Math 231 — Hw 6
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1. Prove or disprove if this is a vector space using theorem 1.34 from the textbook:

$$W = \{(x_1, x_2, x_3) \mid x_1 x_2 x_3 = 0, x_i \in \mathbb{R}\}.$$

This is NOT a vector space.

Proof. Let's call this set W and notice $W \subseteq \mathbb{R}^3$. Consider the vectors $v = (1, 0, 2)$ and $w = (0, 3, 1)$, which are both elements of W. Notice that:

$$v + w = (1, 3, 3) \notin W$$

since $1 \cdot 3 \cdot 3 = 9 \neq 0$. Since W is not closed under addition, it is not a subspace and therefore not a vector space. □

2. Construct an example of a vector space W with two subspaces, W_1, W_2 where $W_1 + W_2 \neq W$.

Consider $W = \mathbb{R}^3$. Define:

$$W_1 = \{(x, 0, 0) \mid x \in \mathbb{R}\}, \quad W_2 = \{(0, x, 0) \mid x \in \mathbb{R}\}.$$

The definition of $W_1 + W_2$ is the set of all sums of vectors from these sets. If $v \in W_1$, then $v = (v_1, 0, 0)$ and if $w \in W_2$, then $w = (0, w_2, 0)$. Thus, elements in $W_1 + W_2$ are of the form:

$$(v_1, w_2, 0).$$

However, the vector $(0, 0, 1) \in \mathbb{R}^3$ is not in $W_1 + W_2$ because it does not match that form. Hence, $W_1 + W_2 \neq W$.

3. Let $V = \mathbb{R}^3$, and define two subspaces:

- $V_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$
- $V_2 = \{(0, y, z) \mid y, z \in \mathbb{R}\}$

Prove that $V_1 + V_2$ forms a subspace of V.

Proof. Let $v_1 = (x_1, y_1, 0) \in V_1$ and $v_2 = (0, y_2, z_2) \in V_2$. Then their sum is:

$$v_1 + v_2 = (x_1, y_1, 0) + (0, y_2, z_2) = (x_1, y_1 + y_2, z_2).$$

Since $x_1, y_1, y_2, z_2 \in \mathbb{R}$, we see that $V_1 + V_2$ consists of all vectors of the form (x, y, z), which shows that it is closed under addition and it contains the additive identity $(0, 0, 0)$.
Now, suppose $v \in V_1 + V_2$ and let $c \in \mathbb{R}$. Then:

$$\begin{aligned} cv &= c(v_1 + v_2) \quad \text{for some } v_i \in V_i \\ &= cv_1 + cv_2 \end{aligned}$$

As is stated in the problem V_1 and V_2 are subspaces, so $cv_1 \in V_1$ and $cv_2 \in V_2$. Hence $cv \in V_1 + V_2$. Thus, by theorem 1.34 in the textbook, $V_1 + V_2$ is a subspace. □

4. Prove that $V_1 + V_2 = V$ in the previous problem.

Proof. By the previous problem, it follows that $V_1 + V_2 \subset V$ and is a vector space. Let $(a, b, c) \in V = \mathbb{R}^3$. We can write:

$$(a, b, c) = (a, b, 0) + (0, 0, c).$$

The first vector $(a, b, 0)$ belongs to V_1, and the second vector $(0, 0, c)$ belongs to V_2, since we can choose $y = 0$ in V_2. Since every vector in V can be expressed as a sum of elements in V_1 and V_2, we conclude that $V \subset V_1 + V_2$. Because $V_1 + V_2 \subset V$ and $V \subset V_1 + V_2$, it follows that $V = V_1 + V_2$. □

5. Let $V = \mathbb{R}^3$, and define two subspaces:

- $V_1 = \{(x, y, 0) \mid x + y = 0, x, y \in \mathbb{R}\}$
- $V_2 = \{(0, y, z) \mid y + z = 0, y, z \in \mathbb{R}\}$

Prove or provide a counterexample to the statement: $V_1 + V_2 = V$.

The statement is false.

Proof. Any vector in $V_1 + V_2$, can be expressed as a sum of vectors $v_1 \in V_1$ and $v_2 \in V_2$. Because the two entries of v_1 must add to zero and the same is true for v_2, we can write them as $(x, -x, 0)$ and $(0, y, -y)$, where $-x$ represents the additive inverse of x and similarly for $-y$ and y. Then $v_1 + v_2 = (x, y - x, -y)$, so it *automatically* follows the sum of the three entries of this vector is zero in $\mathbb{R}$. Observe however that $(1, 1, 1) \in V$ but **lacks** this property. Hence these two vector spaces cannot be equal. □