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Modeling the concept of majority opinion in group decision making

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Abstract

In this paper the problem of group decision making is studied. One of the main issues in this context is to define a decision strategy which takes into account the individual opinions of the decision makers. The concept of majority plays in this context a key role: what is often needed is an overall opinion which synthesizes the opinions of the *majority* of the decision makers. The reduction of the individual values into a representative value (which we call the *majority opinion*) is usually performed through an aggregation process. Within fuzzy set theory the concept of majority can be expressed by a linguistic quantifier (such as *most*), which is formally defined as a fuzzy subset. In this paper we propose two distinct approaches to the definition of a majority opinion. We first consider the case where linguistic quantifiers are associated with aggregation operators which allow us to compute a majority opinion by aggregating the individual opinions. In this case the majority opinion corresponds to the aggregated value. To model this semantics of linguistic quantifiers the IOWA operators are used and a new proposal of definition of their weighting vector is presented. A second method is based on the

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consideration of the concept of majority as a vague concept. Based on this interpretation we propose a formalization of a fuzzy majority opinion as a fuzzy subset.

1. Introduction

In group decision making (multi-agent decision making) a set of experts are involved in a decision process concerning the evaluation of a set of alternatives. The first step of this decision process is constituted by the individual evaluations of the decision makers (agents): each agent rates each alternative on the basis of an adopted evaluation scheme [5]. We assume that at the end of this step each alternative has associated a performance judgment (evaluation or opinion) on a predefined scale (either numeric or linguistic). The second step of a multi-agent decision process consists in determining for each alternative a consensual judgment which synthesizes the agents individual opinions. The consensual judgment is representative of a collective evaluation and is usually computed by means of an aggregation of the individual agents' opinions. Usually also a consensus degree is computed for each alternative, with the consequent problem of comparing the decision makers' opinions to verify the consensus among them. In the case of unanimous consensus, the evaluation process ends with the selection of the best alternative(s). As in real situations humans rarely come to an unanimous agreement, in the literature some fuzzy approaches to evaluate a majority guided aggregation have been proposed. In these approaches full consensus (degree = 1) is not necessarily the result of unanimous agreement, but it can be obtained even in case of agreement among a fuzzy majority of the decision makers [2,7–10].

In this paper we consider the problem of constructing a *majority opinion*, intended as the collective evaluation of a majority of the agents involved in the decision problem. A majority opinion is a consensual judgment of a majority of the decision makers who have similar opinions. Formally we consider we have n agents who have expressed individual judgments (opinions) on an alternative. We need only consider the case of a single alternative, because in the case of multiple alternatives the process of construction of a majority opinion can be independently applied to each alternative. So we have n judgments (opinions) $a_1, \ldots, a_n$ which have to be reduced to an overall majority opinion.

In the fuzzy approaches to multi-agent decision making the concept of majority is usually modeled by means of linguistic quantifiers such as *at least 80%* and *most*. A linguistic quantifier is formally defined as a fuzzy subset of a numeric domain (either non-negative real numbers or the unit interval); the semantics of such a fuzzy subset is described by a membership function which describes the compatibility of a given absolute or percentage quantity to the concept expressed by the linguistic quantifier. By this interpretation a linguistic quantifier is seen as a fuzzy concept referred to the quantity of elements of a considered reference set.

In group decision making linguistic quantifiers are used to indicate a fusion strategy to guide the process of aggregating the members' opinions. The formal mathematical definition of the resulting aggregation operator encodes the semantics of the linguistic quantifier. The notion of quantifier guided aggregation has been formally defined by means of ordered weighted averaging operators [16,20] and by means of the concept of fuzzy integrals [6]. In this paper we consider only the use of OWA operators. An example of linguistic expression which employs a quantifier guided aggregation is the following: *Q agents are satisfied by solution a*, where Q denotes a linguistic quantifier, for example *most*, which expresses a majority. To evaluate the satisfaction of this proposition we aggregate the agents' opinions using the formal aggregation operator which captures the semantics of the concept expressed by the quantifier Q. To associate a linguistic quantifier Q with an OWA aggregation operator, an approach was suggested in [19,20] which makes use of the definition of the linguistic quantifier Q as a fuzzy subset. In this paper we propose two distinct approaches to the definition of a *majority opinion*.

In the first approach we use an aggregation operator, specifically an induced ordered weighted averaging (IOWA) operator, to obtain a scalar value for a majority opinion. Our second approach is based upon the calculation of the concept of the majority opinion as an imprecise value. Under this interpretation we propose a formalization of the idea of a fuzzy majority as a fuzzy subset. As we shall see this approach provides in addition to a value for a majority opinion an indication of the strength of that value as the majority opinion. Our goal here is to obtain a value which we can consider as the opinion of a majority, that is, some value that is similar for any large group of people. Both methods require we have both information about the similarity between the experts opinions, and some information about what quantity constitutes the idea of a majority.

The paper is structured as follows: in Section 2 the notion of linguistic quantifier is synthetically introduced, and the use of a linguistic quantifier to specify an aggregation operator is explained. In Section 3 the problem of the definition of aggregation operators with a semantics of majority is presented, and a couple of new approaches to construct induced ordered weighted averaging operators for linguistic quantifiers with a semantics of majority is proposed. In Sections 4–6 a formal definition of the concept of fuzzy majority opinion is described.

2. OWA operators and linguistic quantifiers in aggregation

In natural languages, expressions such as *at least one*, *most*, *a few*, *more than 80%*, are employed to denote in an approximate way a quantity of the elements belonging to a reference set (the universe of discourse). From a formal point of view the notion of generalized quantifier has been introduced to denote a formal generalization of the existential and universal quantifiers of classical logic [1,11,13]. Within fuzzy logic the concept of generalized quantifier has been formally defined through linguistic quantifiers [24]. In this section the notion of linguistic quantifier and its use to guide an aggregation process are presented; although in literature distinct approaches have been defined to this aim [3,6,14] we focus in this paper on the use of OWA and IOWA operators [16,20]. In Section 2.1 the concept of a linguistic quantifier in fuzzy logic is briefly introduced, and in Section 2.2 the OWA operators and their use in quantifier guided aggregation are described.

2.1. The concept of linguistic quantifier in fuzzy logic

In fuzzy logic the concept of linguistic quantifier has been formalized to generalize the concept of quantification of classical logic. In classical logic the propositions can be quantified either in the universal or in the existential way; in these cases either *all* (for all) or to *at least one* (there exists) of the elements of the considered universe of discourse are characterized. Linguistic quantifiers allow us to refer to a variable number of elements of the domain of discourse, either in a crisp (such as for example *at least k*, *half*) or in a vague way (such as for example *most*, *some*, *approximately k*) [24].

In fuzzy logic two kinds of fuzzy quantified propositions are distinguished: "Q X are Y", i.e., Q elements of set X satisfy the fuzzy predicate Y, and "Q B X are Y", i.e., Q elements of set X which satisfy the fuzzy predicate B also satisfy the fuzzy predicate Y, in which Q is a linguistic quantifier and Y and B are fuzzy predicates. Examples of such statements are:

$$\textit{Most of the criteria are satisfied by alternative } A_i$$

in which $Q = \textit{most}$, $X =$ the set of the *criteria*, and $Y = \textit{satisfies alternative } A_i$ and

$$\textit{Most of the important criteria are satisfied by alternative } A_i$$

in which $B = \textit{important}$.

In fuzzy logic the fuzzy quantifiers have been defined as fuzzy subsets [24] of two main types: absolute and proportional. Absolute quantifiers, such as *about 7*, *almost 6*, etc. are defined as fuzzy subsets with membership function $\mu_Q : \Re^+ \to [0,1]$, where $\forall x \in \Re^+$; $\mu_Q(x)$ indicates the degree to which the amount x satisfies the concept Q. Proportional quantifiers like *most*, or *about 70%*, are defined as fuzzy subsets of the unit interval: $\mu_Q : [0,1] \to [0,1]$, where $\forall x \in [0,1]$, $\mu_Q(x)$ indicates the degree to which the proportion x satisfies the concept Q. For sake of simplicity from now on we will indicate μ_Q by Q and $\mu_Q(x)$ by $Q(x)$. In Zadeh's definition of fuzzy quantifiers, the concept of cardinality of a fuzzy set, defined as Σ-count, is central. The truth of the fuzzy proposition "QX are Y" is the value of Q applied to the Σ-count of the fuzzy set Y when Q is absolute ($\Sigma\text{-count}(Y) = \Sigma\mu_Y(x_i)$ in which $\mu_Y(x_i)$ is the membership degree of $x_i \in X$ to the fuzzy set Y), or to the relative count of Y in the case of proportional quantifier (defined as $\Sigma\text{-count}(Y)/n$ with n the number of elements in X).

In the next subsection we introduce the use of linguistic quantifiers for defining ordered weighted averaging aggregation operators.

2.2. Ordered weighted averaging operators

An OWA operator on unit interval is a mapping OWA: $[0,1]^n \to [0,1]$ that has associated a weighting vector $W = [w_1, w_2, \ldots, w_n]$ such that $w_i \in [0,1]$, and $\sum_{i=1}^{n} w_i = 1$, and for any arguments $a_1, a_2, \ldots, a_n \in [0,1]$:

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} b_i w_i$$

with b_i being the ith largest element of the a_j [20].

More formally if *a-index* is an index function such that *a-index*(i) is the index of the ith largest argument, then we can write:

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{M} w_i a_{a\text{-index}(i)}$$

Furthermore, if W is a vector whose components are w_i and B is a vector whose components are the ordered argument values, $b_i = a_{a\text{-index}(i)}$, then:

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = W^{\mathrm{T}} B$$

A number of approaches have been suggested for determining the weights used in the OWA operator. Here we shall focus on the one that allows us to obtain the weights from a functional form of the linguistic quantifier. Let $Q : [0,1] \to [0,1]$ be a function such that $Q(0) = 0$, $Q(1) = 1$ and $Q(x) \geqslant Q(y)$ for $x > y$ corresponding to a fuzzy set representation of a proportional monotone quantifier. For a given value $x \in [0,1]$, the $Q(x)$ is the degree to which x satisfies the fuzzy concept being represented by the quantifier. Based on function Q, the OWA vector is determined from Q by defining the weights in the following way:

$$w_i = Q(i/n) - Q((i-1)/n)$$

In this case w_i represents the increase of satisfaction in getting i with respect to $i-1$ criteria satisfied.

Let us define for example the weighting vector of the OWA operator associated with the linguistic quantifier *most*. In Fig. 1 a possible membership function of the *most* quantifier is presented.

$$\mu_{\text{most}}(x) = \begin{cases} 1 & x \geq 0.9 \\ 2x - 0.8 & 0.4 < x < 0.9 \\ 0 & x \leq 0.4 \end{cases}$$

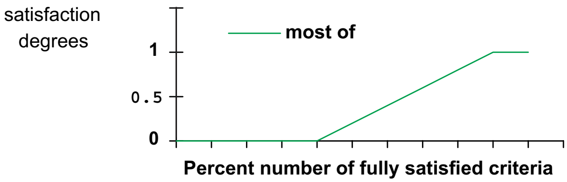

Fig. 1. A possible definition of the linguistic quantifier *most*.

Based on this function, the weights of the weighting vector are computed as explained above; in the case we want to aggregate six elements the weights of the weighting vector are presented in Table 1.

The degree of orness of an OWA aggregation operator expresses its closeness to the OR behavior, and it is defined as:

$$\text{orness}(\mathrm{W}) = \left(\frac{1}{n-1}\right)\sum_{j=1}^{n}((n-j) * w_j)$$

The OWA operator with the weighting vector W_* defined as $[1,0,\ldots,0]$ corresponds to the OR operator, i.e., the max. In this case, orness(W^*) = 1. The OWA operator with the weighting vector W_* defined as $[0,\ldots,0,1]$ corresponds to the AND operator, i.e., the min. In this case, orness(W_*) = 0.

2.3. Induced ordered weighted averaging operators

In [21] Yager and Filev introduced an extension of the OWA called the induced ordered weighted averaging (IOWA) operators. Here as in the case of the ordinary OWA operator we have a collection of argument values $(a_1,\ldots,a_n)$ which we are trying to aggregate, and an OWA weighting vector W. However in the case of IOWA operator in addition we have associated with each of the argument values another value v_j called the order inducing value. In the case of the induced OWA we calculate:

$$I - F(a_1,\ldots,a_n) = \sum_{i=1}^{n} w_i a_{v\text{-index}(i)}$$

Table 1
The weights of the weighting vector associated with the linguistic quantifier in Fig. 1

	w_1	w_2	w_3	w_4	w_5	w_6
W	0	0	0.2	0.33	0.33	0.14

where v-index(i) is the index of the ith largest v_i. Here then while we are still aggregating the a_j values our ordering is done with respect to the v_j value. Here then we have pairs (a_j, v_j), a_j being the argument value and v_j being the order inducing value. If W is a vector of weights and B_V is a vector whose values are the argument values ordered by the order inducing values:

$$I-F(a_1, a_2, \ldots, a_n) = W^{\mathrm{T}} B_V$$

As a simple example consider the case when we want to aggregate the following four pairs (a_j, v_j): (0.7, 0.4), (0.9, 0.3), (0.2, 1) and (0.6, 0.7). In this case the inducing values are $v_1 = 0.4$, $v_2 = 0.3$, $v_3 = 1$, and $v_4 = 0.7$, which are ordered $1 > 0.7 > 0.4 > 0.3$; hence v-index is such that v-index(1) = 3, v-index(2) = 4, v-index(3) = 1, v-index(4) = 2.

If our weighting vector has the following values: $w_1 = 0.1$, $w_2 = 0.2$, $w_3 = 0.3$, $w_4 = 0.4$, we obtain the following aggregated value:

$$I-F(a_1, a_2, a_3, a_4) = (0.1)(0.2) + (0.2)(0.6) + (0.3)(0.7) + (0.4)(0.9) = 0.71$$

3. Linguistic quantifiers as soft specifications of majority-based aggregation

In this section we consider monotonic non-decreasing linguistic quantifiers, such as *most* and *at least 80%*. In particular we are interested in the use of such linguistic quantifiers in guiding an aggregation process aimed at computing a value which synthesizes the majority of values to be aggregated: we call this aggregated value the "*majority opinion*". In multi-agent decision making the synthesis of a majority opinion is a very important question. As illustrated in Section 2.2 OWA operators can be constructed starting by the fuzzy definition of a linguistic quantifier. However in this section we will see when we aggregate a collection of values with a quantifier corresponding to the concept of a *majority*, the resulting aggregated value may not be representative of the majority of values. In this section we first analyze the semantics of the quantifier guided aggregation based on the procedure explained in Section 2.2. Then, we propose a couple of new strategies aimed at constructing OWA operators which allow us to obtain a majority opinion over a set of values to be aggregated by a linguistic quantifier corresponding to the concept of a *majority*.

3.1. The semantics of OWA operators in an aggregation guided by "majority" linguistic quantifiers

Objectively, the OWA operator is an aggregation operator taking a collection of argument values and returning a single value. The weights of the OWA weighting vector determine the behaviour of the aggregation operator.

These weights have the effect of emphasizing or demphasizing different components in the aggregation. Subjectively, there are a number of different semantics that can be associated with an OWA operator and which determine strategies of construction of its weighting vector.

One semantics that can be associated with the OWA operators is as a generalization of the idea of an averaging or summarizing operator. Here for example $w_i = 1/n$ for all i gives us the simple average.

Another semantics that has been associated with the OWA is a generalization of the logical quantifiers *there exists* and *for all*. The weighting vector with $w_1 = 1$ and $w_j = 0$ for $j \neq i$ corresponds to the quantifier "*there exists*" and the one with $w_n = 1$ and $w_j = 0$ for $j \neq n$ corresponds to the quantifiers "*for all*". In this framework the arguments are seen as truth values or degrees of satisfaction. If Q is a quantifier then the OWA aggregation provides a value which can be seen as the truth value of the statement "Q of the elements being aggregated are satisfied". It is this semantics that has been most often used in applications of OWA operators to multi-criteria and multi-agent decision making. As outlined in [16], the weights of the weighting vector of an OWA operator are interpreted as the increase in satisfaction in having $i + 1$ criteria "fully" satisfied with respect to having "fully" satisfied i criteria. We try to make clear this consideration with an example. Let us suppose we define the weighting vector of the aggregation operator associated with the linguistic quantifier *at least* 80%, in the case in which we have to aggregate five values. A crisp definition of this aggregation operator can be: $W_{\text{at least 80\%}} = [0\ 0\ 0\ 1\ 0]$. If we apply this aggregation operator to the values [1 1 1 0.1 0] we obtain the aggregated value 0.1. We would obtain the same value if aggregating the values [0.1 0.1 0.1 0.1 0]. These results highlight the fact that the linguistic quantifier is intended to guide the aggregation with a semantics like "It is true that at least 80% of the criteria are '*fully*' satisfied". In both previous examples, the aggregated value corresponds to the 4th decreasing value, which corresponds to the evaluation of the 80% of the values, independently on the degrees of satisfaction of the previous values, which are greater than or equal to the fourth value.

The semantics of the aggregation guided by the quantifier *at least 80%* modeled with the previous definition of the weighting vector is then not aimed at producing a synthesis of the most similar values in the quantity specified by the quantifier. This semantics is particularly useful in the context group decision making. While with the original semantics of the weighting vector of an OWA operator the aggregated value is like a degree of satisfaction (truth) of the proposition "Q of the values are fully satisfied", an operator with the semantics of calculating a majority opinion should produce a value which is representative of the 80% of the most similar values. In the example above this "representative" value could be 0.75, because the 80% of the most similar values is around the value 0.75. In other words what we want to obtain is an aggregation of the most similar opinions held by a quantity of decision makers specified by the linguistic quantifier Q. This situation appears to bring us closer in spirit to an interpretation of the OWA operator as averaging operator rather then as a generalized quantifier. In fact what we want is an average of "most of the similar values".

We now present an example which better shows the two distinct semantics of a quantifier guided aggregation; in this example we define the weighting vector of an OWA operator associated with the linguistic quantifier *most*. A simple interpretation of this linguistic quantifier can be that I desire that *at least 70%* of the criteria are satisfied (in other words at least 70% of the values to be aggregated have to be greater than 0). This quantifier has an and-like semantics. If we consider six elements to be aggregated a possible weighting vector associated with *most* can be [0 0 0 0.7 0.2 0.1], which means that the concept of majority corresponds to having at least four elements satisfied. Let us now suppose to have the following values to aggregate: (1 1 1 0.5 0 0). The result of the aggregation under the previous weighting vector produces the value 0.35; this aggregated value does not characterize the value of the majority of the most similar values, which is intuitively a value closer to 1.

Let us now consider another possible definition of the linguistic quantifier *most*, the one defined in Fig. 1, with the weighting vector shown in Table 1. Also in this case the aggregation of the values [1 1 1 0.5 0 0] produces the value 0.35. Thus, also by this definition the aggregated value is not representative of the opinion of a majority of the aggregated values. As outlined before the main reason for this result is that the OWA aggregation produces a value which reflects the satisfaction of the proposition "*most of the criteria have to be satisfied*" instead of "*the satisfaction value of most of the criteria*".

It is important to notice that the semantics of the linguistic quantifier is strongly affected by the non-linear component of the aggregation operator (i.e. by the way in which the arguments are reordered). In the example before with the usual construction of the weighting vector the obtained low aggregated value is due to the fact that the values to be aggregated are in decreasing order.

This kind of semantics does not naturally model the meaning of the concept of majority as typically used in group decision making applications. When we use linguistic quantifiers to express our intent in aggregating the opinions of the group of decision makers, the implicit aim in stating *most* is to remind the fact that we want an evaluation that correspond to a majority of the experts holding a similar opinion, where by majority we intend *most*. This means that we need an aggregation operator that takes an average like aggregation of a majority of values that are similar. To this aim what should be effectively aggregated are the most similar values. The concept of similarity plays here a crucial role. In the next subsection we suggest a possible approach to solve this problem.

3.2. Using IOWA operators to obtain a majority opinion

To produce an aggregation with a majority semantics, we propose to use the IOWA operators with an inducing ordering variable which is based on a proximity metric over the elements to be aggregated. The basic idea is that the most similar values must have close positions in the induced ordering in order to appropriately be aggregated. We also suggest a new strategy for constructing the weighting vector so as to better model the new "majority-based" semantics of the aggregation.

What we want is to aggregate a set of given values in order to produce a value which synthesizes the opinion of the majority; to this aim our intent is to take the most similar values in the quantity specified by the quantifier and apply to them an averaging operator. What we need is a computation of the similarities between the opinion values. The values of the inducing variable of the IOWA operator are obtained by means of a function of the similarities between pairs of the opinion values. We define such a function using a support function, as defined in [22]. A support function Sup is a binary function which computes a value $\mathrm{Sup}(a,b)$ which expresses the support from b for a; the more similar, close are two values the more they support each other. A simple example of support function is the following:

$$\mathrm{Sup}(a_i, a_j) = \begin{cases} 1 & \text{if } |a_i - a_j| < \alpha \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

If we consider a set of values to be aggregated, and we want to order them in increasing order of support we compute for each value the sum of its support values with respect to all the others values to be aggregated. Then for each expert's opinion we sum all the supports it has in order to obtain its overall support. These overall supports for an expert's opinion are used as the values of the order inducing variable.

We now present an example of an application of the above support function with $\alpha = 0.4$. Let us suppose we have the following values to aggregate:

$$a_1 = 0.9, \quad a_2 = 0.7, \quad a_3 = 0.6, \quad a_4 = 0.1, \quad a_5 = 0$$

On the basis of the previous support function, we compute the values of the supports for each pairs of values:

$$\begin{array}{llll}
\mathrm{Sup}(a_1,a_2) = 1 & \mathrm{Sup}(a_1,a_3) = 1 & \mathrm{Sup}(a_1,a_4) = 0 & \mathrm{Sup}(a_1,a_5) = 0 \\
\mathrm{Sup}(a_2,a_1) = 1 & \mathrm{Sup}(a_2,a_3) = 1 & \mathrm{Sup}(a_2,a_4) = 0 & \mathrm{Sup}(a_2,a_5) = 0 \\
\mathrm{Sup}(a_3,a_1) = 1 & \mathrm{Sup}(a_3,a_2) = 1 & \mathrm{Sup}(a_3,a_4) = 0 & \mathrm{Sup}(a_3,a_5) = 0 \\
\mathrm{Sup}(a_4,a_1) = 0 & \mathrm{Sup}(a_4,a_2) = 0 & \mathrm{Sup}(a_4,a_3) = 0 & \mathrm{Sup}(a_4,a_5) = 1 \\
\mathrm{Sup}(a_5,a_1) = 0 & \mathrm{Sup}(a_5,a_2) = 0 & \mathrm{Sup}(a_5,a_3) = 0 & \mathrm{Sup}(a_5,a_4) = 1
\end{array}$$

The overall support for each a_i is obtained by adding the support values for a_i; we denote this value by s_i:

$$s_1 = 2, \quad s_2 = 2, \quad s_3 = 2, \quad s_4 = 1, \quad s_5 = 1$$

We can see that we have two main "clusters" of similar values. In fact the use of the adopted support function induces a clustering of the arguments which can be controlled by the choice of the threshold parameter α in function (1). In the example we obtain two clusters with some ties of the overall support values. If we want to "solve" the ties we can impose a "stricter" condition by setting $\alpha = 0.3$; in this way we obtain:

$$s_1 = 1, \quad s_2 = 2, \quad s_3 = 1, \quad s_4 = 1, \quad s_5 = 1$$

This result, combined with the previous one allows to order the elements to be aggregated in the following increasing order of similarity:

induced similarity order: $I = [0\ 0.1\ 0.6\ 0.9\ 0.7]$

So we see that the use of an appropriate support function allows us to induce an ordering based on proximity.

Let us suppose that we want now to obtain a majority-based aggregation of the previous values. The selected IOWA operator should then correspond to the linguistic quantifier *most*. We first consider the definition of *most* proposed in Fig. 1; in this case the linguistic quantifier is defined by means of a non-decreasing function. Starting from this definition we construct the weighting vector of five elements (as five are the elements of the example above); it is $W = [0\ 0\ 0.4\ 0.4\ 0.2]$. Let us notice that the fifth element of this vector is smaller than the fourth element. By aggregating the vector I above we obtain: $IW = 0.74$. We note that this value is a much better representative of the majority of the values to be aggregated, than the value which we would obtain with the usual OWA operator associated with *most* (with the elements in decreasing order of their value): $B = [0.9\ 0.7\ 0.6\ 0.1\ 0]$, and $BW = 0.28$.

However, as outlined before, what can be noticed in the considered W vector is that the last weight (on the right hand side of the vector W) is smaller than the previous value; this is coherent with the interpretation of the weights as increase in satisfaction in having $i + 1$ with respect to having i criteria satisfied. However, in an aggregation with the semantics of majority what would be expected is that the weights of the weighting vector are non-decreasing; in fact as in the induced order of the arguments the top value is the most "supported" one from the all the other values (the most representative) it should be more emphasized than the others, or at least not less emphasized. For this reason, and in order to obtain a value which better represents a majority of the aggregated elements, we propose a new strategy for the construction of the weighting vector. This strategy has the aim of emphasizing in the aggregation the most supported values; in other words the values which appear on the right hand side of the vector of values to be aggregated have more influence in the aggregation. In the following we suggest a procedure for the construction of the weighting vector which produces a weighting vector with non-decreasing weights. First let us consider the overall support (similarity) values computed for the n values to be aggregated: $s_1, s_2, \ldots, s_n$. In order to compute the non-decreasing weights of the weighting vector, we first define the values $t_1, t_2, \ldots, t_n$ based on a modification of the $s_1, s_2, \ldots, s_n$ values: $t_i = s_i + 1$. In this way the similarity of the value a_i with itself (similarity value equal to 1) is also included in the definition of the overall support for a_i. The t_i values are in increasing order, that is t_1 is the smallest value among the t_i.

On the basis of the t_j values, the weights of the weighting vector are computed as follows:

$$w_i = Q(t_1/n) \Big/ \sum_{i=1,\ldots,n} Q(t_i/n)$$

The value $Q(t_1/n)$ denotes the degree to which a given member of the considered set of values represents the majority.

Based on this formula we define the weighting vector of the OWA operator associated with the quantifier *most* presented in Fig. 1; if we want to aggregate five elements we obtain: $W_1 = [0\ 0\ 0.333\ 0.333\ 0.333]$.

Let us now aggregate with this weighting vector the induced ordered elements in the example illustrated before $I = [0\ 0.1\ 0.6\ 0.9\ 0.7]$; we obtain $W_1 I = 0.733$.

Although this value is smaller than the value obtained with the weighting vector W above, it is closer to 0.7 that is the most representative value among the values to be aggregated.

Let us now aggregate the six elements $a_1 = 1$, $a_2 = 1$, $a_3 = 1$ $a_4 = 0.5$, $a_5 = 0$, $a_6 = 0$. First of all we have to induce their similarity-based ordering. We perform this step by adopting the simple support function defined above. If we set a value $\alpha = 0.4$ we obtain $s_1 = 2$, $s_2 = 2$, $s_3 = 2$, $s_4 = 0$, $s_5 = 1$, $s_6 = 1$. As it can be noticed, in this case the fact that 0.5 has an overall support equal to 0 is due to the fact that it is a value equidistant from both 0 and 1, which are the other values to be aggregated. With this choice of the parameter α, the induced ordering of the values is then $I = [0.5\ 0\ 0\ 1\ 1\ 1]$. We now compute the weights of the weighting vector of six elements, based on the computation procedure shown above: $W_1 = [0\ 0\ 0\ 0.33\ 0.33\ 0.33]$. Let us now aggregate the vector I: $W_1 I = 0.33 + 0.33 + 0.33 = 1$. In this case if we aggregate the vector I with the weighting vector W shown in Table 1 we obtain the lower aggregated value 0.8, which could be considered as worst reflecting the majority of the considered elements. We finally notice that the result 0.366 produced by the classical definition of the OWA operator (based on the definition of the linguistic quantifier *most* presented in Fig. 1) applied to the same values [1 1 1 0.5 0 0] is very far from an interpretation based on a majority-oriented aggregation.

4. The concept of fuzzy majority opinion

In the previous section we suggested some approaches to the calculation of a value which synthesizes the majority of a collection of values; we have called this value the *majority opinion*. In the following we shall suggest an approach that is based upon the idea of a *fuzzy majority opinion*. Under this interpretation the majority opinion is no longer represented as a value, but as a fuzzy subset. As we shall see this will provide in addition to a value for the majority opinion an indication of the strength of that value as a representative of the majority opinion.

In the following we shall let $A = \{a_1, \ldots, a_n\}$ be a set of values which constitute the opinions of a group of people. Our definition of a fuzzy majority opinion requires that we have information about the similarity between the values provided. It requires also some information about what quantity constitutes the idea of a majority.

Here we shall assume the availability of a relation on the space from which the values to be aggregated are drawn indicating how similar two values are. In particular we assume a relationship Sim on the domain of A such that for any a_i and a_j $\mathrm{Sim}[a_i, a_j] \in [0,1]$ and that Sim satisfies the properties $\mathrm{Sim}(a_i, a_i) = 1$ and $\mathrm{Sim}(a_i, a_j) = \mathrm{Sim}(a_j, a_i)$. We note that the relationship Sim is not a formal similarity relationship as introduced by Zadeh [23], it lacks transitivity, but formally a proximity relationship [12]. However for linguistic and intuitive convenience we shall refer to $\mathrm{Sim}(x, y)$ as indicating the degree of similarity between x and y. Another tool we need is a formal definition of the quantity we consider to constitute a majority. The concept of a majority is a user and context dependent idea, however there are certain features common to any definition. We shall assume a user provided definition of a majority in terms of a fuzzy subset Q on the unit interval. In particular $Q : [0, 1] \to [0, 1]$ such that $Q(0) = 0$, $Q(1) = 1$ and $Q(\mathrm{x}) \geqslant Q(y)$ if $x > y$. The monotonicity of Q implies that if $Q(y) = 1$ then for all $x > y$ we have $Q(x) = 1$. We shall call the point x^*, the smallest value for which $Q(x) = 1$, the point of realization, POR.

The concept of majority always has some POR, typically POR < 1. As we shall subsequently see often it will be useful to have definitions of Q that are strictly monotonic, if $x > y$ then $Q(x) > Q(y)$. This strict monotonicity requires $Q(x) < 1$ if $x < 1$ and hence POR $= 1$. This type of strictly monotonic definition of Q allows us to always be able to distinguish between sets of different cardinalities. One solution to this conflict between desiring strict monotonicity and POR < 1 is to use a concept of "effective point of realization" EPOR. The quantifier displayed below illustrates this idea (see Fig. 2).

Here we allow our definition to be such that x^* is our EPOR and we consider 0.99 to effectively denote complete satisfaction. However for x from x^* to 1 we use a straight line such that $Q(x) = 1 - \dfrac{0.1(1 - x)}{1 - x^*}$. We now proceed

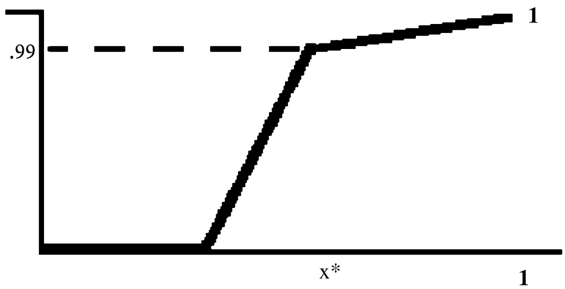

Fig. 2. Implementing an EPOR.

to use these ideas to introduce a concept of a majority opinion that will be a fuzzy subset that can be interpreted as a possibility distribution on the numeric majority opinions [4].

Let E be a crisp subset of A. Our first step is determine the degree to which this is a subset holding a majority opinion. We shall say a subset E *holds* a majority opinion if all the elements in E are similar and the cardinality of E satisfies our idea of being a majority of elements from A. We shall refer to a subset of values holding a majority opinion, as a *gang*. Thus a gang is a subset of E that contains a majority of elements having similar values. Let Majop(E) indicate the degree to which the elements in E, constitute a majority opinion, are a majority of elements from A with similar values. We define

$$\text{Majop(E)} = Q\left(\frac{|E|}{n}\right) \wedge \underset{a_i,a_j \in E}{\text{Min}}[\text{Sim}(a_i, a_j)]$$

where $\wedge$ stands for the min operator.

For simplicity we shall denote $\text{Sim}(E) = \text{Min}_{a_i,a_j \in E}[\text{Sim}(a_i, a_j)]$ and hence

$$\text{Majop}(E) = Q\left(\frac{|E|}{n}\right) \wedge \text{Sim}(E)$$

We now express the *opinion* of the elements in E as

$$\text{Op}(E) = \text{Ave}(E) = \frac{\sum_{a_i \in E} a_i}{|E|}$$

It is the average value of the elements in E.

Using the concepts Op(E) and Majop(E) we can define a fuzzy subset F indicating the majority opinion of the set of elements in A

$$F = \text{Majority Opinion} = \bigcup_{E \subseteq A} \left\{\frac{\text{Majop}(E)}{\text{Op}(E)}\right\}$$

So for each subset E , the value Majop(E) indicates the degree to which the quantity Op(E) is a majority opinion.

With F the fuzzy subset corresponding to the fuzzy majority opinion we see the $\text{Max}_E[\text{Majop}(E)]$, the maximal membership grade in F, indicates the degree to which there exists a majority opinion.

Here we shall provide an example to illustrate the construction of a fuzzy majority opinion.

Example. We assume that our values are drawn from a scale of 0–10. We assume the following simple similarity relation:

$$\begin{aligned} &\text{Sim}(x,y) = 1 && \text{if } |x-y| \leqslant 2 \\ &\text{Sim}(x,y) = \frac{1}{2}(4-|x-y|) && \text{if } 2 \leqslant |x-y| \leqslant 4 \\ &\text{Sim}(x,y) = 0 && \text{if } |x-y| \geqslant 4 \end{aligned}$$

We assume the situation in which our concept majority, Q, is defined as shown in Fig. 3. Thus

$$\begin{aligned} &Q(x) = 0 && \text{if } x \leqslant 0.4 \\ &Q(x) = 5(x-0.4) && 0.4 < x \leqslant 0.6 \\ &Q(x) = 1 && x \geqslant 0.6 \end{aligned}$$

(I) Let us consider the case where $A = \{1,4,5,5,6,9\}$. Since $n = 6$ we have 2^6 possible subsets. However any subset having 2 or less elements has $Q\left(\frac{|E|}{6}\right) = 0$. In addition any subset having elements with a distance between any two of its members of four or more has $\text{Sim}(E) = 0$. Thus the following are the only subsets for which $\text{Majop}(E) \neq 0$

$$\begin{aligned} E_1 &= \{4,5,5,6\} \\ E_2 &= \{4,5,5\} \\ E_3 &= \{4,5,6\} \\ E_4 &= \{5,5,6\} \end{aligned}$$

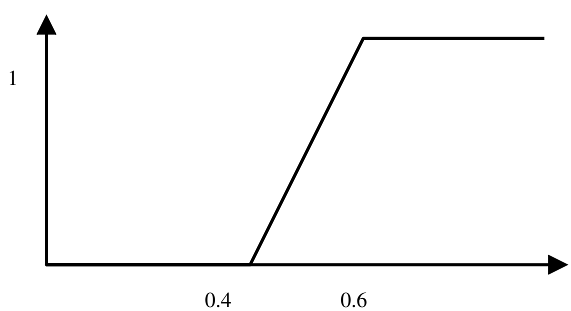

Fig. 3. Definition of the quantity majority.

E	Ave(E)	$Q\left(\frac{\lvert E\rvert}{n}\right)$	Sim(E)	Majop(E)
E_1	5	1	1	1
E_2	4.666	0.5	1	0.5
E_3	5	0.5	1	0.5
E_4	5.333	0.5	1	0.5

Thus in this case our fuzzy majority F is $F = \left\{\frac{0.5}{4.666}, \frac{1}{5}, \frac{0.5}{5.333}\right\}$, which we can see can be expressed as *about* 5.

(II) Let us consider the case where $A = \{1, 1, 4.5, 6.5, 10, 10\}$Here again if we eliminate subsets with two or less elements and those which have elements at a distance of four from each other we only two subsets:

$E_1 = \{1, 1, 4.5\}$

$E_2 = \{6.5, 10, 10\}$

E	Ave(E)	$Q\left(\frac{\lvert E\rvert}{n}\right)$	Sim(E)	Majop(E)
E_1	2.166	0.5	0.25	0.25
E_2	8.833	0.5	0.25	0.25

In this case we get as our majority opinion $F = \left\{\frac{0.25}{2.166}, \frac{0.25}{8.833}\right\}$. Here we see very little support for any fuzzy majority opinion. As matter of fact $\text{Max}_x F(x) = 0.25$, there is no gang, no subset of A constituting a majority of people with similar opinions.

We note that at a formal level F is a fuzzy subset of the real line such that

$$F(r) = \underset{E \subseteq A \text{ s.t. } \text{Ave}(E)=r}{\text{Max}} (\text{Majop}(E))$$

In describing the subset F use can be made of the connection between fuzzy subsets and natural languages to allow, if possible, the expression of F as a linguistic term. We also again note that $\text{Max}_{E\subseteq A}[\text{Majop}(E)]$ indicates the degree to which there exists a majority opinion.

5. The uniqueness of the majority opinion

With the use of the fuzzy majority we get a fuzzy subset F where for each value Op(E) we have Majop(E) indicating the degree to which Op(E) can be considered as a majority opinion. As we noted $\text{Max}_E[\text{Majop}(E)]$ indicates the degree to which there exists at least one value that can be considered as a majority opinion. The possibility exists that there exists multiple majority opinions.

In the following we shall suggest a measure which we call the clarity or uniqueness of the majority opinion. Here we want to calculate the degree to which there seems to be some unique value that is a majority opinion. We note this unique value can be a cluster of similar values.

In the following it will be convenient to view the subset F corresponding to the majority opinion as consisting of a collection of pairs, (u_i, r_i) where $u_i = \text{Majop}(E)$ and $r_i = \text{Op}(E)$. We shall refer to u_i as the strength of the pair and r_i as the value of the pair. Since we obtained F using subsets where $\text{Majop}(E) > 0$ here we assume $u_i > 0$. We let q denote the number of pairs.

As first step in our formulation of this uniqueness measure we order the pairs (u_i, r_i) by these u_i values in descending order. Tied values can be arbitrarily adjudicated. Using this ordering we let index(j) be the index of the jth largest of the u_i. Thus $(u_{\text{index}(j)}, r_{\text{index}(j)})$ is the pair having the jth largest degree of membership in F.

Using this we now introduce the idea of Unique(F) as the degree to which F has a unique majority opinion. This concept which we are trying to capture is closely related to the idea of specificity introduced by Yager [15,18]. We recall specificity tries to measure the degree to which a fuzzy subset has one and only element. We also recall that specificity can be denied if a subset has to many members or no members, a low value for largest membership grade. A feature distinguishing our measure of uniqueness from that of specificity relates to the assumption of an underlying similarity relationship in the case of uniqueness. Essentially here we want to consider two members of F that are similar as corresponding to element when calculating the degree of uniqueness. We note that in [17] Yager looked at closely related issues in investigating measures of specificity in the face of similarity relations. We now turn to the formal definition of Unique(F). We first introduce a related concept called OverShadowed which we denote OS. We define OS as a mapping on $\{1, 2, \ldots, q\}$ into the unit interval such that

$$\begin{aligned}
&\text{OS}(1) = 0\\
&\text{OS}(j) = \text{Max}_{i=1 \text{ to } j-1}[(\text{Sim}(r_{\text{index}(j)}, r_{\text{index}(i)})) \wedge (1 - \text{OS}(i))]\\
&\quad \text{for } j = 2 \text{ to } q
\end{aligned}$$

We see that OS(j) essentially measures the degree to which $r_{\text{index}(j)}$, the value corresponding to the element with j largest membership grade in F, is similar to some other value which has a higher membership grade in F and which has not been overshadowed. Using this we express our measure of uniqueness.

$$\text{Unique}(F) = u_{\text{index}(1)} - \text{Max}_{j=2 \text{ to } q}[u_{\text{index}(j)} \wedge (1 - \text{OS}(j))]$$

Let us apply this concept to our preceding example.

Example. *Case I*: Here $A = \{1, 4, 5, 5, 6, 9\}$ and we got $F = \left(\frac{0.5}{4.666}, \frac{1}{5}, \frac{0.5}{5.333}\right)$. From this we get three pairs $\langle(0.5, 4.7), (1, 5), (0.5, 5.3)\rangle$. Using this we get

j	$u_{\text{index}(j)}$	$r_{\text{index}(j)}$	OS(j)	$1 - \text{OS}(j)$	$u_{\text{index}(1)} \wedge (1 - \text{OS}(r_j))$
1	1	5	–	–	–
2	0.5	5.333	1	0	0
3	0.5	4.666	1	0	0

Thus in this case we get Unique(F) = 1. There is one unique majority opinion.

Case II: Here $A = \{1, 1, 4.5, 6.5, 10, 10\}$ and we got $F = \left\{\frac{0.25}{2.166}, \frac{0.25}{8.833}\right\}$. From this we get two pairs $\langle(0.25, 2.16), (0.25, 8.833)\rangle$. Using this we get

j	$u_{\text{index}(j)}$	$r_{\text{index}(j)}$	OS(j)	$1 - \text{OS}(j)$	$u_{\text{index}(1)} \wedge (1 - \text{OS}(r_j))$
1	0.25	2.166	–	–	–
2	0.25	8.833	0	1	0.25

Thus here Unique(F) = 0.25 − 0.25 = 0. Here we see that does not exist any majority opinion.

We note that Unique(F) $\leqslant u_{\text{index}(1)}$ thus if the strongest element in F is small then Unique(F) will be small, we will have no unique majority opinion. On the other hand if $u_{\text{index}(1)}$ is large this does not assure us a unique majority opinion as we may have multiple diverse majority opinions.

As defined Unique(F) indicates the degree to which F contains a single gang, a single majority opinion. In particular the measure Unique(F) does provide much understanding in the situation when it is, was this caused by having no majority opinion or by having multiple majority opinions. We introduce a related concept which helps provide some understanding. Let us first define g_j for $j = 1$ to q such that

$$g_1 = u_{\text{index}(1)}$$
$$g_j = u_{\text{index}(j)} \wedge (1 - \text{OS}(j)) \quad \text{for } j > 1$$

We now define h_i as the ith largest of g_j. We can now use h_i to indicate the degree to which there exists *at least i distinct majority opinions.* Since $g_1 \geqslant g_j$ for all j then $h_1 = g_1$ and hence g_1 is the degree to which there exists at least one majority opinion.

In some sense our idea of fuzzy majority is related to, although more general then, the concept of the mode of the set of observations A. We recall the mode indicates the value in A occurring the most times. Let us see this relationship here between F and the mode. Instead of considering one value we consider collections of similar values. Furthermore we assign a degree of modeness of one to the subset with the most equal elements. With F we use the concept Q to determine how satisfactory is a subset of a given cardinality.

The relationship Sim indicates our idea of what scores are considered as compatible. Let us look at some special cases of Sim. First consider the case of a strong condition for compatibility here $\mathrm{Sim}(x,x)=1$ and $\mathrm{Sim}(x,y)=0$ for $x\neq y$. In this case $\mathrm{Sim}(E)=0$ except in the case in which all elements are the same, in which case $\mathrm{Sim}(E)=1$. Furthermore if all elements in E are equal to a then $\mathrm{Op}(E)=a$. In this case of similarity the fuzzy majority opinion takes a very interesting form. Let $A=\{a_1,\ldots,a_n\}$ be our data and let $D=\{d_1,\ldots,d_t\}$ be the set of distinct values in A, here of course $t\leqslant n$. Furthermore let n_j be the number of elements in A having the value d_j. Using this we get

$$F=\bigcup_{j=1}^{t}\left\{\frac{Q\left(\frac{n_j}{n}\right)}{d_j}\right\}=\bigcup_{j=1}^{t}\left\{\frac{u_j}{d_j}\right\}$$

Here we clearly see the connection with the mode. Furthermore since Q is monotonic no element in A will have a higher membership grade in F then the one with the biggest count in A. Actually we can formally obtain the mode by a using a special definition for Q. In the preceding we defined Q as a pointwise function of its argument, $Q(x)$ just depends upon x. Here we must define Q not in a pointwise fashion. In particular we define $F=\bigcup_{j=1}^{t}\left\{\frac{Q(n_j,N)}{d_j}\right\}$ thus $u_j=Q(n_j,N)$. Here $N=\{n_1,\ldots,n_t\}$, it is the set of all arguments. We now define $Q(n_j,N)$ as follows:

$$\text{If } n_j\geqslant \mathrm{Max}(N) \text{ then } Q(n_j,N)=1$$
$$\text{If } n_j< \mathrm{Max}(N) \text{ then } Q(n_j,N)=0$$

Using this definition for Q and the preceding definition for Sim we get $F=\mathrm{Mode}$.

Let us now consider the other extreme for Sim where we assume all the values are compatible, $\mathrm{Sim}(x,y)=1$. In this case $\mathrm{Sim}(E)=1$ for all E. In this case $\mathrm{Majop}(E)=Q\left(\frac{|E|}{n}\right)$. Thus

$$F=\bigcup_{E\subseteq A}\left\{\frac{Q\left(\frac{|E|}{n}\right)}{\mathrm{Ave}(E)}\right\}$$

In this case since $\mathrm{Sim}(x,y)=1$ for all x and y, then $\mathrm{Sim}(\mathrm{Ave}(E_1),\mathrm{Ave}(E_2))=1$ for all E_1 and E_2.

Further since $Q\left(\frac{|A|}{n}\right) = 1$ then we easily show $\mathrm{OS}(j) = 1$ for all $j > 1$. Thus in this case

$$\mathrm{Unique}(F) = u_{\mathrm{index}(1)} - \mathrm{Max}_{j=2 \text{ to } q}[u_{\mathrm{index}(j)} \wedge (1 - \mathrm{OS}(j))] = u_{\mathrm{index}(1)} = 1$$

Thus in this case there appears one unique majority opinion, $\mathrm{Ave}(A)$ the average of the observations.

We further note that if we define F where we define Q using the rule base above, $F(E) = Q(|E|, N)$ where $N = \{|B| | \emptyset \neq B \subseteq A\} = \{1, \ldots, |A|\}$ then $F = \bigcup_{E \subseteq A}\{\frac{Q(|E|,N)}{Ave(E)}\}$. In this case with $\mathrm{Sim}(x, y) = 1$ we get F is the average, $F = \mathrm{Ave}(E)$.

6. Ordinal environment

In the preceding we considered the situation in which the values to be aggregated where assumed to be numbers. Here we shall consider the problem of calculating the majority opinion in the case in which the individual opinions are assumed only to have an ordered nature. We let $S = \{s_1, s_2, \ldots, s_m\}$ be an ordinal scale such that $s_i > s_j$ if $i > j$. We shall assume that the opinions to be aggregated $A = \{a_1, a_2, \ldots, a_n\}$ are drawn from S. A prototypical situation of this kind is the case in which opinions are expressed using linguistic terms such as *good*, *very good*, *perfect*.

In order to develop our method for obtaining a majority opinion we need to provide some information about our idea of what is a majority as well as information about the similarity of the objects in S. In order to accomplish this, all we need is an ordinal scale for expressing this information. While this scale can be the scale S we need not require it be the same scale. Here we shall assume a scale $T = \{t_1, t_2, \ldots, t_p\}$ which only need be ordinal, thus we assume $t_i > t_j$ if $i > j$. We emphasize that the assumption that T is not necessarily the same as S is less restrictive then assuming them to be the same. We note that we can provide a negation on this scale as $\mathrm{Neg}(t_j) = t_{p+1-j}$. We also point out the assumption that this is an ordinal scale does not preclude us from using a numeric scale such as the unit interval. We also point out that while the information about the definition of similarity and the concept of majority do not have to be on the same scale as the opinions being aggregated the information about the definition of similarity and the concept of majority do have to be on the same scale.

Here then we assume the availability of a relationship Sim on S such for any $s_i, s_j \in S$ we have $\mathrm{Sim}(s_i, s_j) \in T$. Here we assume that $\mathrm{Sim}(s_i, s_j) = \mathrm{Sim}(s_j, s_i)$ and $\mathrm{Sim}(s_j, s_j) = t_p$. In addition we assume the availability of a definition of majority, Q such that $Q: [0, 1] \to T$ where $Q(0) = t_1, Q(1) = t_p$ and $Q(x) \geqslant Q(y)$ if $x > y$.

Using these tools we can build a concept of a majority opinion as a fuzzy subset in a manner analogous to the preceding. Here we again define our majority opinion as a fuzzy subset F such that

$$F = \bigcup_{E \subseteq A} \left\{ \frac{\mathrm{Majop}(E)}{\mathrm{Op}(E)} \right\}$$

Again for any subset E of A we define

$$\mathrm{Majop}(E) = Q\left(\frac{|E|}{n}\right) \wedge \mathrm{Sim}(E)$$

where $\mathrm{Sim}(E) = \mathrm{Min}_{a_i, a_j \in E}[\mathrm{Sim}(a_i, a_j)]$. Here Majop determines the degree to which the subset E constitutes a majority of values that are compatible, similar. We note that since the elements of E are chosen from an ordinal scale S we can indicate $E^* = \mathrm{Max}_{a_i \in E}[a_i]$ and $E_* = \mathrm{Min}_{a_i \in E}[a_i]$. Since $\mathrm{Sim}(E)$ is the minimum similarity between any two elements in E, it is the similarity between the two most distant elements in E with respect to the scale S. From this we see that $\mathrm{Sim}(E) = \mathrm{Sim}(E^*, E_*)$.

The term $\mathrm{Op}(E)$ is the aggregated opinion of the elements in the subset E. In the preceding we used the average of the elements in E for $\mathrm{OP}(E)$, however here the ordered nature of the elements precludes our using this operation. In this case to calculate the aggregated opinion of the elements E we shall use the median of E, thus $\mathrm{Op}(E) = \mathrm{Med}(E)$. We recall that the median of E is obtained by ordering the elements in E and then taking the middle element. Without loss of generality assume the ordered elements in E are $b_1 \geqslant b_2 \geqslant b_{|E|}$. If W is odd then we can obtain the middle element, $\mathrm{Med}(E) = b_{\frac{|E|+1}{2}}$. If $|E|$ is even then the $\mathrm{Med}(E)$ is not unique, it is between $b_{\frac{|E|}{2}}$ and $b_{\frac{|E|}{2}+1}$. One protocol is to take one of these values as the median. For example we can take the bigger thus if $|E|$ is even $\mathrm{Med}(E) = b_{\frac{|E|}{2}+1}$.

Using the median we get as our majority opinion the fuzzy subset

$$F = \bigcup_{E \subseteq A} \left\{ \frac{\mathrm{Majop}(E)}{\mathrm{Med}(E)} \right\}$$

We shall now turn to some pragmatic issues related to this ordinal environment. First we note that occurrence of a scale S on which we obtain the individual opinions is generally a natural phenomenon and is not difficult to formulate. It is assumed often of a linguistic nature and as noted it may involve terms like *very poor*, *good*, *very good*. The requirement that the measure of similarity between the observed elements and the definition of the concept Q be on the same scale is required because of the need to perform the operator $Q\left(\frac{|E|}{n}\right) \wedge \mathrm{Sim}(E)$.

This requirement can be somewhat relaxed in a pragmatic spirit. In particular if in determining Sim(E) we can be satisfied in only establishing whether the elements in E are compatible with each other or not we greatly reduce the information required with respect to the similarity. That is here we need only assign a value 1 or 0 (true or false) to Sim(E). Furthermore since $\mathrm{Sim}(E) = \mathrm{Sim}(E^*, E_*)$ all we need to determine is if the boundary elements in E are compatible or not. At a formal level the use of this binary type of measurement of similarity can be seen as defining the relationship Sim on a sub scale of T, the subset scale being $\{t_1, t_p\}$. In particular if the elements in E are compatible $\mathrm{Sim}(E) = t_p$ and if they are not compatible $\mathrm{Sim}(E) = t_1$. Thus in the case in which $\mathrm{Sim}(E) = t_p$ we get $\mathrm{Majop}(E) = Q\left(\frac{|E|}{n}\right)$ and if elements in E are not compatible, $\mathrm{Sim}(E) = t_1$ then $\mathrm{Majop}(E) = t_1$. It will be convenient to refer to these special elements in T as **0** and **1**.

Using this more simplified measurement of the similarity we get

$$F = \bigcup_{\mathrm{Sim}(E)=t_p} \left\{ \frac{\mathrm{Majop}(E)}{\mathrm{Med}(E)} \right\}$$

Let us look more carefully at the collection of subsets of A with compatible elements, those with $\mathrm{Sim}(E) = t_p$. We first note that if $E \subseteq E'$ then if the elements in E are not compatible then those in E' are not compatible. On the other if the elements in E' are compatible then all those in E are compatible. We now consider the idea of maximal compatibility set in this environment. Without loss of generality, assume the elements in A have been indexed such that $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$ if $i < j$. Let us start with a_1 and find the largest a_j such that $\mathrm{Sim}(a_1, a_j) = t_p = \mathbf{1}$ it is the farthest element in A still compatible with a_1. Let us denote $E_1 = \{a_j | j \geqslant 1, \mathrm{Sim}(a_1, a_j) = \mathbf{1}\}$ all the elements in E_1 are compatible and all the subsets of E_1 are made of compatible elements. More generally for any $a_i \in A$ let us denote $E_i = \{a_j | j \geqslant i,\ \mathrm{Sim}(a_j, a_i) = \mathbf{1}\}$.

Consider a compatible set E_i and let G and H be any subsets of E_i then since all elements in G and H are compatible and since $\mathrm{Med}(G) \in G$ and $\mathrm{Med}(H) \in H$ then $\mathrm{Sim}(\mathrm{Med}(G),\ \mathrm{Med}(H)) = 1$. In particular $\mathrm{Sim}(\mathrm{Med}(G),\ \mathrm{Med}(E_i)) = \mathbf{1}$. Thus the similarity between the compatible set E_i and any of its subset is one.

Let us denote C_A as the collection of compatible subsets of A. We see that

$$C_A = \bigcup_{i=1}^{n} 2^{E_i}$$

It is the union of the power sets of all of the E_i. We can further refine this definition. For any compatibility set E_i let us denote e_i^* as the maximal element, it is the element farthest from a_i, the seed of E_i. Let E_i and E_j be two compatible sets where $j > i$, $a_j > a_i$. We first note that $e_j^* \geqslant e_i^*$ that is the largest element in E_j must be at least as large as e_i^*. Further we note that if $e_j^* = e_i^*$ then $E_j \subseteq E_i$. We shall call E_i a *maximal compatible set* if $e_j^* > e_i^*$ for all $i < j$.

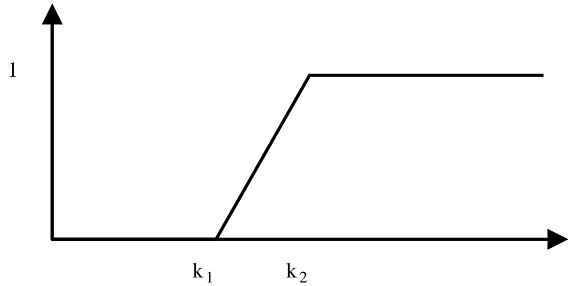

Fig. 4. Basic form for Q.

We shall denote the collection of indices corresponding to maximal some sets as K. We see that we can express $C_A = \bigcup_{i\in K} 2^{E_i}$.

Thus using this binary similarity relationship we have obtained $F = \bigcup_{E\subseteq C_A}\left\{\frac{Q\left(\frac{|E|}{n}\right)}{\mathrm{Med}(E)}\right\}$ where $C_A = \bigcup_{i\in K} 2^{E_i}$ and E is the collection of maximal compatible sets.

Let us now consider the definition of Q. First we recall that Q is monotonic. We can consider Q to be of the form shown as Fig. 4.

Thus there is some quantity of elements k_1 below which the degree of majority is zero and some quantity of elements k_2 for which we assume complete concept of majority. Using this we easily define Q in a natural manner

$$\begin{array}{ll} Q(|E|) = 0 & |E| \leqslant k_1 \\ Q(|E|) = \frac{|E|-k_1}{k_2-k_1} & k_1 < |E| < k_2 \\ Q(|E|) = 1 & |E| \geqslant k_2 \end{array}$$

Furthermore we note that if $B \subset E_i$ then $Q(|B|) \leqslant Q(|E_i|)$ this implies that for any maximal compatible set $Q(|E_i|)$ is at least as large as $Q(|B|)$ for its subsets. Furthermore since if $B \subseteq E_i$ then $\mathrm{Sim}(\mathrm{Med}(E_i), \mathrm{Med}(B)) = \mathbf{1}$ we can effectively represent the majority opinion as

$$F = \bigcup_{i\in K}\left\{\frac{Q(E_i)}{\mathrm{Med}(E_i)}\right\}$$

Thus all we need do is to find all the maximal compatibility sets, the degree to which each constitutes a majority of elements and obtain their respective medians.

7. Conclusions

In this paper the context of multi-agent decision making is considered, where distinct agents express distinct opinions about each one of a set of alternatives. One of the main problems related to this decisional environment is to synthesize an overall opinion shared by the majority of the agents. This requires an aggregation of the individual opinions into an overall value reflecting the concept of majority. Within fuzzy set theory the concept of majority has been modeled by means of fuzzy quantifiers defined as fuzzy subsets of the unit interval. In this paper two possible definitions of a majority opinion related to a linguistic quantifier over a considered set of values have been presented. The first proposal is aimed at constructing an aggregation operator whose semantics reflects the concept of majority. In this context a possible definition of a majority-based aggregation has been proposed and formalized by means of induced ordered weighted averaging operators; when applied to aggregate the considered set of values (opinions) this operator produces an aggregated value which is a representative of the majority of the values to be aggregated (what we call the majority opinion). The second proposal formalizes the concept of majority opinion as a fuzzy subset. Both the case of numeric and ordinal opinions is considered. Connected with this latter formalization is the concept of uniqueness of majority opinion.

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