Math 231 — Hw 1
Sara Jamshidi, Jan 15, 2025

1. **Floating-point** numbers are a way to represent real numbers in computers using a *finite* number of bits, expressed in scientific notation as a combination of a sign, a significand, and an exponent. This format allows representation of a wide range of values but introduces rounding errors and precision limits, leading to potential inaccuracies in arithmetic operations. As was stated in class, this space of numbers are not associative.

   Within your python terminal, demonstrate that floating point numbers are not associative using the following values: $a = 1.0, b = 10^8, c = -10^8$.

   Solution: For this problem to work, you may have had to increase the exponent. Typically, 8 bits are used for the exponent of the number and one bit is used for the sign. The remaining is used for the significand in scientific notation. Due to limited precision, when we add $a + b = 1.0 + 10^8 = 10^8$. Why? Over the real numbers, the sum should be 10000001. In binary, this number requires 24 bits. If your system does not have enough memory allocated for this, then it would simply store it as 1.0×10^8. Once c is added, the answer becomes 0.0.

   If we switch the order, $a + (b + c) = 1.0 + 0.0 = 1.0$. Since $1.0 \neq 0.0$, floating point numbers are not associative.

2. The example given in class used $\{1, 2, 3\}$ with $a \circ b = \max\{a, b\}$ and $a * b = \min\{a, b\}$. Demonstrate why 2 has no inverse under both operations (you can do this exhaustively).

   Solution: For $\circ$ the identity is 1. Observe that

   (a) $2 \circ 1 = 2$
   (b) $2 \circ 2 = 2$
   (c) $2 \circ 3 = 3$

   None of these values is the identity.

   For $*$ the identity is 3. Observe that

   (a) $2 \circ 1 = 1$
   (b) $2 \circ 2 = 2$
   (c) $2 \circ 3 = 2$

   None of these values is the identity.

   Hence, 2 has no inverse under either operation.

3. Consider the space with $\{0,1,2\}$ with operation $a \circ b = (a+b) \mod 3$. What axioms are satisfied? (You can skip the distributive property since there is only one operation.)

First, let's verify that the set is closed under the operation. We do this usually with an operation table. This shows us all possible combinations and the results should all be the set. If there exists a result that is not in the set, then the set is not closed under the operation.

$\circ$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

So the set is closed under the operation.

The axioms satsified by this operation are associativity, commutativity, identity, and inverses. Using the table above, the easiest to demonstrate is the identity. We clearly see 0 is the identity of this operation. Using the table again, we can identiy the inverses of each element:

(a) the inverse of 0 is 0,

(b) the inverse of 1 is 2, and

(c) the inverse of 2 is 1.

Although we didn't need to prove these, let's prove the last two (just to show what it would look like).

Commutativity: First, let's observe that for any choice of a and b in our set, we know that a and b are also in $\mathbb{Z}$. If $a, b \in \mathbb{Z}$, $a + b = 3k + r$ where $r \in \{0, 1, 2\}$, so $a + b \equiv r \mod 3$. Because addition is commutative over $\mathbb{Z}$, the same is true for $b + a$. Hence $a \circ b = b \circ a$.

Associativity: Now let's show associativity. Notice that as elements of $\mathbb{Z}$,

$$a + b + c = 3x + r$$

for some $x \in \mathbb{Z}$ and $r \in \{0, 1, 2\}$. Associativity holds for $\mathbb{Z}$, meaning

$$(a + b) + c = a + (b + c).$$

This also means that $(a + b) + c \mod 3 = a + (b + c) \mod 3 = r$. To show that our operation is associative, we need to show that

$$(a + b) + c \mod 3 = (a + b \mod 3) + c \mod 3$$

because the portion to the right of the equals sign corresponds to $(a \circ b) \circ c)$–our target operation. Since $a, b, c \in \{0, 1, 2\}$, $a + b = 3y + s$ where $y \leq x$. And it must be true that $3y + s + c = 3x + r$. Modulo 3, this equation shows that $s + c \equiv r$ and since $s = a + b \mod 3$, we have that

$$(a + b) + c \mod 3 = (a + b \mod 3) + c \mod 3.$$

The argument to show

$$a + (b + c) \mod 3 = a + (b + c \mod 3) \mod 3$$

is exactly the same and therefore omitted. Thus, we have that

$$(a \circ b) \circ c = (a + b) + c \mod 3 = a + (b + c) \mod 3 = a \circ (b \circ c).$$

Math 231 — Hw 2
Sara Jamshidi, Jan 16, 2025

1. Today we looked at an example of a finite field, a field with finitely many objects: $\mathbb{Z}_p$. Such structures are always fields when p is a prime number. For $\mathbb{Z}_5$, find all the additive and multiplicative inverse of the elements in the field: $\{0, 1, 2, 3, 4\}$. (Note that 0 will have no multiplicative inverse.)

   A table would illustrate the elements well, but here, I will just list them:

   0: The additive inverse is 0. There is no multiplicative inverse for the additive identity.

   1: The additive inverse is 4: $1 + 4 \mod 5 \equiv 0$. The multaplicative inverse is itself.

   2: The additive inverse is 3: $2 + 3 \mod 5 \equiv 0$. The multaplicative inverse is 3: $2 \times 3 = 6 \equiv_5 1$

   3: The additive inverse is 2: $3 + 2 \mod 5 \equiv 0$. The multaplicative inverse is 2: $3 \times 2 = 6 \equiv_5 1$

   4: The additive inverse is 1: $4 + 1 \mod 5 \equiv 0$. The multaplicative inverse is itself: $4 \times 4 = 16 \equiv_5 1$

2. For finite fields, p must be a prime number. To illustrate why $\mathbb{Z}_4$ is not a field, construct its multiplication table.

   Recall that a multiplication table is a table where the header row and first column list the elements of the set, and each cell contains the product of the corresponding row and column elements.

   In our solution below, we see that both 2 and 3 lack multaplicative inverses, so $\mathbb{Z}_4$ could not form a field.

$\times_p$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	3

Math 231 — Hw 3
Sara Jamshidi, Jan 17, 2025

1. Let $S = \{(x,y) \in \mathbb{R}^2 \mid x + y = 1\}$ be a space defined over the field $\mathbb{R}$ with addition defined as
$$(a,b) + (c,d) = (a+c, b+d)$$
and scalar multiplication as $x(a,b) = (xa, xb)$ where $x \in \mathbb{R}$ and $(a,b) \in S$. Show why this is **not** a vector space.

   The main problem here is that our operations take us outside the set. We would say that "the set is not closed under these operations." For example, $(1,0)+(0,1) = (1,1)$ but $(1,1) \notin S$. since $1+1 = 2 \neq 1$.

2. Let $U = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$, with vector addition and scalar multiplication defined as the previous case. Show why this is **not** a vector space.

   The main problem here is that we have no additive inverses. The additive identity for this space is $(0,0)$. But there is no element in U that I can add to $(1,1)$ that gives me this additive identity back. This is because both entries of my vector have to be 0 or larger.

3. Define a set $W = \mathbb{R}^2$ with addition defined as $(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$ and scalar multiplication defined as $c \cdot (x,y) = (cx, y)$. Show why this is **not** a vector space.

   As you might guess, the issue is with scalar multiplication. The problem arises with the distributive property as discussed in the textbook: $(c_1+c_2)\cdot(x,y) \neq c_1 \cdot (x,y) + c_2 \cdot (x,y)$. The left-hand side yields $((c_1+c_2)x, y)$ while the right-hand size produces $(c_1x, y) + (c_2x, y) = ((c_1+c_2)x, 2y)$.

4. Let $X = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0\}$, with vector addition and scalar multiplication defined as usual. Show why this is **not** a vector space.

   This was a mistake–this turns out to be a vector space! Let's show that the space is closed under addition: For $(u_1,u_2,u_3),(v_1,v_2,v_3) \in X$, we know:
$$u_1+u_2+u_3 = 0 \quad \text{and} \quad v_1+v_2+v_3 = 0.$$
   Adding these equations, we see the sum is still 0, ensuring the sum of the vectors is in X:
$$(u_1+v_1)+(u_2+v_2)+(u_3+v_3) = 0.$$
   So X is closed under addition. Now let's show closure under scalar multiplication. For $(x,y,z) \in X$ and $c \in \mathbb{R}$
$$c \cdot (x,y,z) = (cx, cy, cz).$$

Observe that

$$cx + cy + cz = c(x + y + z) = c(0) = 0.$$

So X is closed under scalar multiplication. It is clear that we have the zero vector in the space, so there is an additive identity. Moreover is $(x, y, z) \in X$ then so must be $(-1)(x, y, z)$ by the closure of scalar multiplication. Hence we are closed under additive inverses as well. Finally, all standard operations (associativity of addition, distributivity of scalar multiplication, etc.) are inherited from $\mathbb{R}^3$. These properties hold because the operations are defined in the usual way. So this is a vector space.

Math 231 — Hw 4
Sara Jamshidi, Jan 23, 2025

1. Show that $V = \mathbb{Z}_5^2$ is a vector space (with addition being modulo 5 and scalar multiplication also being modulo 5).

While the table approach is tempting for this problem, you will quickly realize that we have 25 elements–so it gets quite large. If you are feeling lazy like me, we're going to go the abstract route!

V is closed under addition.

Proof. Let $(a_1, b_1), (a_2, b_2) \in V$. Then $(a_1, b_1) +_5 (a_2, b_2) = (a_1 + a_2 \mod 5, b_1 + b_2 \mod 5)$. Because $\mathbb{Z}_5$ is closed under addition mod 5, this vector is in V. Hence, V is closed under addition. □

V is closed under scalar multiplication.

Proof. Let $\lambda \in \mathbb{Z}_5$ and $(a, b) \in V$. Then $\lambda(a, b) = (\lambda \cdot a \mod 5, \lambda \cdot b \mod 5)$. Because $\mathbb{Z}_5$ is closed under multiplication mod 5, this vector is in V. Hence, V is closed under scalar multiplication. □

V satsifies commutativity.

Proof. Because addition modulo 5 over $\mathbb{Z}_5$ is commutative, it follows that

$$\begin{aligned}(a_1, b_1) +_5 (a_2, b_2) &= (a_1 + a_2 \mod 5, b_1 + b_2 \mod 5) \text{ (by vector addition)} \\ &= (a_2 + a_1 \mod 5, b_2 + b_1 \mod 5) \text{ (by commutativity of the field)} \\ &= (a_2, b_2) +_5 (a_1, b_1) \text{ (by vector addition)}\end{aligned}$$

for any choice of $(a_1, b_1), (a_2, b_2) \in V$. □

V satsifies associativity.

Proof. Because addition modulo 5 is associative, it follows that so is vector addition. Details are omitted here, but the format is similar to proving commutativity. □

V has an additive identity.

Proof. Consider $(0, 0) \in \mathbb{Z}_5^2$. Observe that for any $(a, b) \in \mathbb{Z}_5^2$, we have that

$$\begin{aligned}(0, 0) +_5 (a, b) &= (0 + a \mod 5, 0 + b \mod 5) \text{ (by vector addition)} \\ &= (a, b) \text{ (by additive identity of the field)}\end{aligned}$$

This, coupled with commutativity, demonstrate that $(0, 0)$ is an additive identity of the vector space. □

V is closed under additive inverses.

Proof. Let $(a, b) \in \mathbb{Z}_5^2$. Since $\mathbb{Z}_5^2$ is a field, there exists elements c, d in the field such that $a + c \mod 5 = 0$ and $b + d \mod 5 = 0$. It follows that $(a, b) +_5 (c, b) = (0, 0)$. Hence V is closed under additive inverses. $\square$

V has a corresponding scalar identity.

Proof. Consider $1 \in \mathbb{Z}_5$. For any $(a, b) \in V$, $1(a, b) = (1a, 1b) = (a, b)$ by scalar multiplication and by the fact that 1 is the multiplicative identity of $\mathbb{Z}_5$. Hence 1 is the scalar identity of V. $\square$

V satisfies the distributive properties.

Proof. Let $\alpha, \beta \in \mathbb{Z}_5$ and $(a, b), (c, d) \in V$.

$$\begin{aligned}
\alpha((a, b) +_5 (c, d)) &= \alpha(a + c \mod 5, b + d \mod 5) \text{ (by vector addition)} \\
&= (\alpha(a + c) \mod 5, \alpha(b + d) \mod 5) \text{ (by scalar multiplication)} \\
&= (\alpha a + \alpha c) \mod 5, \alpha b + \alpha d \mod 5) \text{ (by distributive property of the field)} \\
&= (\alpha a, \alpha b) +_5 (\alpha c, \alpha d) \text{ (by vector addition)} \\
&= \alpha(a, b) +_5 \alpha(c, d) \text{ (by scalar multiplication)}
\end{aligned}$$

Hence the first distributive property holds. We can do something similar for the second.

$$\begin{aligned}
(\alpha + \beta)(a, b) &= ((\alpha + \beta)a \mod 5, (\alpha + \beta)b \mod 5) \text{ (by scalar multiplication)} \\
&= (\alpha a + \beta a \mod 5, \alpha b + \beta b \mod 5) \text{ (by distributive property of the field)} \\
&= (\alpha a \mod 5, \alpha b \mod 5) +_5 (\beta a \mod 5, \beta b \mod 5) \text{ (by vector addition)} \\
&= \alpha(a, b) +_5 \beta(a, b) \text{ (by scalar multiplication).}
\end{aligned}$$

Hence the second distributive property holds as well. $\square$

2. Let $P^3 = \{ax^3 + bx^2 + cx + d \mid a, b, c, d \in \mathbb{R}\}$ be the space of polynomials up to degree 3 over the field $\mathbb{R}$. Prove that P^3 is a vector space. (In other words, show that vector addition and scalar multiplication is closed. Then, do your best to show the other properties hold: associativity, additive identity, additive inverse, multiplicative identity, and distributivity.)

   To get you started, here is the proof of additive identity and the *start* of the proof for additive inverses:

   P^3 has an additive identity.

Proof. Consider $0x^3 + 0x^2 + 0x + 0$, which we will write as 0_P and call the "zero polynomial." Observe that $0_P \in P^3$ since $0 \in \mathbb{R}$. In addition, for any choice of $ax^3 + bx^2 + cx + d \in P^3$,

$$\begin{aligned}\left(ax^3 + bx^2 + cx + d\right) + 0_P &= (a+0)x^3 + (b+0)x^2 + (c+0)x + (d+0) \quad \text{(by vector addition)}\\ &= ax^3 + bx^2 + cx + d \quad \text{(by additive identity of the field)}\end{aligned}$$

Hence 0_P is the additive identity of P^3. □

P^3 is closed under additive inverses.

Proof. Let $ax^3 + bx^2 + cx + d \in P^3$ be an arbitrary element in the space. Because $\mathbb{R}$ is a field, there exists $-a, -b, -c \in \mathbb{R}$ such that $a + (-a) = 0$, $b + (-b) = 0$, and $c + (-c) = 0$. $\cdots$ finish the argument □

P^3 is closed under addition.

Proof. Let $p(x) = a_1x^3 + b_1x^2 + c_1x + d_1 \in P^3$ and $q(x) = a_2x^3 + b_2x^2 + c_2x + d_2 \in P^3$. Then:

$$p(x) + q(x) = (a_1 + a_2)x^3 + (b_1 + b_2)x^2 + (c_1 + c_2)x + (d_1 + d_2).$$

Since $a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2 \in \mathbb{R}$, we have $p(x) + q(x) \in P^3$. Therefore, P^3 is closed under addition. □

P^3 is closed under scalar multiplication.

Proof. Let $c \in \mathbb{R}$ and $p(x) = ax^3 + bx^2 + cx + d \in P^3$. Then:

$$cp(x) = (ca)x^3 + (cb)x^2 + (cc)x + (cd).$$

Since $ca, cb, cc, cd \in \mathbb{R}$, $cp(x) \in P^3$. Therefore, P^3 is closed under scalar multiplication. □

P^3 satisfies associativity of addition.

Proof. For $p(x), q(x), r(x) \in P^3$:

$$(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x)),$$

because addition of coefficients in $\mathbb{R}$ is associative. Therefore, addition in P^3 is associative. □

P^3 has an additive identity.

Proof. As shown earlier, the zero polynomial $0_P(x) = 0x^3 + 0x^2 + 0x + 0$ is the additive identity because for any $p(x) \in P^3$:

$$p(x) + 0_P(x) = p(x).$$

$\square$

P^3 is closed under additive inverses.

Proof. Let $p(x) = ax^3 + bx^2 + cx + d \in P^3$. Since $\mathbb{R}$ is a field, there exist $-a, -b, -c, -d \in \mathbb{R}$ such that:

$$a + (-a) = 0, \quad b + (-b) = 0, \quad c + (-c) = 0, \quad d + (-d) = 0.$$

Define the additive inverse of $p(x)$ as $-p(x) = -ax^3 - bx^2 - cx - d$. Then:

$$p(x) + (-p(x)) = (a + (-a))x^3 + (b + (-b))x^2 + (c + (-c))x + (d + (-d)) = 0_P(x).$$

Thus, every element of P^3 has an additive inverse in P^3. $\square$

P^3 satisfies the scalar identity.

Proof. Let $p(x) = ax^3 + bx^2 + cx + d \in P^3$. For the scalar identity $1 \in \mathbb{R}$:

$$1p(x) = (1a)x^3 + (1b)x^2 + (1c)x + (1d) = ax^3 + bx^2 + cx + d = p(x).$$

Thus, the scalar identity holds in P^3. $\square$

P^3 satisfies distributive properties.

Proof. Let $\alpha, \beta \in \mathbb{R}$ and $p(x), q(x) \in P^3$.

$$\begin{aligned}
\alpha(p(x) + q(x)) &= \alpha\big((a_1 + a_2)x^3 + (b_1 + b_2)x^2 + (c_1 + c_2)x + (d_1 + d_2)\big), \\
&= (\alpha(a_1 + a_2))x^3 + (\alpha(b_1 + b_2))x^2 + (\alpha(c_1 + c_2))x + (\alpha(d_1 + d_2)). \\
&= (\alpha a_1 + \alpha a_2)x^3 + (\alpha b_1 + \alpha b_2)x^2 + (\alpha c_1 + \alpha c_2)x + (\alpha d_1 + \alpha d_2), \\
&= \big(\alpha p(x)\big) + \big(\alpha q(x)\big).
\end{aligned}$$

Now, let's show the second distributive property:

$$\begin{aligned}
(\alpha + \beta)p(x) &= \big((\alpha + \beta)a\big)x^3 + \big((\alpha + \beta)b\big)x^2 + \big((\alpha + \beta)c\big)x + \big((\alpha + \beta)d\big) \\
&= (\alpha a + \beta a)x^3 + (\alpha b + \beta b)x^2 + (\alpha c + \beta c)x + (\alpha d + \beta d), \\
&= \alpha p(x) + \beta p(x).
\end{aligned}$$

Thus, P^3 satisfies both distributive properties. I leave it as an exercise to you to identify the justification of each piece. Please see problem 1 if you get stuck. $\square$

Math 231 — Hw 5
Sara Jamshidi, Jan 24, 2025

1. Prove that for any $v \in V$, $-(-v) = v$.

   (Hint: Use Thm 1.32).

   Proof. By Theorem 1.32, $-v$ is the additive inverse of v. Similarly $-(-v)$ is the additive inverse of $-v$. Because v and $-(-v)$ are additive inverses of v, it must be the case that $-(-v) = v$ by theorem 1.27. $\square$

2. Let $a \in \mathbb{F}$ and $v \in V$. If $av = 0$, show that either $a = 0$ or $v = 0$.

   (Note: 0 is being used both for the additive identity of the field element and the additive identity of the vector space. This is an "abuse of notation," but you should be able to tell which is which.)

   Theorems 1.30 and 1.31 tell us that if $a = 0$ or $v = 0$, then $av = 0$, but we seek to show the opposite statement. Here, we will prove the contrapositive: if $a \neq 0$ and $v \neq 0$, then $av \neq 0$.

   Proof. Let $a \in \mathbb{F}$ and $v \in V$, neither equal to the additive identities of their respective spaces. Suppose for the sake of contradiction that $av = 0$, the additive identity of V. Since $a \neq 0$ and it is an element of a field, there exists an $a^{-1} \in \mathbb{F}$, the multiplicative inverse of a. It follows that
   $$a^{-1}av = 1v = v.$$
   This implies that $v = 0$, which is a contradiction. Hence $av \neq 0$. $\square$

3. Suppose $-1 \notin \mathbb{F}$. Prove that there exists an element $\lambda \in \mathbb{F}$ such that for any $v \in V$, $v + \lambda v = 0$.

   Proof. Because $\mathbb{F}$ is a field, $1 \in \mathbb{F}$ has an additive inverse. Since $-1 \notin \mathbb{F}$, define the inverse to be λ. Then it follows that $1 + \lambda = 0$. As a result,
   $$0 = 0v = (1 + \lambda)v = v + \lambda v.$$
   Hence $v + \lambda v = 0$. $\square$

Math 231 — Hw 6
Sara Jamshidi, Jan 31, 2025

1. Prove or disprove if this is a vector space using theorem 1.34 from the textbook:

$$W = \{(x_1, x_2, x_3) \mid x_1 x_2 x_3 = 0, x_i \in \mathbb{R}\}.$$

This is NOT a vector space.

Proof. Let's call this set W and notice $W \subseteq \mathbb{R}^3$. Consider the vectors $v = (1, 0, 2)$ and $w = (0, 3, 1)$, which are both elements of W. Notice that:

$$v + w = (1, 3, 3) \notin W$$

since $1 \cdot 3 \cdot 3 = 9 \neq 0$. Since W is not closed under addition, it is not a subspace and therefore not a vector space. $\square$

2. Construct an example of a vector space W with two subspaces, W_1, W_2 where $W_1 + W_2 \neq W$.

Consider $W = \mathbb{R}^3$. Define:

$$W_1 = \{(x, 0, 0) \mid x \in \mathbb{R}\}, \quad W_2 = \{(0, x, 0) \mid x \in \mathbb{R}\}.$$

The definition of $W_1 + W_2$ is the set of all sums of vectors from these sets. If $v \in W_1$, then $v = (v_1, 0, 0)$ and if $w \in W_2$, then $w = (0, w_2, 0)$. Thus, elements in $W_1 + W_2$ are of the form:

$$(v_1, w_2, 0).$$

However, the vector $(0, 0, 1) \in \mathbb{R}^3$ is not in $W_1 + W_2$ because it does not match that form. Hence, $W_1 + W_2 \neq W$.

3. Let $V = \mathbb{R}^3$, and define two subspaces:

- $V_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$
- $V_2 = \{(0, y, z) \mid y, z \in \mathbb{R}\}$

Prove that $V_1 + V_2$ forms a subspace of V.

Proof. Let $v_1 = (x_1, y_1, 0) \in V_1$ and $v_2 = (0, y_2, z_2) \in V_2$. Then their sum is:

$$v_1 + v_2 = (x_1, y_1, 0) + (0, y_2, z_2) = (x_1, y_1 + y_2, z_2).$$

Since $x_1, y_1, y_2, z_2 \in \mathbb{R}$, we see that $V_1 + V_2$ consists of all vectors of the form (x, y, z), which shows that it is closed under addition and it contains the additive identity $(0, 0, 0)$.
Now, suppose $v \in V_1 + V_2$ and let $c \in \mathbb{R}$. Then:

$$\begin{aligned} cv &= c(v_1 + v_2) \quad \text{for some } v_i \in V_i \\ &= cv_1 + cv_2 \end{aligned}$$

As is stated in the problem V_1 and V_2 are subspaces, so $cv_1 \in V_1$ and $cv_2 \in V_2$. Hence $cv \in V_1 + V_2$. Thus, by theorem 1.34 in the textbook, $V_1 + V_2$ is a subspace. □

4. Prove that $V_1 + V_2 = V$ in the previous problem.

Proof. By the previous problem, it follows that $V_1 + V_2 \subset V$ and is a vector space. Let $(a, b, c) \in V = \mathbb{R}^3$. We can write:

$$(a, b, c) = (a, b, 0) + (0, 0, c).$$

The first vector $(a, b, 0)$ belongs to V_1, and the second vector $(0, 0, c)$ belongs to V_2, since we can choose $y = 0$ in V_2. Since every vector in V can be expressed as a sum of elements in V_1 and V_2, we conclude that $V \subset V_1 + V_2$. Because $V_1 + V_2 \subset V$ and $V \subset V_1 + V_2$, it follows that $V = V_1 + V_2$. □

5. Let $V = \mathbb{R}^3$, and define two subspaces:

- $V_1 = \{(x, y, 0) \mid x + y = 0, x, y \in \mathbb{R}\}$
- $V_2 = \{(0, y, z) \mid y + z = 0, y, z \in \mathbb{R}\}$

Prove or provide a counterexample to the statement: $V_1 + V_2 = V$.

The statement is false.

Proof. Any vector in $V_1 + V_2$, can be expressed as a sum of vectors $v_1 \in V_1$ and $v_2 \in V_2$. Because the two entries of v_1 must add to zero and the same is true for v_2, we can write them as $(x, -x, 0)$ and $(0, y, -y)$, where $-x$ represents the additive inverse of x and similarly for $-y$ and y. Then $v_1 + v_2 = (x, y - x, -y)$, so it *automatically* follows the sum of the three entries of this vector is zero in $\mathbb{R}$. Observe however that $(1, 1, 1) \in V$ but **lacks** this property. Hence these two vector spaces cannot be equal. □

Math 231 — Hw 7
Sara Jamshidi, Feb 3, 2025

1. Determine whether the following set is a vector space. Justify your answer using theorem 1.34 from the textbook:

$$W = \{(x, y, z) \mid x - (y + 1) + 2(z + 1) = 1,\ x, y, z \in \mathbb{R}\}.$$

Solution: First, we can rewrite the given equation as

$$x - y + 2z = 0.$$

Studying this question, we need to assess if the additive identity satisfies this rule and if the rule allows for closure of both operations. Thinking it through, it looks like it does. So this is likely a vector space. let's prove it.

Proof. It is clear that $W \subset \mathbb{R}^3$. First observe that $(0, 0, 0) \in W$ since $0-(0+1)+2(0+1) = -1 + 2 = 1$. Next, consider $v, w \in W$. If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$, then

$$v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3).$$

To see if the sum is in W, we need to determining if the resulting sum satisfies the rule defining the space.

$$\begin{aligned}
&v_1 + w_1 - (v_2 + w_2 + 1) + 2(v_3 + w_3 + 1)\\
&= v_1 + w_1 - (v_2 + w_2 + 2 - 1) + 2(v_3 + w_3 + 2 - 1)\\
&= v_1 + w_1 - (v_2 + 1) - (w_2 + 1) + 1 + 2(v_3 + 1) + 2(w_3 + 1) - 2\\
&= v_1 - (v_2 + 1) + 2(v_3 + 1) + w_1 - (w_2 + 1) + 2(w_3 + 1) - 1\\
&= 1 + 1 - 1 = 1
\end{aligned}$$

Therefore W is closed under vector addition. Let $\lambda \in \mathbb{R}$ and $v \in W$. If $v = (v_1, v_2, v_3)$, then $\lambda v = (\lambda v_1, \lambda v_2, \lambda v_3)$. To see if this product is in W, we need to check if the resulting vector satisfies the rule defining that space.

$$\begin{aligned}
&\lambda v_1 - (\lambda v_2 + 1) + 2(\lambda v_3 + 1)\\
&= \lambda v_1 - [\lambda v_2 + \lambda - (\lambda - 1)] + 2[\lambda v_3 + \lambda - (\lambda - 1)]\\
&= \lambda v_1 - (\lambda v_2 + \lambda) + (\lambda - 1) + 2(\lambda v_3 + \lambda) - 2(\lambda - 1)\\
&= \lambda v_1 - \lambda(v_2 + 1) + 2\lambda(v_3 + 1) - (\lambda - 1)\\
&= \lambda (v_1 - (v_2 + 1) + 2(v_3 + 1)) - (\lambda - 1)\\
&= \lambda - (\lambda - 1) = 1
\end{aligned}$$

By theorem 1.34 in the textbook, W is a subspace of $\mathbb{R}^3$, making it a vector space. $\square$

2. Construct an example of a vector space W with two subspaces, W_1, W_2, where you know $W_1 + W_2 = W$. Attempt to prove this.

   Out of laziness, let's define $W = \mathbb{R}^4$ and $W_1 = \{(x, y, 0, 0) \mid x, y \in \mathbb{R}\}$ and $W_2 = \{(0, 0, z, w) \mid z, w \in \mathbb{R}\}$. The next problem states that these are subspaces and the answer proves the sum is a subspace. Problem 4 proves that $W_1 + W_1 = W$.

3. Let $V = \mathbb{R}^4$, and define two subspaces:

   - $V_1 = \{(x, y, 0, 0) \mid x, y \in \mathbb{R}\}$
   - $V_2 = \{(0, 0, z, w) \mid z, w \in \mathbb{R}\}$

   Prove that $V_1 + V_2$ forms a subspace of V.

   Proof. Since $(0, 0, 0, 0) \in V_1, V_2$, it follows that $(0, 0, 0, 0) + (0, 0, 0, 0) = (0, 0, 0, 0) \in V_1 + V_2$.
   Let $u, w \in V_1 + V_2$. Then there must exist $u_1 \in V_1$, $u_2 \in V_2$, $w_1 \in W_1$ and $w_2 \in W_2$ such that $u = u_1 + u_2$ and $w = w_1 + w_2$. Consider $u + w = u_1 + u_2 + w_1 + w_2 = (u_1 + w_1) + (u_2 + w_2)$. Since V_1 and V_2 are closed under addition, it follows that $(u_1 + w_1) \in V_1$ and $(u_2 + w_2) \in V_2$. Therefore, $u + w \in V_1 + V_2$ and the sum space is closed under vector addition.
   Finally, let $u \in V_1 + V_2$ and $\lambda \in \mathbb{R}$. There must exist $u_1 \in V_1$, $u_2 \in V_2$ such that $u = u_1 + u_2$, so $\lambda u = \lambda u_1 + \lambda u_2$ by the distributive property of $\mathbb{R}^4$. Since V_1 and V_2 are subspaces, $\lambda u_1 \in V_1$ and $\lambda u_2 \in V_2$. Hence, $V_1 + V_2$ is closed under scalar multiplication. By theorem 1.34, $V_1 + V_2$ forms a subspace of $\mathbb{R}^4$. □

4. Prove that $V_1 + V_2 = V$ in the previous problem.

   Proof. Because $V_1 + V_2$ is a subspace, it is clear that $V_1 + V_2 \subset V$. The other inclusion, however, needs justification.
   Let $v \in V$. We seek to show that v is also in $V_1 + V_2$. Since $v \in \mathbb{R}^4$, there exists $v_1, v_2, v_3, v_4 \in \mathbb{R}$ such that $v = (v_1, v_2, v_3, v_4)$. Leveraging the additive identity of the field, notice that $v = (v_1 + 0, v_2 + 0, 0 + v_3, 0 + v_4)$. By vector addition, this equals $(v_1, v_2, 0, 0) + (0, 0, v_3, v_4)$. Since $(v_1, v_2, 0, 0) \in V_1$ and $(0, 0, v_3, v_4) \in V_2$, we must conclude that $v \in V_1 + V_2$. Because v was an arbitrary element, we have shown that $V \subset V_1 + V_2$.
   Hence $V_1 + V_2 = V$. □

5. Let $V = \mathbb{R}^2$, and define two subspaces:

   - $V_1 = \{(w_1, w_2) \mid w_1 + 2w_2 = 0, w_1, w_2 \in \mathbb{R}\}$
   - $V_2 = \{(v_1, v_2) \mid v_1 + v_2 = 0, v_1, v_2 \in \mathbb{R}\}$

Prove or provide a counterexample to the statement: $V_1 + V_2 = V$.

Using the theorem on sums of spaces from the book, we know that V_1+V_2 is a subspace of V and, by definition, $V_1 + V_2 \subset V$. We need to know the opposite containment for equality to hold. This means starting with an arbitrary element in V and showing that that same element is in $V_1 + V_2$. As we discussed in Monday's class, we want to see if there is a restriction on the space $V_1 + V_2$. Elements in V_1 are of the form $(-2w, w)$ and elements in V_2 are of the form $(u, -u)$. So elements constructed from their sums will be $(u - 2w, w - u)$. Could I define any vector (x, y) in this way?
This is a system of equations where

$$\begin{aligned} x &= u - 2w \\ y &= -u + w \end{aligned}$$

So to solve, we would pick $w = -x-y$ and $u = -2y-x$. So now we know $V_1+V_2 = V$ and we have a solid way to prove it. Let's write the formal argument.

Proof. By theorem 1.40, $V_1 + V_2$ is a subspace of V and we know that $V_1 + V_2 \subset V$. Now suppose $v \in V$. Then there exists $x, y, \in \mathbb{R}$ such that $v = (x, y)$. Define $u = (-2y - x, 2y + x)$ and define $w = (2x + 2y, -x - y)$. Notice that $u \in V_2$ and $w \in V_1$. Consider $w + u$:

$$\begin{aligned} w + u &= (2x + 2y, -x - y) + (-2y - x, 2y + x) \\ &= (x, y) \\ &= v \end{aligned}$$

Hence $v \in V_1 + V_2$. Thus $V = V_1 + V_2$. $\square$