Hw 21 - linear Algebra

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1) $\det\left(\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = 0$

eigenvalues:

$\Rightarrow (4-\lambda)(3-\lambda) - 2 = 10 - 7\lambda + \lambda^2 = (\lambda-5)(\lambda-2) = 0 \Rightarrow \boxed{\lambda_1 = 5,\ \lambda_2 = 2}$

to calculate eigenvector of $\lambda_1 = 5$:

$\begin{bmatrix} 4-5 & 1 \\ 2 & 3-5 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{Bmatrix} -a+b=0 \\ 2a+2b=0 \end{Bmatrix} \quad a=b \Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

now for $\lambda_2 = 2$.

$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{Bmatrix} 2a+b=0 \\ 2a+b=0 \end{Bmatrix} \quad b = -2a \Rightarrow \boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$

2) (a) $\det\begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = 6 - 5\lambda + \lambda^2 = 0 \Rightarrow \lambda_1 = 3,\ \lambda_2 = 2.$

and $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \xRightarrow{y=0} \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow x = -y \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$?

(c) $P = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

(d) $P^{-1} = \frac{1}{\det(P)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

(e) $PD = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & -2 \end{bmatrix}. \quad PD \cdot P^{-1} = \begin{bmatrix} 3 & 2 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

wait wait wait that's B! so $\underline{\underline{PDP^{-1} = B}}$.

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3) $D^2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$

$PD^2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 9 & 4 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 0 & 4 \end{bmatrix} = PD^2P^{-1}$

now observe

$B^2 = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 0 & 4 \end{bmatrix} \therefore PD^2P^{-1} = B^2$

4) since $B^2 = PD^2P^{-1}$. so we have to show $B^{k+1} = PD^{k+1}P^{-1}$

$B^{k+1} = B \cdot B^k = B \cdot (PD^kP^{-1}) = (PD^1\underline{P^{-1})(P}D^kP^{-1}) = PDID^kP^{-1}$

this has to $= PDD^kP^{-1} = PD^{k+1}P^{-1}$. but how does I get ignored?!

but in the end since $B^{k+1} = PD^{k+1}P^{-1}$, then $B^n = PD^nP^{-1}$. $\square$

5) overview:

$v = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ P_1: listed, P_2: not listed., $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (listed)

it cannot be $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. since P is the probabilities,

$v_1 = Pv_0 \Rightarrow v_2 = Pv_1 = P \cdot Pv_0 = P^2 v_0$

So in general $v_k = P^k v_0$ so $v_{30} = P^{30} v_0$.

→ this will give us $v_{30} = \begin{pmatrix} 1 > P_1 > 0 \\ 1 > P_2 > 0 \end{pmatrix}$ which will give us intuition on the probabilities.

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