HW13 – linear Algebra

Sepehr Akbari

Q1) I'm assuming you meant the initial $p_0, \dots, p_3$ can be degree $\geq$ if so;

let p_0 have degree 0, p_1 degree 1, etc. and let $x \in P$:

for example consider $B_0 = \{1, x, x^2, x^3\}$, then B_0 is a basis as all elements are lin.indep and is spanning.

to remove x^2 we can replace it by $x^2 + x^3$ which will make it into a 3rd degree poly: $B_1 = \{1, x, x^2+x^3, x^3\}$. lets prove its basis.:

~~For~~ let $a_i \in \mathbb{F}$, for $a_1(1) + a_2(x) + a_3(x^2+x^3) + a_4(x^3) = 0$

$\Rightarrow a_1 + x a_2 + x^2 a_3 + x^3(a_3 + a_4) = 0.$

$$\begin{cases} a_1 = 0 \\ x a_2 = 0 \\ a_3(x^2+x^3) = 0 \\ x^3 a_4 = 0 \end{cases} \Rightarrow a_1 = a_2 = a_3 = a_4 = 0 \text{ so its lin.indep.}$$

let $(x, y, z, q) \in V$. $\therefore \begin{cases} a_1 = x \\ a_2 = y \\ a_3 = z \\ a_4 + a_3 = q \end{cases} \Rightarrow a_4 = q - z \Rightarrow$ so $p(x) = x + y\tilde{x} + z(\tilde{x}^2 + \tilde{x}^3) + q(\tilde{x}^3)$

so its spanning the space.

hence, B_1 is a valid basis. $\square$

Q2) ① linear independance:

let $a, b, c, d \in \mathbb{F}$.

for $a(v_1+v_2)+b(v_2+v_3)+c(v_3+v_4)+d(v_4) = (0,0,0,0)$

$= av_1 + av_2 + bv_2 + bv_3 + cv_3 + cv_4 + dv_4 = 0$

$= v_1 a + v_2(a+b) + v_3(b+c) + v_4(c+d) = 0$

Since $\{v_1, v_2, v_3, v_4\}$ is lin. indep, $a = a+b = b+c = c+d = 0$

Since $\boxed{a=0}$, $a+b=0$, then $\boxed{b=0}$, $\boxed{c=0}$, $\boxed{d=0}$.

so $\{v_1+v_2, v_2+v_3, v_3+v_4, v_4\}$ is linearly independent.

② span:

let $\{v_1+v_2, v_2+v_3, v_3+v_4, v_4\} = \{w_1, w_2, w_3, w_4\}$.

we have to show ech v_i in $\{v_1, \dots, v_4\}$ is a linear combination of $w_1, \dots, w_4$.

$$\begin{cases} w_1 = v_1 + v_2 \\ w_2 = v_2 + v_3 \\ w_3 = v_3 + v_4 \\ w_4 = v_4 \end{cases} \Rightarrow$$

so $\boxed{v_4 = w_4}$, so $v_3 = w_3 - v_4 = \boxed{w_3 - w_4 = v_3}$

so $w_2 = w_2 - v_3 = \boxed{w_2 - w_3 + w_4 = v_2}$

and $\boxed{v_1 = w_1 - w_2 + w_3 - w_4}$

since all v_is are lin combinations of w_is.

$\{v_1+v_2, v_2+v_3, v_3+v_4, v_4\}$ spans the space.

hence, it is also a basis of V □

Q3) let $B_0 = \{v_1, v_2, v_3, v_4\}$ be a basis of V.

let $B_1 = \{v_1, v_1+v_2, v_1+v_2+v_3, v_1+v_2+v_3+v_4\} = \{w_1, w_2, w_3, w_4\}$

lets see if B_1 is a basis of V...

① lin independance.

let $a,b,c,d \in \mathbb{F}$.

for $aw_1 + bw_2 + cw_3 + dw_4 = (0,0,0,0)$

$= v_1(a) + v_1 b + v_2 b + v_1 c + v_2 c + v_3 c + v_1 d + v_2 d + v_3 d + v_4 d.$

$= v_1(a+b+c+d) + v_2(b+c+d) + v_3(c+d) + v_4(d)$

Since B_0 is a basis, $a+b+c+d = b+c+d = c+d = d = 0$

so $a=b=c=d=0$. so B_1 is lin. indep.

② spanning V.

let $(n,m,k,l) \in V$.

so for B_1 to span V, this has to hold;

$v_1(a+b+c+d) + v_2(b+c+d) + v_3(c+d) + v_4(d) = (n,m,k,l)$

$$\text{So}\begin{cases} \boxed{d=l} \\ c+d=k \\ b+c+d=m \\ a+b+c+d=n \end{cases} \qquad \begin{aligned} &\Rightarrow \boxed{c = k-d = k-l} \\ &b = m-\cancel{d}-c = m-l-k+l = \boxed{m-k=b} \\ &a = n-b-c-d = n-m+k-k+l-l. \\ &\Rightarrow \boxed{a=n-m} \end{aligned}$$

Since B_1 is representable by n,m,k,l its a spanning set of V.

hence, B_1 is a basis $\square$