

Inductive step: Assume $P(w)$ is true for some arbitrary $w \in S$ *(i.h.)*. We want to show that $P(auavb) \wedge P(buavaw)$ is true.

Let $u, v, w \in S$. Then,

$$\begin{aligned} \#_a(auavb) &= 2 + \#_a(u) + \#_a(v) && \text{(lemma } l_2\text{)} \\ &= 2 + 2\#_b(u) + 2\#_b(v) && \text{(i.h)} \\ &= 2(1 + \#_b(u) + \#_b(v)) && \text{(arithmetic)} \\ &= 2\#_b(auavb) && \text{(by definition of } \#_b\text{)} \end{aligned} \tag{7}$$

$$\begin{aligned} \#_a(buavaw) &= 2 + \#_a(u) + \#_a(v) && \text{(lemma } l_2\text{)} \\ &= 2 + 2\#_b(u) + 2\#_b(v) && \text{(i.h)} \\ &= 2(1 + \#_b(u) + \#_b(v)) && \text{(arithmetic)} \\ &= 2\#_b(buavaw) && \text{(by definition of } \#_b\text{)} \end{aligned} \tag{8}$$

$\therefore \forall w \in S.P(w)$. □

3

Question: Let $\Sigma = \{0\}$ and define $f, g, h \in \Sigma^* \to \mathbb{N}$ recursively in the following way:

- $g(\varepsilon) = 1$
  $g(0w) = |w| + g(w) + -h(w)$
- $h(\varepsilon) = 0$
  $h(0w) = |w| + g(w)$
- $f(\varepsilon) = 1$
  $f(0w) = h(w) = 2g(w)$

3.1

Question: Compute the values of $f(00), g(00), h(00), f(000), g(000), h(000), f(0000), g(0000), h(0000)$

Solution:

$$\begin{aligned} f(00) &= 3 & g(00) &= 1 & h(00) &= 2 \\ f(000) &= 4 & g(000) &= 1 & h(000) &= 3 \\ f(0000) &= 5 & g(0000) &= 1 & h(0000) &= 4 \end{aligned} \tag{9}$$

3.2

Question: Prove that $\forall n \in \mathbb{N}. f(0^n) = 1 + n$.

Hint: First prove that $\forall n \in \mathbb{N}. g(0^n) = 1 \wedge h(0^n) = n$.

Solution:

Proof of lemma from hint

Lemma to prove: $l_3 : \forall n \in \mathbb{N}. g(0^n) = 1 \wedge h(0^n) = n$.

Let $P(n) \coloneqq g(0^n) = 1 \wedge h(0^n) = n$.

Proof: By induction on n.

Basis: $P(0)$ is true since $g(0^0) = g(\varepsilon) = 1 = 1 \wedge h(0^0) = 1 \wedge h(\varepsilon) = 1 \wedge 1 = 1$.

Inductive step: Assume $P(n)$ is true for some arbitrary $n \in \mathbb{N}$ *(i.h.)*. We want to show that $P(n+1)$ is true.

Let $n \in \mathbb{N}$. Then,

$$\begin{aligned} g\left(0^{\{n+1\}}\right) &= |0^n| + g(0^n) + -h(0^n) && \text{(by definition of } g\text{)} \\ &= n + 1 + 1 + -n && \text{(i.h)} \\ &= 1 \end{aligned} \tag{10}$$

$\therefore \forall n \in \mathbb{N}. g(0^n) = 1 \wedge h(0^n) = n$ □

Proof of main statement

Statement to prove: $\forall n \in \mathbb{N}. f(0^n) = 1 + n$.

Let $P(n) \coloneqq f(0^n) = 1 + n$.

Proof: By induction on n.

Basis: $P(0)$ is true since $f(0^0) = f(\varepsilon) = 1 = 1 + 0$.

Inductive step: Assume $P(n)$ is true for some arbitrary $n \in \mathbb{N}$ *(i.h.)*. We want to show that $P(n+1)$ is true.

Let $n \in \mathbb{N}$. Then,

$$\begin{aligned} f\left(0^{\{n+1\}}\right) &= h(0^n) + 2g(0^n) && \text{(by definition of } f\text{)} \\ &= n + 2 && \text{(lemma } l_3\text{)} \\ &= 1 + (n + 1) \end{aligned} \tag{11}$$

$\therefore \forall n \in \mathbb{N}. f(0^n) = 1 + n$ □