Learning with Hypergraphs

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1 Hypergraph Laplacian generalization

1.1 Non-normalized Laplacian

About the notation used, refer to *Zhou and al.* section 2.
Define the **hyperedge derivative** $\frac{\partial f}{\partial_e} : \mathcal{E} \to ??$ for a signal $f : \mathcal{V} \to \mathbb{R}$ as :

$$\frac{\partial f}{\partial_e}(e) = \left(\sqrt{w(e)}(f(u) - f(v))\right)_{u,v \in e}$$

Then obtain the **gradient** $\nabla_{HG} f : \mathbb{R}^{|\mathcal{V}|} \to ??$ of a hypergraph signal f :

$$\nabla_{HG} f = \left(\frac{\partial f}{\partial_e}(e)\right)_{e \in \mathcal{E}}$$

The **divergence**, the adjoint of the gradient operator, satisfies $\langle \nabla f, g_e \rangle = \langle f, \mathrm{div} g_e \rangle$ for a signal g on the hyperedges $g_e(e) = \big(g(u,v)\big)_{\{u,v\} \in e}$.

$$\begin{aligned}
\langle \nabla f, g \rangle &= \sum_{e} \sum_{\{u,v\} \subseteq e} \sqrt{w(e)} (f(u) - f(v)) g(u,v) \\
&= \frac{1}{2} \sum_{e} \sqrt{w(e)} \sum_{u \in e} \sum_{v \in e} (f(u) g(u,v) - f(v) g(u,v)) \\
&= \frac{1}{2} \sum_{e} \sqrt{w(e)} \big(\sum_{u \in e} \sum_{v \in e} f(u) g(u,v) - \sum_{u \in e} \sum_{v \in e} f(v) g(u,v) \big) \\
&= \frac{1}{2} \sum_{e} \sqrt{w(e)} \big(\sum_{u \in e} \sum_{v \in e} f(u) g(u,v) - \sum_{v \in e} \sum_{u \in e} f(u) g(v,u) \big) \\
&= \frac{1}{2} \sum_{u \in e} f(u) \sum_{e} \sqrt{w(e)} \big(\sum_{v \in e} g(u,v) - \sum_{v \in e} g(v,u) \big) \\
&= \frac{1}{2} \sum_{u} f(u) \sum_{e} \sqrt{w(e)} \big(\sum_{v} g(u,v) h(e,v) - \sum_{v} g(v,u) h(e,v) \big) h(e,u)
\end{aligned}$$

Thus the divergence is then defined as:

$$(\mathrm{div} g)(u) = \frac{1}{2} \sum_{e} \sqrt{w(e)} \big(\sum_{v} \big(g(u,v) - g(v,u) \big) \big) h(e,v) h(e,u)$$

We now derive the **hypergraph Laplacian**: $\mathcal{L} : \mathbb{R}^{|\mathcal{V}|} \to \mathbb{R}^{|\mathcal{V}|}$ defined as usual by: $\mathcal{L} f \mapsto -\mathrm{div} \nabla f$

$$
\begin{aligned}
\mathcal{L}f(u) &= -\mathrm{div}\nabla f \\
&= \frac{1}{2}\sum_e \big(\sum_v \big(\sqrt{w(e)}\sqrt{w(e)}(f(u)-f(v)) - \sqrt{w(e)}\sqrt{w(e)}(f(v)-f(u))\big)\big)h(e,v)h(e,u) \\
&= \frac{1}{2}\big(\sum_e\sum_v w(e)(f(u)-f(v) - \sum_e\sum_v w(e)(f(v)-f(u))\big)h(e,v)h(e,u) \\
&= \frac{1}{2}\big(\sum_e\sum_v w(e)(f(u)-f(v) + \sum_e\sum_v w(e)(f(u)-f(v))\big)h(e,v)h(e,u) \\
&= \sum_e\sum_v w(e)(f(u)-f(v))h(e,v)h(e,u) \\
&= \sum_e\sum_v w(e)f(u)h(e,v)h(e,u) - \sum_e\sum_v w(e)f(v)h(e,v)h(e,u) \\
&= f(u)\sum_e w(e)h(e,u)\underbrace{\sum_v h(e,v)}_{\delta(e)} - \sum_v f(v)\underbrace{\sum_e w(e)h(e,v)h(e,u)}_{\mathbf{A}(u,v)}
\end{aligned}
$$

And the combinatorial Laplacian follows:

$$\mathcal{L} = \mathbf{D}'_v - \mathbf{A}$$

where $\mathbf{D}'_v$ is a diagonal matrix similar to $\mathbf{D}_v$ but using weights $w'(e) = w(e)\delta(e)$ instead and $\mathbf{A} = \mathbf{HWH}^T$

1.2 Normalized Laplacian

Define the **hyperedge derivative** $\frac{\partial f}{\partial_e} : \mathcal{E} \to ??$ for a signal $f : \mathcal{V} \to \mathbb{R}$ as :

$$\frac{\partial f}{\partial_e}(e) = \left(\frac{\sqrt{w(e)}}{\sqrt{\delta(e)}}\left(\frac{f(u)}{d(u)} - \frac{f(v)}{d(v)}\right)\right)_{u,v\in e}$$

Then obtain the **gradient** $\nabla_{HG} f : \mathbb{R}^{|\mathcal{V}|} \to ??$ of a hypergraph signal f :

$$\nabla_{HG} f = \left(\frac{\partial f}{\partial_e}(e)\right)_{e\in\mathcal{E}}$$

The **divergence**, the adjoint of the gradient operator, satisfies $\langle \nabla f, g_e \rangle = \langle f, \mathrm{div} g_e \rangle$ for a signal g on the hyperedges $g_e(e) = \big(g(u,v)\big)_{\{u,v\} \in e}$.

$$
\begin{aligned}
\langle \nabla f, g \rangle &= \sum_e \sum_{\{u,v\} \subseteq e} \frac{\sqrt{w(e)}}{\sqrt{\delta(e)}} \left(\frac{f(u)}{d(u)} - \frac{f(v)}{d(v)} \right) g(u,v) \\
&= \frac{1}{2} \sum_e \frac{\sqrt{w(e)}}{\sqrt{\delta(e)}} \sum_{u \in e} \sum_{v \in e} \big(\frac{f(u)}{d(u)} g(u,v) - \frac{f(v)}{d(v)} g(u,v) \big) \\
&= \frac{1}{2} \sum_e \frac{\sqrt{w(e)}}{\sqrt{\delta(e)}} \big(\sum_{u \in e} \sum_{v \in e} \frac{f(u)}{d(u)} g(u,v) - \sum_{u \in e} \sum_{v \in e} \frac{f(v)}{d(v)} g(u,v) \big) \\
&= \frac{1}{2} \sum_e \frac{\sqrt{w(e)}}{\sqrt{\delta(e)}} \big(\sum_{u \in e} \sum_{v \in e} \frac{f(u)}{d(u)} g(u,v) - \sum_{v \in e} \sum_{u \in e} \frac{f(u)}{d(u)} g(v,u) \big) \\
&= \frac{1}{2} \sum_{u \in e} \frac{f(u)}{d(u)} \sum_e \frac{\sqrt{w(e)}}{\sqrt{\delta(e)}} \big(\sum_{v \in e} g(u,v) - \sum_{v \in e} g(v,u) \big) \\
&= \frac{1}{2} \sum_u \frac{f(u)}{d(u)} \sum_e \frac{\sqrt{w(e)}}{\sqrt{\delta(e)}} \big(\sum_v g(u,v) h(e,v) - \sum_v g(v,u) h(e,v) \big) h(e,u)
\end{aligned}
$$

Thus the divergence is then defined as:

$$
(\mathrm{div} g)(u) = \frac{1}{2d(u)} \sum_e \frac{\sqrt{w(e)}}{\sqrt{\delta(e)}} \big(\sum_v \big(g(u,v) - g(v,u) \big) \big) h(e,v) h(e,u)
$$

We now derive the **hypergraph Laplacian**: $\mathcal{L} : \mathbb{R}^{|\mathcal{V}|} \to \mathbb{R}^{|\mathcal{V}|}$ defined as usual by: $\mathcal{L} f \mapsto -\mathrm{div} \nabla f$

$$\begin{aligned}
\mathcal{L}f(u) &= -\mathrm{div}\nabla f \\
&= \frac{1}{2}\sum_{e}\Big(\sum_{v}\Big(\frac{\sqrt{w(e)}}{\sqrt{\delta(e)}}\frac{\sqrt{w(e)}}{\sqrt{\delta(e)}}\Big(\frac{f(u)}{d(u)}-\frac{f(v)}{d(v)}\Big)-\frac{\sqrt{w(e)}}{\sqrt{\delta(e)}}\frac{\sqrt{w(e)}}{\sqrt{\delta(e)}}\Big(\frac{f(v)}{d(v)}-\frac{f(u)}{d(u)}\Big)\Big)\Big)h(e,v)h(e,u) \\
&= \frac{1}{2}\Big(\sum_{e}\sum_{v}\frac{w(e)}{\delta(e)}\Big(\frac{f(u)}{d(u)}-\frac{f(v)}{d(v)}-\sum_{e}\sum_{v}\frac{w(e)}{\delta(e)}\Big(\frac{f(v)}{d(v)}-\frac{f(u)}{d(u)}\Big)\Big)h(e,v)h(e,u) \\
&= \frac{1}{2}\Big(\sum_{e}\sum_{v}\frac{w(e)}{\delta(e)}\Big(\frac{f(u)}{d(u)}-\frac{f(v)}{d(v)}+\sum_{e}\sum_{v}\frac{w(e)}{\delta(e)}\Big(\frac{f(u)}{d(u)}-\frac{f(v)}{d(v)}\Big)\Big)h(e,v)h(e,u) \\
&= \sum_{e}\sum_{v}\frac{w(e)}{\delta(e)}\Big(\frac{f(u)}{d(u)}-\frac{f(v)}{d(v)}\Big)h(e,v)h(e,u) \\
&= \sum_{e}\sum_{v}\frac{w(e)}{\delta(e)}\frac{f(u)}{d(u)}h(e,v)h(e,u)-\sum_{e}\sum_{v}\frac{w(e)}{\delta(e)}\frac{f(v)}{d(v)}h(e,v)h(e,u) \\
&= \frac{f(u)}{d(u)}\sum_{e}\frac{w(e)}{\delta(e)}h(e,u)\underbrace{\sum_{v}h(e,v)}_{\delta(e)}-\sum_{v}\frac{f(v)}{d(v)}\sum_{e}\frac{w(e)}{\delta(e)}h(e,v)h(e,u) \\
&= \frac{f(u)}{d(u)}\sum_{e}w(e)h(e,u)-\sum_{v}f(v)\sum_{e}\frac{1}{\sqrt{d(v)}}h(e,u)\frac{w(e)}{\delta(e)}h(e,v)\frac{1}{\sqrt{d(v)}}
\end{aligned}$$

And the combinatorial Laplacian follows:

$$\mathcal{L} = \mathbf{I} - \Theta$$

where $\mathbf{I}$ is the identity matrix and $\Theta = D_V^{-\frac{1}{2}}HWD_e^{-1}H^TD_V^{-\frac{1}{2}}$

1.3 Another Laplacian

Using the same procedure using the following hyperedge derivative

$$\frac{\partial f}{\partial_e}(e) = \left(\frac{\sqrt{w(e)}}{\sqrt{\delta(e)}}(f(u)-f(v))\right)_{u,v\in e}$$

We can derive the Laplacian

$$\mathcal{L} = \mathbf{D}_v - \mathbf{A'}$$

where $\mathbf{A'} = HWD_e^{-1}H^T$