Math 231 — Hw 5
Sara Jamshidi, Jan 24, 2025

1. Prove that for any $v \in V$, $-(-v) = v$.

   (Hint: Use Thm 1.32).

   Proof. By Theorem 1.32, $-v$ is the additive inverse of v. Similarly $-(-v)$ is the additive inverse of $-v$. Because v and $-(-v)$ are additive inverses of v, it must be the case that $-(-v) = v$ by theorem 1.27. □

2. Let $a \in \mathbb{F}$ and $v \in V$. If $av = 0$, show that either $a = 0$ or $v = 0$.

   (Note: 0 is being used both for the additive identity of the field element and the additive identity of the vector space. This is an "abuse of notation," but you should be able to tell which is which.)

   Theorems 1.30 and 1.31 tell us that if $a = 0$ or $v = 0$, then $av = 0$, but we seek to show the opposite statement. Here, we will prove the contrapositive: if $a \neq 0$ and $v \neq 0$, then $av \neq 0$.

   Proof. Let $a \in \mathbb{F}$ and $v \in V$, neither equal to the additive identities of their respective spaces. Suppose for the sake of contradiction that $av = 0$, the additive identity of V. Since $a \neq 0$ and it is an element of a field, there exists an $a^{-1} \in \mathbb{F}$, the multiplicative inverse of a. It follows that

   $$a^{-1}av = 1v = v.$$

   This implies that $v = 0$, which is a contradiction. Hence $av \neq 0$. □

3. Suppose $-1 \notin \mathbb{F}$. Prove that there exists an element $\lambda \in \mathbb{F}$ such that for any $v \in V$, $v + \lambda v = 0$.

   Proof. Because $\mathbb{F}$ is a field, $1 \in \mathbb{F}$ has an additive inverse. Since $-1 \notin \mathbb{F}$, define the inverse to be λ. Then it follows that $1 + \lambda = 0$. As a result,

   $$0 = 0v = (1 + \lambda)v = v + \lambda v.$$

   Hence $v + \lambda v = 0$. □