Math 231 — Hw 7
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1. Determine whether the following set is a vector space. Justify your answer using theorem 1.34 from the textbook:

$$W = \{(x, y, z) \mid x - (y + 1) + 2(z + 1) = 1,\ x, y, z \in \mathbb{R}\}.$$

Solution: First, we can rewrite the given equation as

$$x - y + 2z = 0.$$

Studying this question, we need to assess if the additive identity satisfies this rule and if the rule allows for closure of both operations. Thinking it through, it looks like it does. So this is likely a vector space. let's prove it.

Proof. It is clear that $W \subset \mathbb{R}^3$. First observe that $(0, 0, 0) \in W$ since $0-(0+1)+2(0+1) = -1 + 2 = 1$. Next, consider $v, w \in W$. If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$, then

$$v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3).$$

To see if the sum is in W, we need to determining if the resulting sum satisfies the rule defining the space.

$$\begin{aligned}
&v_1 + w_1 - (v_2 + w_2 + 1) + 2(v_3 + w_3 + 1)\\
&= v_1 + w_1 - (v_2 + w_2 + 2 - 1) + 2(v_3 + w_3 + 2 - 1)\\
&= v_1 + w_1 - (v_2 + 1) - (w_2 + 1) + 1 + 2(v_3 + 1) + 2(w_3 + 1) - 2\\
&= v_1 - (v_2 + 1) + 2(v_3 + 1) + w_1 - (w_2 + 1) + 2(w_3 + 1) - 1\\
&= 1 + 1 - 1 = 1
\end{aligned}$$

Therefore W is closed under vector addition. Let $\lambda \in \mathbb{R}$ and $v \in W$. If $v = (v_1, v_2, v_3)$, then $\lambda v = (\lambda v_1, \lambda v_2, \lambda v_3)$. To see if this product is in W, we need to check if the resulting vector satisfies the rule defining that space.

$$\begin{aligned}
&\lambda v_1 - (\lambda v_2 + 1) + 2(\lambda v_3 + 1)\\
&= \lambda v_1 - [\lambda v_2 + \lambda - (\lambda - 1)] + 2\,[\lambda v_3 + \lambda - (\lambda - 1)]\\
&= \lambda v_1 - (\lambda v_2 + \lambda) + (\lambda - 1) + 2(\lambda v_3 + \lambda) - 2(\lambda - 1)\\
&= \lambda v_1 - \lambda(v_2 + 1) + 2\lambda(v_3 + 1) - (\lambda - 1)\\
&= \lambda\,(v_1 - (v_2 + 1) + 2(v_3 + 1)) - (\lambda - 1)\\
&= \lambda - (\lambda - 1) = 1
\end{aligned}$$

By theorem 1.34 in the textbook, W is a subspace of $\mathbb{R}^3$, making it a vector space. $\square$

2. Construct an example of a vector space W with two subspaces, W_1, W_2, where you know $W_1 + W_2 = W$. Attempt to prove this.

   Out of laziness, let's define $W = \mathbb{R}^4$ and $W_1 = \{(x, y, 0, 0) \mid x, y \in \mathbb{R}\}$ and $W_2 = \{(0, 0, z, w) \mid z, w \in \mathbb{R}\}$. The next problem states that these are subspaces and the answer proves the sum is a subspace. Problem 4 proves that $W_1 + W_1 = W$.

3. Let $V = \mathbb{R}^4$, and define two subspaces:

   - $V_1 = \{(x, y, 0, 0) \mid x, y \in \mathbb{R}\}$
   - $V_2 = \{(0, 0, z, w) \mid z, w \in \mathbb{R}\}$

   Prove that $V_1 + V_2$ forms a subspace of V.

   Proof. Since $(0, 0, 0, 0) \in V_1, V_2$, it follows that $(0, 0, 0, 0) + (0, 0, 0, 0) = (0, 0, 0, 0) \in V_1 + V_2$.
   Let $u, w \in V_1 + V_2$. Then there must exist $u_1 \in V_1$, $u_2 \in V_2$, $w_1 \in W_1$ and $w_2 \in W_2$ such that $u = u_1 + u_2$ and $w = w_1 + w_2$. Consider $u + w = u_1 + u_2 + w_1 + w_2 = (u_1 + w_1) + (u_2 + w_2)$. Since V_1 and V_2 are closed under addition, it follows that $(u_1 + w_1) \in V_1$ and $(u_2 + w_2) \in V_2$. Therefore, $u + w \in V_1 + V_2$ and the sum space is closed under vector addition.
   Finally, let $u \in V_1 + V_2$ and $\lambda \in \mathbb{R}$. There must exist $u_1 \in V_1$, $u_2 \in V_2$ such that $u = u_1 + u_2$, so $\lambda u = \lambda u_1 + \lambda u_2$ by the distributive property of $\mathbb{R}^4$. Since V_1 and V_2 are subspaces, $\lambda u_1 \in V_1$ and $\lambda u_2 \in V_2$. Hence, $V_1 + V_2$ is closed under scalar multiplication. By theorem 1.34, $V_1 + V_2$ forms a subspace of $\mathbb{R}^4$. $\square$

4. Prove that $V_1 + V_2 = V$ in the previous problem.

   Proof. Because $V_1 + V_2$ is a subspace, it is clear that $V_1 + V_2 \subset V$. The other inclusion, however, needs justification.
   Let $v \in V$. We seek to show that v is also in $V_1 + V_2$. Since $v \in \mathbb{R}^4$, there exists $v_1, v_2, v_3, v_4 \in \mathbb{R}$ such that $v = (v_1, v_2, v_3, v_4)$. Leveraging the additive identity of the field, notice that $v = (v_1 + 0, v_2 + 0, 0 + v_3, 0 + v_4)$. By vector addition, this equals $(v_1, v_2, 0, 0) + (0, 0, v_3, v_4)$. Since $(v_1, v_2, 0, 0) \in V_1$ and $(0, 0, v_3, v_4) \in V_2$, we must conclude that $v \in V_1 + V_2$. Because v was an arbitrary element, we have shown that $V \subset V_1 + V_2$.
   Hence $V_1 + V_2 = V$. $\square$

5. Let $V = \mathbb{R}^2$, and define two subspaces:

   - $V_1 = \{(w_1, w_2) \mid w_1 + 2w_2 = 0, w_1, w_2 \in \mathbb{R}\}$
   - $V_2 = \{(v_1, v_2) \mid v_1 + v_2 = 0, v_1, v_2 \in \mathbb{R}\}$

Prove or provide a counterexample to the statement: $V_1 + V_2 = V$.

Using the theorem on sums of spaces from the book, we know that V_1+V_2 is a subspace of V and, by definition, $V_1 + V_2 \subset V$. We need to know the opposite containment for equality to hold. This means starting with an arbitrary element in V and showing that that same element is in $V_1 + V_2$. As we discussed in Monday's class, we want to see if there is a restriction on the space $V_1 + V_2$. Elements in V_1 are of the form $(-2w, w)$ and elements in V_2 are of the form $(u, -u)$. So elements constructed from their sums will be $(u - 2w, w - u)$. Could I define any vector (x, y) in this way?
This is a system of equations where

$$\begin{aligned} x &= u - 2w \\ y &= -u + w \end{aligned}$$

So to solve, we would pick $w = -x-y$ and $u = -2y-x$. So now we know $V_1+V_2 = V$ and we have a solid way to prove it. Let's write the formal argument.

Proof. By theorem 1.40, $V_1 + V_2$ is a subspace of V and we know that $V_1 + V_2 \subset V$. Now suppose $v \in V$. Then there exists $x, y, \in \mathbb{R}$ such that $v = (x, y)$. Define $u = (-2y - x, 2y + x)$ and define $w = (2x + 2y, -x - y)$. Notice that $u \in V_2$ and $w \in V_1$. Consider $w + u$:

$$\begin{aligned} w + u &= (2x + 2y, -x - y) + (-2y - x, 2y + x) \\ &= (x, y) \\ &= v \end{aligned}$$

Hence $v \in V_1 + V_2$. Thus $V = V_1 + V_2$. $\square$