Math 231 — Hw 4
Sara Jamshidi, Jan 23, 2025

1. Show that $V = \mathbb{Z}_5^2$ is a vector space (with addition being modulo 5 and scalar multiplication also being modulo 5).

While the table approach is tempting for this problem, you will quickly realize that we have 25 elements–so it gets quite large. If you are feeling lazy like me, we're going to go the abstract route!

V is closed under addition.

Proof. Let $(a_1, b_1), (a_2, b_2) \in V$. Then $(a_1, b_1) +_5 (a_2, b_2) = (a_1 + a_2 \mod 5, b_1 + b_2 \mod 5)$. Because $\mathbb{Z}_5$ is closed under addition mod 5, this vector is in V. Hence, V is closed under addition. □

V is closed under scalar multiplication.

Proof. Let $\lambda \in \mathbb{Z}_5$ and $(a, b) \in V$. Then $\lambda(a, b) = (\lambda \cdot a \mod 5, \lambda \cdot b \mod 5)$. Because $\mathbb{Z}_5$ is closed under multiplication mod 5, this vector is in V. Hence, V is closed under scalar multiplication. □

V satsifies commutativity.

Proof. Because addition modulo 5 over $\mathbb{Z}_5$ is commutative, it follows that

$$\begin{aligned}(a_1, b_1) +_5 (a_2, b_2) &= (a_1 + a_2 \mod 5, b_1 + b_2 \mod 5) \text{ (by vector addition)}\\ &= (a_2 + a_1 \mod 5, b_2 + b_1 \mod 5) \text{ (by commutativity of the field)}\\ &= (a_2, b_2) +_5 (a_1, b_1) \text{ (by vector addition)}\end{aligned}$$

for any choice of $(a_1, b_1), (a_2, b_2) \in V$. □

V satsifies associativity.

Proof. Because addition modulo 5 is associative, it follows that so is vector addition. Details are omitted here, but the format is similar to proving commutativity. □

V has an additive identity.

Proof. Consider $(0, 0) \in \mathbb{Z}_5^2$. Observe that for any $(a, b) \in \mathbb{Z}_5^2$, we have that

$$\begin{aligned}(0, 0) +_5 (a, b) &= (0 + a \mod 5, 0 + b \mod 5) \text{ (by vector addition)}\\ &= (a, b) \text{ (by additive identity of the field)}\end{aligned}$$

This, coupled with commutativity, demonstrate that $(0, 0)$ is an additive identity of the vector space. □

V is closed under additive inverses.

Proof. Let $(a, b) \in \mathbb{Z}_5^2$. Since $\mathbb{Z}_5^2$ is a field, there exists elements c, d in the field such that $a + c \mod 5 = 0$ and $b + d \mod 5 = 0$. It follows that $(a, b) +_5 (c, b) = (0, 0)$. Hence V is closed under additive inverses. $\square$

V has a corresponding scalar identity.

Proof. Consider $1 \in \mathbb{Z}_5$. For any $(a, b) \in V$, $1(a, b) = (1a, 1b) = (a, b)$ by scalar multiplication and by the fact that 1 is the multiplicative identity of $\mathbb{Z}_5$. Hence 1 is the scalar identity of V. $\square$

V satisfies the distributive properties.

Proof. Let $\alpha, \beta \in \mathbb{Z}_5$ and $(a, b), (c, d) \in V$.

$$\begin{aligned}
\alpha((a,b) +_5 (c,d)) &= \alpha(a + c \mod 5, b + d \mod 5) \text{ (by vector addition)}\\
&= (\alpha(a+c) \mod 5, \alpha(b+d) \mod 5) \text{ (by scalar multiplication)}\\
&= (\alpha a + \alpha c) \mod 5, \alpha b + \alpha d \mod 5) \text{ (by distributive property of the field)}\\
&= (\alpha a, \alpha b) +_5 (\alpha c, \alpha d) \text{ (by vector addition)}\\
&= \alpha(a,b) +_5 \alpha(c,d) \text{ (by scalar multiplication)}
\end{aligned}$$

Hence the first distributive property holds. We can do something similar for the second.

$$\begin{aligned}
(\alpha+\beta)(a,b) &= ((\alpha+\beta)a \mod 5, (\alpha+\beta)b \mod 5) \text{ (by scalar multiplication)}\\
&= (\alpha a + \beta a \mod 5, \alpha b + \beta b \mod 5) \text{ (by distributive property of the field)}\\
&= (\alpha a \mod 5, \alpha b \mod 5) +_5 (\beta a \mod 5, \beta b \mod 5) \text{ (by vector addition)}\\
&= \alpha(a,b) +_5 \beta(a,b) \text{ (by scalar multiplication)}.
\end{aligned}$$

Hence the second distributive property holds as well. $\square$

2. Let $P^3 = \{ax^3 + bx^2 + cx + d \mid a, b, c, d \in \mathbb{R}\}$ be the space of polynomials up to degree 3 over the field $\mathbb{R}$. Prove that P^3 is a vector space. (In other words, show that vector addition and scalar multiplication is closed. Then, do your best to show the other properties hold: associativity, additive identity, additive inverse, multiplicative identity, and distributivity.)

   To get you started, here is the proof of additive identity and the *start* of the proof for additive inverses:

   P^3 has an additive identity.

Proof. Consider $0x^3 + 0x^2 + 0x + 0$, which we will write as 0_P and call the "zero polynomial." Observe that $0_P \in P^3$ since $0 \in \mathbb{R}$. In addition, for any choice of $ax^3 + bx^2 + cx + d \in P^3$,

$$\begin{aligned}\big(ax^3 + bx^2 + cx + d\big) + 0_P &= (a+0)x^3 + (b+0)x^2 + (c+0)x + (d+0) \quad \text{(by vector addition)}\\ &= ax^3 + bx^2 + cx + d \quad \text{(by additive identity of the field)}\end{aligned}$$

Hence 0_P is the additive identity of P^3. □

P^3 is closed under additive inverses.

Proof. Let $ax^3 + bx^2 + cx + d \in P^3$ be an arbitrary element in the space. Because $\mathbb{R}$ is a field, there exists $-a, -b, -c \in \mathbb{R}$ such that $a + (-a) = 0$, $b + (-b) = 0$, and $c + (-c) = 0$. $\cdots$ finish the argument □

P^3 is closed under addition.

Proof. Let $p(x) = a_1x^3 + b_1x^2 + c_1x + d_1 \in P^3$ and $q(x) = a_2x^3 + b_2x^2 + c_2x + d_2 \in P^3$. Then:

$$p(x) + q(x) = (a_1 + a_2)x^3 + (b_1 + b_2)x^2 + (c_1 + c_2)x + (d_1 + d_2).$$

Since $a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2 \in \mathbb{R}$, we have $p(x) + q(x) \in P^3$. Therefore, P^3 is closed under addition. □

P^3 is closed under scalar multiplication.

Proof. Let $c \in \mathbb{R}$ and $p(x) = ax^3 + bx^2 + cx + d \in P^3$. Then:

$$cp(x) = (ca)x^3 + (cb)x^2 + (cc)x + (cd).$$

Since $ca, cb, cc, cd \in \mathbb{R}$, $cp(x) \in P^3$. Therefore, P^3 is closed under scalar multiplication. □

P^3 satisfies associativity of addition.

Proof. For $p(x), q(x), r(x) \in P^3$:

$$(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x)),$$

because addition of coefficients in $\mathbb{R}$ is associative. Therefore, addition in P^3 is associative. □

P^3 has an additive identity.

Proof. As shown earlier, the zero polynomial $0_P(x) = 0x^3 + 0x^2 + 0x + 0$ is the additive identity because for any $p(x) \in P^3$:

$$p(x) + 0_P(x) = p(x).$$

$\square$

P^3 is closed under additive inverses.

Proof. Let $p(x) = ax^3+bx^2+cx+d \in P^3$. Since $\mathbb{R}$ is a field, there exist $-a, -b, -c, -d \in \mathbb{R}$ such that:

$$a + (-a) = 0, \quad b + (-b) = 0, \quad c + (-c) = 0, \quad d + (-d) = 0.$$

Define the additive inverse of $p(x)$ as $-p(x) = -ax^3 - bx^2 - cx - d$. Then:

$$p(x) + (-p(x)) = (a + (-a))x^3 + (b + (-b))x^2 + (c + (-c))x + (d + (-d)) = 0_P(x).$$

Thus, every element of P^3 has an additive inverse in P^3. $\square$

P^3 satisfies the scalar identity.

Proof. Let $p(x) = ax^3 + bx^2 + cx + d \in P^3$. For the scalar identity $1 \in \mathbb{R}$:

$$1p(x) = (1a)x^3 + (1b)x^2 + (1c)x + (1d) = ax^3 + bx^2 + cx + d = p(x).$$

Thus, the scalar identity holds in P^3. $\square$

P^3 satisfies distributive properties.

Proof. Let $\alpha, \beta \in \mathbb{R}$ and $p(x), q(x) \in P^3$.

$$\begin{aligned}\alpha(p(x) + q(x)) &= \alpha\big((a_1 + a_2)x^3 + (b_1 + b_2)x^2 + (c_1 + c_2)x + (d_1 + d_2)\big),\\ &= (\alpha(a_1 + a_2))x^3 + (\alpha(b_1 + b_2))x^2 + (\alpha(c_1 + c_2))x + (\alpha(d_1 + d_2)).\\ &= (\alpha a_1 + \alpha a_2)x^3 + (\alpha b_1 + \alpha b_2)x^2 + (\alpha c_1 + \alpha c_2)x + (\alpha d_1 + \alpha d_2),\\ &= \big(\alpha p(x)\big) + \big(\alpha q(x)\big).\end{aligned}$$

Now, let's show the second distributive property:

$$\begin{aligned}(\alpha + \beta)p(x) &= \big((\alpha + \beta)a\big)x^3 + \big((\alpha + \beta)b\big)x^2 + \big((\alpha + \beta)c\big)x + \big((\alpha + \beta)d\big)\\ &= (\alpha a + \beta a)x^3 + (\alpha b + \beta b)x^2 + (\alpha c + \beta c)x + (\alpha d + \beta d),\end{aligned}$$
$$= \alpha p(x) + \beta p(x).$$

Thus, P^3 satisfies both distributive properties. I leave it as an exercise to you to identify the justification of each piece. Please see problem 1 if you get stuck. $\square$