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Positive polynomials on compact sets

Abstract. The aim of this note is to give short algebraic proofs of theorems of Handelman, Pólya and Schmüdgen concerning the algebraic structure of polynomials being positive on certain subsets of $\mathbb{R}^n$. The main ingredient of the proofs is the representation theorem of Kadison–Dubois. The proof of the latter is elementary and algebraic but tricky.

1. Introduction

Let $f_1, \ldots, f_m \in \mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$ be polynomials and

$$S = \{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \ldots, f_m(x) \geq 0\}$$

the associated closed semialgebraic set. In this paper we are interested in the polynomials which are strictly positve on S. The Positivstellensatz of semialgebraic geometry (see e.g. [KS]) gives a characterization of these polynomials. Namely, let

$$\langle f_1, \ldots, f_m \rangle_2 := \left\{ \sum_{\varepsilon \in \{0,1\}^m} g_\varepsilon f_1^{\varepsilon_1} \cdot \ldots \cdot f_m^{\varepsilon_m} \mid g_\varepsilon \in \sum \mathbb{R}[X]^2 \right\},$$

where $\sum \mathbb{R}[X]^2$ denotes the set of sums of squares in $\mathbb{R}[X]$. Given $f \in \mathbb{R}[X]$, then

$$f_{|S} > 0 \iff \exists p, q \in \langle f_1, \ldots, f_m \rangle_2 : f = \frac{1+p}{1+q}. \tag{1}$$

Let us first consider the case $S = \mathbb{R}^n$. Then (1) tells us that every polynomial f which is strictly positve on $\mathbb{R}^n$ admits a representation

$$f = \frac{1 + \sum g_i^2}{1 + \sum h_i^2}$$

for some $g_i, h_i \in \mathbb{R}[X]$. If $n = 1$, then obviously every strictly positive polynomial on $\mathbb{R}$ is a sum of squares of polynomials. Hilbert already proved that for $n \geq 2$ this

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fails but he didn't give counterexamples. In 1967 such were given by Motzkin for all $n \geq 2$ (see [Mo]). The simplest one is

$$f = 2 + X^2 Y^2 (X^2 + Y^2 - 1).$$

One readily checks that f is strictly positve on $\mathbb{R}^2$. Now assume by way of contradiction that there is a representation

$$f = \sum_{i=1}^{k} g_i^2 \qquad (2)$$

with $g_i \in \mathbb{R}[X]$. From $f(0, Y) = 2 = f(X, 0)$ we infer that g_i is of the form

$$g_i = \varepsilon_i + XY(a_i X + b_i Y + c_i).$$

Hence the coefficient of $X^2 Y^2$ on the right hand side of (2) is $\sum c_i^2$ which gives the desired contradiction.

Therefore, in general the denominator that appears in the representation (1) is necessary. This leads to the problem whether we can get more information about the occuring denominators or in which situations they are not needed at all. These questions are of interest for semialgebraic geometry. But they play a role in other branches of mathematics as well.

As an example let us consider the K-moment problem. Let $K \subset \mathbb{R}^n$ be a nonempty closed set. The problem is to characterize those $\mathbb{R}$-linear functionals

$$L : \mathbb{R}[X] \longrightarrow \mathbb{R}$$

for which there exists a non-negative Borel-measure μ on K such that

$$L(f) = \int_K f \, d\mu$$

for all $f \in \mathbb{R}[X]$. By Haviland's theorem (cf. [B, Theorem 1']) such a measure exists if and only if

$$L(f) \geq 0 \text{ for all } f \in \{g \in \mathbb{R}[X] \mid g_{|K} \geq 0\}. \qquad (3)$$

But one easily verifies that L satisfies (3) if and only if $L(f) \geq 0$ for all $f \in \mathbb{R}[X]$ which are strictly positive on K. Hence a characterization of the strictly positive polynomials is of interest for the K-moment problem as well.

In 1991 K. Schmüdgen solved the K-moment problem for compact semialgebraic subsets of $\mathbb{R}^n$. As a consequence he obtained for these sets a surprising improvement of (1). Assume that our semialgebraic set

$$S = \{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \ldots, f_m(x) \geq 0\}$$

is compact and let $f \in \mathbb{R}[X] = \mathbb{R}[X_1, \ldots, X_n]$. Then

$$f_{|S} > 0 \Leftrightarrow \exists \, 0 < \varepsilon \in \mathbb{Q}, \, p \in \langle f_1, \ldots, f_m \rangle_2 : \ f = \varepsilon + p. \qquad (4)$$

In this paper we will give short algebraic proofs for this theorem and other "Stellensätze", where more information about the appearing denominator can be found. It is our goal to show that all these results are consequences of the representation theorem of Kadison and Dubois (cf. [BS]).

2. Preliminaries

All rings R are commutative with 1 and contain $\mathbb{R}$. $\sum R^2$ denotes the set of all sums of squares in R. A subset $P \subset R$ is called a semiring iff

$$0, 1 \in P, \ P + P \subset P, \ P \cdot P \subset P.$$

P is called an (infinite) preprime if $-1 \notin P$. A preordering $P \subset R$ (of level 1) is a preprime which contains R^2.

Example 1. Let $S \subset \mathbb{R}^n$ be a non-empty set and $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$. Then

$$P(S)^> := \{f \in \mathbb{R}[X] \mid f_{|S} > 0\} \cup \{0\}$$

is a preprime and

$$P(S) := \{f \in \mathbb{R}[X] \mid f_{|S} \geq 0\}$$

is a preordering of $\mathbb{R}[X]$.

A preprime $P \subset R$ is called archimedean if for all $a \in R$ there exists an $N \in \mathbb{N}$ such that

$$N \pm a \in P.$$

Example 2. Let $S \subset \mathbb{R}^n$ be a non-empty closed set. Then $P(S)$ and $P(S)^>$ are archimedean if and only if S is compact.

Given a preprime $P \subset R$ let

$$X(P) := \{\phi \in \operatorname{Hom}(R, \mathbb{R}) \mid \phi(P) \subset \mathbb{R}_+\}.$$

Then every element $f \in R$ defines a function

$$\hat{f} : X(P) \longrightarrow \mathbb{R} : \phi \mapsto \phi(f).$$

We impose on $X(P)$ the weak topology with respect to all functions $\hat{f}$. Then we get a representation

$$\Phi : R \longrightarrow C(X(P), \mathbb{R}) : f \mapsto \hat{f}$$

whose properties are characterized by the representation theorem of Kadison–Dubois in the case of archimedean preprimes. We will use only the following special aspect of this principal result:

Theorem 1 (Kadison–Dubois). *Let $P \subset R$ be an archimedean preprime and $f \in R$. Then the following statements hold:*

(1) $X(P)$ *is a nonempty and compact space.*
(2) $\forall \phi \in X(P) : \ \phi(f) \geq 0 \Leftrightarrow \forall N \in \mathbb{N} \, \exists k \in \mathbb{N} : \ k(1 + Nf) \in P.$

Proof. See [BS, Hauptsatz]. □

Given an affine $\mathbb{R}$-algebra $R = \mathbb{R}[X_1, \ldots, X_n]/\mathfrak{a}$ whith $\mathfrak{a} = (a_1, \ldots, a_k)$ and a preprime $P \subset R$ we set

$$V_{\mathbb{R}}(\mathfrak{a}) = \{x \in \mathbb{R}^n \mid a_1(x) = \ldots = a_k(x) = 0\}$$

and

$$S(P) := \{x \in V_{\mathbb{R}}(\mathfrak{a}) \mid f(x) \geq 0 \text{ for all } f \in P\}.$$

Then the representation space $X(P)$ can be canonically identified with the closed set $S(P)$ via

$$S(P) \longrightarrow X(P) : x \mapsto e_x,$$

where $e_x : R \to \mathbb{R}$, $f \mapsto f(x)$. In this situation we get the following consequence of Theorem 1:

Corollary 1. *Let $R = \mathbb{R}[X_1, \ldots, X_n]/(a_1, \ldots, a_k)$ and $P \subset R$ an archimedean preprime with $\mathbb{Q}_+ \subset P$. Then for all $f \in R$ we have:*

$$f_{|S(P)} > 0 \iff \exists\, 0 < \varepsilon \in \mathbb{Q},\ p \in P : f = \varepsilon + p.$$

Proof. As just mentioned, $S(P)$ is homeomorphic to $X(P)$. Hence $S(P)$ is non-empty and compact, by Theorem 1. Now assume that f is strictly positive on $S(P)$. Then there exists $N \in \mathbb{N}$ s.t. $(f - \frac{1}{N})_{|S(P)} > 0$. By Theorem 1 we find $k \in \mathbb{N}$ such that

$$k\left(1 + N\left(f - \frac{1}{N}\right)\right) = k \cdot N \cdot f \in P.$$

By assumption $\mathbb{Q}_+ \subset P$. Hence every g strictly positive on $S(P)$ belongs to P. In particular, $f - \frac{1}{N} \in P$, by the choice of N. Thus

$$f = \frac{1}{N} + p$$

for some $p \in P$. The converse direction is trivial. $\square$

Remark 1. If P does not contain $\mathbb{Q}_+$, the same argument as above shows:

$$f_{|S(P)} > 0 \iff \exists\, 0 < \varepsilon \in \mathbb{Q},\ k \in \mathbb{N},\ p \in P : kf = \varepsilon + p.$$

Lemma 1. *Let $R = \mathbb{R}[y_1, \ldots, y_n]$ be an affine $\mathbb{R}$-algebra, $P \subset R$ a preprime. Then*

$$P \text{ is archimedean} \iff \exists N \in \mathbb{N}\ \forall i = 1 \ldots n : N \pm y_i \in P.$$

Proof. We only prove the nontrivial direction. Let $a, b \in R$ with

$$N \pm a \in P, M \pm b \in P$$

for some $N, M \in \mathbb{N}$ (a and b are called P-bounded). Then

$$(N + M) \pm (a + b) \in P$$

and

$$NM \mp ab = \frac{1}{2}\left((N \pm a) \cdot (M - b) + (N \mp a) \cdot (M + b)\right).$$

This means, products and sums of P-bounded elements are P-bounded. Since the y_i are P-bounded, every polynomial expression in the y_i is P-bounded, hence P is archimedean. $\square$

Lemma 2. *Let* $R = \mathbb{R}[y_1, \ldots, y_n]$ *be an affine* $\mathbb{R}$-*algebra and* $P \subset R$ *a preordering. Then*

$$P \text{ is archimedean} \iff \exists N \in \mathbb{N}: \; N - \sum y_i^2 \in P.$$

Proof. The nontrivial direction follows from Lemma 1, the identity

$$\frac{1}{2}N + \frac{1}{2} \pm y_j = \frac{1}{2N} \left[(N \pm y_j)^2 + (N - \sum_{i=1}^{n} y_i^2) + \sum_{i=1, i\neq j}^{n} y_i^2 \right] \in P$$

and the fact $\mathbb{Q}_+ = \sum \mathbb{Q}^2 \subset P$. □

3. The theorems

In this section we will show that several results on positive polynomials – proven in different branches of mathematics – are in fact a consequence of Corollary 1.

Given $g_1, \ldots, g_m \in R$ we let $\langle g_1, \ldots, g_m \rangle$ (resp. $\langle g_1, \ldots, g_m \rangle_2$) denote the semiring generated by $g_1, \ldots, g_m$ and $\mathbb{R}_+$ (resp. R^2), i.e.

$$\langle g_1, \ldots, g_m \rangle = \left\{ \sum_{i \in I} a_i g_1^{i_1} \cdot \ldots \cdot g_m^{i_m} \mid I \subset \mathbb{Z}_+^m \text{ finite }, a_i \in \mathbb{R}_+ \right\}.$$

and

$$\langle g_1, \ldots, g_m \rangle_2 = \left\{ \sum_{\epsilon \in \{0,1\}^m} h_\epsilon g_1^{\epsilon_1} \cdot \ldots \cdot g_m^{\epsilon_m} \mid h_\epsilon \in \sum R^2 \right\}.$$

As an immediate consequence of Corollary 1 and Lemma 2 we get

Corollary 2. *Let* $f \in \mathbb{R}[X_1, \ldots, X_n]$ *and* $D^n \subset \mathbb{R}^n$ *the unit ball. Then the following statements hold:*

(i) $f_{|[-1,1]^n} > 0 \Longrightarrow f \in \langle 1 - X_1^2, \ldots, 1 - X_n^2 \rangle_2$.
(ii) $f_{|D^n} > 0 \Longrightarrow f \in \langle 1 - \sum_{i=1}^{n} X_i^2 \rangle_2$.

The following theorem is due to D. Handelman (see [H1, Theorem VI.4] for this and a generalization to compact convex polyhedra).

Theorem 2. *Let* $f \in \mathbb{R}[X_1, \ldots, X_n]$. *Then the following statements hold:*

(i) $f_{|[-1,1]^n} > 0 \Rightarrow f \in \langle 1 - X_1, \ldots, 1 - X_n, 1 + X_1, \ldots, 1 + X_n \rangle$.
(ii) *Let* $S := \{x \in \mathbb{R}^n \mid x_1 \geq 0, \ldots, x_n \geq 0, \; 1 - \sum_{i=1}^{n} x_i \geq 0\}$.

$$f_{|S} > 0 \Rightarrow f \in \langle X_1, \ldots, X_n, 1 - \sum_{j=1}^{n} X_j \rangle.$$

Proof. The preprimes above are archimedean by Lemma 1. The sets $[-1, 1]^d$ and S are the associated $S(P)'s$. Let $f \in \mathbb{R}[X_1, \ldots, X_n]$. If f is positive on an $S(P)$ then f is in the according preprime by Corollary 1. □

Remark 2. Handelman's proof uses techniques developed in the context of actions of compact groups on AF algebras. In particular it uses a result on certain parially ordered groups ([H1, Theorem I.1]) which is similar to Theorem 1 above.

Remark 3. Let $S \subset \mathbb{R}^n$ be still the standard simplex. If f is a polynomial with integer coefficients, then Theorem 3 can be refined. Let $R = \mathbb{Z}[X_1, \ldots, X_n]$ and $P \subset R$ the preprime generated by

$$X_1, \ldots, X_n \text{ and } 1 - \sum_{i=1}^{n} X_i.$$

Handelman shows that (R, P) can be regarded as a bounded subring of a limit of dimension groups. From this fact he derives that for all $f \in R$ and $k \in \mathbb{N}$:

$$k \cdot f \in P \Rightarrow f \in P. \tag{*}$$

This implies the following result:

$$\forall f \in R : f_{|S} > 0 \Rightarrow f \in P.$$

In view of Remark 1 and (∗) this follows from Theorem 1 as well.

Handelman's result can also be deduced from a theorem of Pólya ([P]). Pólya's original proof is elementary but tedious. We will give another proof of this theorem which is again based on Theorem 1.

Theorem 3 (Pólya). *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be homogeneous and $f(x) > 0$ for all $x \in S'$,*

$$S' := \{x \in \mathbb{R}^n \mid x_1 \geq 0, \ldots x_n \geq 0, \sum x_i \neq 0\}.$$

Then for some $N \in \mathbb{N}$, $\left(\sum X_i\right)^N \cdot f$ is a positive linear combination of power products of the X_i. In other words

$$\left(\sum X_i\right)^N \cdot f \in < X_1, \ldots, X_n > .$$

Proof. Consider the ring $R := \mathbb{R}[Y_1, \ldots, Y_n]/(1 - \sum Y_i)$ and the preprime $P := < y_1, \ldots, y_n >$ generated by the residues y_i of the Y_i. By Lemma 1 P is archimedean and we have

$$S(P) = \left\{y \in \mathbb{R}^n \mid y_1 \geq 0, \ldots, y_n \geq 0, \sum y_i = 1\right\}.$$

With Corollary 1 we get $f(y_1, \ldots, y_n) \in P$. This means that as functions on S we have the equality

$$f(y_1, \ldots, y_n) = \sum a_i y_1^{i_1} \cdot \ldots \cdot y_n^{i_n} \ (a_i \in \mathbb{R}_+),$$

with $a_i = 0$ for almost all i. Now $x / \sum x_i \in S(P)$ for all $x \in S'$. Plugging this in for y and clearing denominators, we get:

$$\left(\sum x_i\right)^N \cdot f(x) = \sum b_i x_1^{i_1} \cdot \ldots x_n^{i_n} \text{ (for some } b_i \geq 0)$$

as functions on S'. Since S' is a Zariski dense subset of $\mathbb{R}^n$, this is also an identity of polynomials, hence Pólya's theorem.   □

Theorem 4 (Schmüdgen). *Assume the basic closed semialgebraic set* $S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m \geq 0\}$ *to be compact. Then for any* $f \in \mathbb{R}[X_1, \ldots, X_n] =: R$ *we have*

$$f_{|S} > 0 \Rightarrow f \in P := \langle g_1, \ldots, g_n \rangle_2.$$

Proof. Let us assume first that $S = \emptyset$. Then -1 is strictly positive on S, and by the Positivstellensatz we find $p, p' \in P$ s.t. $(1 + p) \cdot (-1) = 1 + p'$. Hence $-1 = 1 + p' + p \in P$. Since every element a of R is a difference of two squares $(a = \frac{(a+1)^2}{4} - \frac{(a-1)^2}{4})$, we get $R = P$ and the statement of the theorem is trivially true.

Now consider the case $S \neq \emptyset$. Then P is a preordering. Assume w.l.o.g. that

$$S \subset \{x \in \mathbb{R}^n \mid \sum x_i^2 < 1\}.$$

The extended preorder

$$P' := P + (1 - \sum X_i^2)P$$

is archimedean by Lemma 2. Corollary 1 assures us that every f positive on S belongs to P'. Due to the Positivstellensatz, there exist $p, p' \in P$ with

$$(1 + p) \cdot (1 - \sum X_i^2) = 1 + p'.$$

By adding $p \cdot \sum X_i^2$ to the left-hand side we see that

$$1 - \sum X_i^2 + p \in P. \tag{*}$$

On the other hand we have seen that any polynomial g positive on S belongs to P'. This shows that $(1 + p)g$ belongs to P for any such polynomial g. Since S is compact, there exists $N \in \mathbb{N}$ such that $(N - p)_{|S} > 0$. This yields

$$(1 + p)(N - p) = N + (N - 1)p - p^2 \in P. \tag{**}$$

Finally, the following square lies in P

$$(N/2 - p)^2 = N^2/4 - Np + p^2 \in P. \tag{***}$$

Adding these three elements of P, we get

$$(*) + (**) + (***) = (1 + N + N^2/4) - \sum X_i^2 \in P.$$

Now Lemma 2 shows that P itself is archimedean. Therefore, by Corollary 1 any polynomial positive on S belongs to P. $\square$

Remark 4. This result was originally proved by Schmüdgen in [S]. He obtained it as a consequence of his solution of the K-moment problem for compact semialgebraic sets which is essentially based on the theory of Hilbert spaces and spectral measures.

As A. Prestel observed, the solution of the K-moment problem for compact semialgebraic sets is in fact equivalent to Theorem 4.

Corollary 3. *Let* $L : \mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n] \longrightarrow \mathbb{R}$ *be a linear functional and*

$$K := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m \geq 0\}$$

a compact semialgebraic set. Then the follwing statements are equivalent:

(1) *There exists a nonnegative Borel measure μ on K such that for all $f \in \mathbb{R}[X]$:*

$$L(f) = \int_K f \, d\mu.$$

(2) $L(h) \geq 0$ *for all* $h \in \langle g_1, \dots, g_k \rangle_2$.

Proof. Obviously (1) $\Rightarrow$ (2). So assume that L is nonnegative on $\langle g_1, \dots, g_k \rangle_2$. Let $h \in \mathbb{R}[X]$ be nonnegative on K. By [B, Theorem 1', p.118] it is suffcent to show $L(h) \geq 0$. Of course we may assume $\varepsilon_0 := |L(h)| > 0$. Pick $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < \varepsilon_0$. Then $h + \varepsilon$ is strictly positive on K. Hence Theorem 4 shows

$$h + \varepsilon \in \langle g_1, \dots, g_m \rangle_2.$$

By our assumption we get $L(h + \varepsilon) \geq 0$. From $0 < \varepsilon < \varepsilon_0$ we deduce $L(h) > 0$. $\square$

   Jacobi and Prestel have shown that in certain situations the last result can be sharpened (see [JP, Corollary 4.2]).

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