Fund. Math. Topic 2: Probability

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Introduction

Having become acquainted with the basic concepts of set theory, we will now put additional structure on sets in the form of a probabilistic measure. This forms the basis of probability theory, a mathematical framework to discuss situations that are subject to uncertainty. We will start with some formal definitions and study basic examples, before we discuss Bayes' rule and apply it to the famous Monty Hall problem.

NB: to avoid having to define a σ-algebra, we will only consider finite sample spaces in this course, i.e. sets with a finite number of elements.

Lecture 1: Definitions

1.1 Probability space

Before we can quantify how likely a certain event is to happen, we need to know what all the possibilities are. To this end, we define the **sample space**.

Definition 1. *The* ***sample space*** Ω *is the set of all possible outcomes of a random experiment.*

The prototypical example of a sample space is the set of all possible outcomes of the throw of a die.

Example 1. *The sample space of a die throw is* $\Omega = \{1, 2, 3, 4, 5, 6\}$*, i.e. the set containing the different numbers on a die. An* ***event*** *is a subset of* Ω*, i.e. a set of outcomes that we are interested in. For example, the event of throwing an even number is* $E = \{2, 4, 6\}$.

The collection of all events of interest is referred to as the event space $\mathcal{F}$, a collection of subsets of Ω. Often, the event space will simply be the power set of Ω, i.e. the set of all subsets of Ω. While this does not need to be the case - and sometimes it cannot be the case[1] - we will assume this in the rest of the course unless explicitly stated otherwise.

The question we now want to answer is: what is the probability of a certain event A happening? In the case or our example above, the probability of throwing an even number can be calculated straightforwardly.

[1] Our restriction to finite sample spaces means we need not worry about this. Extending the topics here to infinite spaces requires the introduction of a σ-algebra, a topic beyond the scope of this course.

Under the assumption that all outcomes are equally likely, we can simply count the number of outcomes in the event E and divide it by the total number of outcomes in the sample space Ω. We will denote the probability with $P(E)$, and we have that

$$P(E) = \frac{\#E}{\#\Omega} = \frac{3}{6} = \frac{1}{2}\,.$$

However, this would not have been the case if the die was not fair, i.e. if the outcomes were not equally likely. In that case, we would need more knowledge about the die to determine the probability of an event. The probabilities of different events are determined by the **probability measure**.

Definition 2. *A **probability measure** $P : \mathcal{F} \to [0,1]$ on a sample space Ω is a function from the event space $\mathcal{F}$ to the real numbers, such that*

- *$P(\Omega) = 1$, i.e. the probability of the sample space is 1,*
- *$P(A) \geq 0$ for all $A \in \mathcal{F}$, i.e. the probability of an event is non-negative,*
- *$P(A_1 \cup A_2) = P(A_1) + P(A_2)$ for all $A_1, A_2 \in \mathcal{F}$ such that $A_1 \cap A_2 = \emptyset$, i.e. the probability of the union of two disjoint events is the sum of their probabilities.*

Definition 3. *A **probability space** is a triple $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is the event space and P is the probability measure.*

We can now tackle the problem of determining probabilities for a loaded die.

Example 2. *Consider a die with the following probabilities for each outcome, where we understand $P(i) = P(\{i\})$:*

$$\begin{aligned} &P(1) = 1/8, && P(2) = 1/8, && P(3) = 1/8, \\ &P(4) = 1/8, && P(5) = 1/8, && P(6) = 3/8. \end{aligned}$$

We check that this is a valid probability measure, i.e. the probabilities are non-negative and sum to 1:

$$\begin{aligned} P(1) + P(2) + P(3) + P(4) + P(5) + P(6) &= 1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 3/8 \\ \Leftrightarrow P(\{1,2,3,4,5,6\}) &= 1\,. \end{aligned}$$

We can now calculate the probability of throwing an even number:

$$\begin{aligned} P(E) &= P(\{2,4,6\}) \\ &= P(2) + P(4) + P(6) \\ &= 1/8 + 1/8 + 3/8 \\ &= 5/8\,. \end{aligned}$$

Note that this is different from the case of a fair die, where we had $P(E) = 1/2$.

Exercise 1. *Consider a die with the following probabilities:*

$$\begin{aligned} &P(1) = 1/16, && P(2) = 1/16, && P(3) = 1/8, \\ &P(4) = 1/8, && P(5) = 3/8\,. \end{aligned}$$

1. *What does $P(6)$ have to be for this to be a valid probability measure?*
2. *What is the probability of throwing an even number?*
3. *What is the probability of throwing a number that is not a multiple of 3?*

Exercises p.9

1.2 Conditional probability

Lecture 2: Bayes' rule

Define Bayes rule, rule of total probability.

2.1 The Monty Hall problem

Only basic version here. In class we might handle extensions too.

Lecture 3: Stochastic variables

In case there is time left on day 5.

References

These notes are based on my own knowledge of these basic mathematical concepts, and the writing has been accelerated by the use of *GitHub copilot* and its implementation in VSCode. Inspiration has been taken from the course notes for *"Kansrekenen I"* (Probabilistic calculus), used in the first year of the Bachelor of Mathematics at the KU Leuven: Prof. Tim Verdonck is the author of the lecture notes.