Math 231 — Hw 20

Sara Jamshidi, Apr 11, 2025

1. Suppose S is the half circle of radius 1 under a basis $\{v_1, v_2\}$ of $\mathbb{R}^2$. Let A is a matrix that changes this basis to the standard normal basis. If the determinant of A is 4, what is the area of AS?

   Solution: The area of the half unit circle is $\pi/2$. Under this map, we need to multiply this area by 4, which means the area of AS is 2π.

2. What is the determinant of the following matrix:

$$\begin{pmatrix} 1 & 1 & -2 \\ 2 & 1 & -1 \\ 1 & 2 & -1 \end{pmatrix}.$$

   Based on the determinant, compute the null space of this map.

   Solution: To find the determinant, we use the cofactor expansion along the first row:

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + (-2) \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 1 \cdot (-1 + 2) - 1 \cdot (-2 + 1) + (-2) \cdot (4 - 1)$$

$$= 1 \cdot 1 - 1 \cdot (-1) + (-2) \cdot 3$$

$$= 1 + 1 - 6 = -4$$

   Since the determinant is non-zero, the null space of the matrix is $\{0\}$.

3. Suppose you wish to find the area of the ellipse

$$\left(\frac{x + 2y}{3}\right)^2 + \left(\frac{y - x}{2}\right)^2 = 1.$$

   After thinking about this for a moment, you realize that you can characterize this as a linear change of coordinates (a change of basis!!) (see the figures below). You first define a linear transformation T that would characterize transforming the ellipse into the unit circle: from $\left(\frac{x+2y}{3}, \frac{y-x}{2}\right)$ to (x, y). This is equivalent to "translating" the ellipse's basis, which we can denote as $\{v_1, v_2\}$, to the standard normal basis $\{e_1, e_2\}$:

$$Tv_1 = \frac{1}{3}e_1 - \frac{1}{2}e_2$$

   and

$$Tv_2 = \frac{2}{3}e_1 + \frac{1}{2}e_2$$

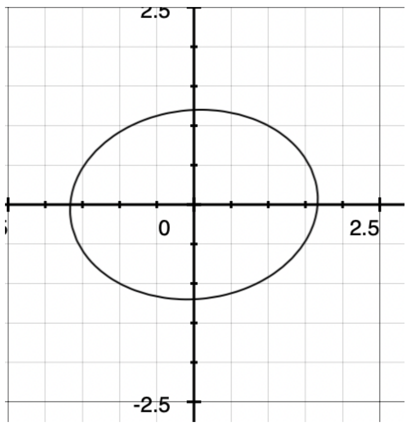

Figure 1: Original Ellipse

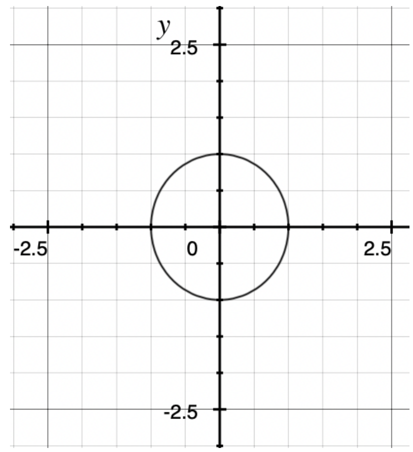

Figure 2: Unit Circle

Here, the standard normal basis helps us represent the cartesian coordinate system with points being (x, y).

Find a matrix representation of T and compute its determinant. Use its determinant to find the area of the ellipse. (Hint: What is the area of the image? How should you use the determinant to figure out the ellipse, which is your domain?)

Solution:

$$M(T) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then

$$\det(M(T)) = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) - \left(\frac{2}{3}\right)\left(-\frac{1}{2}\right) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$

The area of the unit circle is π. That means when we apply T to the ellipse, the resulting area is π. Because the determinant is $1/2$, the original had to be 2π. Thus the area of the ellipse is 2π.

4. The inverse of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has the following formula:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Based on the matrix you found in the previous problem, find the matrix representation of T^{-1}.

Solution: From the previous problem, we have:

$$M(T) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The determinant of $M(T)$ is $\frac{1}{2}$. Therefore, the inverse of $M(T)$ is:

$$M(T)^{-1} = \frac{1}{\frac{1}{2}} \begin{pmatrix} \frac{1}{2} & -\frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} = 2 \begin{pmatrix} \frac{1}{2} & -\frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{4}{3} \\ 1 & \frac{2}{3} \end{pmatrix}$$

5. Verify the formula of the inverse matrix by multiplying the matrices from the two previous problems and show that you get the identity matrix:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Solution: We need to verify that $M(T) \cdot M(T)^{-1} = I$.

$$M(T) \cdot M(T)^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & -\frac{4}{3} \\ 1 & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 1 & \frac{1}{3} \cdot -\frac{4}{3} + \frac{2}{3} \cdot \frac{2}{3} \\ -\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 & -\frac{1}{2} \cdot -\frac{4}{3} + \frac{1}{2} \cdot \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} + \frac{2}{3} & -\frac{4}{9} + \frac{4}{9} \\ -\frac{1}{2} + \frac{1}{2} & \frac{2}{3} + \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, the formula for the inverse matrix is verified.

~~###~~ HW # 21

1.) $\det\begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = (4-\lambda)(3-\lambda) - 2$

$$\Rightarrow 12 - 7\lambda + \lambda^2 - 2 = \lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda-5)(\lambda-2) = 0$$

$$\lambda = 5, \ \lambda = 2$$

$\underline{\lambda=5}$: $\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad x_1 = x_2$

<u>eigenvector</u>: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{\lambda=2}$: $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad 2x_1 - x_2 = 0$

<u>eigenvector</u>: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2.) a.) $\det\begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 3, 2$

$\underset{\lambda=3}{\mathrm{null}}\left(\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right) \Rightarrow \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$; $\underset{\lambda=2}{\mathrm{null}}\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

b.) $D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

c.) $P = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

d.) P^{-1}: $\det P = -1 \Rightarrow P^{-1} = -\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

e.) $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ ✓

3.) $B^2 = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 5 \\ 0 & 4 \end{pmatrix}$ ✓

$$D^2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$$

$$PD^2P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 4 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 5 \\ 0 & 4 \end{pmatrix}$$ ✓

4.) We prove this by induction. Since $B = PDP^{-1}$, then this holds for $n=1$.
Suppose this holds for some $n-1$. That is,

$$B^{n-1} = PD^{n-1}P^{-1}.$$ ✓

Then $$B^n = PD^{n-1}P^{-1} \cdot B = PD^{n-1}P^{-1} \cdot PDP^{-1}$$

$$= PD^{n-1}\, I\, DP^{-1}$$

Since I is the identity map, we have that

$$B^n = PD^{n-1}DP^{-1} = PD^nP^{-1}. \quad \square$$

5. $P = \begin{pmatrix} .99 & .05 \\ .01 & .95 \end{pmatrix}$ then $\det\begin{pmatrix} .99-\lambda & .05 \\ .01 & .95-\lambda \end{pmatrix} = (.99-\lambda)(.95-\lambda) - (.05)(.1) = 0$

$$= (.99)(.95) - 1.94\lambda + \lambda^2 - .0005$$
$$= 0.9405 - 1.94\lambda + \lambda^2 - 0.0005 = \boxed{0.94 - 1.94\lambda + \lambda^2}$$

5.) continued

$$0.94 - 1.94\lambda + \lambda^2 = 0$$

$$\lambda = \frac{1.94 \pm \sqrt{0.0036}}{2} = 1,\ 0.94$$

$\lambda = 1$ $\quad \begin{pmatrix} -.01 & .05 \\ .01 & -.05 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow +.01x_1 = +.05x_2$

$x_1 = 5x_2$

eigenvector: $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$

$\lambda = 0.94$ $\quad \begin{pmatrix} .05 & .05 \\ .01 & .01 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = -x_2$

eigenvector: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 0.94 \end{pmatrix}$ $\quad Q = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad Q^{-1} = \frac{-1}{6}\begin{pmatrix} -1 & -1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1/6 & 1/6 \\ 1/6 & -5/6 \end{pmatrix}$

$P^{30}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is what we want to compute. To make this easier, we will use the decomposition.

$$QD^{30}Q^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1^{30} & 0 \\ 0 & 0.94^{30} \end{pmatrix}\begin{pmatrix} 1/6 & 1/6 \\ 1/6 & -5/6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.16 \end{pmatrix}\begin{pmatrix} 1/6 & 1/6 \\ 1/6 & -5/6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 0.16 \\ 1 & -0.16 \end{pmatrix}\begin{pmatrix} 1/6 & 1/6 \\ 1/6 & -5/6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} .86 & .69 \\ .14 & 0.3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} .69 \\ .3 \end{pmatrix}$$

So if a house is listed for sale, the likelihood that it is still listed for sale is 30%. Otherwise, there's a 69% (really 70%) it will be off market

rounding error

Math 231 — Hw 22

Sara Jamshidi, Apr 14, 2025

1. Suppose you are modeling a a chemical system where molecules of 2-butene can exist in either the cis or trans form. The transition matrix T describes the probabilities of converting from one form to the other every second.

$$T = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$$

The first row and column correspond to the cis form, and the second row and column correspond to the trans form.

Compute the eigenvalues of this matrix. Based on your eigenvalues, which state ends up being favored over time?

We need to solve

$$\det(T - \lambda I) = \det \begin{pmatrix} 0.7 - \lambda & 0.4 \\ 0.3 & 0.6 - \lambda \end{pmatrix} = (0.7 - \lambda)(0.6 - \lambda) - 0.12 = 0.3 - 1.3\lambda + \lambda^2 = 0$$

Solving this means $\lambda_1 = 1$ and $\lambda_2 = 0.3$. As time goes to infinity, we should see the system favoring the cis form of the molecule.

2. If 50% of the system is in cis form and 50% is in trans form, what is the predicted proportion after 60 seconds?

Plugging in λ_1 gives

$$\begin{pmatrix} -0.3 & 0.4 \\ 0.3 & -0.4 \end{pmatrix},$$

and plugging in λ_2 gives

$$\begin{pmatrix} 0.4 & 0.4 \\ 0.3 & 0.3 \end{pmatrix}.$$

The corresponding eigenvectors are

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We need to compute the following

$$\begin{aligned}
T^{60} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} &= \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1^{60} & 0 \\ 0 & 0.3^{60} \end{pmatrix} \begin{pmatrix} 1/7 & 1/7 \\ 3/7 & -4/7 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \\
&\approx \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/7 & 1/7 \\ 3/7 & -4/7 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \\
&= \begin{pmatrix} 1/7 & 1/7 \\ 3/7 & 3/7 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \\
&= \begin{pmatrix} 4/7 \\ 3/7 \end{pmatrix}
\end{aligned}$$

3. Consider a 2×2 diagonal matrix. Prove why the diagonal values are the eigenvectors and the standard normal basis is an eigenbasis.

   Let
   $$D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$
   The eigenvalues are computed from
   $$\det(D - \lambda I) = (a - \lambda)(b - \lambda) = 0.$$
   The only solutions are $\lambda_1 = a$ and $\lambda_2 = b$. Hence the diagonal elements are the eigenvalues. The eigenvector for λ_1 is the null space of
   $$\begin{pmatrix} 0 & 0 \\ 0 & b - a \end{pmatrix}.$$
   Notice that
   $$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
   is in that space. Similarly, the eigenvector for λ_2 is the null space of
   $$\begin{pmatrix} a - b & 0 \\ 0 & 0 \end{pmatrix}.$$
   Notice that
   $$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
   is in that space. Hence the standard normal basis is an eigenbasis.

Math 231 — Hw 23
Sara Jamshidi, Apr 18, 2025

Suppose you are analyzing student performances in a school. Below is your sample of students and their grades in Math and Science:

Student	**Math**	**Science**
A	85	90
B	78	82
C	92	88
D	88	94
E	70	75
Means	82.6	85.8

You'd like to compress your analysis to one input instead of two and decide to use PCA to achieve this.

1. Compute the variance of each feature.

$$\text{Var(Math)} \approx 60.72, \text{Var(Science)} \approx 44.16$$

2. Compute the covariance between math and science.

$$\text{Covariance(Math, Science)} = 45.72$$

3. Construct the variance-covariance matrix.

$$\begin{pmatrix} 60.72 & 45.72 \\ 45.72 & 44.16 \end{pmatrix}$$

4. Find the eigenvalues of the matrix. Do you think you should reduce the matrix to 2 dimensions?

$$\lambda_1 = 98.935, \lambda_2 = 5.945$$

5. Suppose you choose to reduce the dimension to 1. If you did, find the vector that represents your feature compression.

$$v_1 \approx \begin{pmatrix} 0.8 \\ .67 \end{pmatrix}$$

In other words, the math score needs to be weighted more than the science score.

6. Write down the new scores for the students in this one dimensional space.

Student	**Math**	**Science**	**New Score**
A	85	90	128.3
B	78	82	117.34
C	92	88	132.56
D	88	94	133.38
E	70	75	106.25
Means	82.6	85.8	

Note that this is just a method of ordering and separating points. There is no reason to believe that the score we compute here actually means something. It is a mathematical artifact. But there are some who would *interpret* this new score to be representative of something. All it represents is a space where the points maximally separate and can be ranked. That's it!

Math 231 — Hw 24
Sara Jamshidi, Apr 18, 2025

1. Explain how PCA transforms the data into a new coordinate system.

   PCA (Principal Component Analysis) transforms the data into a new coordinate system by diagonalizing the variance-covariance matrix.

   (a) **Standardize the Data (Optional)**: Subtract the mean and scale to unit variance. Note that we skip this step in our class.

   (b) **Compute the Covariance Matrix**: This matrix captures how much each dimension varies from the mean with respect to each other.

   (c) **Compute the Eigenvectors and Eigenvalues**: The eigenvectors of the covariance matrix represent the directions of the new coordinate system, and the eigenvalues represent the magnitude of variance along these directions.

   (d) **Sort Eigenvectors by Eigenvalues**: The eigenvectors are sorted in descending order of their corresponding eigenvalues.

   (e) **Project the Data**: The original data is projected onto the new coordinate system defined by the top eigenvectors, resulting in a transformed dataset.

2. How can PCA be used for noise reduction in data?

   PCA can be used for noise reduction by retaining only the principal components that capture the most significant variance in the data. This is done by:

   (a) **Performing PCA**: Transform the data into the new coordinate system.

   (b) **Selecting Principal Components**: Choose the top k principal components that capture the majority of the variance.

   (c) **Reconstructing the Data**: Project the data back into the original space using only the selected principal components. This process filters out the noise, which is typically captured by the components with lower variance.

3. Suppose you are working with a two-state system.

   - The element $A_{11} = 0.8$ represents the probability of staying in State 1 after one time step.
   - The element $A_{21} = 0.2$ represents the probability of transitioning from State 1 to State 2 after one time step.
   - The element $A_{12} = 0.3$ represents the probability of transitioning from State 2 to State 1 after one time step.
   - The element $A_{22} = 0.7$ represents the probability of staying in State 2 after one time step.

Write down the transition matrix for A. Suppose you begin with an item in state 2. After n timesteps, what is the probability that this item is now in state 1.

The transition matrix A is:

$$A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

To find the eigenvalues, we need to solve the characteristic equation: $(0.8-\lambda)(0.7-\lambda) - 0.06 = 0$. The solution is $\lambda_1 = 1$ and $\lambda_2 = 0.5$. The corresponding eigenvectors for each value is

$$v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We can define

$$Q = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

As a result,

$$Q^{-1} = \begin{pmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{pmatrix}.$$

Therefore, $A = QDQ^{-1}$, so $A^n = QD^nQ^{-1}$, by an exercise in a previous homework. To solve the problem we must solve the computation

$$\begin{aligned}
\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & (1/2)^n \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & (1/2)^n \\ 2 & -(1/2)^n \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (1/5)(3+(1/2)^{n-1}) & (3/5)(1-(1/2)^n) \\ (2/5)(1-(1/2)^n) & (1/5)(2+3(1/2)^n) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (3/5)(1-(1/2)^n) \\ (1/5)(2+3(1/2)^n) \end{pmatrix}
\end{aligned}$$

Based on our calculation, we get $(3/5)(1-(1/2)^n)$ as the probability we will be in state 1 after n timesteps. As n gets larger, this probability increases. The first few terms are $0.3, 0.45, 0.525, 0.5625, 0.58125$.

4. Suppose you are analyzing customer retention for a subscription service with two states: *Subscribed* (State 1) and *Unsubscribed* (State 2).

   - The element $A_{11} = 0.9$ represents the probability of a customer remaining subscribed after one month.
   - The element $A_{21} = 0.1$ represents the probability of a customer transitioning from subscribed to unsubscribed after one month.
   - The element $A_{12} = 0.2$ represents the probability of a customer transitioning from unsubscribed to subscribed after one month.
   - The element $A_{22} = 0.8$ represents the probability of a customer remaining unsubscribed after one month.

Suppose you begin with 450 subscribers and 50 who were previously subscribed but now have unsubscribed. After 12 months, what do you project to be the new proportion?

The transition matrix A is:

$$A = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$$

The initial state vector is:

$$X_0 = \begin{pmatrix} 450 \\ 50 \end{pmatrix}$$

To find the new proportion after 12 months, we must compute $A^{12}X_0$. We first need to diagonalize, using the eigenvalues and eigenvectors. The eigenvalues solve $(0.9 - \lambda)(0.8 - \lambda) - 0.02 = 0$. Hence, $\lambda_1 = 1$ and $\lambda_2 = 0.7$. The corresponding eigenvectors for each value is

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We can define

$$Q = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}.$$

As a result,

$$Q^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}.$$

Therefore, $A = QDQ^{-1}$, so $A^{12} = QD^{12}Q^{-1}$, by an exercise in a previous homework. To solve the problem we must solve the computation

$$\begin{aligned}
&\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1^{12} & 0 \\ 0 & 0.7^{12} \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 450 \\ 50 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.014 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 450 \\ 50 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 2 & 0.014 \\ 1 & -0.014 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 450 \\ 50 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 0.67 & 0.66 \\ 0.33 & 0.34 \end{pmatrix} \begin{pmatrix} 450 \\ 50 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 334.5 \\ 165.5 \end{pmatrix}
\end{aligned}$$

Based on our calculation, we estimate 334 subscribed, 165 unsubscribed, and 1 in one of the two categories (it is not clear based on our rounding).

5. Suppose you are analyzing weather patterns in a region with two states: *Sunny* (State 1) and *Rainy* (State 2).

   - The element $A_{11} = 0.9$ represents the probability that a sunny day is followed by another sunny day.

- The element $A_{21} = 0.1$ represents the probability that a sunny day is followed by a rainy day.
- The element $A_{12} = 0.4$ represents the probability that a rainy day is followed by a sunny day.
- The element $A_{22} = 0.6$ represents the probability that a rainy day is followed by another rainy day.

Today is a rainy day. What is the probability that it will be sunny in 2 weeks?

The transition matrix A is:

$$A = \begin{pmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{pmatrix}$$

The eigenvalues solve $(0.9 - \lambda)(0.6 - \lambda) - 0.04 = 0$. To find the probability that it will be sunny in 2 weeks (14 days), we must compute A^{14}. This works like the other exercises. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0.5$. The corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We can define

$$Q = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

As a result,

$$Q^{-1} = \begin{pmatrix} 1/5 & 1/5 \\ 1/5 & -4/5 \end{pmatrix}.$$

Therefore, $A = QDQ^{-1}$, so $A^{14} = QD^{14}Q^{-1}$, by an exercise in a previous homework.

$$\begin{aligned}
\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1^{14} & 0 \\ 0 & 0.5^{14} \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \\ 1/5 & -4/5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \\
\approx \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \\ 1/5 & -4/5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \\
= \begin{pmatrix} 4 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \\ 1/5 & -4/5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \\
= \begin{pmatrix} 4/5 & 4/5 \\ 1/5 & 1/5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \\
= \begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix} &
\end{aligned}$$

Based on our model, there is a 80% chance it will be sunny.

6. Find the determinant of the following matrix:

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & 4 & 1 \\ 1 & 2 & 5 \end{pmatrix}$$

The determinant of the matrix can be calculated using the cofactor expansion along the first row:

$$\det(A) = 2\begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 5 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix}$$

Calculating the 2x2 determinants:

$$\begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} = (4)(5) - (1)(2) = 20 - 2 = 18$$

$$\begin{vmatrix} 0 & 1 \\ 1 & 5 \end{vmatrix} = (0)(5) - (1)(1) = 0 - 1 = -1$$

$$\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = (0)(2) - (4)(1) = 0 - 4 = -4$$

Substituting back:

$$\det(A) = 2(18) - 1(-1) + 3(-4) = 36 + 1 - 12 = 25$$

Thus, the determinant is 25.

7. Suppose you have a vector $(1, 2)$. Write its representation in the bases $\{(1, 1), (1, -1)\}$. Write down a map that converts this basis to the standard normal basis.

We know the representation of this vector under the standard normal basis is

$$V = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We can convert from the given basis to the standard normal basis using the linear transformation

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

So the inverse map

$$P^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

takes us from the standard normal basis and the given basis. This is useful to us! To compute the representation of this vector under the given basis, we simply multiply

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}.$$

You can verify this answer by computing

$$(3/2)(1, 1) - (1/2)(1, -1) = (1, 2).$$