Course Overview

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1 Introduction

This is the tentative overview for the Mathematics course I will be teaching for Immerse Education. The course spans 2 weeks, with 4 hours of class everyday, totalling 40 hours. I have aimed to design a course with following objectives in mind:

- My goal is *not* to simply teach the students the things they would see in a standard high school curriculum. Of course, some of the topics will be similar, but I also want to touch on topics that may usually be taught at an undergraduate level.
- For every broad topic covered, I also want to discuss an application that shows how these abstract concepts are actually applicable.
- My focus is not on making the students experts in certain fields, but rather to develop their mathematical thinking by showing them problems from different fields, and how one can think about them in different ways.
- Finally, I want to prepare them for studying mathematics at an undergraduate level. This means I will also teach them skills like LaTeXand proofwriting.

2 Syllabus Overview

This Mathematics course aims to develop your mathematical thinking by introducing you to a variety of topics, ranging from number theory to geometry. Some of these subjects may be familiar to you, but others are more advanced and are usually only taught at an undergraduate level. You will discover basic theoretical concepts for each of these topics, and consolidate them in the context of a more tangible application afterwards, like cryptography or Einstein's Theory of Relativity.

During the course, you will also visit the Centre for Mathematical Sciences at the University of Cambridge and discover LaTeX, a typesetting system that is widely used amongst mathematicians and physicists to write scientific papers. By the end of the course, you will present a brief research project on a topic that you can choose yourself, and for which you will receive guidance from the instructor during some of the sessions.

3 Topics

Set theory and mathematical proofwriting

We will start of the course by introducing the basics of set theory, a branch of mathematics that lies at the foundation of all other mathematical concepts. We will discuss basic set operations, concepts like cardinality and the power set, and study maps and functions between sets. With this very basic setup, we will then explore the basis of proofwriting by discussing some of the most commonly used methods to prove mathematical theorems - a skill that any good mathematician needs to master.

We finish this topic by making sense of the statement that "there are as many natural numbers as there are rational numbers" in the context of cardinality of infinite sets, and then proceed to show that there really are *more real numbers than rational numbers.*

Set theory: This is the branch of mathematics that studies sets, which can be informally described as collections of objects. All topics in the course will use the concept of a set in one way or another.

Mathematical proof: A formal argument for a mathematical statement that shows that, given certain assumptions, a certain conclusion is guaranteed to follow logically.

Cardinality: Informally described as the "size" of a set. For finite sets, this is simply the number of elements in the set, but for infinite sets, like the natural numbers, this concept becomes more subtle.

- **Lecture 1:** Introduction to sets. Operations on sets, cardinality, the power set, Cartesian product, relations, functions.
- **Lecture 2:** Mathematical reasoning. quantifiers, basics of proof writing, the method of induction, direct proof, proof by contradiction. Application: cardinality of infinite sets.

Group theory and linear algebra

Having studied sets in general, we will now impose some extra structure in the form of an operation on the elements of a set. This leads to the mathematical concept of a group, which is a fundamental concept in mathematics and has many applications in physics and computer science.

We then move on to linear algebra, which is the study of vector spaces and linear transformations between them. A vector space is a group with yet even more structure, which we will define in a precise way. Linear algebra is also inherently associated with matrices, which also have their dedicated lecture.

Linear algebra and groups have tons of applications, some of which we will study, like Google's PageRank algorithm and if time permits the Standard Model of particle physics and the Rubik's Cube.

Group theory: The study of groups, which are sets equipped with a binary operation that satisfies certain properties.

Linear algebra: The study of vector spaces and linear transformations between them.

Matrix: A table of numbers arranged in rows and columns with specific prescriptions for multiplication and inversion. Matrices are ubiquitous in advanced mathematics, physics, computer science and engineering.

- **Lecture 1:** Introduction to groups. Definition, examples, subgroups.
- **Lecture 2:** Linear algebra. Vector spaces, linear transformations.
- **Lecture 3:** Linear algebra. Matrices, determinants, eigenvalues and eigenvectors.
- **Lecture 4:** Application: Google's PageRank algorithm.

Number theory and modular arithmetic

In this shorter topic we will address the properties of numbers, and in particular divisibility and prime numbers. We will focus on modular arithmetic, which is a way of doing arithmetic with a fixed number of integers that wrap around like the numbers on a clock. We then utilize these concepts to discuss some of the basics of cryptography, which is largely based on the special properties that prime numbers have in this modular arithmetic.

Prime number: A natural number greater than 1 for which the only divisors are 1 and itself.

Modular arithmetic: A system of arithmetic for integers, where numbers "wrap around" upon reaching a certain value. This can be compared to the numbers of a clock, where 12 is followed by 1 again instead of 13.

Cryptography: The practice and study of techniques for secure communication in the presence of third parties.

- **Lecture 1:** Divisibility, prime numbers, Modular arithmetic.
- **Lecture 2:** Application: Cryptography.

Fundamentals of calculus

Calculus is the branch of mathematics that revolves around derivatives and integrals. It is fundamental in physics and engineering, and forms the basis of any advanced mathematical curriculum. Parts of this topic will be familiar from high school, like the concept of limits and derivatives (and potentially integrals). We start by bringing everyone on the same page, and then discover some more advanced topics as time permits: partial derivatives, multidimensional integrals, differential equations and their applications in physics ...

Derivative: The rate of change of a function at a certain point. It is defined as the limit of the difference quotient as the difference in the input approaches zero.

Integral: The Riemann integral of a function is the area under the curve of the function. It is defined as the limit of Riemann sums, which approximate the area under the curve by summing up the areas of rectangles.

Differential equation: An equation that relates a function to its derivatives. They are ubiquitous in physics and engineering.

- **Lecture 1:** Limits, continuity, differentiability.
- **Lecture 2:** Mean value theorem, Taylor's theorem, partial derivatives.
- **Lecture 3:** Integrals, fundamental theorem of calculus, higher dimensional integrals.
- **Lecture 4:** Applications: differential equations in physics; Schrödinger equation, (gravitational) wave equation,

Differential geometry and Relativity

The final topic of the course focusses on geometry, and in particular the geometry of curved spaces. We review the basics of Euclidean geometry, which should be familiar from high school, and then move on the the more general setting of differential geometry - where we will need the tools from calculus. We discuss the concept of a manifold, and with it Riemannian geometry and curvature. We then apply all of this to the theories of Special and General Relativity, Einstein's greatest achievements, and discuss how differential geometry lies at its heart.

Riemannian geometry: Branch of differential geometry that studies Riemannian manifolds, which are smooth manifolds equipped with a Riemannian metric.

Special Relativity: The theory of physics that describes the behaviour of objects moving at high speeds, close to the speed of light.

General Relativity: The theory of physics that describes the behaviour of objects in the presence of a gravitational field. At its heart lies the concept of a curved, four-dimensional spacetime manifold.

- **Lecture 1:** Euclidean geometry, metric spaces.
- **Lecture 2:** Manifolds, Riemannian geometry, curvature.
- **Lecture 3:** Application: Lorentzian geometry and Special / General Relativity.

4 LaTeX

Intro to LaTeX, using the Overleaf tutorial. I will then assign the students a small exercise to hand in, that they have to typeset in Latex, e.g. a short proof. Document in Overleaf with details.

5 Preparatory

Materials

- We will try to get the basics down of LaTeX, a typesetting system that is widely used in academia. Make an account on `https://www.overleaf.com/`, and go through the following Tutorial `https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`.
- For most of the material I will assume very little prior knowledge, with the exception being the topic of Calculus. Check Chapters 2,3(1,2, 4-7, 10),4(1-4),7(1-3) of `https://people.math.wisc.edu/~angenent/Free-Lecture-Notes/free221.pdf` to refresh your memory on limits, derivatives, integrals in case you have seen this OR try to get a feeling for the basic concept if you have not seen this before. We will go over this in the classes, but might be high-paced if you have never seen this before.
- Watch the following Veritasium video `https://youtu.be/lFlu60qs7_4?si=wTdhMoXerqpfGbQV` to get a feeling for the concepts we will discover in the Geometry topic. Can you explain why the path a plane follows from London to New York does not look straight on a world map?

Questions

We will briefly discuss the following questions in the first session, to get to know each other.

- What problem in Mathematics do you find most interesting? Why? This can be an open / closed problem. Briefly research this problem.
- What do you hope to gain from the Mathematics Summer School?
- Do you want to study Mathematics at university? Why / why not? Do you want to do something with Mathematics later in life?

6 Schedule

- Day 1: Set Theory and Mathematical Proofwriting.
- Day 2: Algebra Lecture 1 + project exploration.

- Day 3: Algebra Lecture 2 + 3.
- Day 4: Algebra lecture 4 + LaTeX intro.
- Day 5: Number Theory: lecture 1 + 2
- WEEKEND
- Day 6: Calculus: lecture 1 + visit to CMS.
- Day 7: Calculus: lecture 2 + 3.
- Day 8: Calculus lecture 4 + project work.
- Day 9: Geometry: lecture 1 + 2.
- Day 10: Geometry: lecture 3 + presentations.

7 Personal Projects

I will ask the students to give a brief (less than 10 minutes) presentation on a topic of their choice. Below are suggested topics, but the students are free to choose their own topic. The presentation will be followed by a short round of questions, as is usually done in academia. Aside from the presentation, I will also ask them to hand in a short report, typeset in LaTeX. This report should contain

- A bibliography, as well as proper referencing and acknowledgement throughout the entire text.
- A brief introduction / abstract on the topic.
- At least 3 equations / formulae / expressions in *math mode*, to show that they have a basic working of LaTeX
- The discussion of a proof. This does not need to be completely rigorous, as this will be impossible for some topics, but I want to see the outline of an argument of some sorts, i.e. the report cannot just be descriptive. Ideally, however, it should be a complete proof.

7.1 Monty Hall problem

The Monty Hall problem is a famous probability puzzle named after the host of the game show "Let's Make a Deal," Monty Hall. In the problem, a contestant is presented with three doors, behind one of which is a valuable prize, while the other two doors hide goats. The contestant chooses one door, and then Monty, who knows what is behind each door, opens one of the remaining doors to reveal a goat. The contestant is then given the opportunity to switch their choice to the other unopened door or stick with their original choice. Surprisingly, it is statistically advantageous for the contestant to switch doors, as doing so doubles their chances of winning the prize. This counterintuitive result can be explained using conditional probability.

This topic can be studied using Bayes' Theorem, and extensions of the Monty Hall problem can be explored.

7.2 Complex numbers

Complex numbers are ubiquitous in mathematics, physics and engineering. A student can explore the basics of this topic, learning how to use them in calculations. Particularly interesting notions include the complex conjugate, Euler's formula, the fundamental theorem of algebra and the quaternions.

7.3 Continuum hypothesis

7.4 Euler's constant

One could spend an entire course on Euler's number e (not to be confused with the Euler-Mascheroni constant). A student could investigate different formula's for e, and perhaps prove that it is irrational, as done in e.g. `https://youtu.be/xOXsDfMMTjs?si=WElwTf8XjSvhx6Rr`.

7.5 Determinants in general

I have briefly mentioned determinants of matrices, and given formulae for the case of 2×2 and 3×3 matrices. This project could explore the general calculation of determinants, as well as delve deeper into their usecases, in particular in matrix inversion.

7.6 Fields as an algebraic structure

We have explored several algebraic structures, like groups, vector spaces and rings. Fields are another such structure, essentially rings for which every element also has a multiplicative inverse, and are widely studied as well. A student could explore their usecases, and show for example that $\mathbb{Z}_p$ is a field if and only if p is prime. One could also explore vector spaces over general fields, rather than the real numbers.

7.7 Topology

Topology is an abstract branch of mathematics, concerned with *open* and *closed* sets. Open and closed intervals in the real numbers may be familiar, and topology extends this to more general spaces. The student could for example compare the definitions of continuous functions between the fields of topology and calculus. Particularly interesting could be the notion of the *genus*, and how this leads to *a mug being equivalent to a donut*.

7.8 Dual vector spaces

We have briefly discussed the notion of a vector space. It turns out that one can associate a *dual vector space* to every vector space. A student can study this definition, the properties of the dual vector space, and if time permits the bidual vector space and its isomorphism to the original vector space.

7.9 Lotka-Volterra equations

These are a coupled set of equations that form a simple model to describe predator-prey systems in biology. Prior knowledge of derivatives could be useful for this topic.