**Math 240 — Project 2 Prompt**
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**Objective:**
In this project, you will write a Python program to implement the Euler method and Runge-Kutta for solving an ordinary differential equation (ODE). You will apply the method to a real-world scenario and estimate the error of the Euler method. Then you will compare your estimate to an exact solution.

1. **ODE Definition:**

   - Solve the following initial value problem (IVP) which models population growth with a carrying capacity (logistic growth):

     $$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right), \quad y(0) = y_0$$

     where:

     - $r$ is the growth rate,
     - $K$ is the carrying capacity of the environment (the maximum population allowable),
     - $y_0$ is the initial population size at $t = 0$.

     The logistic growth model is used in biology, ecology, and other fields to describe how populations grow in an environment with limited resources.

   - Construct example parameters from a real-world case relevant to a field you care about. For example,

     $$r = 0.2, \quad K = 500, \quad y_0 = 50$$

     translates to a growth rate of 20%, with a capacity of $K = 500$, and an initial population of 50.

2. **Euler Method Implementation:**

   - Implement the Euler method to approximate the solution to the logistic growth model over a specified time interval $[t_0, t_f]$.
   - The Euler update rule is given by:

     $$y_{i+1} = y_i + hf(t_i, y_i)$$

     where $h$ is the step size, and $y_i$ is the approximate solution at time $t_i$.

   - Implement the method for 5 different step sizes.

3. **Runge-Kutta Method Implementation:**

- Implement the 4th-order Runge-Kutta method (RK4) to approximate the solution to the logistic growth model over a specified time interval $[t_0, t_f]$.
- The RK4 update rule is given by:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

  where:

$$k_1 = f(t_n, y_n), \quad k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$

$$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2), \quad k_4 = f(t_n + h, y_n + hk_3)$$

  Here, $h$ is the step size, and $f(t, y) = ry(1 - y/K)$.

4. **Exact Solution:**For the logistic growth equation, the exact solution is known and given by:

$$y(t) = \frac{Ky_0e^{rt}}{K + y_0\left(e^{rt} - 1\right)}$$

   Use this exact solution to compare against your Euler and RK4 approximation.

5. **Report:** Write a brief report explaining:
   - The ODE being solved and its real-world application,
   - How the Runge-Kutta and Euler methods were implemented,
   - A discussion of the error behavior as the step size changes for both methods,
   - Observations on the efficiency and accuracy of the RK4 method compared to the Euler method.