HW 24 – linear Algebra

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1) Using the variance and covariance of data points, and by obtaining the var-cover matrix we compute eigenvalues that will result in eigenvectors which can each explain some variance of the data; Each eigenvector shows us the direction of the variation and the size tells us how important that direction is.

2) Noise is seen in the non-strong patterns of the data, usually captured by low-variance components (eigenvectors with low eigenvalues). By analyzing eigenvalues and dropping the lower ones, we can get rid of unimportant variance or noise.

3) $A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$ since the second row responds to state 2; state 2: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, $a_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so the probability of us being is state 1 after n step is $A^n \cdot a_0 = a_n = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x\%$

So now we have diagnolize A^n so we can actually get a good calculation:

1. finding eigen values & eigenvectors (Q)
2. finding Q^{-1} and D.
3. computing $QD^nQ^{-1}(a_0)$

① $\det(A - \lambda I) = 0$

$$= \begin{pmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{pmatrix} = 0 \Rightarrow (.8-\lambda)(.7-\lambda) - (.3)(.2) = \lambda^2 - 1.5\lambda + 0.5 = 0$$

$$\Rightarrow \lambda_1 = 1 \;,\; \lambda_2 = 0.5.$$

So $D = \begin{pmatrix} 1 & 0 \\ 0 & .5 \end{pmatrix}$.

② for $\lambda_1 = 1$: $\begin{pmatrix} .8-1 & .3 \\ .2 & .7-1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} -0.2a + 0.3b = 0 \\ 0.2a + (-0.3)b = 0 \end{matrix}\right\} a = \frac{.3}{.2} b, \vec{v}_1 = \begin{pmatrix} .3 \\ .2 \end{pmatrix}$

for $\lambda_2 = 0.5$: $\begin{pmatrix} .3 & .3 \\ .2 & .2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} 0.3a + 0.3b = 0 \\ 0.2a + 0.2b = 0 \end{matrix}\right\} a = -b, \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

So $Q = \begin{pmatrix} .3 & -1 \\ .2 & 1 \end{pmatrix}$, $Q^{-1} = \frac{1}{.3+.2}\begin{pmatrix} 1 & +1 \\ -.2 & .3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -.4 & .6 \end{pmatrix}$

③ $a^n = A^n a_0 = \begin{pmatrix} .3 & -1 \\ .2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & .5 \end{pmatrix}^n \begin{pmatrix} 2 & 2 \\ -.4 & .6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} .3 & -1 \\ .2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/2^n \end{pmatrix}\begin{pmatrix} 2 \\ .6 \end{pmatrix} = \begin{pmatrix} .3 & -1 \\ .2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ \frac{10 \cdot 0.2^n}{6} \end{pmatrix}$$

$$= \begin{pmatrix} .6 - \frac{.6}{2^n} \\ .4 + \frac{.6}{2^n} \end{pmatrix} = \begin{pmatrix} \frac{.6\,2^n - .6}{2^n} \\ \frac{.4\,2^n + .6}{2^n} \end{pmatrix}$$

the probabitly of beig in state 1 is

$$\frac{0.6 \cdot 2^n - 0.6}{2^n}$$

4) $A = \begin{pmatrix} .9 & .2 \\ .1 & .8 \end{pmatrix}$, $a_0 = \begin{pmatrix} 450/500 \\ 50/500 \end{pmatrix} = \begin{pmatrix} .9 \\ .1 \end{pmatrix}$

$a^{12} = A^{12} a_0 = QD^{12}Q^{-1}a_0$

$\det\begin{pmatrix} .9-\lambda & .2 \\ .1 & .8-\lambda \end{pmatrix} = 0 \Rightarrow (.9-\lambda)(.8-\lambda) - (.2)(.1) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 0.7$

$\Rightarrow D = \begin{pmatrix} 1 & 0 \\ 0 & .7 \end{pmatrix}$

for λ_1: $\begin{pmatrix} .9-1 & .2 \\ .1 & .8-\lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -.1 & .2 \\ .1 & -.2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \left.\begin{matrix} -.1a + .2b = 0 \\ .1a - .2b = 0 \end{matrix}\right\} \begin{matrix} .2b = .1a \\ 2b = a \end{matrix}$, $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

for λ_2: $\begin{pmatrix} .2 & .2 \\ .1 & .1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{matrix} .2a + .2b = 0 \\ .1a + .1b = 0 \end{matrix} \Rightarrow a = -b$, $\vec{v}_2 = \begin{pmatrix} -1 \\ +1 \end{pmatrix}$

$Q = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $Q^{-1} = \frac{1}{2+1}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}$

$\lambda^{12} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & .7 \end{pmatrix}^{12}\begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} .9 \\ .1 \end{pmatrix} = \begin{pmatrix} 0.669 \\ 0.33 \end{pmatrix}$

So aftr 12 months 33% of users will unsubscribe, and 67% will remain subscribed!

5) $A = \begin{pmatrix} .9 & .4 \\ .1 & .8 \end{pmatrix}$, $a_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$a^{14} = A^{14} a_0 = Q D^{14} Q^{-1} a_0$

$\det \begin{pmatrix} .9-\lambda & .4 \\ .1 & .8-\lambda \end{pmatrix} = (.9-\lambda)(.8-\lambda)-(.4)(.1) = 0 \Rightarrow \lambda_1 = 1, \ \lambda_2 = 0.5$

so, $D = \begin{pmatrix} 1 & 0 \\ 0 & .5 \end{pmatrix}$

for λ_1: $\begin{pmatrix} -.1 & .4 \\ .1 & -.4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \left.\begin{matrix} -.1a + .4b = 0 \\ .1a - .4b = 0 \end{matrix}\right\} \begin{matrix} .1a = .4b \\ a = 4b \end{matrix}$, $\vec{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

for λ_2: $\begin{pmatrix} .4 & .4 \\ .1 & .1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \left.\begin{matrix} .4a + .4b = 0 \\ .1a + .1b = 0 \end{matrix}\right\} a = -b$, $\vec{v}_2 = \begin{pmatrix} -1 \\ +1 \end{pmatrix}$

$Q = \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}$, $Q^{-1} = \frac{1}{4+1}\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1/5 & 1/5 \\ -1/5 & 4/5 \end{pmatrix}$

so $a^{14} = \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & .5 \end{pmatrix}^{14}\begin{pmatrix} 1/5 & 1/5 \\ -1/5 & 4/5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \approx \begin{pmatrix} .8 \\ .2 \end{pmatrix}$

So in two weeks (14 days),
its 80% gonna be sunny,
20% rainy.

6) $$\begin{pmatrix} 2 & 1 & 3 \\ 0 & 4 & 1 \\ 1 & 2 & 5 \end{pmatrix} = 2(20-2) - 1(0-1) + 3(0-4)$$
$$= 2(18) - 1(-1) + 3(-4) = 25$$

7) $$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} b \\ -b \end{pmatrix} = \begin{pmatrix} a+b \\ a-b \end{pmatrix}$$

$$\left.\begin{aligned} a+b &= 1 \\ a-b &= 2 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= 2+b \\ b &= 1-a \end{aligned} \Rightarrow a = 2 + 1 - a$$
$$2a = 3 \Rightarrow a = \frac{3}{2},\ b = -\frac{1}{2}$$

So $\begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the other basis!

but another way is let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

and $A^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

So now

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix} \checkmark$$

So the map is $A^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$