Math 231 — Hw 8
Sara Jamshidi, Feb 10, 2025

1. Let V be a vector space and $0 \in V$ the additive identity. Prove that $0 + 0 = 0$. Then prove that $0 + \ldots + 0 = 0$ for any finite number of sums.

Proof. We will prove this using induction.
Base case: Since 0 is the additive identity in V, we know that for any $v \in V$, we have:

$$v + 0 = v.$$

Since 0 is a vector as well, it follows that:

$$0 + 0 = 0.$$

Inductive step: Suppose $0+0+\ldots+0$ (k times) equals 0. Now consider $k+1$ terms:

$$0 + 0 + \ldots + 0 + 0 = (0 + 0 + \ldots + 0) + 0.$$

By the inductive hypothesis, the first k terms sum to 0, so this reduces to

$$= 0 + 0.$$

This was shown in the base case to be 0. □

2. Let $V = \mathbb{R}^3$ and consider the subspaces:

$$W_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}, \quad W_2 = \{(0, 0, z) \mid z \in \mathbb{R}\}.$$

Prove that $V = W_1 \oplus W_2$ using the last theorem from class.

To prove that $V = W_1 \oplus W_2$, we must show two things:

(a) $V = W_1 + W_2$, meaning every vector in V can be written as the sum of a vector from W_1 and a vector from W_2.

(b) $W_1 \cap W_2 = \{0\}$.

Proof. Because W_1, W_2 are subspaces, it follows that $W_1 + W_2 \subseteq V$. Let $(a, b, c) \in V = \mathbb{R}^3$. We can write:

$$(a, b, c) = (a, b, 0) + (0, 0, c),$$

where $(a, b, 0) \in W_1$ and $(0, 0, c) \in W_2$. Hence, we have the opposite containment. Therefore, $V = W_1 + W_2$.

The intersection is the additive identity.
Suppose $(x, y, 0) \in W_1$ is also in W_2. Then it must also be of the form $(0, 0, z)$ for some z. The only vector that satisfies both forms is $(0, 0, 0)$. Therefore:

$$W_1 \cap W_2 = \{0\}.$$

By theorem 1.46,

$$V = W_1 \oplus W_2.$$

□

3. Let $V = \mathbb{R}^3$. Consider the subspace $U = \{(x, y, 0) \mid x + y = 0\}$. Find a space W such that $V = U \oplus W$.

For this problem, we need only define W to be any space that covers a different area in the xy-plane as well as the z direction. We will define it to be

$$W = \{(0, y, z) \mid y, z \in \mathbb{R}\}.$$

Claim: $V = U \oplus W$.

Proof. Both W and U are subspaces of V, so the same holds for $W + U$, meaning $W + U \subseteq V$. Let $v \in V$. Then $v = (a, b, c)$ for some $a, b, c, \in \mathbb{R}$. Define vectors $u = (a, -a, 0)$ and $w = (0, b + a, c)$, which are vectors in U and W respectively. Observe that $v = u + w$. Hence $v \in U + W$. Lastly, observe that any vector in both U and W must be equal to $(0, 0, 0)$ since the first and last entry must be 0 to be in space W and U respectively. Consequently, the middle value must then be 0 for the vector to be in U. Thus, $U + W = W \oplus W$. □

Math 231 — Hw 9

Sara Jamshidi, Feb 12, 2025

1. Find a list of four distinct vectors whose span equals the space $\{(x, y, z) \mid x+y+z = 0\}$ within a vector space $\mathbb{F}^3$. ($\mathbb{F}$ could be any field)

   Let's call the space W. All the vectors in W must satisfy the rule that defines the space. We could characterize these vectors using only two variables, $(x, y, -x - y)$, which suggests that geometrically, the space is two dimensional. Given how it is written two distinct vectors whose sum is what we see above, we could pick $(1, 0, -1)$, and $(0, 1, -1)$. Their linear combination should give us the previous vector. We now know enough to write the answer to this problem.

   Proof. Consider the set of distinct vectors $\{(1, 0, -1), (0, 1, -1), (-2, 1, 1), (-1, 2, -1)\}$. We claim the span of these vectors is equal to $W = \{(x, y, z) \mid x + y + z = 0\}$. Let $(x, y, z) \in W$. Then it must be the case that $z = -x - y$. Hence we can rewrite the vector as $(x, y, -x - y)$. Observe that

   $$x(1, 0, -1) + y(0, 1, -1) + 0(-2, 1, 1) + 0(-1, 2, -1) = (x, y, -x - y)$$

   $\square$

2. If v_1, v_2, v_3, v_4 span V, is it true that $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ will as well? Prove your response.

   If v_1, v_2, v_3, v_4 span V, then for $x \in V$, there exists coefficients where

   $$x = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4.$$

   If $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ also spans, then there should be a set of coefficients I can construct using $\{a_i\}$ to describe x. If I can, then I can prove it.

   (a) For $v_1 - v_2$, the coefficient would have to be a_1.
   (b) Then for $v_2 - v_3$, we would need to have $a_2 + a_1$ (the a_1 here is needed to cancel out with the first term).
   (c) For $v_3 - v_4$, we would need the coefficient $a_3 + a_2 + a_1$.
   (d) For v_4, we would need $a_4 + a_3 + a_2 + a_1$.

   Hence

   $$x = a_1(v_1 - v_2) + (a_2 + a_1)(v_2 - v_3) + (a_3 + a_2 + a_1)(v_3 - v_4) + (a_4 + a_3 + a_2 + a_1)(v_4).$$

   Now we can prove this.

Proof. Suppose v_1, v_2, v_3, v_4 span V. Consider the the set of vectors in V: $B = \{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$. We aim to prove that the span V as well. Let $x \in V$. Then

$$x = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$$

for some $\{a_1, \ldots, a_4\}$. Observe that

$$x = a_1(v_1 - v_2) + (a_2 + a_1)(v_2 - v_3) + (a_3 + a_2 + a_1)(v_3 - v_4) + (a_4 + a_3 + a_2 + a_1)(v_4).$$

Hence B spans V as well. $\square$

Math 231 — Hw 10
Sara Jamshidi, Feb 13, 2025

1. Is the following set of vectors from $\mathbb{R}^3$ a linearly independent set?

$$\{(1,2,3),(4,5,6),(7,8,9)\}$$

Prove or disprove.

To determine if the set of vectors $\{(1,2,3),(4,5,6),(7,8,9)\}$ is linearly independent, we need to check that there is only one linear combination for $(0,0,0)$ with coefficients $c_1 = c_2 = c_3 = 0$. Consider

$$c_1(1,2,3) + c_2(4,5,6) + c_3(7,8,9) = (0,0,0).$$

Solving the system of equations, we can see that when $c_1 = c_3$ and $c_2 = -2c_3$, we can get $(0,0,0)$. This means that there are non-trivial solutions (i.e., not all c_i are zero), indicating that the vectors are NOT linearly independent.

2. Remember that vectors are just elements of a vector space. Since P_2, the space of polynomials up to degree 2, is a vector space, then below is a set of vectors from that space.

$$S = \{2, x-1, x^2 - x\}$$

Is it true that $\textbf{span}(S) = P_2$. Prove your answer.

To determine if $\textbf{span}(S) = P_2$, where $S = \{2, x-1, x^2 - x\}$, we need to check if any polynomial of the form $ax^2 + bx + c$ can be written as a linear combination of the vectors in S. We want to express $ax^2 + bx + c$ as:

$$k_1 \cdot 2 + k_2 \cdot (x-1) + k_3 \cdot (x^2 - x)$$

Expanding and combining like terms, we get:

$$k_3 x^2 + (k_2 - k_3)x + (2k_1 - k_2)$$

We need this to equal $ax^2 + bx + c$. Thus, we equate the coefficients:

$$k_3 = a$$

$$k_2 - k_3 = b$$

$$2k_1 - k_2 = c$$

Solving this system of equations, we find:

$$k_1 = \frac{a}{2} + \frac{b}{2} + \frac{c}{2}$$

$$k_2 = a + b$$

$$k_3 = a$$

Since we can find such k_1, k_2, k_3 for any a, b, c. So $\textbf{span}(S) = P_2$.

Math 231 — Hw 11
Sara Jamshidi, Feb 14, 2025

1. In this video, the speaker explains that historically, rainbows in Western art were depicted with only three colors until Isaac Newton expanded the concept to seven. This shift highlights how different cultural and scientific perspectives influence our understanding of color.

   In modern computing, colors are represented mathematically using vector spaces such as RGB (Red-Green-Blue) and CMYK (Cyan-Magenta-Yellow-Black). We can approximate these models using the vector spaces $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively, where each coordinate represents a component's intensity. For example, in RGB, the vector $(1, 0, 0)$ represents pure red, while in CMYK, the vector $(1, 0, 0, 0)$ represents pure cyan.

   The RGB and CMYK models describe color in fundamentally different ways:

   (a) RGB is an additive color model, where colors are created by combining light.

   (b) CMYK is a subtractive model, used in printing, where colors result from absorbing certain wavelengths.

   (c) What do you think it means in this context if a given set spans a set? Mathematically, could three distinct shades of a color we recognize, say three shades of red, be enough to span the entire space of colors?

   Pick three vectors in the RGB space with values between 0 and 1 (e.g., $(.33, .25, 0)$). Prove that your three vectors span $\mathbb{R}^3$. With the provided code, convert them to CMYK. Prove that the result does not span $\mathbb{R}^4$.

   Suppose we pick the vectors $(0.33, 0.25, 0)$, $(0.5, 0.75, 0.25)$, and $(0.1, 0.4, 0.6)$. Notice that this set is linearly independent. Why? Consider

$$a_1(0.33, 0.25, 0) + a_2(0.5, 0.75, 0.25) + a_3(0.1, 0.4, 0.6) = (0, 0, 0).$$

   Solving the system of equations, we found that the only solution is indeed $a_1 = 0$, $a_2 = 0$, and $a_3 = 0$. This confirms that the vectors are linearly independent.To prove that the vectors $(0.33, 0.25, 0)$, $(0.5, 0.75, 0.25)$, and $(0.1, 0.4, 0.6)$ span $\mathbb{R}^3$, we need to show that any vector (x, y, z) in $\mathbb{R}^3$ can be written as a linear combination of these vectors. This means we need to find scalars c_1, c_2, and c_3 such that:

$$c_1(0.33, 0.25, 0) + c_2(0.5, 0.75, 0.25) + c_3(0.1, 0.4, 0.6) = (x, y, z)$$

   This leads to the system of linear equations:

$$\begin{cases} 0.33c_1 + 0.5c_2 + 0.1c_3 = x \\ 0.25c_1 + 0.75c_2 + 0.4c_3 = y \\ 0.0c_1 + 0.25c_2 + 0.6c_3 = z \end{cases}$$

We can represent this system as a matrix equation $A\mathbf{c} = \mathbf{b}$, where A is the matrix formed by the vectors, $\mathbf{c}$ is the vector of coefficients, and $\mathbf{b}$ is the vector (x, y, z). We already know from the previous calculation that the determinant of the matrix A is non-zero, which means A is invertible. Therefore, for any (x, y, z), there exists a unique solution for (c_1, c_2, c_3). This confirms that the vectors span $\mathbb{R}^3$.
Next, we converted these RGB vectors to CMYK space and checked if they span $\mathbb{R}^4$. These vectors are now $(0.0, 0.24242424, 1.0, 0.67)$, $(0.33333333, 0.0, 0.66666667, 0.25)$, and $(0.83333333, 0.33333333, 0.0, 0.4)$.

$$\begin{aligned}&b_1(0.0, 0.24242424, 1.0, 0.67) + b_2(0.33333333, 0.0, 0.66666667, 0.25)\\&+b_3(0.83333333, 0.33333333, 0.0, 0.4) = (0, 0, 0, 0)\end{aligned}$$

is $b_1 = b_2 = b_3 = 0$. Solving the system of equations, we found that the only solution is indeed $b_1 = 0$, $b_2 = 0$, and $b_3 = 0$. This confirms that the vectors are linearly independent. But they do not span because there aren't enough elements.

2. In music theory, chords can be represented as vectors in a musical space, where each coordinate represents the presence or absence of a particular note in a chord. Let's look at the 5 chords within the B-minor scale famously used in the song "15 Step" from *In Rainbows*, Radiohead's 7th album.[1]

Chord	B	$C\#$	D	E	$F\#$	G	A
Bm	1	0	1	0	1	0	0
Cdim5	0	1	0	1	0	1	0
A6	0	0	1	0	1	0	1
Gmaj7	0	0	1	0	1	1	1

(a) Are the given chord vectors linearly independent in $\mathbb{R}^7$? Justify your answer.

(b) Do these chords span the space of all possible chords in the B minor scale (i.e., all possible vectors in $\mathbb{R}^7$ with only 0s and 1s)? If not, add additional chords to the set in the provided code. Alternatively, feel free to create an entirely new set of chords from the basis that span.

(c) What do you think it means in this context if a given set is linearly dependent?

Linear Independence

To determine if the given chord vectors are linearly independent in $\mathbb{R}^7$, we need to check that the vectors can only form the zero vector with coefficients of only 0.

[1]Fun fact: They released this album themselves (without a record label) and allowed customers to pay what they wanted for it. The band made a much bigger profit on this album than their previous ones.

Math 231 — Hw 12

Sara Jamshidi, Feb 21, 2025

1. Consider the vector space $\mathbb{F}_2^2$. What are all the possible bases for this space? (You do not need to prove this).

    - $\{(1,0),(0,1)\}$
    - $\{(1,1),(0,1)\}$
    - $\{(1,0),(1,1)\}$

2. Consider a linearly independent set in $\mathbb{R}^3$ and construct a basis by adding one element to it. Then prove that it is a basis. This question was clearly a typo. My bad. Here's just a random set of vectors in the space and a proof that they are a basis.

    Proof. Assume $a(1,0,1)+b(-1,0,1)+c(0,1,0)=(0,0,0)$. This gives us the system of equations:

    $$\begin{cases} a-b=0 \\ c=0 \\ a+b=0 \end{cases}$$

    Solving this system, we get $a=0$, $b=0$, and $c=0$. Thus, the vectors are linearly independent. Any vector $(x,y,z)\in\mathbb{R}^3$ can be written as a linear combination of these vectors:

    $$(x,y,z)=\frac{x+z}{2}(1,0,1)+\frac{(z-x)}{2}(-1,0,1)+y(0,1,0)$$

    $\square$

3. Consider the linearly independent set in $\mathbb{R}^3$ and construct a basis by adding one element to it. Then prove that it is a basis.

    $$\{(1,-1,1),(0,1,1)\}$$

    Proof. Assume $a(1,-1,1)+b(0,1,1)+c(1,0,0)=(0,0,0)$. This gives us the system of equations:

    $$\begin{cases} a+c=0 \\ -a+b=0 \\ a+b=0 \end{cases}$$

Solving this system, we get $a = 0$, $b = 0$, and $c = 0$. Thus, the vectors are linearly independent. Any vector $(x, y, z) \in \mathbb{R}^3$ can be written as a linear combination of these vectors:

$$(x, y, z) = \left(\frac{z - y}{2}\right)(1, -1, 1) + \left(\frac{y + z}{2}\right)(0, 1, 1) + \left(x + \frac{y - z}{2}\right)(1, 0, 0)$$

$\square$

4. In the previous two examples, you wrote two distinct bases of $\mathbb{R}^3$. Given a vector $(x, y, z) \in \mathbb{R}^3$, write a set of functions $f_i(a_1, a_2, a_3) = b_i$ where each function takes in the coefficients of the first basis and produces the i^{th} coefficient of the second basis. What kind of functions are these?

   In the first basis, (x, y, z) is computed using the coefficients

   $$a_1 = \frac{x + z}{2}, a_2 = \frac{(z - x)}{2}, a_3 = y.$$

   The second basis has coefficients

   $$b_1 = \left(\frac{z - y}{2}\right), b_2 = \left(\frac{y + z}{2}\right), b_3 = \left(x + \frac{y - z}{2}\right)$$

   This question asks us to write each of the coefficients of the second basis in terms of the first set of coefficients. Here are the solutions:

   - $b_1 = \frac{1}{2}(a_1 + a_2 - a_3)$
   - $b_2 = \frac{1}{2}(a_1 + a_2 + a_3)$
   - $b_3 = \frac{1}{2}(a_1 - 3a_2 + a_3)$

5. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

   Consider the vector space $V = \mathbb{R}^4$ with the basis:

   $$v_1 = (0, 0, 1, -1), \quad v_2 = (0, 0, 1, 1), \quad v_3 = (0, 1, 0, 0), \quad v_4 = (1, 0, 0, 0)$$

   Let U be the subspace of V defined as:

   $$U = \{(0, x, y, z) \mid x + y + z = 0\}$$

   Clearly, $v_1, v_2 \in U$ and $v_3, v_4 \notin U$. However, v_1 and v_2 do not form a basis of U. Consider the element $(0, 1, -1, 0)$. This cannot be constructed from v_1, v_2.

Math 231 — Hw 13
Sara Jamshidi, Feb 26, 2025

1. Suppose p_0, p_1, p_2, p_3 is a basis of the space of polynomials of degree 3. Construct a basis where none of the polynomials are degree 2.

   Recall: the degree of a polynomial is the term with the maximum degree, for example this polynomial is degree 3: $x^3 + x^2 + x + 1$.

   We can choose the following polynomials as a basis where none of the polynomials are of degree 2:

   - $p_0 = 1$ (degree 0)
   - $p_1 = x$ (degree 1)
   - $p_2 = x^3$ (degree 3)
   - $p_3 = x^3 + x^2$ (degree 3, not 2)

   We know this will be a basis because the function $x^2 = p_3 - p_2$.

2. Suppose $\{v_1.v_2, v_3, v_4\}$ is basis of V. Prove that

   $$\{v_1 + v_2,\ v_2 + v_3,\ v_3 + v_4,\ v_4\}$$

   is also a basis of V.

   Proof. To prove that $\{v_1 + v_2,\ v_2 + v_3,\ v_3 + v_4,\ v_4\}$ is a basis, we need to show that these vectors are linearly independent and span V. Assume $a(v_1 + v_2) + b(v_2 + v_3) + c(v_3 + v_4) + d(v_4) = 0$. This gives us:

   $$av_1 + (a + b)v_2 + (b + c)v_3 + (c + d)v_4 = 0$$

   Since $\{v_1, v_2, v_3, v_4\}$ is a basis, the coefficients must all be zero:

   $$a = 0, \quad a + b = 0, \quad b + c = 0, \quad c + d = 0$$

   Solving this system, we get $a = b = c = d = 0$. Thus, the vectors are linearly independent. Since there are four linearly independent vectors in a four-dimensional space, they span V. $\square$

3. Suppose $v_1.v_2, v_3, v_4$ is basis of V. Prove that

   $$\{v_1,\ v_1 + v_2,\ v_1 + v_2 + v_3,\ v_1 + v_2 + v_3 + v_4\}$$

   is also a basis of V.

Spanning $\mathbb{R}^7$

The chords do not span the space of all possible chords in the B minor scale (i.e., all possible vectors in $\mathbb{R}^7$ with only 0s and 1s) because there are only 4 elements and not 7. To span $\mathbb{R}^7$, we would need 7 linearly independent vectors.

Linear Dependence in Context

In this context, if a given set of chord vectors is linearly dependent, it implies that at least one chord can be expressed as a linear combination of the others. This means that some chords are redundant in terms of the musical space they occupy.

Given that each note in a chord is either present or absent, we can represent the set of chords as a vector space over the binary field $\mathbb{Z}_2^7$. In this framework, each chord is a binary vector, and the musical space is defined by the presence or absence of notes. By focusing on 7 linearly independent combinations, we can generate a diverse range of musical expressions while ensuring that each chord contributes uniquely to the overall sound.

The expressiveness of a piece of music can be encoded through the complexity of these chord combinations and we have the ability to develop a mathematical framework to create and analyze music.

Proof. To prove that $\{v_1,\ v_1+v_2,\ v_1+v_2+v_3,\ v_1+v_2+v_3+v_4\}$ is a basis, we need to show that these vectors are linearly independent and span V. Assume $a(v_1)+b(v_1+v_2)+c(v_1+v_2+v_3)+d(v_1+v_2+v_3+v_4)=0$. This gives us:

$$(a+b+c+d)v_1+(b+c+d)v_2+(c+d)v_3+dv_4=0$$

Since $\{v_1,v_2,v_3,v_4\}$ is a basis, the coefficients must all be zero:

$$a+b+c+d=0, \quad b+c+d=0, \quad c+d=0, \quad d=0$$

Solving this system, we get $a=b=c=d=0$. Thus, the vectors are linearly independent. Since there are four linearly independent vectors in a four-dimensional space, they span V. $\square$

Math 231 — Hw 15
Sara Jamshidi, Mar 19, 2025

In class, we discussed how we to use matrix representations of linear maps between vector spaces. We did two examples in class. Below is the complete version of the second one with a small change to make the math easier.

Suppose we have two vector spaces, $V = \mathbb{Z}_7^3$ and $W = \mathbb{R}^2$, with the bases

$$B_V = \{(1,1,1), (1,6,1), (0,1,1)\}$$

and $B_W = \{(1,0),(0,1)\}$. I will use v_i and w_i to denote these elements. And we define the following linear map T between them:

- $Tv_1 = 3w_1$
- $Tv_2 = w_1 + 2w_2$
- $Tv_3 = -w_2$

The matrix representation of T between these two spaces with those bases is

$$M(T) = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & \text{-1} \end{pmatrix}$$

The basis used for $\mathbb{Z}_7^3$ in defining T is not ideal, so we'd like to construct a map $U : \mathbb{Z}_7^3 \to \mathbb{Z}_7^3$ where we change the basis from the standard basis, $\{(1,0,0),(0,1,0),(0,0,1)\}$ to B_V. Then we can construct a map by composing $T \circ U$ to construct a map from $\mathbb{Z}_7^3$ to $\mathbb{R}^3$.
So we need to construct $(1,0,0)$ as a linear combination of the elements in B_V. In other words, we need to find elements x, y, z such that $x + y = 1$, $x + 6y + z = 0$, $x + y + z = 0$. From the first and third equation, we can deduce that $z = 6$. And then see that $x = 1$ and $y = 0$ works for our problem. Hence

$$U((1,0,0)) = v_1 + 6v_3.$$

By similar logic, we can get the remaining two:

- $U((1,0,0)) = v_1 + 6v_3$.
- $U((0,1,0)) = 4v_1 + 3v_2$.
- $U((0,0,1)) = v_3$.

The matrix representation of U is

$$M(U) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Suppose we have the element $v = (2, 3, 1)$ in $\mathbb{Z}_7^3$. Where does T map this element to? We can use the matrix representations and compute $T \circ U$:

$$\begin{aligned} TUv &= \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & \text{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & \text{-1} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 17 \\ 3 \end{pmatrix} \end{aligned}$$

A tricky element here is that U is happening modulo 7 because of the spaces it is going between. Now it is your turn to try.

1. Consider the vector spaces $V = \mathbb{Z}_5^2$ and $W = \mathbb{R}^3$ with bases

$$B_V = \{(1, 2), (0, 1)\}$$

   and $B_W = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.Define the linear map $T : V \to W$ such that

   - $Tv_1 = 2w_1 + w_3$
   - $Tv_2 = w_2 - w_3$

   Let $v = (3, 4)$ be a vector in V. Compute $Tv \in W$ using the matrix representation method.

1. The matrix representation of T with respect to the bases B_V and B_W is:

$$M(T) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

2. Express $v = (3, 4)$ as a linear combination of the basis vectors in B_V. Let $v = a(1, 2) + b(0, 1)$. This gives us the system of equations:

$$\begin{cases} a = 3 \\ 2a + b = 4 \end{cases}$$

   Solving this system, we get $a = 3$ and $b = -2$. Thus, $v = 3v_1 - 2v_2$.

3. Compute Tv using the matrix representation of T:

$$Tv = M(T) \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 5 \end{pmatrix}$$

   So our answer is $(6, -2, 5)$.

Math 231 — Hw 16
Sara Jamshidi, Mar 20, 2025

In class, we continued to discuss how we to use matrix representations of linear maps between vector spaces. Suppose we define the horrible map you came up with in class, $T : \mathbb{R}^4 \to \mathbb{R}^3$

- $T(1, 2, 3, 4) = (5, 6, 7)$
- $T(11, 10, 9, 8) = (2, 3, 1)$
- $T(1, 5, 7, 2) = (7, 8, 6)$
- $T(0, 0, 0, 1) = (9, 1, 1)$

In this document, we will write down the full method of constructing the representation from $\mathbb{R}^4 \to \mathbb{R}^3$ but beginning from the basis

$$\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$$

to the basis

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

The first step is to define a map from

$$\mathcal{B}_1 = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$$

to

$$\mathcal{B}_2 = \{(1, 2, 3, 4), (11, 10, 9, 8), (1, 5, 7, 2), (0, 0, 0, 1)\}.$$

This map is derived by solving a system of equations that allow us to express each element in $\mathcal{B}_1$ in terms of the elements in $\mathcal{B}_2$.
Let's write out how to solve for first element $(1, 0, 0, 0)$:

- $1 = x + 11y + z + 0w$
- $0 = 2x + 10y + 5z + 0w$
- $0 = 3x + 9y + 7z + 0w$
- $0 = 4x + 8y + 2z + w$

To solve this by hand (which is not ideal), means eliminating variables and using substitution. Omitting this work (which is really better done with a calculator or a computer), we get the answers:

$$x = \frac{25}{24}, y = \frac{1}{24}, z = -\frac{1}{2}, w = -\frac{7}{2}$$

So if we want to build a map $U : \mathbb{R}^4 \to \mathbb{R}^4$, translating between these bases, we will define this part as

- $U(1,0,0,0) = \frac{25}{24}v_1 + \frac{1}{24}v_2 - \frac{1}{2}v_3 - \frac{7}{2}v_4$

We would need to repeat this process three more times to get all the expressions. Once we have these equations, we can construct the matrix representation based on these coefficients we find. That representation ends up being the following[1]:

$$M(U) = \begin{pmatrix} \frac{25}{24} & \frac{-17}{6} & \frac{15}{8} & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{-1}{8} & 0 \\ \frac{-1}{2} & 1 & \frac{-1}{2} & 0 \\ \frac{-7}{2} & 8 & \frac{-11}{2} & 1 \end{pmatrix}$$

Notice the first column matches the work we did!

Now, given our current map, we need to define a matrix representing T. To do this properly (and not embed into $\mathbb{R}^4$ like we did in class), we need to recognize a convenient basis for $\mathbb{R}^3$ based on the image vectors of T. First, we would check that three of them, say $\mathcal{B}_3 = \{(5,6,7),(2,3,1),(7,8,6)\}$ form a basis. Then we would express the remaining vector, $(9,1,1)$, in terms of these three.

- $T(1,2,3,4) = (5,6,7)$
- $T(11,10,9,8) = (2,3,1)$
- $T(1,5,7,2) = (7,8,6)$
- $T(0,0,0,1) = (9,1,1) = -\frac{16}{3}(5,6,7) - \frac{163}{15}(2,3,1) + \frac{41}{5}(7,8,6)$

The resulting matrix representation of T is

$$M(T) = \begin{pmatrix} 1 & 0 & 0 & -\frac{16}{3} \\ 0 & 1 & 0 & -\frac{163}{15} \\ 0 & 0 & 1 & \frac{41}{5} \end{pmatrix}$$

The last map we would want to make converts from this basis $\mathcal{B}_3 = \{(5,6,7),(2,3,1),(7,8,6)\}$ and into the more preferred basis $\mathcal{B}_4 = \{(1,0,0),(0,1,0),(0,0,1)\}$. If we call this map $S : \mathbb{R}^3 \to \mathbb{R}^3$, it will be

- $S(5,6,7) = 5(1,0,0) + 6(0,1,0) + 7(0,0,1)$
- $S(2,3,1) = 2(1,0,0) + 3(0,1,0) + 1(0,0,1)$
- $S(7,8,6) = 7(1,0,0) + 8(0,1,0) + 6(0,0,1)$

The matrix representation of this map (the exercise for this homework) is

$$M(S) = \begin{pmatrix} 5 & 2 & 7 \\ 6 & 3 & 1 \\ 7 & 1 & 6 \end{pmatrix}$$

[1]Note, it would take about two hours to do this by hand–you will not have to do something like this on any exam

The result? If I take an arbitrary vector $v \in \mathbb{R}^4$, I can represent v using basis $\mathcal{B}_1$, which is easy. Then $M(U)v$ will produce the representation of v using the basis $\mathcal{B}_2$. Then $M(T)M(U)v$ will produce the image of T of v as represented with the basis $\mathcal{B}_3$. But when I apply $M(S)$ to $M(T)M(U)v$, I get a representation of v with basis $\mathcal{B}_4$.

For example, suppose I want to know what happens to vector $v = (1, 0, 1, 1) \in \mathbb{R}^4$. The representation of v under basis $\mathcal{B}_1$ is

$$v = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

To find its representation under the basis $\mathcal{B}_2$, we multiply by $M(U)$:

$$\begin{pmatrix} \frac{25}{24} & -\frac{17}{6} & \frac{15}{8} & 0 \\ \frac{1}{24} & \frac{1}{6} & -\frac{1}{8} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{7}{2} & 8 & -\frac{11}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{35}{12} \\ -\frac{1}{12} \\ -1 \\ -8 \end{pmatrix}$$

What this tells us is that $(1, 0, 1, 1) = \frac{35}{12}(1, 2, 3, 4) - \frac{1}{12}(11, 10, 9, 8) - (1, 5, 7, 2) - 8(0, 0, 0, 1)$. Now that we have this new representation of v under $\mathcal{B}_2$, we can apply T using its matrix representation.

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{16}{3} \\ 0 & 1 & 0 & -\frac{163}{15} \\ 0 & 0 & 1 & \frac{41}{5} \end{pmatrix} \begin{pmatrix} \frac{35}{12} \\ -\frac{1}{12} \\ -1 \\ -8 \end{pmatrix} = \begin{pmatrix} \frac{547}{12} \\ \frac{1737}{20} \\ -\frac{333}{5} \end{pmatrix}$$

This is the image of v under T but based on the basis $\mathcal{B}_3$. So this result says

$$Tv = \frac{547}{12}(5, 6, 7) + \frac{1737}{20}(2, 3, 1) - \frac{333}{5}(7, 8, 6).$$

We could do that math or rely on $M(S)$ to do it for us.

$$\begin{pmatrix} 5 & 2 & 7 \\ 6 & 3 & 1 \\ 7 & 1 & 6 \end{pmatrix} \begin{pmatrix} \frac{547}{12} \\ \frac{1737}{20} \\ -\frac{333}{5} \end{pmatrix} = \begin{pmatrix} -\frac{775}{12} \\ \frac{18698}{40} \\ -\frac{76}{12} \end{pmatrix}$$

In other words, the image of $(1, 0, 1, 1)$ under T is

$$\left(-\frac{775}{12}, \frac{18,698}{40}, -\frac{76}{12} \right).$$

Math 231 — Hw 17
Sara Jamshidi, Mar 21, 2025

The null space is the space of all vectors that are sent to 0 by a matrix. For example, the null space of

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$$

is the set of vectors of the form

$$\begin{pmatrix} 2x \\ -x \end{pmatrix}.$$

To demonstrate this, we see that

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

1. Consider the following matrices. What is their null space? Based on their null space, do their column vectors form a basis?

(a)

$$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$$

The null space of this matrix is the set of vectors $\mathbf{v}$ such that:

$$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This simplifies to $3v_1 + 3v_2 = 0$, which implies $v_1 = -v_2$. Therefore, the null space is:

$$\text{null space} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Since the null space is nontrivial, the column vectors do not form a basis.

(b)

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

The null space of this matrix is the set of vectors $\mathbf{v}$ such that:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This simplifies to the system:

$$\begin{cases} v_1 + v_3 = 0 \\ v_2 + v_3 = 0 \\ v_1 + v_2 + 2v_3 = 0 \end{cases}$$

Solving this system, we find $v_1 = v_2 = -v_3$. Therefore, the null space is:

$$\text{null space} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}.$$

Since the null space is nontrivial, the column vectors do not form a basis.

2. If a null space has more than just the 0 vector, we call it "nontrivial." Give the basis of the nontrivial null space of the following matrix

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \end{pmatrix}$$

The null space of this matrix is the set of vectors $\mathbf{v}$ such that:

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This simplifies to the system:

$$\begin{cases} v_1 - v_2 + 2v_3 = 0 \\ 3v_1 - 3v_2 + 6v_3 = 0 \end{cases}$$

The second equation is a multiple of the first, so we only need to solve $v_1 - v_2 + 2v_3 = 0$. Let $v_2 = t$ and $v_3 = s$, then $v_1 = t - 2s$. Therefore, the null space is:

$$\text{null space} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

3. Suppose M is a 3×3 matrix. We said in class that M can be thought of as changing the basis of the matrix. For this reason, the columns of M represent a basis. If the null space is nontrivial, then the vectors don't form a basis of the 3 dimensional vector space. What does that mean about dimensionality of the range of M?

If the null space of M is nontrivial, it means that there exists at least one non-zero vector $\mathbf{v}$ such that $M\mathbf{v} = \mathbf{0}$. This implies that the columns of M are linearly dependent and do not span the entire 3-dimensional space. Therefore, the range (or column space) of M has a dimension less than 3.

4. Based on your answer to the previous question, what does it mean geometrically if a matrix has a nontrivial null space?

Geometrically, if a matrix has a nontrivial null space, it means that the transformation represented by the matrix "collapses" or "squashes" the space into a lower-dimensional subspace. For example, in 3-dimensional space, the range of the matrix would be a plane or a line, rather than filling the entire space.

5. To capture rotations in two dimensions, we can use the following matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Suppose a camera is pointed downward looking at a specimen located at $(2, 3)$. If the camera is rotated by 240 degrees in the positive direction, what is the vector that represents its location with this new orientation?

A rotation by 240 degrees corresponds to $\theta = 240°$. The rotation matrix is:

$$\begin{pmatrix} \cos 240° & -\sin 240° \\ \sin 240° & \cos 240° \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}.$$

Applying this rotation to the vector $(2, 3)$, we get:

$$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1/2 \cdot 2 - \sqrt{3}/2 \cdot 3 \\ \sqrt{3}/2 \cdot 2 - 1/2 \cdot 3 \end{pmatrix} = \begin{pmatrix} -1 - 3\sqrt{3}/2 \\ \sqrt{3} - 3/2 \end{pmatrix}.$$

Thus, the new location vector is:

$$\begin{pmatrix} -1 - 3\sqrt{3}/2 \\ \sqrt{3} - 3/2 \end{pmatrix}.$$

Math 231 — Hw 18
Sara Jamshidi, Mar 31, 2025

1. In class, we discussed the differential operator, D, over the space P_3, the space of polynomials up to degree 3. The differential operator takes polynomials to their derivatives. Solve the following equation: $D(ax^3 + bx^2 + cx + d) =$

   The differential operator D takes a polynomial and returns its derivative. For a polynomial $ax^3 + bx^2 + cx + d$, the derivative is:

$$D(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c$$

2. If $ax^3 + bx^2 + cx + d$ is represented as the column vector

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

   write out $M(D)$. (Hint: Use your previous answer.)

   The polynomial $ax^3 + bx^2 + cx + d$ can be represented as the column vector $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$. The differential operator D can be represented as a matrix $M(D)$ that acts on this vector to produce the coefficients of the derivative. From the derivative $3ax^2 + 2bx + c$, we see that:

$$M(D) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3. In class, we stated that the null D is the space of constant functions. What is the representation of this null space? In other words, what is null $M(D)$?

   The null space of D consists of all polynomials that are mapped to zero by D. These are the constant functions, since the derivative of a constant is zero. Thus, the null space of D is:

$$\text{null } D = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

4. Suppose S is a map that represents a shift in vectors over $\mathbb{R}^3$. $S(a, b, c) = (b, c, 0)$. Describe its null space and give a representation $M(S)$.

   The shift operator S maps (a, b, c) to $(b, c, 0)$. The matrix representation $M(S)$ is:

$$M(S) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

   The null space of S consists of all vectors that are mapped to zero by S. This is the space spanned by $(1, 0, 0)$:

$$\text{null } S = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

5. Now suppose we define a function P that represents a permutation over the vector space $\mathbb{R}^3$. $P(a, b, c) = (b, c, a)$. Describe its null space and give a representation $M(P)$.

   The permutation operator P maps (a, b, c) to (b, c, a). The matrix representation $M(P)$ is:

$$M(P) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

   The null space of P is the zero vector, since P is a bijection and does not map any non-zero vector to zero:

$$\text{null } P = \{\mathbf{0}\}$$

Math 231 — Hw 19
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1. Give an example of a linear map T with $\dim\ \operatorname{null} T = 3$ and $\dim\ \operatorname{range} T = 2$.

   Let's define $T : \mathbb{R}^5 \to W$, where $\dim W \geq 2$. Define the basis of $\mathbb{R}^5$ to be

   $$\{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1)\}.$$

   We define T to work as follows:

   - $T(1,0,0,0,0) = 0_W$
   - $T(0,1,0,0,0) = 0_W$
   - $T(0,0,1,0,0) = 0_W$
   - $T(0,0,0,1,0) = w_1$
   - $T(0,0,0,0,1) = w_2$

   where w_1 and w_2 are linearly independent in W. Then T has a 3 dimensional null space and a 2 dimensional image. Note: there are *many* possible answers this question.