Linear Algebra Overview

Intelligent Systems and Control

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Scalars

- A scalar is a single number.
- Integers, real number, rational numbers, etc.
- Often denoted with *italic* font: a, n, x.

Vectors

- A vector is a 1-D array of numbers:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} .$$

- Can be real, binary, integer, etc.
- Notation to indicate type and size: $\mathbf{x} \in \mathbb{R}^n$

Matrices

- A matrix is a 2-D array of numbers:

$$\begin{array}{cl} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} & \longrightarrow \text{Row} \\ \downarrow & \\ \text{Column} & \end{array}$$

- Example notation for type and shape: $\mathbf{A} \in \mathbb{R}^{m \times n}$.

Tensors

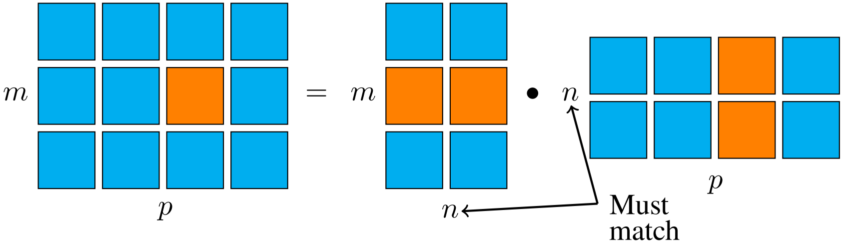
A tensor is an array of numbers, that may have:

- Zero dimensions, and be a scalar ;
- One dimension, and be a vector ;
- Two dimensions, and be a matrix ;
- Or more dimensions .

Matrix (Dot) Product

Consider two matrices $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times p}$. their product, $\mathbf{C} \in \mathbb{R}^{m \times p}$, is denoted by $\mathbf{C} = \mathbf{AB}$ and defined by:

$$C_{ij} = \sum_k A_{ik} B_{kj} \ .$$



Identity Matrix

A rectangular matrix $\mathbf{I}_n$ is defined as an identity matrix, if and only if:

$$\forall \mathbf{X} \in \mathbb{R}^{m \times n},\ \mathbf{X}\,\mathbf{I}_n = \mathbf{X}\ .$$

Example:

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Inversion

$\mathbf{A} \in \mathbb{R}^{n \times n}$ is called *invertible* (non-singular), if there exists $\mathbf{B} \in \mathbb{R}^{n \times n}$, such that:

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n \ .$$

Non-square matrices do not have an inverse. However a non-square matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ has:

- A left inverse if $rank(\mathbf{A}) \leq m$ (i.e., $\exists \mathbf{B} \in \mathbb{R}^{n \times m} | \mathbf{BA} = \mathbf{I}_n$) ;
- A right inverse if $rank(\mathbf{A}) \leq n$ (i.e., $\exists \mathbf{B} \in \mathbb{R}^{n \times m} | \mathbf{AB} = \mathbf{I}_m$) .

Matrix rank: The maximum number of linearly independent column/rows.

Norms

Functions that measure how "large" a vector is.

- $f(\mathbf{x}) = 0 \Longrightarrow \mathbf{x} = \mathbf{0}$
- $f(\mathbf{x}) + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ (the *triangle inequality*)
- $\forall \alpha \in \mathbb{R}, f(\alpha \mathbf{x}) = |\alpha| f(\mathbf{x})$

Norms

L^p norm is defined by:

$$||\mathbf{x}||_p = \left(\sum_i |x_i|^p \right)^{\frac{1}{p}}$$

Some popular norms are:

- $L^2 (L2)$ norm, where $p = 2$
- $L1$ norm, where $p = 1 \implies ||\mathbf{x}||_1 = \sum_i |x_i|$
- Infinity norm:
$$||\mathbf{x}||_\infty = \max_i |x_i|$$

System of Equations

The linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ expands to:

$$\mathbf{A}_{[1,:]}\mathbf{x} = b_1$$

$$\mathbf{A}_{[2,:]}\mathbf{x} = b_2$$

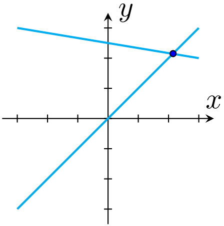
$$\mathbf{A}_{[3,:]}\mathbf{x} = b_3$$

$$\dots$$

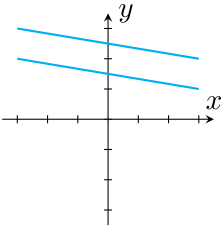
$$\mathbf{A}_{[m,:]}\mathbf{x} = b_m$$

Solution of System of Equations

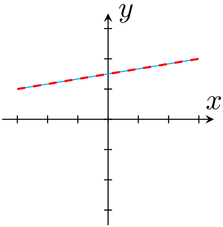
The linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$, can have:



Exactly one solution



No solutions



Infinite solutions

Moore-Penrose Pseudoinverse

Pseudoinverse of $\mathbf{A}$ is denoted by $\mathbf{A}^\dagger$ and defined by:

- If $\mathbf{A}$ has linearly independent columns (making $\mathbf{A}^\top\mathbf{A}$ invertible): $\mathbf{A}^\dagger = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top$.
- If $\mathbf{A}$ has linearly independent rows (making $\mathbf{A}\mathbf{A}^\top$ invertible): $\mathbf{A}^\dagger = \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}$.

Considering solutions $\mathbf{x} = \mathbf{A}^\dagger\mathbf{b}$, if exist, where the system of equations has:

- Exactly one solution: this is the same as the inverse.
- No solutions: Gives the solution with the smallest error in terms of an L2 norm, i.e., $||\mathbf{A}\mathbf{x} - \mathbf{b}||_2$.
- Infinite solutions: Gives the solution with the smallest norm of $\mathbf{x}$.

Special Matrices and Vectors

- Unit vector:

$$||\mathbf{x}||_2 = 1 \ .$$

- Symmetric matrix:

$$\mathbf{A} = \mathbf{A}^\top \ .$$

- Orthogonal matrix:

$$\mathbf{A}^\top \mathbf{A} = \mathbf{A}\mathbf{A}^\top = \mathbf{I}$$
$$\mathbf{A}^{-1} = \mathbf{A}^\top$$

Some References

- Introduction to Linear Algebra – Gilbert Strang, 1993

- Linear Algebra Done Right – Sheldon Axler, 1995