Math 231 — Hw 3

Sara Jamshidi, Jan 17, 2025

1. Let $S = \{(x, y) \in \mathbb{R}^2 \mid x + y = 1\}$ be a space defined over the field $\mathbb{R}$ with addition defined as
$$(a, b) + (c, d) = (a + c, b + d)$$
and scalar multiplication as $x(a, b) = (xa, xb)$ where $x \in \mathbb{R}$ and $(a, b) \in S$. Show why this is **not** a vector space.

   The main problem here is that our operations take us outside the set. We would say that "the set is not closed under these operations." For example, $(1, 0) + (0, 1) = (1, 1)$ but $(1, 1) \notin S$. since $1 + 1 = 2 \neq 1$.

2. Let $U = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$, with vector addition and scalar multiplication defined as the previous case. Show why this is **not** a vector space.

   The main problem here is that we have no additive inverses. The additive identity for this space is $(0, 0)$. But there is no element in U that I can add to $(1, 1)$ that gives me this additive identity back. This is because both entries of my vector have to be 0 or larger.

3. Define a set $W = \mathbb{R}^2$ with addition defined as $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and scalar multiplication defined as $c \cdot (x, y) = (cx, y)$. Show why this is **not** a vector space.

   As you might guess, the issue is with scalar multiplication. The problem arises with the distributive property as discussed in the textbook: $(c_1 + c_2) \cdot (x, y) \neq c_1 \cdot (x, y) + c_2 \cdot (x, y)$. The left-hand side yields $((c_1 + c_2)x, y)$ while the right-hand size produces $(c_1 x, y) + (c_2 x, y) = ((c_1 + c_2)x, 2y)$.

4. Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$, with vector addition and scalar multiplication defined as usual. Show why this is **not** a vector space.

   This was a mistake–this turns out to be a vector space! Let's show that the space is closed under addition: For $(u_1, u_2, u_3), (v_1, v_2, v_3) \in X$, we know:
$$u_1 + u_2 + u_3 = 0 \quad \text{and} \quad v_1 + v_2 + v_3 = 0.$$
   Adding these equations, we see the sum is still 0, ensuring the sum of the vectors is in X:
$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = 0.$$
   So X is closed under addition. Now let's show closure under scalar multiplication. For $(x, y, z) \in X$ and $c \in \mathbb{R}$
$$c \cdot (x, y, z) = (cx, cy, cz).$$

Observe that

$$cx + cy + cz = c(x + y + z) = c(0) = 0.$$

So X is closed under scalar multiplication. It is clear that we have the zero vector in the space, so there is an additive identity. Moreover is $(x, y, z) \in X$ then so must be $(-1)(x, y, z)$ by the closure of scalar multiplication. Hence we are closed under additive inverses as well. Finally, all standard operations (associativity of addition, distributivity of scalar multiplication, etc.) are inherited from $\mathbb{R}^3$. These properties hold because the operations are defined in the usual way. So this is a vector space.