Adv. Math. Topic 1: Integrals

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Introduction

Calculus is the branch of mathematics that revolves around derivatives and integrals. It is fundamental in physics and engineering, and forms the basis of any advanced mathematical curriculum.

We focus on the concept of integrals, which is the inverse of derivatives - covered in last week's lectures. We define the Riemann integral, and understand its motivation by approximating the area under a curve using rectangles. Afterwards, we turn towards calculating integrals in practice, which we then will be able to exploit in applications.

Lecture 1: Integrals (intuitive)

Just like subtraction is the inverse of addition and division is the inverse of multiplication, taking derivatives of a function will also have an "inverse operation". This inverse operation is called the integral, and it is used to calculate the area under a curve.

1.1 The area under a curve

Suppose we have a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on an interval $[a, b]$. We want to calculate the area under the curve of f on this interval, as shown in Figure 1. We can approximate the area under the curve by using rectangles, as shown in the right part of the same figure. For simplicity, we take the interval $[a, b]$ to be divided into n equal parts, each of length $\Delta x = \frac{b-a}{n}$. The end points of these intervals are then $x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_n = a + n\Delta x = b$. The midpoints of these intervals are then

$$\overline{x}_1 = \frac{x_0 + x_1}{2}, \overline{x}_2 = \frac{x_1 + x_2}{2}, \ldots, \overline{x}_n = \frac{x_{n-1} + x_n}{2}.$$

If we evaluate the function f at these midpoints, we get the values $f(\overline{x}_1), f(\overline{x}_2), \ldots, f(\overline{x}_n)$. Summing the areas of the rectangles, with height $f(\overline{x}_i)$ and width Δx, we get an approximation of the area under the curve of f on the interval $[a, b]$:

$$A_n = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x. \tag{1}$$

This is illustrated in Figure 1, for the interval $[0, 4]$ and $n = 7$. It should be clear that the more rectangles we use, the better the approximation of the area under the curve will be. Figure 2 shows the area under the curve of f on the interval $[0, 4]$, and an approximation of this area using 20 rectangles.

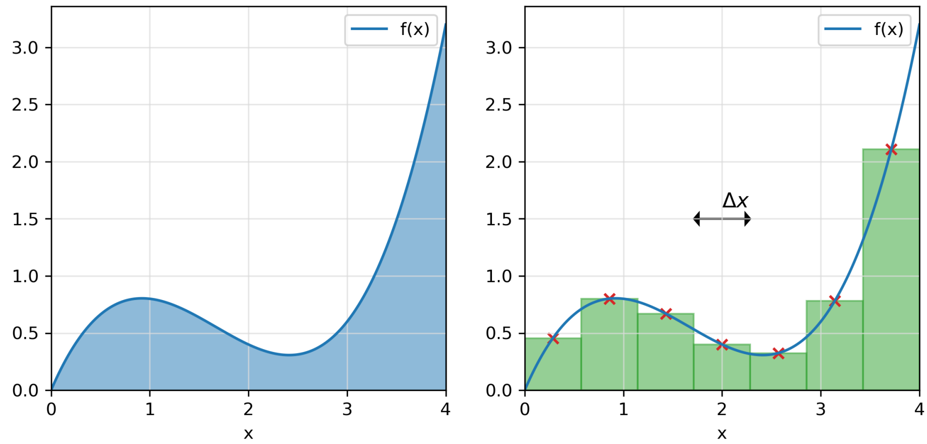

Figure 1: (*left*)The area under the curve of f on the interval $[0, 4]$, and (*right*) an approximation of this area using 7 rectangles. The red crosses indicate the midpoints of the rectangles, at which the function is evaluated.

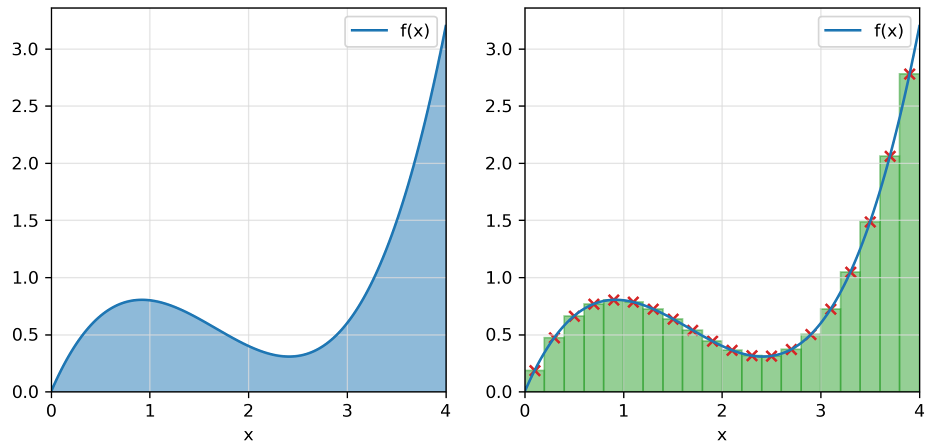

Figure 2: (*left*)The area under the curve of f on the interval $[0, 4]$, and (*right*) an approximation of this area using 20 rectangles. The red crosses indicate the midpoints of the rectangles, at which the function is evaluated.

Indeed, in the limit where n goes to infinity, the approximation of the area under the curve will become exact.

1.2 The Riemann integral

This is not the most general way to define the Riemann integral, but if we restrict to continuous functions - as we do - this limit does the trick. We can now define the Riemann integral of a function f on an interval $[a, b]$.

Definition 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on an interval $[a, b]$. We define the Riemann integral of f on $[a, b]$ as*

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x, \tag{2}$$

where $\Delta x = \frac{b-a}{n}$ and $\overline{x}_i = \frac{x_{i-1}+x_i}{2}$.

Again, bear in mind that this is not the most general definition of the Riemann integral, which can be given for functions that are not continuous, and uses the concept of partition of an interval and refinement of the latter.

Example 1. *Let $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x) = x^2$. We want to calculate the integral of f on the interval $[0, 4]$.*

Using the definition, we have that $\Delta x = \frac{4}{n}$ and $\overline{x}_i = \frac{x_{i-1}+x_i}{2} = \frac{i-1+i}{2}\frac{4}{n} = \frac{2(2i-1)}{n}$. Hence, we have that

$$\begin{aligned}
\int_0^4 x^2 \mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2(2i-1)}{n}\right)^2 \frac{4}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{16(2i-1)^2}{n^3} \\
&= \lim_{n\to\infty} \frac{16}{n^3} \sum_{i=1}^{n} \left(4i^2 - 4i + 1\right) \\
&= \lim_{n\to\infty} \frac{16}{n^3} \left(4\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n\right).
\end{aligned}$$

To obtain the last line we used the formulas for the sum of the first n squares and the first n numbers:

$$\begin{aligned}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2}, \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

We can now simplify the expression to find the integral.

$$\begin{aligned}
\int_0^4 x^2 \mathrm{d}x &= \lim_{n\to\infty} \frac{16}{n^3} \left(\frac{8n^3 + 12n^2 + 4n - 12n^2 - 12n + 6n}{6}\right) \\
&= \lim_{n\to\infty} \frac{16}{n^3} \left(\frac{8n^3 - 2n}{6}\right) \\
&= \lim_{n\to\infty} \frac{16}{6} \left(8 - \frac{2}{n^2}\right) \\
&= \frac{64}{3}.
\end{aligned}$$

Note that this is equal to $\frac{4^3}{3}$, which is the more familiar result for people that have studied integrals before.

1.3 The fundamental theorem of calculus

This section clarifies in what sense the integral is the inverse operation of the derivative. The fundamental theorem of calculus actually consists of two parts, which we will state here.

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Define the function $F : [a, b] \to \mathbb{R}$ as*

$$F(x) = \int_a^x f(t)\mathrm{d}t. \tag{3}$$

Then F is differentiable on (a, b), and $F'(x) = f(x)$ for all $x \in (a, b)$.

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on an interval $[a, b]$, and let $F : \mathbb{R} \to \mathbb{R}$ be a function such that $F'(x) = f(x)$ for all $x \in (a, b)$. Then we have that*

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a). \tag{4}$$

To prove these theorems, we would need a couple more tools, like the mean value theorem and a general treatment of integrals. A full treatment of these topics should be available in any analysis course at university level.

1.4 Indefinite integrals

From the linearity of the derivative, it follows that if two functions f, g have the same derivative, i.e. $f'(x) = g'(x)$ for all x, then they differ by a constant. Indeed, if $f'(x) = g'(x)$, then $(f - g)'(x) = f'(x) - g'(x) = 0$, and thus $f - g$ is a constant function. Therefore, we can write that $f(x) = g(x) + C$ for some constant C.

For a given function f, we definite a **primitive function** F as a function such that $F'(x) = f(x)$ for all x. From the above, it thus follows that all primitive functions of f differ by a constant.

This motivates the definition of the **indefinite integral** of a function f as the set of all primitive functions of f:

Definition 2. *The **indefinite integral** of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as*

$$\int f(x)\mathrm{d}x = \{F : F'(x) = f(x) \text{ for all } x \in \mathbb{R}\}\,. \tag{5}$$

Suppose we know a primitive function F of f, then we can write the indefinite integral as

$$\int f(x)\mathrm{d}x = F(x) + C\,, \tag{6}$$

where $C \in \mathbb{R}$ is an arbitrary constant.

The integral we defined in the previous section is called a **definite integral**, as it has limits of integration a, b. Note that the definite integral can be calculated using *any* primitive function F of f, as the constant C will cancel out when we calculate $F(b) - F(a)$. Therefore, the definite integral can always be calculated from the indefinite integral, no matter the primitive function we choose.

Lecture 2: Calculating integrals

While the calculation in Example 1 was still tractable, it becomes more cumbersome for more general functions. In order to avoid this, a series of tools has been developed to calculate integrals of complex functions from more *elementary integrals.*

2.1 Elementary integrals

Calculating any integral boils down to reducing it to a sum of elementary integrals. Elementary integrals are integrals of functions that can be expressed in terms of elementary functions, and they should be considered basic knowledge. The following list contains some of the most common elementary (indefinite) integrals, which we will not prove here.

Lemma 1 (Elementary integrals)**.** *Let $C \in \mathbb{R}$ be an arbitrary constant:*

$$\begin{aligned}
\int x^n \,\mathrm{d}x &= \frac{x^{n+1}}{n+1} + C, && n \neq -1,\\
\int \frac{1}{x} \,\mathrm{d}x &= \ln|x| + C,\\
\int e^x \,\mathrm{d}x &= e^x + C,\\
\int a^x \mathrm{d}x &= \frac{a^x}{\ln a} + C, && a > 0, a \neq 1,\\
\int \sin x \,\mathrm{d}x &= -\cos x + C,\\
\int \cos x \,\mathrm{d}x &= \sin x + C\,.
\end{aligned}$$

2.2 Rules for calculating integrals

The following rules can be used to calculate integrals of more complex functions, and are based on the properties of the Riemann integral. These rules have counterparts for derivatives, and are often called the *rules of integration.*

The first one follows straightforwardly from the linearity of the derivative:

Theorem 3 (Linearity of the integral)**.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions, and let $a, b \in \mathbb{R}$ be constants. Then we have that*

$$\int (af(x) + bg(x))\mathrm{d}x = a\int f(x)\mathrm{d}x + b\int g(x)\mathrm{d}x. \tag{7}$$

Example 2. *Let $f(x) = 2x^2 + 3x - 5$. Then we find that*

$$\begin{aligned}
\int f(x)\mathrm{d}x &= \int (2x^2 + 3x - 5)\mathrm{d}x\\
&= 2\int x^2 \,\mathrm{d}x + 3\int x \,\mathrm{d}x - 5\int 1 \,\mathrm{d}x\\
&= 2\left(\frac{x^3}{3} + C_1\right) + 3\left(\frac{x^2}{2} + C_2\right) - 5(x + C_3)\\
&= \frac{2}{3}x^3 + \frac{3}{2}x^2 - 5x + C,
\end{aligned}$$

where $C = 2C_1 + 3C_2 - 5C_3$ is an arbitrary constant. Note that we can also write this as

$$\int f(x)\,\mathrm{d}x = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 5x + C, \tag{8}$$

where C is an arbitrary constant. In the case that linearity is used, we will henceforth immediately combine the constants into one.

While linearity is straightforward, it is still restricted to functions that can be expressed as a linear combination of elementary functions. For more complex functions, substitution is often a very powerful tool. It is intimately related to the chain rule for derivatives, which we repeate here for convenience:

Theorem 4 (Chain rule)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions, and let g be differentiable at x. Then we have that*

$$(F \circ g)'(x) = F'(g(x))g'(x). \tag{9}$$

Theorem 5 (Substitution)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be a function, whose derivative we denote with f, and let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $g'(x) \neq 0$ for all x in the domain of g. Then we have that*

$$\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u\Big|_{u=g(x)}, \tag{10}$$

or equivalently

$$\int f(g(x))g'(x)\mathrm{d}x = F(g(x)) + C, \tag{11}$$

It should be noted that for definite integrals, the limits of integration will change when we apply substitution, i.e. Eq. (10) will become

$$\int_a^b f(g(x))g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u. \tag{12}$$

Example 3. *Let $f(x) = \sin^2 x \cos x$, and we want to calculate the integral of f on the interval $[0, \frac{\pi}{2}]$. We start by calculating the indefinite integral of f:*

$$\int f(x)\,\mathrm{d}x = \int \sin^2 x \cos x\,\mathrm{d}x.$$

We can use substitution here, by letting $u = g(x) = \sin x$. Indeed, note that $g'(x) = \cos x$, which according to Thm. 5 allows us to write

$$\int f(x)\,\mathrm{d}x = \int u^2 \mathrm{d}u,$$

where we used that $g'(x)\mathrm{d}x = \mathrm{d}u$. We can now calculate the integral of u^2 by using Lemma 1:

$$\begin{aligned}\int f(x)\,\mathrm{d}x &= \frac{u^3}{3} + C \\ &= \frac{\sin^3 x}{3} + C\,,\end{aligned}$$

where we replaced u by $g(x) = \sin x$ in the last step.

We can now calculate the definite integral of f on the interval $[0, \frac{\pi}{2}]$:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \sin^2 x \cos x\,\mathrm{d}x \\ &= \left[\frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{2}} \\ &= \frac{\sin^3 \frac{\pi}{2}}{3} - \frac{\sin^3 0}{3} \\ &= \frac{1^3}{3} - 0 \\ &= \frac{1}{3}.\end{aligned}$$

Note that we could have also calculated the definite integral directly, by using Eq. (12)*:*

$$\begin{aligned}\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \sin^2 x \cos x\,\mathrm{d}x \\ &= \int_{\sin 0}^{\sin \frac{\pi}{2}} u^2 \mathrm{d}u \\ &= \left[\frac{u^3}{3}\right]_{\sin 0}^{\sin \frac{\pi}{2}} \\ &= \frac{1^3}{3} - 0 \\ &= \frac{1}{3}.\end{aligned}$$

The hardest part of this method usually is to find the right substitution, which can be a bit of a trial-and-error process. Additionally, multiple substitutions could work, although different substitutions could significantly alter the complexity of the subsequent calculations.

Finally, we look at *integration by parts*, linked to the product rule for derivatives - which we also repeat here for convenience:

Theorem 6 (Product rule)**.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions, and let f and g be differentiable at x. Then we have that*

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x). \tag{13}$$

Theorem 7 (Integration by parts)**.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions, and let f and g be differentiable at x. Then we have that*

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x. \tag{14}$$

Example 4. *Let $f(x) = x^2$ and $g(x) = e^x$. We want to calculate the integral of f times the derivative of g:*

$$\int f(x)g'(x)\,\mathrm{d}x = \int x^2 e^x\,\mathrm{d}x.$$

We can now apply integration by parts, with $f(x) = x^2$ and $g'(x) = e^x$, which gives us

$$\int x^2 e^x\,\mathrm{d}x = x^2 e^x - \int 2xe^x\,\mathrm{d}x.$$

We can now apply integration by parts again, with $f(x) = 2x$ and $g'(x) = e^x$, which gives us

$$\begin{aligned}\int x^2 e^x \, \mathrm{d}x &= x^2 e^x - (2xe^x - 2\int e^x \, \mathrm{d}x) \\ &= x^2 e^x - 2xe^x + 2e^x + C,\end{aligned}$$

where we used Lemma 1 to calculate the last integral.

Again, the hardest part of this method is to find the right functions f, g to apply integration by parts. Calculating integrals in general is a trial-and-error process, as opposed to taking derivatives, and it requires a lot of practice and a bit of creativity.

We provide a couple of example integrals that can be calculated using the methods we discussed above.

Exercise 1. *Calculate the integrals below. You may need to use multiple methods, as well as the list of elementary integrals in Lemma 1.*

1. $\int x^3 \, \mathrm{d}x$.
2. $\int \frac{1}{x^2} \, \mathrm{d}x$.
3. $\int \left(e^{2x} + 4\sin x\right) \, \mathrm{d}x$.
4. $\int (\sin(3x) + 3) \, \mathrm{d}x$.
5. $\int \cos^2(2x)\sin(2x) \, \mathrm{d}x$.
6. $\int xe^{2x} \, \mathrm{d}x$.
7. $\int x^2 \sin(x) \, \mathrm{d}x$.
8. $\int \frac{1}{\sqrt{x}} \, \mathrm{d}x$.
9. $\int \frac{\cos x}{\sqrt{\sin x}} \, \mathrm{d}x$.
10. $\int e^{3\cos x} \sin x \, \mathrm{d}x$.

Lecture 3: Applications of integrals

In this Lecture, we briefly illustrate the importance of integral calculus in various settings, by applying it to simple problems.

3.1 Physics: a falling object

Some students may be familiar with Newton's second law, which states that the force F acting on an object is equal to the mass m of the object times its acceleration a:

$$F = ma. \tag{15}$$

The acceleration a is the derivative of the velocity v with respect to time t, i.e. $a = \frac{\mathrm{d}v}{\mathrm{d}t}$. Give that the velocity v is the derivative of the position x with respect to time, i.e. $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, we can rewrite Eq. (15) as

$$F = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}. \tag{16}$$

This is really an equation relating two vectors, $\mathbf{F}$ and $\mathbf{x}$, but for the sake of simplicity we will restrict to an example in one dimension. Consider an object of mass m, initially at rest at height h above the ground, and subject to the force of gravity. The gravitational force has a constant magnitude $F = mg$, and points towards the ground. Using $y(t)$ to denote the height of the object at time t, we can write Eq. (16) as

$$-mg = m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\,, \tag{17}$$

i.e. the gravitational force provides a negative acceleration, pulling the object to the ground. The masses cancel out (this is the famous result that, in vacuum, all objects fall the same way), and we find that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -g\,. \tag{18}$$

Recasting this in terms of the velocity $v(t) = \frac{\mathrm{d}y}{\mathrm{d}t}$, we find that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g\,. \tag{19}$$

Integrating both sides of this equation with respect to t, we find that

$$\begin{aligned} \int \frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t &= \int -g\,\mathrm{d}t \\ \Leftrightarrow v(t) &= -gt + C\,, \end{aligned}$$

where we implicitly used the fundamental theorem of calculus, Thm. 2, using that the integral of a derivative is the original function. The constant C can be understood as being the initial velocity of the object, as $v(0) = C$. Given that we assumed that the object is initially at rest, we have that $C = 0$. Recasting the equation in terms of the height $y(t)$, we now have that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -gt\,. \tag{20}$$

Integrating both sides of this equation with respect to t again, we find

$$\begin{aligned} \int \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t &= \int -gt\,\mathrm{d}t \\ \Leftrightarrow y(t) &= -\frac{1}{2}gt^2 + C'\,, \end{aligned}$$

where we again used the fundamental theorem of calculus, Thm. 2, and one of the fundamental integrals in Lemma 1. Again, the constant C' can be understood as being the initial height of the object, as $y(0) = C'$.

The initial height is what we denoted by h at the beginning of this section, so we have that $C' = h$. We thus find that the height of the object at time t is given by

$$y(t) = -\frac{1}{2}gt^2 + h\,. \tag{21}$$

As a consequence, we find that the object will hit the ground when $y(t) = 0$, i.e. when

$$0 = -\frac{1}{2}gt^2 + h\,. \tag{22}$$

Rearranging this equation, we find that the time it takes for the object to hit the ground is given by $t = \sqrt{\frac{2h}{g}}$, irrespective of the mass of the object.

Exercise 2. *Assume now that the object is not initially at rest, but has an initial velocity v_0 upwards.*

1. *Calculate the height of the object as a function of time.*
2. *Calculate the time it takes for the object to hit the ground, as a function of h and v_0.*
3. *Can you explain your result? What happens if v_0 is negative, i.e. if the object is thrown downwards?*
4. *Can you throw a ball sufficiently hard so that it never hits the ground?*[a]

[a] Note that this question still assumes that the force of gravity is constant as a function of height, which is not the case if we consider large height differences. If we would treat the problem in full generality, with the force of gravity decreasing with height, we would find a different answer to this question.

3.2 Economics

Lecture 4: First-order differential equations

Differential equations (DE) relate functions to their derivative(s). They govern the laws of physics and model stock markets. Multiple university-level courses can be dedicated to differential equations, and we obviously won't have time to cover everything.

We first focus on simple first-order differential equations that can be solved by separation of variables. If time permits, we will expand to more general first-order differential equations.

4.1 First-order DEs

As a basic example of a DE, consider the following equation for the function $f(x)$:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 2fx\,. \tag{23}$$

This equation prescribes the derivative of the function f not only in terms of x, but also in terms of f itself. Assuming for now that $f(x) > 0$ for all $x \in \mathbb{R}$, we can rewrite Eq. (23) as

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = 2x\,. \tag{24}$$

Let's integrate both sides of this equation over x:

$$\begin{aligned} \int \frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x &= \int 2x\mathrm{d}x \\ \Leftrightarrow \int \frac{1}{f}\mathrm{d}f &= x^2 + C \\ \Leftrightarrow \ln f(x) &= x^2 + C\,. \end{aligned}$$

We can now exponentiate both sides of this equation to find and expression for the function $f(x)$:

$$f(x) = Ae^{x^2}\,, \tag{25}$$

where we defined $A = e^C$ and used the rule for the exponent of a sum, $e^{a+b} = e^a e^b$. The method we used is called *separation of variables*, and is only applicable because we were able to separate f, x to different signs of the equality sign in Eq. (24).

Exercise 3. *Solve the following first-order differential equations by separation of variables:*

1. $\frac{\mathrm{d}f}{\mathrm{d}x} = 3fx^2$.
2. $\frac{\mathrm{d}f}{\mathrm{d}x} = -f^2$.
3. $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{f}$.
4. $\frac{\mathrm{d}f}{\mathrm{d}x} = 2f + 3$.

References

These notes are based on my own knowledge of these basic mathematical concepts, and the writing has been accelerated by the use of *Github copilot* and its implementation in VSCode.