Hw12 - linear Algebra Sepehr Akbari

Q1) considering $\mathbb{F}$ means elements between 0 and 1 : $\{0,1\}$, and mod 2,

all its elements are : $(0,0), (1,0), (0,1), (1,1)$

since $\mathbb{F}_2^2$ has dimentionality of 2, its basis has to b at least ? 2 elemnts.

so for example $\{(1,0),(0,1)\}$ are both lin.indep and span $\mathbb{F}_2^2$.

Q2) consider $\{(1,0,1),(-1,0,1),(0,1,0)\}$ (I chose this bcs it seems correct)

① ~~Basis~~. it has to span $\mathbb{R}^3$.

let $a,b,c \in \mathbb{R}$, $(x,y,z) \in \mathbb{R}^3$

then, $a(1,0,1)+b(-1,0,1)+c(0,1,0)=(x,y,z)$

$=(a,0,a)+(-b,0,b)+(0,c,0)=(a-b,c,a+b)$

$\Rightarrow \boxed{y=c}$, $x=a-b \Rightarrow a=x+b$, $z=a+b \Rightarrow b=z-a=z-x-b$

$a=x+\frac{z-x}{2}=\boxed{\frac{z+x}{2}}$ $\Rightarrow \boxed{b=\frac{z-x}{2}}$

So since $\left(\left(-\frac{z-x}{2}+\frac{z+x}{2}\right), y, \frac{z+x}{2}+\frac{z-x}{2}\right)=(x,y,z)$ it spans $\mathbb{R}^3$.

② lin.indep.

for $a(1,0,1)+b(-1,0,1)+c(0,1,0)=(0,0,0)$

$=(a-b,c,a+b)=(0,0,0) \Rightarrow \boxed{c=0}$

$, a-b=a+b=0 \Rightarrow a=b \Rightarrow a+b=0 \Rightarrow \boxed{a=b=0}$

So its lin.indep.

so its a valid basis □

Q3) my approach is to add a vector which is non-constructable, through using Os. so $B = \{(1,-1,1),(0,1,1),(1,0,0)\}$.

① spanning $\mathbb{R}^3$.

let $a,b,c \in \mathbb{R}$, $(x,y,z) \in \mathbb{R}^3$.

$$a(1,-1,1) + b(0,1,1) + c(1,0,0) = (x,y,z)$$

$$\Rightarrow (a,-a,a) + (0,b,b) + (c,0,0) = (a+c,\ b-a,\ a+b)$$

$$\begin{cases} x = a+c \\ y = b-a \\ z = a+b \end{cases} \Rightarrow \begin{array}{l} a = x-c = z-b = b-y \\ b = y+a = z-a \\ \boxed{c = x-a = x+y} \end{array} \Rightarrow \boxed{b = y+(b-y) = 0} \Rightarrow a = z = -y$$

So $(z+x+y,\ -z,\ z) = (x,y,z)$ so it does span $\mathbb{R}^3$.

② lin. indep.

for $(a+c,\ b-a,\ a+b) = (0,0,0)$,

$$\begin{cases} a+c = 0 \\ b-a = 0 \\ a+b = 0 \end{cases} \quad a=b \Rightarrow a+a=0 \Rightarrow \boxed{a=b=0},\ 0+c=0 \Rightarrow \boxed{c=0}$$

So v is lin. indep.

hence v is a basis. □

$$v_1 \qquad v_2 \qquad v_3 \qquad\qquad w_1 \qquad w_2 \qquad w_3$$

Q4) $B_1 = \{(1,0,1), (-1,0,1), (0,1,0)\}$, $B_2 = \{(1,-1,1), (0,1,1), (1,0,0)\}$

let $(x,y,z) \in \mathbb{R}^3$ and $(x,y,z) = av_1 + bv_2 + cv_3 = \tilde{a}w_1 + \tilde{b}w_2 + \tilde{c}w_3$

* I'm not sure to express v_i as w_i or vice-versa so I'll do one *

$$\boxed{v_i = n_1 w_1 + n_2 w_2 + n_3 w_3 \quad \text{or} \quad w_i = m_1 v_1 + m_2 v_2 + m_3 v_3.}$$

v_1 $(1,0,1) = n_1(1,-1,1) + n_2(0,1,1) + n_3(1,0,0) = (n_1+n_3, n_2-n_1, n_1+n_2)$

$$\begin{cases} n_1+n_3=1 \\ n_2-n_1=0 \Rightarrow n_1=n_2 \\ n_1+n_2=1 \end{cases} \Rightarrow 2n_1=1 \Rightarrow n_1 = 1/2 = n_2 \qquad \Rightarrow 1/2 + n_3 = 1 \Rightarrow n_3 = 1/2$$

$\Rightarrow$ ~~(1,0,1)~~ $\boxed{v_1 = \frac{1}{2}w_1 + \frac{1}{2}w_2 + \frac{1}{2}w_3}$

v_2 $(-1,0,1) \Rightarrow \begin{cases} n_1+n_3=-1 \\ n_2-n_1=0 \\ n_1+n_2=1 \end{cases} \Rightarrow n_2 = n_1 \Rightarrow 2n_1=1 \Rightarrow n_1 = 1/2 = n_2$ $\qquad 1/2 + n_3 = -1 = -\frac{3}{2}$

$\Rightarrow \boxed{v_2 = \frac{1}{2}w_1 + \frac{1}{2}w_2 - \frac{3}{2}w_3}$

v_3 $\begin{cases} n_1+n_3=0 \Rightarrow n_2 = -n_1 \\ n_2-n_1=1 \\ n_1+n_2=0 \Rightarrow n_1 - n_1 = 0 \end{cases}$ $\quad -n_1-n_1=1 \Rightarrow n_1=-1/2 \quad \Rightarrow n_2 = 1/2, \ -1/2+n_3=0 \Rightarrow n_3 = 1/2$

$\Rightarrow \boxed{v_3 = -\frac{1}{2}w_1 + \frac{1}{2}w_2 + \frac{1}{2}w_3}$

w_1 $(1,-1,1) = (m_1-m_2, m_3, m_1+m_2)$ $\quad \begin{cases} m_1-m_2=1 \\ m_3=-1 \\ m_1+m_2=1 \end{cases} \Rightarrow -m_2 = m_2 = 0 \quad m_3=-1 \quad \Rightarrow m_1 = 1$

$\Rightarrow \boxed{w_1 = 1v_1 + 0v_2 + (-1)v_3}$

w_2 $\begin{cases} m_1-m_2=0 \\ m_3=1 \\ m_1+m_2=1 \end{cases} \Rightarrow m_1 = m_2 = 1/2 \Rightarrow \boxed{w_2 = \frac{1}{2}v_1 + \frac{1}{2}v_2 + 1v_3}$

w_3 $\begin{cases} m_1-m_2=1 \\ m_3=0 \\ m_1+m_2=0 \end{cases} \Rightarrow m_1 = -m_2 \Rightarrow -m_2-m_2=1 \Rightarrow m_2=-1/2, \ m_1 = 1/2$

$\Rightarrow \boxed{w_3 = \frac{1}{2}v_1 + (-\frac{1}{2})v_2 + 0v_3}$

now, $(x,y,z) = a_1v_1 + a_2v_2 + a_3v_3 = a_1(\frac{1}{2}w_1 + \frac{1}{2}w_2 + \frac{1}{2}w_3) + a_2(\frac{1}{2}w_1 + \frac{1}{2}w_2 - \frac{3}{2}w_3)$
$+ a_3(-\frac{1}{2}w_1 + \frac{1}{2}w_2 + \frac{1}{2}w_3) = w_1(\frac{a_1}{2} + \frac{a_2}{2} - \frac{a_3}{2}) + w_2(\frac{a_1}{2} + \frac{a_2}{2} + \frac{a_3}{2})$
$+ w_3(\frac{a_1}{2} + \frac{-3a_2}{2} + \frac{a_3}{2}) = w_1b_1 + w_2b_2 + w_3b_3$

$\boxed{\text{damn, I understand now lol :)}}$

it'll be $\begin{cases} f_1(a_1,a_2,a_3) = \frac{1}{2}(a_1+a_2-a_3) \\ f_2(a_1,a_2,a_3) = \frac{1}{2}(a_1+a_2+a_3) \\ f_3(a_1,a_2,a_3) = \frac{1}{2}(a_1-3a_2+a_3) \end{cases}$

I didn't even have to solve for both, but pretty sure it'll work both ways.

$\square$

Q5) let $\{v_1, v_2, v_3, v_4\} = B$ be a basis of V.

this implies $v_1 \dots v_4$ are lin.indep and span V.

let $\tilde{B} = \{v_1, v_2\}$, we still know v_1, v_2 are lin.indep

so we just have to prove they span $U \subset V$ for $\tilde{B}$ to be a basis of U.

so let $(x, y) \in U^2$, $a, b \in \mathbb{F}$.

for $(x, y) = a(v_1) + b(v_2)$ *

*but this not have enough evidence so $\{v_1, v_2\}$ dont necc span U.

<u>counterexample</u>

let $V = \mathbb{R}^4$, $v_1 = (1,0,0,0)$, $v_2 = (0,1,0,0)$, ..., $v_4 = (0,0,0,1)$ (this make B) which is a basis of V

let's define $U = \{(a, b, 0, 0) \mid a, b \in \mathbb{R}\}$

So $v_1, v_2 \in U$, $v_3, v_4 \notin U$. wait that is a basis. num!

consider $U = \text{span}\{v_1 + v_3, v_2\}$ in this case $v_1, v_2 \in U$, $v_3, v_4 \notin U$

now $U = a(v_1 + v_3) + bv_2 = (a, b, a, 0)$

we'll need $v_3 = (0,0,1,0)$ to construct this but $v_3 \notin U$,

$\Rightarrow \Leftarrow$

So the statement is false $\boxtimes$