Hw4 — linear Algebra

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Q1) ① closure under addition

for $(a,b),(c,d) \in \mathbb{Z}_5^2$; $(a,b)+(c,d) = ((a+c) \bmod 5, (b+d) \bmod 5)$,

$(a+c) \bmod 5$, $(b+d) \bmod 5 \in \mathbb{Z}_5^2$, so their sum is in $\mathbb{Z}_5^2$

hence, addition is closed ☒

② closure under scalar mult.

for $\lambda \in \mathbb{Z}_5$, $(a,b) \in \mathbb{Z}_5^2$; $\lambda(a,b) = (\lambda a \bmod 5, \lambda b \bmod 5) \in \mathbb{Z}_5^2$

hence, scalar mult is closed ☒

③ Commutivity

$(a+c) \bmod 5 = (c+a) \bmod$ and $(b+d) \bmod 5 = (d+b) \bmod 5$

so, $(a,b)+(c,d) = (c,d)+(a,b)$

hence, commutivity is satisfied ☒

④ Inverses

ⓐ additive ; for $(a,b) \in \mathbb{Z}_5^2$, $(a,b)+(-a,-b)$

$= ((a-a) \bmod 5, (b-b) \bmod 5) = (0,0)$

ⓑ scalar mult ; for $c \in \mathbb{F}$, $c^{-1}(c(a,b)) = c^{-1}(ca, cb) = (a,b)$

hence, additive, and mult inverses exist. ☒

⑤ Identity

ⓐ additive ; $(a,b)+(0,0) = ((a+0) \bmod 5, (b+0) \bmod 5) = (a,b)$

ⓑ scalar mult ; $1 \cdot (a,b) = ((1a) \bmod 5, (1b) \bmod 5) = (a,b)$

hence, they both have ids. ☒

⑥ Associativity

derived from the field.

⑦ Distributivity

for $k \in \mathbb{Z}_5$, $(a,b),(c,d) \in \mathbb{Z}_5^2$.

a) $k \cdot ((a,b),(c,d)) = k \cdot ((a+c) \bmod 5, (b+d) \bmod 5)$

$= (k((a+c) \bmod 5), k((b+d) \bmod 5))$

$= (ka \bmod 5 + kc \bmod 5, kb \bmod 5 + kd \bmod 5)$

$= (ka \bmod 5, kb \bmod 5) + (kc \bmod 5, kd \bmod 5)$

$= k(a,b) + k(c,d)$

b) $(k_1+k_2)(a,b) = (a(k_1+k_2) \% 5, b(k_1+k_2) \% 5)$

$= (ak_1 \% 5 + ak_2 \% 5, bk_1 \% 5 + bk_2 \% 5)$

$= (ak_1 \% 5, bk_1 \% 5) + (ak_2 \% 5, bk_2 \% 5)$

$= k_1(a,b) + k_2(a,b)$

hence distributivity holds ☒

Q2) $P^3 = \{ax^3+bx^2+cx+d \mid a,b,c,d \in \mathbb{R}\} \overset{?}{=} V$:

① closure under addition

let $P_1(x) = a_1x^3+b_1x^2+c_1x+d_1$, $P_2(x) = a_2x^3+b_2x^2+c_2x+d_2$

$P_1(x)+P_2(x) = (a_1+a_2)x^3+(b_1+b_2)x^2+\dots+(d_1+d_2)$

since, $(a_1+a_2), \dots, (d_1+d_2) \in \mathbb{R}$, then $P_1(x)+P_2(x) \in P^3$. ☒

② closure under Scalar mult

let $\lambda \in \mathbb{R}$ be the scalar.

$\lambda(p(x)) = (\lambda a)x^3+(\lambda b)x^2+(\lambda c)x+\lambda d$

Since $\lambda a, \lambda b, \lambda c, \lambda d \in \mathbb{R}$, $\lambda p(x) \in P^3$ ☒

③ associativity: derived from the field.

④ additive ID:

let $P_0(x) = 0x^3+0x^2+0x+0=0$

$p(x)+P_0(x) = (a+0)x^3+(b+0)x^2+\dots+(d+0) = p(x)$ ☒

⑤ additive inv:

the inverse of $p(x)$ is $-p(x) = -ax^3-bx^2-cx-d$

$p(x)-p(x) = (a-a)x^3+\dots+(d-d) = 0$ ☒

⑥ Scalar mult id.

$1 \cdot p(x) = p(x)$ [Q: how do I show $1 \in \mathbb{F}$]

⑦ next page!

⑦ Distributive

for $k \in \mathbb{R}$.

a) $k \cdot (P_1(x) + P_2(x)) = k\left((a_1x^3 + b_1x^2 + c_1x + d_1) + (a_2x^3 + b_2x^2 + c_2x + d_2)\right)$

$= k\left((a_1 + a_2)x^3 + \dots + (d_1 + d_2)\right)$

$= (ka_1)x^3 + (ka_2)x^3 + (kb_1)x^2 + \dots + (kd_2)$

$= k(a_1x^3 + b_1x^2 + c_1x + d_1) + k(a_2x^3 + b_2x^2 + c_2x + d_2)$

$= k(P_1(x)) + k(P_2(x))$

b) $(k_1 + k_2)(P(x)) = (k_1 + k_2)(ax^3 + bx^2 + cx + d)$

$= (k_1 + k_2)(ax^3) + (k_1 + k_2)(bx^2) + \dots + (k_1 + k_2)(d)$

$= (ak_1 + ak_2)x^3 + (bk_1 + bk_2)x^2 + (ck_1 + ck_2)x + (dk_1 + dk_2)$

$= k_1(P(x)) + k_2(P(x))$

☒