An Overview of Methods for Determining OWA Weights

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The ordered weighted aggregation (OWA) operator has received more and more attention since its appearance. One key point in the OWA operator is to determine its associated weights. In this article, I first briefly review existing main methods for determining the weights associated with the OWA operator, and then, motivated by the idea of normal distribution, I develop a novel practical method for obtaining the OWA weights, which is distinctly different from the existing ones. The method can relieve the influence of unfair arguments on the decision results by weighting these arguments with small values. Some of its desirable properties have also been investigated. © 2005 Wiley Periodicals, Inc.

1. INTRODUCTION

In Ref. 1, Yager introduced an ordered weighted aggregation (OWA) operator to aggregate information. In the less than two decades since its first appearance, this operator has been investigated in many documents and used in an astonishingly wide range of applications.[2–39] The OWA operator is generally composed of the following three steps:

(1) Reorder the input arguments in descending order.
(2) Determine the weights associated with the OWA operator by using a proper method.
(3) Utilize the OWA weights to aggregate these reordered arguments.

Clearly, one key point in the OWA operator is to determine its associated weights. A number of methods have been developed for obtaining the OWA weights. Yager[1] suggested an interesting way to compute the weights of the OWA operator using linguistic quantifiers. O'Hagan[40,41] developed a procedure to generate the OWA weights that have a predefined degree of orness and maximize the entropy of the OWA weights. Fullér and Majlender[42] used the method of Lagrange multipliers to solve O'Hagan's procedure analytically. Filev and Yager[43] developed two

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procedures, based on the exponential smoothing, to obtain the OWA weights. Yager[44] introduced some families of OWA weights. Yager and Filev[45] suggested an algorithm to obtain the OWA weights from a collection of samples with the relevant aggregated data. Xu and Da[46] established a linear objective-programming model for obtaining the weights of the OWA operator under partial weight information. In this article, we first make a survey of the existing main methods and then develop a novel practical method based on normal distribution for determining the OWA weights.

2. A SURVEY OF THE EXISTING MAIN METHODS

An OWA operator[1] of dimension n is a mapping, $OWA : R^n \to R$, that has an associated n vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_i = 1$. Furthermore,

$$OWA_w(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{1}$$

where b_j is the jth largest element of the collection of the aggregated objects $a_1, a_2, \ldots, a_n$.

Central to this operator is the reordering of the arguments, based upon their values,[45] in particular an argument a_i is not associated with a particular weight w_i but rather a weight w_i is associated with a particular ordered position i of the arguments.[43]

One important issue in the OWA operator is to determine its associated weights. Yager[1,44] suggested an interesting approach to obtaining the OWA weights by using linguistic quantifiers, which, in the case of a nondecreasing proportional fuzzy linguistic quantifier Q, is given by

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1, 2, \ldots, n \tag{2}$$

Zadeh[47] defined Q as follows:

$$Q(r) = \begin{cases} 0, & \text{if } r < a \\ \dfrac{r-a}{b-a}, & \text{if } a \le r \le b \\ 1, & \text{if } r > b \end{cases} \tag{3}$$

with $a, b, r \in [0,1]$. Some examples of nondecreasing proportional fuzzy linguistic quantifiers are shown in Figure 1, where the parameters, (a, b) are (0.3, 0.8), (0, 0.5), and (0.5, 1), respectively.[48–54] The fuzzy linguistic quantifier Q generally represents the concept of fuzzy majority in the aggregation of arguments.[55]

Yager[44] defined three quantifiers, "*for all*" Q_*, "*there exists*" Q^*, and "*identify*" $\dot{Q}$, as follows (shown in Figure 2), respectively:

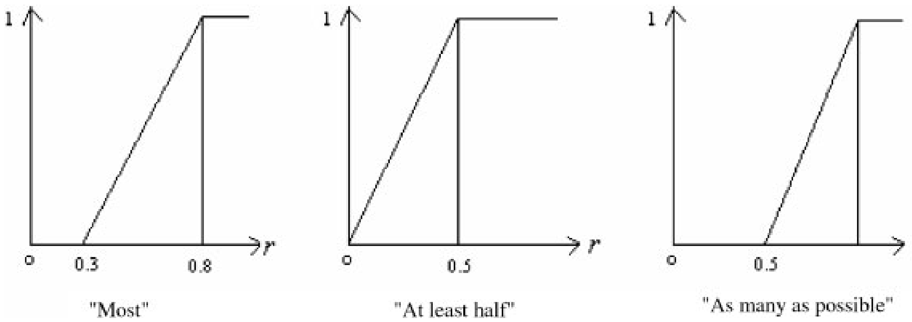

Figure 1. Proportional fuzzy quantifiers.

(1)

$$Q_*(r) = \begin{cases} 0, & \text{for } r < 1 \\ 1, & \text{for } r = 1 \end{cases} \tag{4}$$

By Equations 2 and 4, it yields

$$w_i = \begin{cases} 0, & i < n \\ 1, & i = n \end{cases} \tag{5}$$

(2)

$$Q^*(r) = \begin{cases} 0, & \text{for } r = 0 \\ 1, & \text{for } r > 0 \end{cases} \tag{6}$$

In this case, it can obtained that

$$w_i = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1 \end{cases} \tag{7}$$

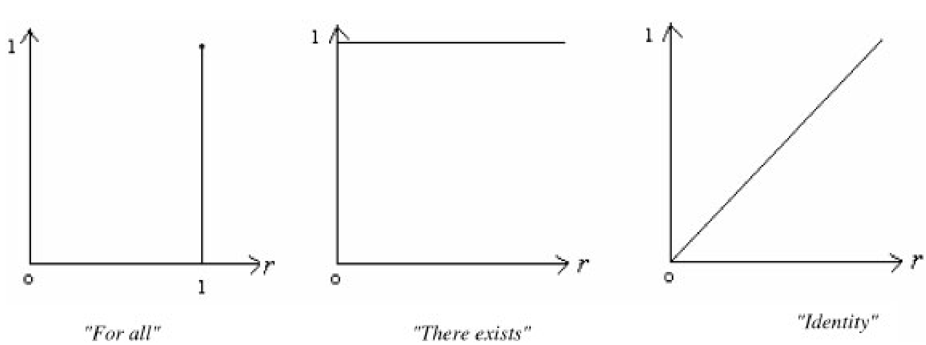

Figure 2. The quantifiers "*for all*," "*there exists*," and "*identify*."

(3)

$$\dot{Q}(r) = r \tag{8}$$

From Equations 2 and 8, it follows that

$$w_i = \frac{1}{n}, \quad i = 1,2,\ldots,n \tag{9}$$

Yager[1] further introduced two characterizing measures called *orness measure* and *dispersion measure*, respectively, associated with the weighting vector w of an OWA operator, where the *orness measure* of the aggregation is defined as

$$orness(w) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i \tag{10}$$

which lies in the unit interval $[0,1]$ and characterizes the degree to which the aggregation is like an *or* operation. The second one, the *dispersion measure* of the aggregation, is defined as

$$disp(w) = -\sum_{i=1}^{n} w_i \ln w_i \tag{11}$$

which measures the degree to which w takes into account the information in the arguments during the aggregation.

O'Hagan[40,41] used these two measures to develop a procedure to generate the OWA weights that have a predefined degree of orness and maximize the entropy. This procedure is based on the solution of the following constrained optimization problem:

$$\begin{aligned}
&\text{Maximize: } -\sum_{i=1}^{n} w_i \ln w_i \\
&\text{Subject to: } \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i = \alpha, \quad 0 \le \alpha \le 1 \\
&\qquad\qquad \sum_{i=1}^{n} w_i = 1, \quad 0 \le w_i \le 1, \quad i = 1,2,\ldots,n
\end{aligned} \tag{12}$$

In light of work by Yager,[56,57] Yager[44] expressed a measure of entropy as $1 - \mathrm{Max}_i[w_i]$, and then extended problem 12 to the following:

$$\begin{aligned}
&\text{Minimize: } \max_i w_i \\
&\text{Subject to: } \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i = \alpha, \quad 0 \le \alpha \le 1 \\
&\qquad\qquad \sum_{i=1}^{n} w_i = 1, \quad 0 \le w_i \le 1, \quad i = 1,2,\ldots,n
\end{aligned} \tag{13}$$

Xu and Da[58] considered the situation where the weight information is available partially and suggested an approach based on the following mathematical programming problem:

$$\text{Maximize: } -\sum_{i=1}^{n} w_i \ln w_i$$

$$\text{Subject to: } \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i = \alpha, \quad 0 \le \alpha \le 1$$

$$w \in H$$

$$\sum_{i=1}^{n} w_i = 1, \quad 0 \le w_i \le 1, \quad i = 1,2,\ldots,n \tag{14}$$

where H is the set of all known weight information, which can be constructed by the following forms,[59–61] for $i \neq j$:

(1) A weak ranking: $\{w_i \ge w_j\}$
(2) A strict ranking: $\{w_i - w_j \ge \alpha_i\}$
(3) A ranking with multiples: $\{w_i \ge \alpha_i w_j\}$
(4) An interval form: $\{\alpha_i \le w_i \le \alpha_i + \varepsilon_i\}$
(5) A ranking of differences: $\{w_i - w_j \ge w_k - w_l\}$, for $j \neq k \neq l$

where α_i and ε_i are nonnegative constants.

Fullér and Majlender[42] used the method of Lagrange multipliers to solve problem 12 analytically and got the following:

(1) If $n = 2$, then $w_1 = \alpha$, $w_2 = 1 - \alpha$.
(2) If $\alpha = 0$ or $\alpha = 1$, then the associated weighting vectors are uniquely defined as $w = (0,0,\ldots,1)^{\mathrm{T}}$ and $w = (1,0,\ldots,0)^{T}$, respectively, with value of dispersion zero.
(3) If $n \ge 3$ and $0 < \alpha < 1$, then

$$w_j = \sqrt[n-1]{w_1^{n-j} w_n^{j-1}} \tag{15}$$

$$w_n = \frac{((n-1)\alpha - n)w_1 + 1}{(n-1)\alpha + 1 - nw_1} \tag{16}$$

$$w_1[(n-1)\alpha + 1 - nw_1]^n = ((n-1)\alpha)^{n-1}[((n-1)\alpha - n)w_1 + 1] \tag{17}$$

Solving Equations 15–17, the optimal OWA weights can be determined.

Yager and Filev[62] introduced some interesting families of OWA weights as follows:

(1)

$$w_i = \begin{cases} \dfrac{1}{n}(1-\alpha) + \alpha, & i = 1; \\ \dfrac{1}{n}(1-\alpha), & i = 2,\ldots,n \end{cases}, \quad \alpha \in [0,1] \tag{18}$$

which results in an "orlike" S-OWA operator, denoted as F_{SO}:

$$F_{SO}(a_1, a_2, \ldots, a_n) = \alpha \operatorname*{Max}_i (a_i) + \frac{1}{n}(1-\alpha)\sum_{i=1}^{n} a_i \tag{19}$$

(2)

$$w_i = \begin{cases} \frac{1}{n}(1-\beta), & i \neq n; \\ \frac{1}{n}(1-\beta) + \beta, & i = n \end{cases}, \quad \beta \in [0,1] \tag{20}$$

which results in an "andlike" S-OWA operator, denoted as F_{SA}:

$$F_{SA}(a_1, a_2, \ldots, a_n) = \beta \operatorname*{Min}_i (a_i) + \frac{1}{n}(1-\beta)\sum_{i=1}^{n} a_i \tag{21}$$

(3)

$$w_i = \begin{cases} \frac{1}{n}(1-(\alpha+\beta)) + \alpha, & i = 1 \\ \frac{1}{n}(1-(\alpha+\beta)), & i = 2, \ldots, n-1 \\ \frac{1}{n}(1-(\alpha+\beta)) + \beta, & i = n \end{cases} \tag{22}$$

where $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$. These weights define a generalized S-OWA operator, denoted as F_S:

$$F_S(a_1, a_2, \ldots, a_n) = \alpha \operatorname*{Max}_i [a_i] + \beta \operatorname*{Min}_i (a_i) + (1-(\alpha+\beta))\sum_{i=1}^{n} a_i \tag{23}$$

(4)

$$w_i = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \tag{24}$$

which defines a Step-OWA operator.

(5)

$$w_i = \begin{cases} 0, & i < k \\ \frac{1}{m}, & k \leq i < k+m \\ 0, & i \geq k+m \end{cases} \tag{25}$$

which defines a Window-OWA operator.

(6)

$$w_i = \frac{b_i^\alpha}{\sum_{i=1}^{n} b_i^\alpha}, \quad i = 1,2,\ldots,n \tag{26}$$

where $\alpha \in (-\infty,+\infty)$, b_i is the ith largest element of the collection of the aggregated objects $a_1, a_2, \ldots, a_n$; clearly, these weights depend on the aggregated objects. In this case, it leads to a neat OWA operator:

$$OWA_w(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n} a_i^{\alpha+1}}{\sum_{i=1}^{n} a_i^\alpha} \tag{27}$$

(7)

$$w_i = \frac{(1-b_i)^\alpha}{\sum_{i=1}^{n} (1-b_i)^\alpha}, \quad i = 1,2,\ldots,n \tag{28}$$

In this case, it yields

$$OWA_w(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n} (1-a_i)^\alpha a_i}{\sum_{i=1}^{n} (1-a_i)^\alpha} \tag{29}$$

which is also a neat aggregation.

(8)

$$w_i = \frac{(b_{n-i+1})^\alpha}{\sum_{i=1}^{n} b_i^\alpha}, \quad i = 1,2,\ldots,n \tag{30}$$

which results in the following OWA operator:

$$OWA_w(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n} (b_{n-i+1})^\alpha b_i}{\sum_{i=1}^{n} b_i^\alpha} \tag{31}$$

Clearly, this aggregation is not neat.

Similar to the exponential smoothing method,[63] Filev and Yager[43] suggested two procedures to obtain the OWA weights as follows:

(1)

$$w_1 = \alpha, \quad w_2 = \alpha(1-\alpha), \quad w_3 = \alpha(1-\alpha)^2, \ldots, w_{n-1} = \alpha(1-\alpha)^{n-2},$$
$$w_n = (1-\alpha)^{n-1} \tag{32}$$

where $\alpha \in [0,1]$. There weights can be expressed recursively as

$$w_i = \begin{cases} \alpha, & i = 1 \\ w_{i-1}(1-w_1), & i = 2, \ldots, n-1 \\ w_{i-1}(1-w_1)/w_1, & i = n \end{cases} \tag{33}$$

(2)

$$w_1 = \alpha^{n-1}, \quad w_2 = (1-\alpha)\alpha^{n-2}, \quad w_3 = (1-\alpha)\alpha^{n-3}, \ldots, w_{n-1} = (1-\alpha)\alpha,$$
$$w_n = (1-\alpha) \tag{34}$$

where $\alpha \in [0,1]$. There weights can also be expressed recursively as

$$w_i = \begin{cases} 1-\alpha, & i = n \\ w_i(1-w_n), & i = 2, \ldots, n-1 \\ w_{i+1}(1-w_n)/w_n, & i = 1 \end{cases} \tag{35}$$

Yager and Filev[43,45,64] suggested an algorithm to obtain the OWA aggregating operator from a collection of samples with the relevant aggregated data, which was described as follows.

Assume there is a collection of m samples (observations) each comprised of a n-tuple of arguments $(a_{k1}, a_{k2}, \ldots, a_{kn})$, and an associated aggregated value, d_k. We denote the reordered objects of the kth sample by $b_{k1}, b_{k2}, \ldots, b_{kn}$ where b_{kj} is the jth largest element of the argument collection $a_{k1}, a_{k2}, \ldots, a_{kn}$. Using these ordered arguments, we need to find a vector of the OWA weights $w = (w_1, w_2, \ldots, w_n)^T$ to satisfy the following condition as faithfully as possible:

$$b_{k1}w_1 + b_{k2}w_2 + \cdots + b_{kn}w_n = d_k, \quad k = 1, 2, \ldots, m \tag{36}$$

We relax the above condition by looking for a vector of OWA weights that approximates the aggregation operator by minimizing the instantaneous errors $e_k (k = 1, 2, \ldots, m)$:

$$e_k = \tfrac{1}{2}(b_{k1}w_1 + b_{k2}w_2 + \cdots + b_{kn}w_n - d_k)^2, \quad k = 1, 2, \ldots, m \tag{37}$$

with respect to the weights $w_i (i = 1, 2, \ldots, n)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$.

To circumvent the constraints on the w_i, Yager and Filev[43,45,64] developed an iterative procedure. Let $\lambda_i (i = 1, 2, \ldots, n)$ be n parameters, and set the initial values $\lambda_i(0) = 0$ $(i = 1, 2, \ldots, n)$; then the procedure used at each iteration is suggested as follows:

(1) Calculate the current estimates of the λ_i, $\lambda_i(l)(i = 1,2,\ldots,n)$, and a new observation consisting of the ordered arguments $b_{k1}, b_{k2}, \ldots, b_{kn}$.
(2) Use the $\lambda_i(l)(i = 1,2,\ldots,n)$ to provide a current estimate of the weights

$$w_i(l) = \frac{e^{\lambda_i(l)}}{\sum_{j=1}^{n} e^{\lambda_j(l)}}, \quad i = 1,2,\ldots,m \tag{38}$$

(3) Utilize the estimated weights along with the ordered arguments to get a calculated aggregated value

$$\hat{d}_k = b_{k1} w_1(l) + b_{k2} w_2(l) + \cdots + b_{kn} w_n(l), \quad k = 1,2,\ldots,m \tag{39}$$

(4) Update the estimates of the λ_i:

$$\lambda_i(l+1) = \lambda_i(l) - \beta w_i(l)(b_{ki} - \hat{d}_k)(\hat{d}_k - d_k), \quad i = 1,2,\ldots,n \tag{40}$$

Clearly, the parameters λ_i determining the OWA weights are updated by propagation of the error $(\hat{d}_k - d_k)$ between the current estimated aggregated value and the actual aggregated value with factors w_i and $(b_{ki} - \hat{d}_k)$.

In many actual situations, however, the information about the OWA weights can be partially known. Xu and Da[46] established a linear objective-programming procedure for obtaining the OWA weights from observational data under partial weight information, which can be elaborated as follows.

First, we relax condition 36 by looking for a vector of OWA weights that approximates the aggregation operator by minimizing the instantaneous errors $e_k(k = 1,2,\ldots,m)$:

$$e_k = \left| \sum_{j=1}^{n} b_{kj} w_j - d_k \right|, \quad k = 1,2,\ldots,m \tag{41}$$

with respect to the weights $w_j(j = 1,2,\ldots,n)$. Then, we construct the following multiobjective programming model:

$$\begin{aligned} &\min e_k = \left| \sum_{j=1}^{n} b_{kj} w_j - d_k \right|, \quad k = 1,2,\ldots,m \\ &\text{s.t. } w \in H \\ &\sum_{i=1}^{n} w_i = 1, \quad 0 \le w_i \le 1, \quad i = 1,2,\ldots,n \end{aligned} \tag{42}$$

where H is the set of all known weight information, which is defined as in problem 14.

The solution to the above minimization problem is found by solving the following linear objective programming model:

$$\min J = \sum_{k=1}^{m} (e_k^+ + e_k^-), \quad k = 1,2,\ldots,m$$

$$\text{s.t.} \sum_{j=1}^{n} b_{kj} w_j - d_k - e_k^+ + e_k^- = 0, \quad k = 1,2,\ldots,m$$

$$w \in H$$

$$w_i \geq 0, \quad i = 1,2,\ldots,n, \quad \sum_{i=1}^{n} w_i = 1$$

$$e_k^+ \geq 0, \quad e_k^- \geq 0, \quad k = 1,2,\ldots,m \tag{43}$$

where, e_k^+ and e_k^- are the upper and lower deviation variables of d_k, respectively.

By solving the above linear objective programming model, we can obtain the vector of the OWA weights.

3. NORMAL DISTRIBUTION BASED METHOD

In real world, a collection of n aggregated arguments $a_1, a_2, \ldots, a_n$ usually takes the form of a collection of n preference values provided by n different individuals. Some individuals may assign unduly high or unduly low preference values to their preferred or repugnant objects. In such a case, we shall assign very low weights to these "false" or "biased" opinions, that is to say, the closer a preference value (argument) is to the mid one(s), the more the weight; conversely, the further a preference value is apart from the mid one(s), the less the weight. However, all the existing main procedures discussed above should be unsuitable to deal with this case. In the following, we shall develop a normal distribution-based method to determine the OWA weights.

The normal distribution is one of the most commonly observed and is the starting point for modeling many natural processes. It is usually found in events that are the aggregation of many smaller, but independent random events. We now first review the concept of normal distribution (or so-called Gaussian distribution) and some of its desirable properties.

Definition 1. *Let x be the continuous random variable, and define its probability density function as*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-[(x-\mu)^2/2\sigma^2]}, \quad -\infty < x < +\infty \tag{44}$$

where μ and $\sigma\,(\sigma > 0)$ are constant. Then x is normally distributed with a mean of μ and a standard deviation of σ.

The normal distribution provides a realistic approximation to the distribution of deviations in many experimental situations, especially for the "central" portion of the deviations. A graph of a normal distribution is shown in Figure 3.

The graph of normal distribution is determined by setting the parameters (μ and σ) of the model to a particular value. Because the parameter μ can take on any real value, and the parameter σ can take on any positive value, this makes the normal distribution able to describe a large number of events occurring in the real world.

The normal distribution has the following well-known properties:

(1) The graph of a normal distribution is symmetrical (see Figure 4); that is,

$$P\{\mu - h < x \leq \mu\} = P\{\mu < x \leq \mu + h\} \tag{45}$$

where

$$P\{\mu - h < x \leq \mu\} = \int_{\mu-h}^{\mu} f(x)\, dx \tag{46}$$

$$P\{\mu < x \leq \mu + h\} = \int_{\mu}^{\mu+h} f(x)\, dx \tag{47}$$

(2) The graph of a normal distribution reaches its maximum when $x = \mu$:

$$f(\mu) = \frac{1}{\sqrt{2\pi\sigma}} \tag{48}$$

Moreover, the further x is apart from μ, the smaller the value of $f(x)$. If μ is fixed, then the smaller the value of σ, the sharper the graph of normal distribution.

Motivated by the idea of the normal distribution, in the following, we shall develop a simple and practical method for determining the weights of the OWA operator.

Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the OWA operator; then we define the following:

$$w_i = \frac{1}{\sqrt{2\pi\sigma_n}}\, e^{-[(i-\mu_n)^2/2\sigma_n^2]}, \quad i = 1, 2, \ldots, n \tag{49}$$

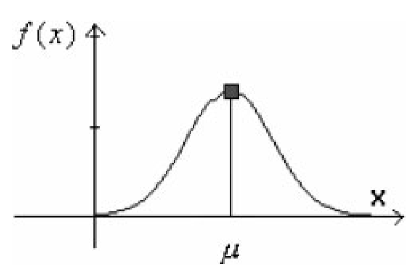

Figure 3. A graph of a normal distribution.

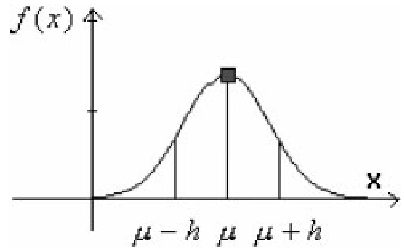

Figure 4. Symmetry property.

where μ_n is the mean of the collection of $1,2,\ldots,n$, and $\sigma_n(\sigma_n > 0)$ is the standard deviation of the collection of $1,2,\ldots,n$. μ_n and σ_n are obtained by the following formulas, respectively:

$$\mu_n = \frac{1}{n}\frac{n(1+n)}{2} = \frac{1+n}{2} \tag{50}$$

$$\sigma_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(i-\mu_n)^2} \tag{51}$$

Consider that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_i = 1$; then by Equation 49, we have

$$w_i = \frac{\dfrac{1}{\sqrt{2\pi\sigma_n}}\, e^{-[(i-\mu_n)^2/2\sigma_n^2]}}{\displaystyle\sum_{j=1}^{n}\frac{1}{\sqrt{2\pi\sigma_n}}\, e^{-[(j-\mu_n)^2/2\sigma_n^2]}} = \frac{e^{-[(i-\mu_n)^2/2\sigma_n^2]}}{\displaystyle\sum_{j=1}^{n} e^{-[(j-\mu_n)^2/2\sigma_n^2]}}, \quad i = 1,2,\ldots,n \tag{52}$$

Because the mean of the collection of $1,2,\ldots,n$ is $(1+n)/2$, then Equation 52 can be rewritten as

$$w_i = \frac{e^{-[(1-(1+n)/2)^2/2\sigma_n^2]}}{\displaystyle\sum_{j=1}^{n} e^{-[(j-(1+n)/2)^2/2\sigma_n^2]}}, \quad i = 1,2,\ldots,n \tag{53}$$

From Equations 51–53, we have the following theorem.

THEOREM 1. *Both $\{\mu_n\}$ and $\{\sigma_n\}$ are strictly monotone increasing sequences, that is,*

$$\mu_{n+1} > \mu_n, \quad \sigma_{n+1} > \sigma_n, \quad \text{for all } n \tag{54}$$

Proof. Because

$$\mu_{n+1} = \frac{1+(n+1)}{2} = \frac{1+n}{2} + \frac{1}{2} > \frac{1+n}{2} = \mu_n, \quad \text{for all } n \tag{55}$$

hence, the sequence $\{\mu_n\}$ is strictly monotone increasing.

To prove $\sigma_{n+1} > \sigma_n$, for all n, we only need to prove that

$$\sigma_n^2 < \sigma_{n+1}^2, \quad \text{for all } n \tag{56}$$

namely,

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\left(i - \frac{(n+1)+1}{2}\right)^2 > \frac{1}{n}\sum_{i=1}^{n}\left(i - \frac{n+1}{2}\right)^2, \quad \text{for all } n \tag{57}$$

$$n\sum_{i=1}^{n+1}\left(i - \frac{(n+1)+1}{2}\right)^2 > (n+1)\sum_{i=1}^{n}\left(i - \frac{n+1}{2}\right)^2, \quad \text{for all } n \tag{58}$$

$$n\sum_{i=1}^{n+1}\left[\left(i - \frac{n+1}{2}\right) - \frac{1}{2}\right]^2 > n\sum_{i=1}^{n}\left(i - \frac{n+1}{2}\right)^2 + \sum_{i=1}^{n}\left(i - \frac{n+1}{2}\right)^2, \quad \text{for all } n \tag{59}$$

$$\begin{aligned} n\sum_{i=1}^{n+1}\left(i - \frac{n+1}{2}\right)^2 - n\sum_{i=1}^{n+1}\left(i - \frac{n+1}{2} - \frac{1}{4}\right) \\ > n\sum_{i=1}^{n}\left(i - \frac{n+1}{2}\right)^2 + \sum_{i=1}^{n}\left(i - \frac{n+1}{2}\right)^2, \quad \text{for all } n \end{aligned} \tag{60}$$

$$\begin{aligned} n\sum_{i=1}^{n}\left(i - \frac{n+1}{2}\right)^2 + n\left(\frac{n+1}{2}\right)^2 - n\sum_{i=1}^{n+1}\left(i - \frac{n+1}{2} - \frac{1}{4}\right) \\ > n\sum_{i=1}^{n}\left(i - \frac{n+1}{2}\right)^2 + \sum_{i=1}^{n}\left(i - \frac{n+}{2}\right)^2 \quad \text{for all } n \end{aligned} \tag{61}$$

$$n\left(\frac{n+1}{2}\right)^2 - n\sum_{i=1}^{n+1}\left(i - \frac{n+1}{2} - \frac{1}{4}\right) > \sum_{i=1}^{n}\left(i - \frac{n+1}{2}\right)^2, \quad \text{for all } n \tag{62}$$

$$\begin{aligned} n\left(\frac{n+1}{2}\right)^2 - \frac{n(n+1)(n+2)}{2} + \frac{n(n+1)^2}{2} + \frac{n(n+1)}{4} \\ > \sum_{i=1}^{n}\left[i^2 - (n+1)i + \left(\frac{n+1}{2}\right)^2\right], \quad \text{for all } n \end{aligned} \tag{63}$$

$$\begin{aligned} \frac{3n(n+1)^2}{4} - \frac{n(n+1)(n+2)}{2} + \frac{n(n+1)}{4} \\ > \frac{1}{6}n(n+1)(2n+1) - \frac{n}{2}(n+1)^2 + \frac{n(n+1)^2}{4}, \quad \text{for all } n \end{aligned} \tag{64}$$

that is,

$$n > -1 \tag{65}$$

It is clear that Equation 65 holds; hence $\{\sigma_n\}$ is a strictly monotone increasing sequence. ■

THEOREM 2.

(1) The weights $w_i (i = 1,2,\ldots,n)$ are symmetrical, that is,

$$w_i = w_{n+1-i}, \quad i = 1,2,\ldots,n \tag{66}$$

(2)

(i) $w_i < w_{i+1}$, for all $i = 1,\ldots,round[(1+n)/2]$;

(ii) If n is odd, then $w_i > w_{i+1}$, for all $i = round[(1+n)/2],\ldots,n$; if n is even, then

$$w_i > w_{i+1}, \quad \text{for all } i = round\left(\frac{1+n}{2}\right) + 1,\ldots,n$$

where round is the usual round operation. Especially, if n is odd, then the weight w_i reaches its maximum when $i = round[(1+n)/2]$; if n is even, then the weight w_i reaches its maximum when $i = round[(1+n)/2]$ or $1 + round[(1+n)/2]$.

(3) $orness(w) = 0.5$.

(4)

$$disp(w) = \begin{cases} -2\sum_{i=1}^{n/2} w_i \ln w_i, & \text{if } n \text{ is even} \\ -2\sum_{i=1}^{(n-1)/2} w_i \ln w_i - w_{(n+1)/2} \ln w_{(n+1)/2}, & \text{if } n \text{ is odd} \end{cases}$$

Proof.

(1) By Equation 53, we have

$$w_{n+1-i} = \frac{e^{-\frac{\left(n+1-i-\frac{1+n}{2}\right)^2}{2\sigma_n^2}}}{\sum_{j=1}^{n} e^{-\frac{\left(n+1-j-\frac{1+n}{2}\right)^2}{2\sigma_n^2}}} = \frac{e^{-\frac{\left[-\left(i-\frac{1+n}{2}\right)\right]^2}{2\sigma_n^2}}}{\sum_{j=1}^{n} e^{-\frac{\left[-\left(j-\frac{1+n}{2}\right)\right]^2}{2\sigma_n^2}}}$$

$$= \frac{e^{-\frac{\left(i-\frac{1+n}{2}\right)^2}{2\sigma_n^2}}}{\sum_{j=1}^{n} e^{-\frac{\left(i-\frac{1+n}{2}\right)^2}{2\sigma_n^2}}} = w_i, \quad i = 1,2,\ldots,n \tag{67}$$

(2) (i) Because

$$\left(i - \frac{1+n}{2}\right)^2 > \left(i + 1 - \frac{1+n}{2}\right)^2, \quad \text{for all } i = 1,\ldots,round\left(\frac{1+n}{2}\right) \tag{68}$$

then

$$e^{-\frac{\left(i-\frac{1+n}{2}\right)^2}{2\sigma_n^2}} < e^{-\frac{\left(i+1-\frac{1+n}{2}\right)^2}{2\sigma_n^2}}, \quad \text{for all } i = 1, \ldots, round\left(\frac{1+n}{2}\right) \tag{69}$$

thus,

$$\frac{e^{-\frac{\left(i-\frac{1+n}{2}\right)^2}{2\sigma_n^2}}}{\sum_{j=1}^{n} e^{-\frac{\left(j-\frac{1+n}{2}\right)^2}{2\sigma_n^2}}} < \frac{e^{-\frac{\left(i+1-\frac{1+n}{2}\right)^2}{2\sigma_n^2}}}{\sum_{j=1}^{n} e^{-\frac{\left(j-\frac{1+n}{2}\right)^2}{2\sigma_n^2}}}, \quad \text{for all } i = 1, \ldots, round\left(\frac{1+n}{2}\right) \tag{70}$$

that is,

$$w_i < w_{i+1}, \quad \text{for all } i = 1, \ldots, round\left(\frac{1+n}{2}\right) \tag{71}$$

Similarly, we can easily prove (ii).

(3) Because $w_i = w_{n+1-i}$, $i = 1, 2, \ldots, n$, then from Equation 10, it follows that

(i) If n is even, then

$$\begin{aligned} orness(w) &= \frac{1}{n-1} \sum_{i=1}^{n} (n-i) w_i \\ &= \frac{1}{n-1} \sum_{i=1}^{n/2} [(n-i) w_i + (n-(n+1-i)) w_{n+1-i}] \\ &= \frac{1}{n-1} \sum_{i=1}^{n/2} [(n-i) + (n-(n+1-i))] w_i \\ &= \frac{1}{n-1} \sum_{i=1}^{n/2} (n-1) w_i = \sum_{i=1}^{n/2} w_i \\ &= \frac{1}{2} \sum_{i=1}^{n} w_i \\ &= 0.5 \end{aligned} \tag{72}$$

(ii) If n is odd, then

$$
\begin{aligned}
orness(w) &= \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i \\
&= \frac{1}{n-1}\left[\sum_{i=1}^{(n-1)/2}[(n-i)w_i + (n-(n+1-i))w_{n+1-i}] \right.\\
&\qquad \left. + \left(n - \frac{n+1}{2}\right) w_{(n+1)/2}\right] \\
&= \frac{1}{n-1}\left[\sum_{i=1}^{(n-1)/2}[(n-i) + (n-(n+1-i))]w_i + \frac{n-1}{2}\, w_{(n+1)/2}\right] \\
&= \frac{1}{n-1}\sum_{i=1}^{(n-1)/2}(n-1)w_i + \frac{1}{n-1}\cdot\frac{n-1}{2}\, w_{(n+1)/2} \\
&= \sum_{i=1}^{(n-1)/2} w_i + \frac{1}{2}\, w_{(n+1)/2} \\
&= \frac{1}{2}\sum_{i\neq(n+1)/2} w_i + \frac{1}{2}\, w_{(n+1)/2} \\
&= \frac{1}{2}\sum_{i=1}^{n} w_i \\
&= 0.5
\end{aligned}
\tag{73}
$$

(4) This result can be derived directly from Equations 11 and 66. ■

Based on normal distribution, we now define a disjunctive function as follows:

$$
F(r) = \begin{cases}
0, & r < \dfrac{1}{n} \\[2ex]
\dfrac{\displaystyle\sum_{k=1/n}^{r} e^{-\frac{n^2\left(k-\frac{1+n}{2n}\right)^2}{2\sigma_n^2}}}{\displaystyle\sum_{k=1/n}^{1} e^{-\frac{n^2\left(k-\frac{1+n}{2n}\right)^2}{2\sigma_n^2}}}, & r = \dfrac{1}{n}, \dfrac{2}{n}, \ldots, \dfrac{n-1}{n}, 1 \\[2ex]
1, & r > 1
\end{cases}
\tag{74}
$$

where σ_n is defined as in Equation 51.

Obviously, the function $F(r)$ has the following characteristics:

(1) $F(0) = 0$
(2) $F(1) = 1$
(3) $F(r_1) \geq F(r_2)$, if $r_1 > r_2$

Furthermore, the OWA weights defined in Equation 53 can be obtained from the function $F(r)$, that is,

$$w_i = F\left(\frac{i}{n}\right) - F\left(\frac{i-1}{n}\right), \quad i = 1,2,\ldots,n \tag{75}$$

It can be shown that these weights satisfy the conditions $w_i \in [0,1]$ and $\sum_{j=1}^{n} w_i = 1$.

The above function has characteristics very similar to those of the basic unit-interval monotonic (BUM) function suggested by Yager.[7,65]

Example 1. Suppose that there are five experts $e_i(i = 1,2,3,4,5)$; these experts provide their individual preferences for a university faculty with respect to the criterion *research*. Assume that the given preference values are as follows:

$$a_1 = 80,\ a_2 = 75,\ a_3 = 100,\ a_4 = 50,\ a_5 = 85$$

By Equations 51 and 52, we have

$$\mu_5 = 3,\ \sigma_5 = \sqrt{2}$$

then from Equation 53, it follows that

$$w_1 = 0.1117,\ w_2 = 0.2365,\ w_3 = 0.3036,\ w_4 = 0.2365,\ w_5 = 0.1117$$

which are shown in Figure 5.

By Equations 10 and 11, we have

$$\begin{aligned} orness(w) &= \frac{1}{5-1}\sum_{i=1}^{5}(5-i)w_i \\ &= \tfrac{1}{4} \times (4 \times 0.1117 + 3 \times 0.2365 + 2 \times 0.3036 \\ &\qquad + 1 \times 0.2365 + 0 \times 0.1117) \\ &= 0.5 \end{aligned}$$

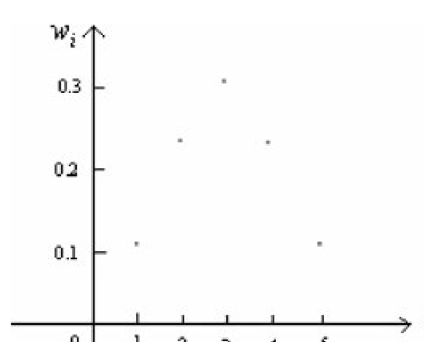

Figure 5. The weights $w_i(i = 1,2,3,4,5)$.

and

$$\begin{aligned} disp(w) &= -\sum_{i=1}^{5} w_i \ln w_i \\ &= -[0.1117 \times \ln(0.1117) + 0.2365 \times \ln(0.2365) + 0.3036 \\ &\quad \times \ln(0.3036) + 0.2365 \times \ln(0.2365) + 0.1117 \times \ln(0.1117)] \\ &= 1.5336 \end{aligned}$$

Because the ordered preference values $a_i (i = 1,2,3,4,5)$ are

$$b_1 = 100,\ b_2 = 85,\ b_3 = 80,\ b_4 = 75,\ b_5 = 50$$

then by Equation 1, we have

$$\begin{aligned} OWA_w(a_1, a_2, \ldots, a_n) &= 0.1117 \times 100 + 0.2365 \times 85 + 0.3036 \times 80 \\ &\quad + 0.2365 \times 75 + 0.1117 \times 50 \\ &= 78.883 \end{aligned}$$

hence, the collective preference value is 78.883.

In the above example, the preference value $a_3 = 100$ is assigned unduly high, and the preference value $a_4 = 50$ is assigned unduly low. To relieve the influence of these unfair arguments on the decision results, we assign the low weight $w_1 = w_5 = 0.1117$ to both of them. Because the preference values $a_2 = 75$ and $a_5 = 85$ are close to the mid value $a_1 = 80$, then we assign more weight $w_2 = w_5 = 0.2365$ to both of them, and assign the most weight $w_3 = 0.3036$ to the mid one. We assign the value 0.5 to the *orness* measure, which is in accordance with Theorem 2, and obtain $disp(w) = 1.5336$.

For the convenience of applications, in the following, we give the weights of the OWA operator by using Equation 53, which take the values of n from 2 to 20, and give the corresponding values of *orness* and *dispersion* measures:

(1) $n = 2$, $\mu_2 = 1.5$, $\sigma_2 = 0.5$, $orness(w) = 0.5$, $disp(w) = 0.6931$,

$w = (0.5, 0.5)^T$

(2) $n = 3$, $\mu_3 = 2$, $\sigma_3 = \sqrt{2/3}$, $orness(w) = 0.5$, $disp(w) = 0.8473$,

$w = (0.2429, 0.5142, 0.2429)^T$

(3) $n = 4$, $\mu_4 = 2.5$, $\sigma_4 = \sqrt{5/4}$, $orness(w) = 0.5$, $disp(w) = 1.3122$,

$w = (0.1550, 0.3450, 0.3450, 0.1550)^T$

(4) $n = 5$, $\mu_5 = 3$, $\sigma_5 = \sqrt{2}$, $orness(w) = 0.5$, $disp(w) = 1.5336$,

$w = (0.1117, 0.2365, 0.3036, 0.2365, 0.1117)^T$

(5) n=6, $\mu_6 = 3.5$, $\sigma_6 = \sqrt{35/12}$, $orness(w) = 0.5$, $disp(w) = 1.7150$,

$w = (0.0865, 0.1716, 0.2419, 0.2419, 0.1717, 0.0865)^T$

(6) $n = 7$, $\mu_7 = 4$, $\sigma_7 = 2$, $orness(w) = 0.5$, $disp(w) = 1.8688$,

$w = (0.0702, 0.1311, 0.1907, 0.2161, 0.1907, 0.1311, 0.0702)^T$

(7) $n = 8$, $\mu_8 = 4.5$, $\sigma_8 = \sqrt{21/4}$, $orness(w) = 0.5$, $disp(w) = 2.0017$,

$w = (0.0588, 0.1042, 0.1525, 0.1845, 0.1845, 0.1525, 0.1042, 0.0588)^T$

(8) $n = 9$, $\mu_9 = 5$, $\sigma_9 = \sqrt{20/3}$, $orness(w) = 0.5$, $disp(w) = 2.1195$,

$w = (0.0506, 0.0855, 0.1243, 0.1557, 0.1678, 0.1557, 0.1243, 0.0855, 0.0506)^T$

(9) $n = 10$, $\mu_{10} = 5.5$, $\sigma_{10} = \sqrt{33/4}$, $orness(w) = 0.5$, $disp(w) = 2.2247$,

$w = (0.0443, 0.0719, 0.1034, 0.1317, 0.1487, 0.1487, 0.1317, 0.1034, 0.0719, 0.0443)^T$

(10) $n = 11$, $\mu_{11} = 6$, $\sigma_{11} = \sqrt{10}$, $orness(w) = 0.5$, $disp(w) = 2.3207$

$w = (0.0393, 0.0617, 0.0875, 0.1124, 0.1305, 0.1372, 0.1305, 0.1124, 0.0875, 0.0617, 0.0393)^T$

(11) $n = 12$, $\mu_{12} = 6.5$, $\sigma_{12} = \sqrt{143/12}$, $orness(w) = 0.5$, $disp(w) = 2.4066$,

$w = (0.0353, 0.0538, 0.0752, 0.0968, 0.1144, 0.1245, 0.1245, 0.1144, 0.0968, 0.0752, 0.0538, 0.0353)^T$

(12) $n = 13$, $\mu_{13} = 7$, $\sigma_{13} = \sqrt{14}$, $orness(w) = 0.5$, $disp(w) = 2.4868$,

$w = (0.0321, 0.0475, 0.0656, 0.0842, 0.1006, 0.1120, 0.1161, 0.1120, 0.1006, 0.0842, 0.0656, 0.0475, 0.0321)^T$

(13) $n = 14$, $\mu_{14} = 7.5$, $\sigma_{14} = \sqrt{65/4}$, $orness(w) = 0.5$, $disp(w) = 2.5608$,

$w = (0.0294, 0.0425, 0.0578, 0.0739, 0.0889, 0.1006, 0.1069, 0.1069, 0.1006, 0.0889, 0.0739, 0.0578, 0.0425, 0.0294)^T$

(14) $n = 15$, $\mu_{15} = 8$, $\sigma_{15} = \sqrt{56/3}$, $orness(w) = 0.5$, $disp(w) = 2.6297$,

$w = (0.0271, 0.0383, 0.0515, 0.0655, 0.0790, 0.0904, 0.0979, 0.1006, 0.0979, 0.0904, 0.0790, 0.0655, 0.0515, 0.0383, 0.0271)^T$

(15) $n = 16$, $\mu_{16} = 8.5$, $\sigma_{16} = \sqrt{85/4}$, $orness(w) = 0.5$, $disp(w) = 2.6942$,

$w = (0.0251, 0.0349, 0.0463, 0.0585, 0.0707, 0.0814, 0.0894, 0.0937, 0.0937, 0.0894, 0.0814, 0.0707, 0.0585, 0.0463, 0.0349, 0.0251)^T$

(16) $n = 17$, $\mu_{17} = 9$, $\sigma_{17} = \sqrt{24}$, $orness(w) = 0.5$, $disp(w) = 2.7550$,

$$w = (0.0234, 0.0320, 0.0419, 0.0527, 0.0636, 0.0736, 0.0816, 0.0869, 0.0887, 0.0869, 0.0816, 0.0736, 0.0636, 0.0527, 0.0419, 0.0320, 0.0234)^T$$

(17) $n = 18$, $\mu_{18} = 9.5$, $\sigma_{18} = \sqrt{323/12}$, $orness(w) = 0.5$, $disp(w) = 2.8120$,

$$w = (0.0219, 0.0295, 0.0382, 0.0478, 0.0575, 0.0667, 0.0746, 0.0804, 0.0834, 0.0834, 0.0804, 0.0746, 0.0667, 0.0575, 0.0478, 0.0382, 0.0295, 0.0219)^T$$

(18) $n = 19$, $\mu_{19} = 10$, $\sigma_{19} = \sqrt{30}$, $orness(w) = 0.5$, $disp(w) = 2.8659$,

$$w = (0.0206, 0.0273, 0.0351, 0.0435, 0.0523, 0.0608, 0.0683, 0.0743, 0.0781, 0.0794, 0.0781, 0.0743, 0.0683, 0.0608, 0.0523, 0.0435, 0.0351, 0.0273, 0.0206)^T$$

(19) $n = 20$, $\mu_{20} = 10.5$, $\sigma_{20} = \sqrt{133/4}$, $orness(w) = 0.5$, $disp(w) = 2.9173$,

$$w = (0.0194, 0.0254, 0.0324, 0.0400, 0.0479, 0.0556, 0.0627, 0.0686, 0.0729, 0.0751, 0.0751, 0.0729, 0.0686, 0.0627, 0.0556, 0.0479, 0.0400, 0.0324, 0.0254, 0.0194)^T$$

From the above numerical results, we know that all the values of the mean μ_n and the standard deviation σ_n of the collection of $1, 2, \ldots, n$ and the values of *dispersion* measure increase as n steadily increases whereas the values of *orness* measure are always equal to 0.5.

4. CONCLUDING REMARKS

In this article, we have surveyed the existing main procedures for determining the OWA weights. We have also briefly reviewed the normal distribution (Gaussian distribution). Based on the normal distribution, we have developed a novel practical method for obtaining the weight vector of the OWA operator. We have then investigated some of its desirable properties in detail. The prominent characteristic of the developed method is that it can relieve the influence of unfair arguments on the decision results by assigning low weights to those "false" or "biased" ones.

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