Topic 4: Calculus / Analysis

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Introduction

Calculus is the branch of mathematics that revolves around derivatives and integrals. It is fundamental in physics and engineering, and forms the basis of any advanced mathematical curriculum. Parts of this topic will be familiar from high school, like the concept of limits and derivatives (and potentially integrals).

We start by defining the concept of a limit in terms of the ϵ-δ definition. We then use this to define what it means for a function to be continuous at a point, and what it means for a function to be differentiable at a point. In particular, we focus on the interpretation of the derivative as the rate of change of a function. Note that we will not focus on calculating derivatives, other than giving a few basic examples, as this is extensively done in high school.

We then move on to the concept of integrals, which is the inverse of derivatives. We define the Riemann integral, and understand its motivation by approximating the area under a curve using rectangles. Again, we focus on interpretation and motivation, rather than calculation.

1 Lecture 1: Limits, continuity, differentiability

1.1 Limits

We want to describe a function $f : \mathbb{R} \to \mathbb{R}$ as x approaches a point a, without looking at the point a itself.

Definition 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$ be a point. We say that the limit of f as x approaches a is L, and write*

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for any $x \in \mathbb{R}$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. In mathematical notation this becomes

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R} : 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon. \tag{1}$$

This is typically a very confusing definition at first. The idea is that we can make $f(x)$ as close to L as we want, by making x close enough to a. If that is the case, we say that the limit of $f(x)$ as x approaches a is L.

Note that this limit is necessarily unique, if it exists.

Proposition 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$ be a point. If the limit of f as x approaches a exists, then it is unique.*

Proof. Suppose that L_1 and L_2 are two limits of f as x approaches a. Take $\epsilon = \frac{1}{2}|L_1 - L_2|$. Then there exists a $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$, then $|f(x) - L_1| < \epsilon$. Similarly, there exists a $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then $|f(x) - L_2| < \epsilon$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then if $0 < |x-a| < \delta$, we have that

$$|L_1 - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < 2\epsilon = |L_1 - L_2|,$$

which is a contradiction. Hence, $L_1 = L_2$. $\square$

1.2 Continuity

Now that we have defined the limit of a function in a point, we can define what it means for a function to be continuous at a point.

Definition 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$ be a point. We say that f is continuous at a if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for any $x \in \mathbb{R}$, if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. In other words, f is continuous at a if $\lim_{x\to a} f(x) = f(a)$. In mathematical notation, this becomes*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R} : |x-a| < \delta \implies |f(x) - f(a)| < \epsilon. \tag{2}$$

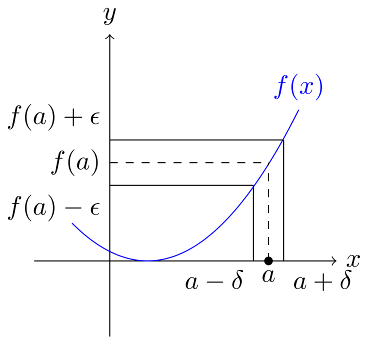

Figure 1: Graphical representation of the ϵ-δ definition of continuity.

This definition of continuity is visualized in Figure 1. Essentially, this definition states that the limit of $f(x)$ as x approaches a is has to be the function value $f(a)$ for f to be continuous at a. We illustrate this definition by means of an example.

Example 1. *The function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x) = x^2$ is continuous at every point $a \in \mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$ be a point. We want to show that f is continuous at a. Take $\epsilon > 0$ at random. We want to find a $\delta > 0$ such that if $|x-a| < \delta$, then $|f(x) - f(a)| < \epsilon$. Assume first that $a \neq 0$. We have that

$$|f(x) - f(a)| = |x^2 - a^2| = |x-a||x+a|.$$

Note that we can bound $|x+a| < |x-a| + 2|a|$. Hence, we want to find a $\delta > 0$ such that $|x-a|(|x-a| + 2|a|) < \epsilon$. Take $\delta = \min\left\{|a|, \frac{\epsilon}{3|a|}\right\}$. If now $|x-a| < \delta$, we have that $|x-a| + 2|a| < 3|a|$. Furthermore, since we have $|x-a| < \frac{\epsilon}{3|a|}$, we have that

$$|x-a|(|x-a| + 2|a|) < \frac{\epsilon}{3|a|} \cdot 3|a| = \epsilon.$$

So, we have found a $\delta > 0$ such that if $|x-a| < \delta$, then $|f(x) - f(a)| < \epsilon$. Hence, f is continuous at all

$a \neq 0$. Consider now $a = 0$. Choose $\delta = \min\{\epsilon, 1\}$. Then if $|x| < \delta$, we have that $|f(x) - f(0)| = |x^2| < |x| < \delta \leq \epsilon$. We used that $x^2 < |x|$ as $|x| < 1$. Therefore, f is also continuous at $a = 0$. □

1.3 Differentiability

The notion of continuity is relatively straightforward, though it does take some time to get used to its definition. However, this is a good starting point to understand the concept of differentiability.

Suppose we want to know the average change of a function over an interval, as shown in Figure 2. We can determine this average change by comparing the points $(a, f(a))$ and $(a + \Delta x, f(a + \Delta x))$. The average change is given by

$$\frac{\Delta f}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}. \tag{3}$$

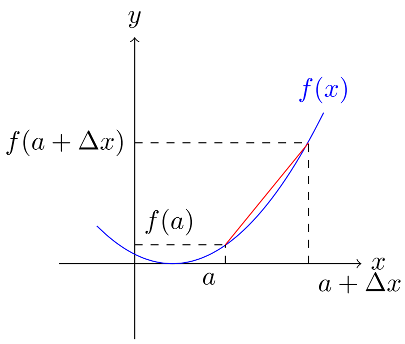

Figure 2: Average change of the function f over the interval $[a, a + \Delta x]$.

We can however decide how large we want the interval Δx to be. The smaller we make Δx, the closer the average change will be to the local change of the function. In the limit where Δx goes to zero, we get the instantaneous rate of change of the function at a. This is the derivative of the function at a.

Given what we have learned about limits, we can define the derivative as follows.

Definition 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$ be a point. We say that f is differentiable at a if the limit*

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \tag{4}$$

exists. The value $f'(a)$ is called the derivative of f at a.

An alternative way to define the derivative is the following

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{5}$$

Exercise 1. *Show that the two definitions of the derivative are equivalent.*

We illustrate the definition of the derivative by means of an example.

Example 2. *The function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x) = x^2$ is differentiable at every point $a \in \mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$ be a point. We want to show that f is differentiable at a. We have that

$$\frac{f(a + \Delta x) - f(a)}{\Delta x} = \frac{(a + \Delta x)^2 - a^2}{\Delta x} = \frac{a^2 + 2a\Delta x + \Delta x^2 - a^2}{\Delta x} = 2a + \Delta x.$$

Hence, we have that

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = 2a.$$

Hence, f is differentiable at a, and the derivative is $f'(a) = 2a$. □

Exercise 2. *Show that the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x) = mx + b$, where $m, b \in \mathbb{R}$ is differentiable at every point $a \in \mathbb{R}$, and find the derivative.*

If we can find the derivative of a function at every point in an interval, we can define the derivative of the function on that interval.

Definition 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that f is differentiable on an interval $I \subseteq \mathbb{R}$ if f is differentiable at every point $a \in I$. We then define the derivative of f on I as the function $f' : I \to \mathbb{R} : x \mapsto f'(x)$.*

As mentioned before, we will not focus on calculating derivatives. Therefore, we will not list the derivatives of elementary functions here, nor will we discuss the rules of differentiation. These can be found in any textbook, on the internet, and are normally taught towards the end of high school.

The final thing we want to mention is the interpretation of the derivative as the slope of the tangent line to the graph of the function at a point. We do not prove the following theorem, but simply mention it.

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$ be a point. If f is differentiable at a, then the tangent line to the graph of f at the point $(a, f(a))$ has slope $f'(a)$.*

As a consequence, the tangent line to a differentiable function at a point a is given by

$$y(x) = f(a) + f'(a)(x - a). \tag{6}$$

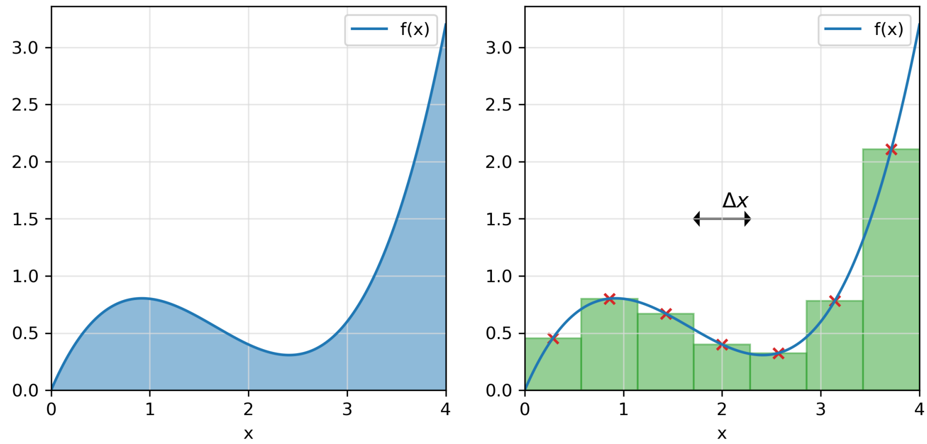

Figure 3: (*left*)The area under the curve of f on the interval $[0, 4]$, and (*right*) an approximation of this area using 7 rectangles. The red crosses indicate the midpoints of the rectangles, at which the function is evaluated.

2 Lecture 2: Integrals and the fundamental theorem of calculus

Just like subtraction is the inverse of addition and division is the inverse of multiplication, taking derivatives of a function will also have an "inverse operation". This inverse operation is called the integral, and it is used to calculate the area under a curve.

2.1 The area under a curve

Suppose we have a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on an interval $[a, b]$. We want to calculate the area under the curve of f on this interval, as shown in Figure 3. We can approximate the area under the curve by using rectangles, as shown in the right part of the same figure.

For simplicity, we take the interval $[a, b]$ to be divided into n equal parts, each of length $\Delta x = \frac{b-a}{n}$. The end points of these intervals are then $x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_n = a + n\Delta x = b$. The midpoints of these intervals are then

$$\overline{x}_1 = \frac{x_0 + x_1}{2}, \overline{x}_2 = \frac{x_1 + x_2}{2}, \ldots, \overline{x}_n = \frac{x_{n-1} + x_n}{2}.$$

If we evaluate the function f at these midpoints, we get the values $f(\overline{x}_1), f(\overline{x}_2), \ldots, f(\overline{x}_n)$. Summing the areas of the rectangles, with height $f(\overline{x}_i)$ and width Δx, we get an approximation of the area under the curve of f on the interval $[a, b]$:

$$A_n = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x. \tag{7}$$

This is illustrated in Figure 3, for the interval $[0, 4]$ and $n = 7$. It should be clear that the more rectangles we use, the better the approximation of the area under the curve will be. Figure 4 shows the area under the curve of f on the interval $[0, 4]$, and an approximation of this area using 20 rectangles.

Indeed, in the limit where n goes to infinity, the approximation of the area under the curve will become exact.

2.2 The Riemann integral

This is not the most general way to define the Riemann integral, but if we restrict to continuous functions - as we do - this limit does the trick. We can now define the Riemann integral of a function f on an interval $[a, b]$.

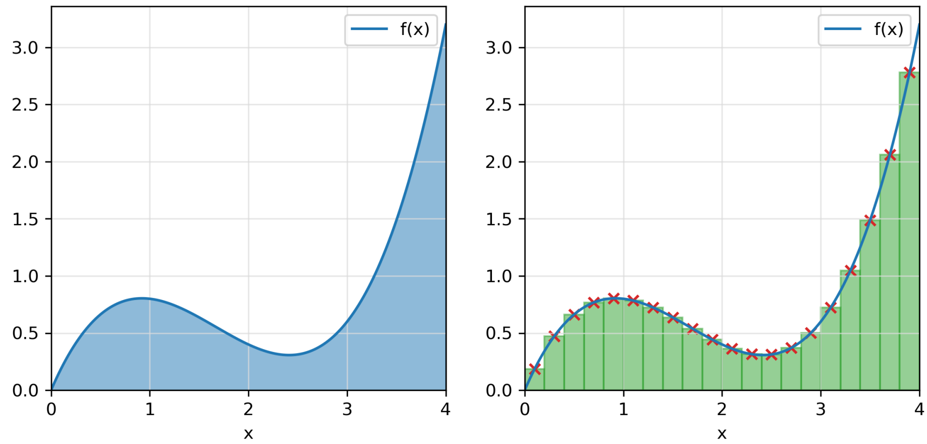

Figure 4: (*left*)The area under the curve of f on the interval $[0, 4]$, and (*right*) an approximation of this area using 20 rectangles. The red crosses indicate the midpoints of the rectangles, at which the function is evaluated.

Definition 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on an interval $[a, b]$. We define the Riemann integral of f on $[a, b]$ as*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x, \tag{8}$$

where $\Delta x = \frac{b-a}{n}$ and $\overline{x}_i = \frac{x_{i-1}+x_i}{2}$.

Again, bear in mind that this is not the most general definition of the Riemann integral, which can be given for functions that are not continuous, and uses the concept of partition of an interval and refinement of the latter.

Example 3. *Let $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x) = x^2$. We want to calculate the integral of f on the interval $[0, 4]$.*

Using the definition, we have that $\Delta x = \frac{4}{n}$ and $\overline{x}_i = \frac{x_{i-1}+x_i}{2} = \frac{i-1+i}{2}\frac{4}{n} = \frac{2(2i-1)}{n}$. Hence, we have that

$$\begin{aligned}
\int_0^4 x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2(2i-1)}{n}\right)^2 \frac{4}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{16(2i-1)^2}{n^3} \\
&= \lim_{n\to\infty} \frac{16}{n^3} \sum_{i=1}^{n} \left(4i^2 - 4i + 1\right) \\
&= \lim_{n\to\infty} \frac{16}{n^3} \left(4\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n\right).
\end{aligned}$$

To obtain the last line we used the formulas for the sum of the first n squares and the first n numbers:

$$\begin{aligned}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2}, \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

We can now simplify the expression to find the integral.

$$\begin{aligned}\int_0^4 x^2\,dx &= \lim_{n\to\infty} \frac{16}{n^3}\left(\frac{8n^3+12n^2+4n-12n^2-12n+6n}{6}\right)\\ &= \lim_{n\to\infty} \frac{16}{n^3}\left(\frac{8n^3-2n}{6}\right)\\ &= \lim_{n\to\infty} \frac{16}{6}\left(8-\frac{2}{n^2}\right)\\ &= \frac{64}{3}.\end{aligned}$$

Note that this is equal to $\frac{4^3}{3}$, *which is the more familiar result for people that have studied integrals before.*

2.3 The fundamental theorem of calculus

This section clarifies in what sense the integral is the inverse operation of the derivative. The fundamental theorem of calculus actually consists of two parts, which we will state here.

Theorem 2. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous. Define the function* $F : [a, b] \to \mathbb{R}$ *as*

$$F(x) = \int_a^x f(t)\,dt. \tag{9}$$

Then F *is differentiable on* (a, b), *and* $F'(x) = f(x)$ *for all* $x \in (a, b)$.

Theorem 3. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function that is continuous on an interval* $[a, b]$, *and let* $F : \mathbb{R} \to \mathbb{R}$ *be a function such that* $F'(x) = f(x)$ *for all* $x \in (a, b)$. *Then we have that*

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{10}$$

To prove these theorems, we would need a couple more tools, like the mean value theorem and a general treatment of integrals. A full treatment of these topics should be available in any analysis course at university level.

3 References

These notes are based on my own knowledge of these basic mathematical concepts, and the writing has been accelerated by the use of *Github copilot* and its implementation in VSCode.