Computational Neuroscience

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Exercise Sheet 3 — 12 May 2025

Send your solution to Pierre by email on June 2nd at 4 PM the latest.

Advanced exercises count for extra points. You can still get a very high grade, but not a maximum, if you do not solve them.

1. **Hopfield Networks – Storing Multiple Patterns.**

   A Hopfield network with N binary neurons ($s_i \in \{-1, +1\}$) can be used to store patterns using the following learning rule. Suppose we want to store two patterns $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, both vectors of length N. The synaptic weight matrix $W = [w_{ij}]$ is defined as:

   $$w_{ij} = \frac{1}{N} (\xi_i \xi_j + \eta_i \eta_j)$$

   The update rule for neuron i in the network is:

   $$y_i = \mathrm{sgn}\left(\sum_{j=1}^{N} w_{ij} y_j \right)$$

   (a) Show that the pattern $\boldsymbol{\xi}$ is a fixed point of the dynamics assuming orthogonality between the two patterns:

   $$\sum_{j=1}^{N} \xi_j \eta_j = 0$$

   (b) Prove that under the same orthogonality assumption, the pattern $\boldsymbol{\eta}$ is also a fixed point of the dynamics.

   (c) Explain intuitively why the orthogonality condition between stored patterns helps reduce interference in the Hopfield network.

2. **Rate-based Dynamics: Firing Rate vs. Membrane Potential.**

   Consider the two alternative formulations of neural dynamics in a recurrent network that we discussed in class:

   $$\dot{r} = -r + \phi(Jr) \qquad \text{or} \qquad \dot{v} = -v + J\phi(v)$$

Show that the two differential equations are mathematically equivalent under the change of variables $v = Jr$.

3. **Leak in Linear Dynamical Systems.**

   A leak term introduces a decay force into the dynamics of neural activity, stabilizing the system. Consider the continuous-time linear dynamical system with leak:

   $$\dot{x}(t) = -x(t) + Jx(t)$$

   As discussed in class, we can rewrite the system in the form $\dot{x}(t) = Ax(t)$, in which case $A = I - J$.

   (a) Explain how the leak term affects the eigenvalues of the system matrix. In particular, if J has eigenvalues λ_i, what are the eigenvalues of A?

   (b) What is the condition on the eigenvalues of J such that the full system is stable (i.e., all modes decay to zero over time)? What happens if the leak is removed (i.e., $\dot{x} = Jx$)?

   tip: use python to generate random matrices and to calculate its eigenvalues to confirm your intuitions.

   (a) **Nonlinear RNN for cognitive tasks.**

   (a) Use the tutorials linked in the class slides to train an RNN on a task of your choice. Describe the task. How do you know if your training worked?

   (d) **(advanced)** After training, do some decoding and/or PCA analyses on the neural activity and try to come up with an explanation of how the network is solving the task.

   (e) **(very advanced)** Use the fixed-point finder toolbox to find fixed points in your RNN dynamics. Can you use it to validate what you found in (a)?