

# Methods in Computational Neuroscience: Firing rate models and dynamical systems

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## Introduction

This is a brief introduction about the assignment topic.

## 1 Neuron with self-connection

### 1.1 Plot the activation function

The activation function  $f(s) = 60(1 + \tanh(s))$  maps the total input to the neuron's firing rate. This sigmoidal function has several key properties: output range of  $[0, 120]$  spikes per second, centered at  $f(0) = 60$  spikes per second, and asymptotic behavior approaching 0 for very negative inputs and 120 for very positive inputs.

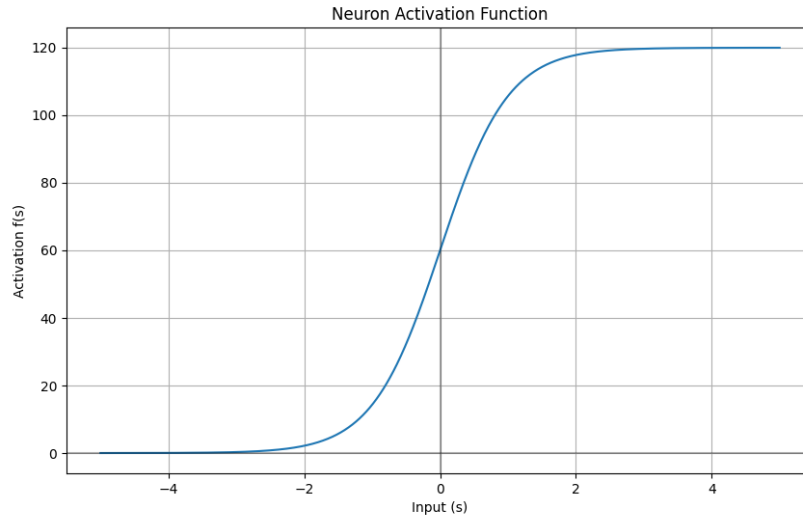


Figure 1: Plot of the activation function  $f(s) = 60(1 + \tanh(s))$  over the input range  $[-5, 5]$ . The function shows the characteristic S-shape of a sigmoid, with output values ranging from 0 to 120 spikes per second.

### 1.2 Study the dynamics

When simulating the system with three initial conditions  $r(0) = 59$ ,  $r(0) = 60$ , and  $r(0) = 61$ , we observe interesting dynamical behavior:

- For  $r(0) = 59$ : The firing rate decreases and settles at the lower stable fixed point.
- For  $r(0) = 60$ : The firing rate remains constant, indicating this is an unstable fixed point.
- For  $r(0) = 61$ : The firing rate increases and settles at the higher stable fixed point.

This behavior indicates the system has three fixed points, with the middle one ( $r \approx 60$ ) acting as a threshold. Initial conditions below this threshold lead to a low firing rate state, while initial conditions above it lead to a high firing rate state. This is characteristic of a bistable system.

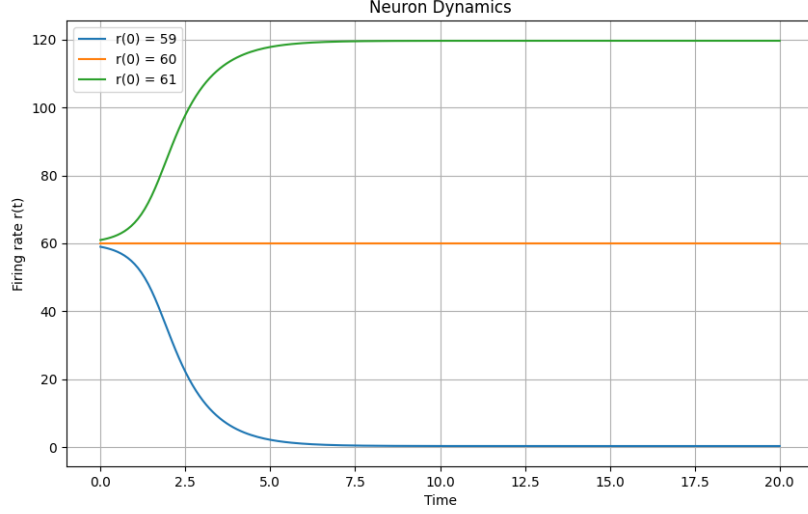


Figure 2: Time evolution of the neuron firing rate  $r(t)$  with three different initial conditions:  $r(0) = 59$ ,  $r(0) = 60$ , and  $r(0) = 61$ . The trajectories demonstrate the bistable nature of the system, with two stable fixed points separated by an unstable threshold at  $r \approx 60$ .

### 1.3 Add noise - NOT FINAL

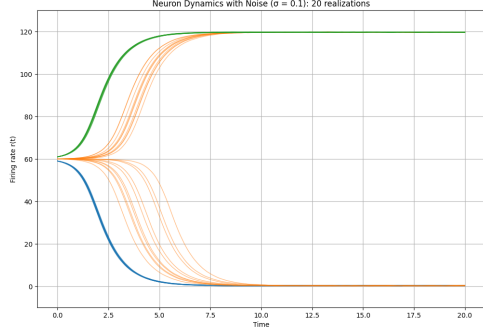
Adding noise to the system reveals additional insights:

$$\frac{dr(t)}{dt} = -r(t) + f(wr(t) + I) + \sigma\eta(t) \quad (1)$$

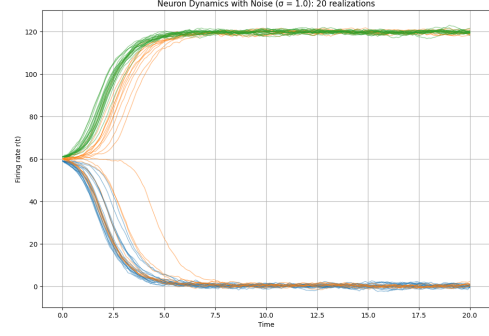
Where  $\eta(t)$  is Gaussian white noise and  $\sigma$  is the noise magnitude.

- **(a) Low noise** ( $\sigma = 0.1$ ): The system mostly maintains its bistable behavior, exhibiting only small stochastic excursions around the two attracting fixed points. A trajectory that starts near  $r_0 \approx 60$  Hz drifts away because this intermediate firing rate is an *unstable* fixed point, so the rate does *not* remain constant there.
- **(b) Medium noise** ( $\sigma = 1.0$ ): Larger fluctuations occasionally push trajectories across the separatrix, so the neuron sometimes switches between low- and high-firing states.
- **(c) High noise** ( $\sigma = 5.0$ ): Noise dominates the dynamics; trajectories hop back and forth between the two attractors so frequently that the system effectively loses any memory of its initial condition.

This demonstrates how noise can induce transitions between stable states in a bistable system, with the frequency of transitions increasing with noise magnitude.



(a) Low noise ( $\sigma = 0.1$ )



(b) Medium noise ( $\sigma = 1.0$ )



(c) High noise ( $\sigma = 5.0$ )

Figure 3: Effect of different noise levels on the neuron dynamics. As noise magnitude increases, transitions between stable states become more frequent, demonstrating how noise can induce state switching in a bistable system.

## 1.4 Study the flux

The flux analysis examines the derivative  $\frac{dr}{dt}$  as a function of the firing rate  $r$ . The plot of this flux function reveals:

- Three zero-crossings (fixed points) where  $\frac{dr}{dt} = 0$ :
  1. Lower fixed point: Stable (negative slope at crossing)
  2. Middle fixed point: Unstable (positive slope at crossing)
  3. Upper fixed point: Stable (negative slope at crossing)

The stability of these fixed points can be determined by the slope of the flux function at each crossing:

- Negative slope: Stable fixed point (perturbing the system slightly causes it to return to the fixed point)
- Positive slope: Unstable fixed point (perturbing the system slightly causes it to move away from the fixed point)

These findings confirm our observations from the time series simulations, providing a deeper understanding of the bistable nature of the system.

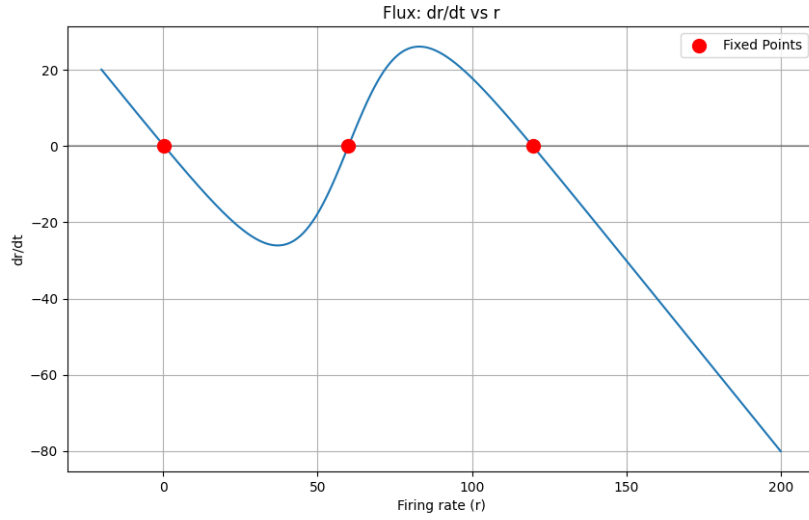


Figure 4: Flux analysis showing  $\frac{dr}{dt}$  as a function of firing rate  $r$ . The zero-crossings represent fixed points of the system. The middle fixed point has a positive slope (unstable), while the other two have negative slopes (stable), confirming the bistable nature of the system.

## 1.5 Bifurcation diagram

The bifurcation diagram explores how the number and nature of fixed points change as system parameters (synaptic strength  $w$  and external input  $I$ ) vary. The analysis reveals:

- For low values of  $w$  (weak self-connection): The system has a single stable fixed point.
- For higher values of  $w$ : The system exhibits bistability with three fixed points (two stable, one unstable).

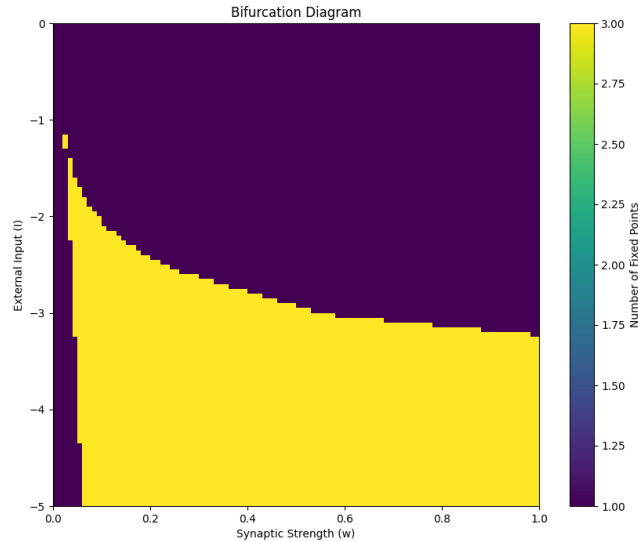


Figure 5: Bifurcation diagram showing how the number of fixed points changes with synaptic strength ( $w$ ) and external input ( $I$ ). Different colors represent different numbers of fixed points. The boundaries between regions represent bifurcation points where the qualitative behavior of the system changes.

## 2 2D network: mutual excitation

### 2.1 Plot the dynamics

The time evolution of the firing rates  $x(t)$  and  $y(t)$  was simulated for different initial conditions. The differential equations were integrated numerically using the Euler method with a time step of  $dt = 0.1$ .

Figure 6 shows the evolution of the firing rates over time for four different initial conditions.

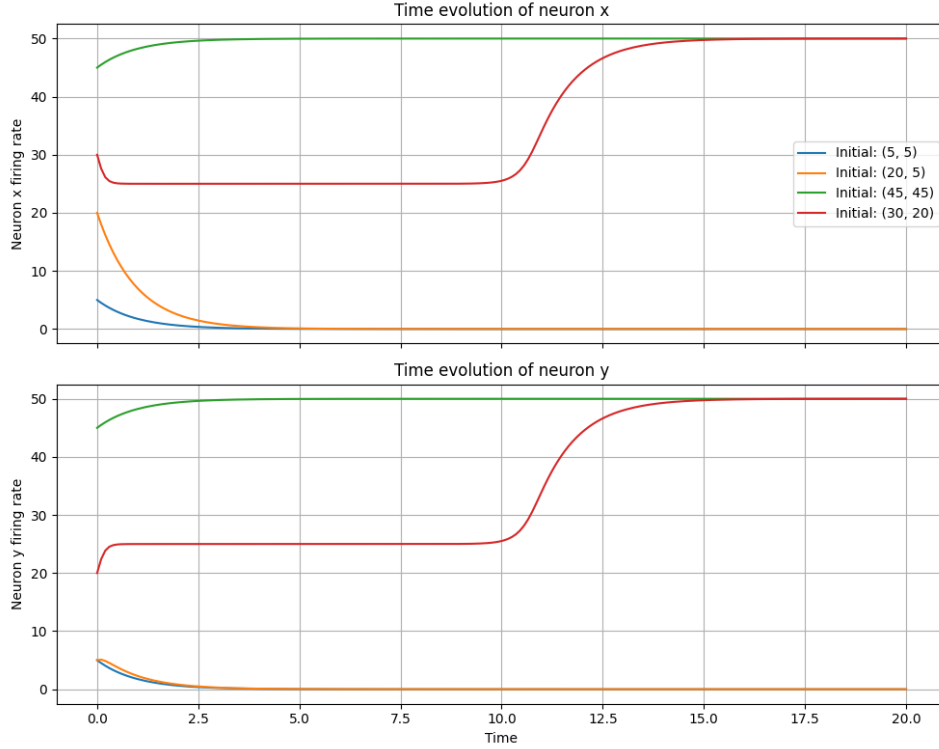


Figure 6: Time evolution of the firing rates  $x(t)$  and  $y(t)$  for different initial conditions. The top panel shows the firing rate of neuron  $x$ , and the bottom panel shows the firing rate of neuron  $y$ .

From the time evolution plots, we can observe that:

- Initial conditions starting near  $[5, 5]$  converge to zero firing rates (both neurons shut down).
- Initial conditions starting near  $[45, 45]$  converge to a high firing rate fixed point.
- The middle-low and mixed initial states show interesting transient dynamics before eventually converging to either zero firing rate or a high firing rate fixed point.

This suggests that the system exhibits bistability, with two stable fixed points (attractors) and a saddle point that separates their basins of attraction.

### 2.2 State space

The trajectories in state space (plotting  $x$  versus  $y$ ) provide a more comprehensive view of the dynamics. Figure 7 shows these trajectories for seven different initial conditions.

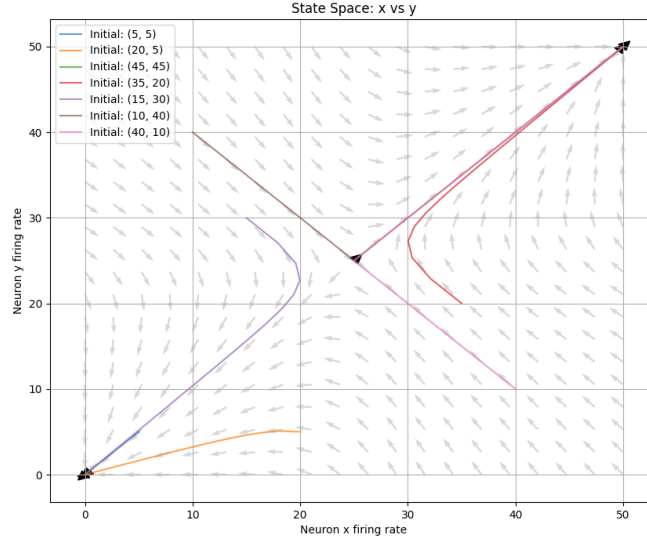


Figure 7: State space trajectories for different initial conditions. Arrows indicate the direction of movement along each trajectory. They show that trajectories starting on one side of the saddle's stable manifold flow to the low-rate rest state, while those on the other side are driven to the high-rate self-excited state, revealing bistability.

### 2.3 Nullclines

Nullclines are curves in the state space where the derivative of one variable is zero. For this system:

- $x$ -nullcline:  $\frac{dx}{dt} = 0 \implies x = f(w_2 y + I) = f(0.4y - 10)$
- $y$ -nullcline:  $\frac{dy}{dt} = 0 \implies y = f(w_1 x + I) = f(0.4x - 10)$

Figure 8 shows the nullclines plotted in the state space.

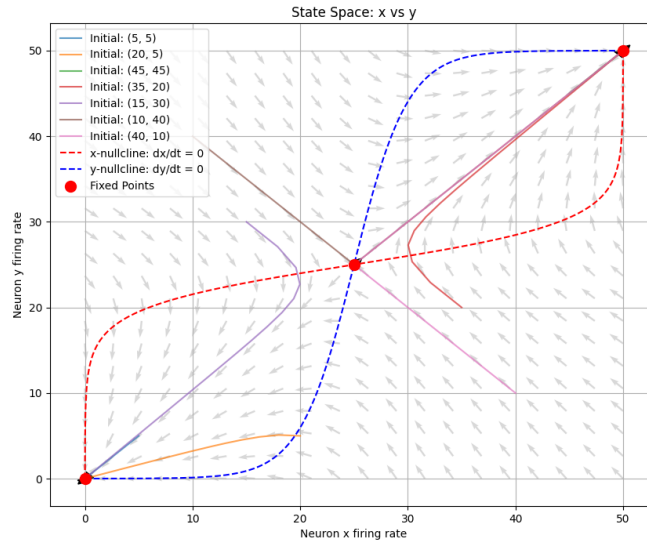


Figure 8: Nullclines of the system. The red dashed line represents the  $x$ -nullcline ( $dx/dt = 0$ ), the blue dashed line represents the  $y$ -nullcline ( $dy/dt = 0$ ), and red dots represent fixed points.

Along the red  $x$ -nullcline ( $\dot{x} = 0$ ) the horizontal flow changes sign— arrows point left ( $\dot{x} < 0$ ) to its left and right ( $\dot{x} > 0$ ) to its right— while along the blue  $y$ -nullcline ( $\dot{y} = 0$ ) the vertical flow is upward ( $\dot{y} > 0$ ) below and downward ( $\dot{y} < 0$ ) above. Because the sigmoid is S-shaped, the nullclines bend sharply around the threshold  $0.4(x \text{ or } y) - 10 = 25$ .

## 2.4 Fixed points

The fixed points of the system are located at the intersections of the nullclines (Figure 8). Solving the system numerically:

$$\begin{cases} x = f(0.4y - 10) \\ y = f(0.4x - 10) \end{cases} \quad (2)$$

We find three fixed points:

1. Low firing rate fixed point: approximately at  $[x, y] \approx [0, 0]$
  2. Middle firing rate fixed point: approximately at  $[x, y] \approx [25, 25]$
  3. High firing rate fixed point: approximately at  $[x, y] \approx [50, 50]$
- **Stable attractors.** Trajectories drift toward the low-rate node near zero and toward the high-rate node close to saturation, demonstrating that both equilibria are stable.
  - **Unstable saddle.** At moderate activity the trajectories reach a fixed point that is stable along one manifold ( $y = -x$ ) and unstable along the orthogonal one ( $y = x$ ). This exactly aligns with the signature of a saddle.

The state-space therefore contains two attracting fixed points—one at low ( $x = y \approx 0$ ) and one at high ( $x = y \approx 50$ ) firing rates separated by an unstable saddle at  $x = y = 25$ . State-space trajectories illustrate how every initial condition eventually settles into one of the two attractors, with the saddle acting as a boundary between them.

## 2.5 Study the stability of the fixed points analitically

To analyze the stability of the fixed points analytically, we compute the Jacobian matrix of the system:

$$J = \begin{bmatrix} \frac{\partial}{\partial x}(-x + f(w_2y + I)) & \frac{\partial}{\partial y}(-x + f(w_2y + I)) \\ \frac{\partial}{\partial x}(-y + f(w_1x + I)) & \frac{\partial}{\partial y}(-y + f(w_1x + I)) \end{bmatrix} = \begin{bmatrix} -1 & w_2f'(w_2y + I) \\ w_1f'(w_1x + I) & -1 \end{bmatrix} \quad (3)$$

where  $f'(s) = \frac{df}{ds} = 50\sigma(s)(1 - \sigma(s))$ , the derivative of our activation function. For each fixed point  $(x^*, y^*)$ , we evaluate the Jacobian and determine its eigenvalues:

$$\lambda_{1,2} = \frac{\text{tr}(J)}{2} \pm \sqrt{\frac{(\text{tr}(J))^2}{4} - \det(J)}$$

The stability of each fixed point is determined by the eigenvalues:

- If all eigenvalues have negative real parts, the fixed point is stable (attractor)
- If any eigenvalue has a positive real part, the fixed point is unstable
- If eigenvalues have opposite signs, the fixed point is a saddle point

Fixed point classification based on the determinant and trace:

- $\det(J) < 0$ : Saddle point (unstable)
- $\det(J) > 0, \text{tr}(J) < 0$ : Stable node or spiral
- $\det(J) > 0, \text{tr}(J) > 0$ : Unstable node or spiral

At the low fixed point  $(x^*, y^*) \approx (0, 0)$ , the derivatives  $f'$  are nearly zero because inputs are far below threshold, making  $\det(J) \approx 1$  and  $\text{tr}(J) \approx -2$ , yielding eigenvalues  $\lambda_{1,2} \approx -1$ . This confirms it's a stable node.

At the middle fixed point  $(x^*, y^*) \approx (25, 25)$ , the derivatives  $f'$  are near maximum, and with  $w_1 = w_2 = 0.4$ , we get  $\det(J) = -24 < 0$ , confirming it's a saddle point with one positive (4) and one negative (-6) eigenvalue.

At the high fixed point  $(x^*, y^*) \approx (50, 50)$ , similar to the low fixed point, the derivatives are small as inputs are far above threshold, giving  $\det(J) > 0$  and  $\text{tr}(J) < 0$ , confirming it's a stable node.

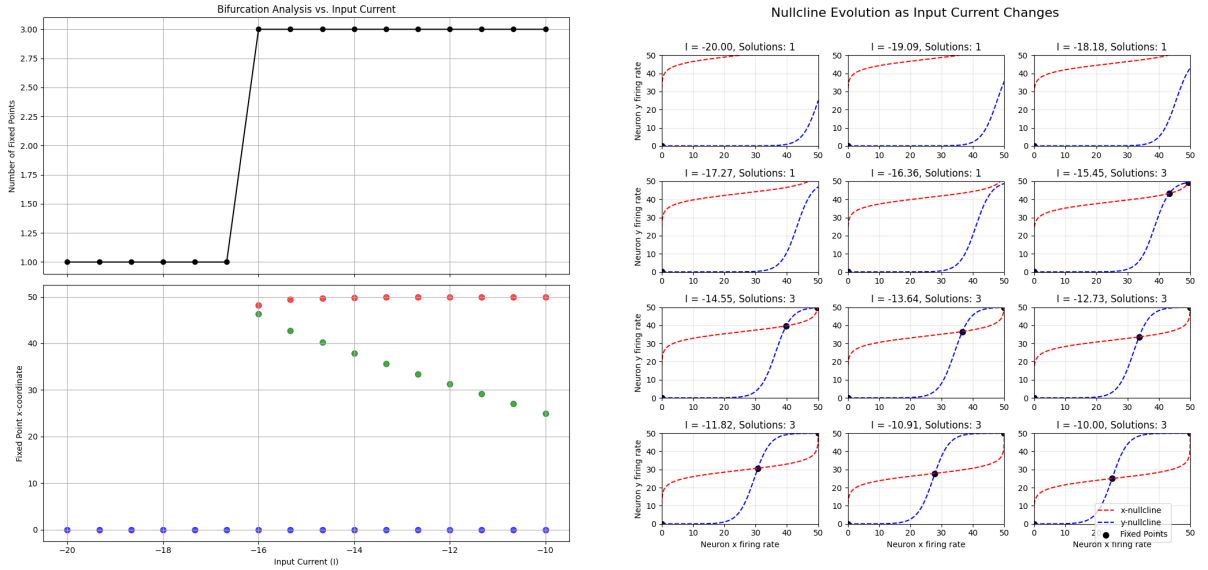
Fixed Point	Eigenvalues	Determinant	Classification
Low (0, 0)	Both negative	Positive	Stable node
Middle (25, 25)	One positive, one negative	Negative	Saddle point
High (50, 50)	Both negative	Positive	Stable node

Table 1: Stability analysis results for the three fixed points. The bistable behavior arises from having two stable nodes separated by a saddle point.

This analytical stability analysis confirms our observations from the state space trajectories and explains the system's bistable behavior, with the saddle point forming the boundary between the basins of attraction of the two stable states.

## 2.6 Bifurcation diagram

The bifurcation diagram shows how the number and position of fixed points change as the external input current  $I$  varies from  $-20$  to  $-10$ .



(a) Bifurcation diagram showing the number of fixed points (top) and the  $x$ -coordinates of fixed points (bottom) as a function of external input  $I$ .

(b) Evolution of nullclines as  $I$  changes. The intersections of nullclines are fixed points of the system, showing how they emerge with changing input.

Figure 9: Analysis of system dynamics as a function of external input current  $I$ .

From the bifurcation diagram, we observe how the fixed point structure changes with input current  $I$ . The bifurcation point occurs at approximately  $I \approx -16$  where saddle-node bifurcation takes place.

From the bifurcation diagram, we can identify distinct regimes:

- For very negative input ( $I < -16$  approximately), there is only one stable fixed point with zero firing rate.



- For less negative input values ( $-16 < I$  approximately), there are three fixed points: two stable (low (zero) and high firing rates) and one unstable (saddle point).

The system exhibits bistability in the intermediate range of input currents.

### **3 ND network: Hopfield neural network**

### **4 Conclusion**

Summary of main findings and their significance.