On an extension of Pólya's Positivstellensatz

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Abstract

In this paper we provide a generalization of a Positivstellensatz by Pólya [Georg Pólya. Über positive darstellung von polynomen vierteljschr. In *Naturforsch. Ges. Zürich*, 73: 141-145, 1928]. We show that if a homogeneous polynomial is positive over the intersection of the non-negative orthant and a given basic semialgebraic cone (excluding the origin), then there exists a "Pólya type" certificate for non-negativity. The proof of this result uses the original Positivstellensatz by Pólya, and a Positivstellensatz by Putinar and Vasilescu [Mihai Putinar and Florian-Horia Vasilescu. Positive polynomials on semialgebraic sets. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 328(7), 1999].

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1 Introduction

A *Positivstellensatz* basically characterizes the polynomials that are positive on a semialgebraic set, i.e. on a set defined by a system of polynomial inequalities with real coefficients. In our opinion the most famous Positivstellensätze are due to Pólya [Pól74, HLP88], Schmüdgen [Sch91] and Putinar [Put93]. We point the reader towards [Sch09] for a nice and comprehensive overview of Positivstellensätze. For complexity issues related with Schmüdgen and Putinar Positivstellensätze see [Sch04, NS07]. Pólya's Positivstellensatz is related to polynomials with non-negative coefficients while Schmüdgen and Putinar Positivstellensätze are based on polynomials that are sums-of-squares. In this paper, we shall consider a new Positivstellensatz based on polynomials with all their coefficients

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being non-negative, which is in this sense a generalization of the Positivstellensatz from Pólya.

Positivstellensätze are highly useful for constructing and proving the convergence of approximation hierarchies for polynomial optimization problems, see for example [Lau09] and [Las10]. Pólya's Positivstellensatz implies linear programming approximation hierarchies for the copositive programming problems [Par00, dKP02], while Putinar-Vasilescu Positivstellensatz (actually already the initial version from Reznick [Rez95]) implies semidefinite programming approximation hierarchies for the copositive programming problems. We suggest the reader to consider also [dKLP06, Bom12, Dür10, Don12, **?**, **?**] for other results about linear and semidefinite programming approximation hierarchies for the linear optimization problems over the cone of copositive or completely positive matrices.

Our new Positivstellensatz has strong potential to construct similar hierarchies for the linear and non-linear optimization problems over the semialgebraic sets, intersected by the non-negative orthant. Some new results in this direction will be presented in the paper [DP13] which is in preparation.

1.1 Notation

We let $\mathbf{e} \in \mathbb{R}^n$ denote the all-ones vector. We use $\mathbb{Z}_+ = \{0, 1, 2 \ldots\}$ and $\mathbb{Z}_{++} = \{1, 2, 3 \ldots\}$. For $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{m} \in \mathbb{Z}_+^n$ we let $\mathbf{x}^{\mathbf{m}} := \prod_{i=1}^n x_i^{m_i}$ (where $0^0 := 1$).

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we define their Hadamard product $(\mathbf{x} \circ \mathbf{y}) \in \mathbb{R}^n$ such that for all i we have $(\mathbf{x} \circ \mathbf{y})_i = x_i y_i$. Note that for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{m} \in \mathbb{Z}_+^n$ we have $(\mathbf{x} \circ \mathbf{x})^{\mathbf{m}} = \mathbf{x}^{2\mathbf{m}} = (\mathbf{x}^{\mathbf{m}})^2$. We call such terms *even terms*.

We let $\mathbb{R}[\mathbf{x}]$ denote the ring of multivariate polynomials on $\mathbb{R}^n$ with real coefficients. For a polynomial $f \in \mathbb{R}[\mathbf{x}]$, we let $\deg(f)$ denote its degree, i.e. the highest degree of its terms, and for $f(\mathbf{x}) \equiv 0$ we define $\deg(f) := 0$. We say that a polynomial is *homogeneous* if all of its terms have the same degree. Note that for a homogeneous polynomial $f \in \mathbb{R}[\mathbf{x}]$ of degree d, we have $f(\lambda\mathbf{x}) = \lambda^d f(\mathbf{x})$ for all $\lambda \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$. Polynomial has even degree if $\deg(f)$ is even number.

For two polynomials $f, g \in \mathbb{R}[\mathbf{x}]$, we write $f \equiv g$, or equivalently $f(\mathbf{x}) \equiv g(\mathbf{x})$, if all the corresponding coefficients of these polynomials are equal.

For a polynomial $f \in \mathbb{R}[\mathbf{x}]$ and a set $\mathcal{M} \subseteq \mathbb{R}$, we let $f^{-1}(\mathcal{M}) := \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \in \mathcal{M}\}$.

A non-zero polynomial $f \in \mathbb{R}[\mathbf{x}]$ is defined to be *sum-of-squares* (SOS), if there exists $p \in \mathbb{Z}_{++}$ and polynomials $h_1, \ldots, h_p \in \mathbb{R}[\mathbf{x}]$ such that $f(\mathbf{x}) \equiv \sum_{i=1}^p \big(h_i(\mathbf{x})\big)^2$. We now note that:

1. if $f, g \in \mathbb{R}[\mathbf{x}]$ are SOS then both $(f + g)$ and fg are SOS,
2. if $f \in \mathbb{R}[\mathbf{x}]$ is SOS then $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,
3. if $f \in \mathbb{R}[\mathbf{x}]$ such that $f(\mathbf{x} \circ \mathbf{x})$ is SOS then $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}_+^n$.

We also consider polynomials having only non-negative coefficients, for which we similarly have:

1. if all the coefficients of $f, g \in \mathbb{R}[\mathbf{x}]$ are non-negative then so are all the coefficients of both $(f + g)$ and fg,
2. if all the coefficients of $f \in \mathbb{R}[\mathbf{x}]$ are non-negative then $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n_+$,
3. if all the coefficients of $f \in \mathbb{R}[\mathbf{x}]$ are non-negative then $f(\mathbf{x} \circ \mathbf{x})$ is SOS.

For $f_1, \ldots, f_m \in \mathbb{R}[\mathbf{x}]$, the set $\{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \ldots, f_m(x) \geq 0\} = \bigcap_{i=1}^m f_i^{-1}(\mathbb{R}_+)$ is referred to as a *semialgebraic set*. If all of the polynomials are homogeneous, then we shall refer to this as a *semialgebraic cone*.

1.2 Contribution

The main contributions of this paper are twofold, and are summarised below. In these, when we say that a homogeneous polynomial f is positive on a set $\mathcal{K}$, we mean that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{K} \setminus \{\mathbf{0}\}$. The origin is excluded as for all homogeneous polynomials f we have $f(\mathbf{0}) = 0$.

1. The central result of this paper is Theorem 3.5, where we prove a non-trivial generalization of the well-known Pólya's theorem [Pól74, HLP88]. Our new Positivstellensatz is closely related to Putinar and Vasilescu's Positivstellensatz [PV99a, PV99b]. The original Pólya's theorem states that there exists a certificate for positivity for all homogeneous polynomials which are positive on the non-negative orthant, while the Putinar-Vasilescu's theorem states that there exists an SOS certificate for non-negativity for all homogeneous polynomials of even degree that are positive on a semialgebraic cone defined by homogeneous polynomials of even degree. Our version states roughly the same, but we do not demand polynomials of even degree and we only consider semialgebraic cones that are subsets of the non-negative orthant.

2. If we allow the semialgebraic cones to be defined by infinitely many polynomial inequalities, then we can prove that a given polynomial is positive on this set if and only if it is positive on a semialgebraic cone which is defined by some finite subset of the polynomial inequalities.

Up to the best of our knowledge, neither of these results have previously been published.

2 Positivstellensätze

In this section we recall few well-known Positivstellensätze which motivated our Positivstellensatz in Theorem 3.5 and are important basis to prove it.

The following theorem is an example of a well known Positivstellensatz, known as Pólya's Positivstellensatz:

Theorem 2.1 ([HLP88, Section 2.24],[Pól74])**.** *Let $f \in \mathbb{R}[\mathbf{x}]$ be a homogeneous polynomial on $\mathbb{R}^n$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$. Then for some $r \in \mathbb{Z}_+$, we have that all the coefficients of $(\mathbf{e}^\mathsf{T}\mathbf{x})^r f(\mathbf{x})$ are non-negative.*

Powers and Reznick [PR01] provided also a tight upper bound on the size of r in the theorem above.

Another well known Positivstellensatz is the following from Reznick, which provides a constructive solution to Hilbert's seventeenth problem for the case of positive definite forms:

Theorem 2.2 ([Rez95])**.** *Let $f \in \mathbb{R}[\mathbf{x}]$ be a homogeneous polynomial of even degree on $\mathbb{R}^n$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Then for some $r \in \mathbb{Z}_+$, we have that $(\mathbf{x}^\mathsf{T}\mathbf{x})^r f(\mathbf{x})$ is SOS.*

Putinar and Vasilescu extended this Positivstellensatz to give the following Positivstellensatz:

Theorem 2.3 ([PV99a, Theorem 1])**.** *Let $m \in \mathbb{Z}_{++}$ and $f_0, \ldots, f_m \in \mathbb{R}[\mathbf{x}]$ be homogeneous polynomials of even degree on $\mathbb{R}^n$ such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \bigcap_{i=1}^m f_i^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$ and $f_1(\mathbf{x}) \equiv 1$. (Note that $f_1^{-1}(\mathbb{R}_+) = \mathbb{R}^n$.) Then for some $r \in \mathbb{Z}_+$, there exists homogeneous SOS polynomials $g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$ such that $(\mathbf{x}^\mathsf{T}\mathbf{x})^r f_0(\mathbf{x}) \equiv \sum_{i=1}^m f_i(\mathbf{x}) g_i(\mathbf{x})$.*

We suggest the reader to consider also [PV99b, Theorem 4.5] where this theorem was generalized for non-homogeneous polynomials of even degree.

In Section 3 we shall prove the following new Positivstellensatz, which can be seen as an extension of Pólya's Positivstellensatz and is closely related to Putinar-Vasilescu's Positivstellensatz:

Theorem 3.5. *Let be $m \in \mathbb{Z}_{++}$ and $f_0, \ldots, f_m \in \mathbb{R}[\mathbf{x}]$ be homogeneous polynomials on $\mathbb{R}^n$ such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n_+ \cap \bigcap_{i=1}^m f_i^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$ and $f_1(\mathbf{x}) \equiv 1$. Then for some $r \in \mathbb{Z}_+$, there exists homogeneous polynomials $g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$ such that all of their coefficients are non-negative and $(\mathbf{e}^\mathsf{T}\mathbf{x})^r f_0(\mathbf{x}) \equiv \sum_{i=1}^m f_i(\mathbf{x}) g_i(\mathbf{x})$.*

Example 2.4. *Consider $n = 3$ and $m = 4$, with*

$$\begin{aligned} f_0(\mathbf{x}) &\equiv 3x_1 - 2x_2 - 2x_3, \\ f_1(\mathbf{x}) &\equiv 1, \\ f_2(\mathbf{x}) &\equiv x_1 - x_2, \\ f_3(\mathbf{x}) &\equiv x_1 - x_3, \\ f_4(\mathbf{x}) &\equiv x_1^2 - 4x_2x_3. \end{aligned}$$

For $r = 1$, a certificate for $f_0(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n_+ \cap \bigcap_{i=1}^4 f_i^{-1}(\mathbb{R}_+)$ is given by:

$$(\mathbf{e}^\mathsf{T}\mathbf{x}) f_0(\mathbf{x}) = (x_1 + 2x_2) f_2(\mathbf{x}) + (x_1 + 2x_3) f_3(\mathbf{x}) + f_4(\mathbf{x}).$$

We can say even more, this is a certificate for positivity of f_0 since there exists no $\mathbf{x} \geq 0$ such that $f_i(\mathbf{x}) = 0$ for $i = 2, 3, 4$.

In Section 4, the final section of this paper, we shall look at how Theorems 2.3 and 3.5 can be extended for infinitely many polynomials.

3 Proof of new Positivstellensatz

In this section we shall consider $m \in \mathbb{Z}_{++}$ and homogeneous polynomials $f_1, \dots, f_m \in \mathbb{R}[\mathbf{x}]$ on $\mathbb{R}^n$, with $f_1(\mathbf{x}) \equiv 1$, and we define the set

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \|\mathbf{x}\|_2 = 1,\ f_i(\mathbf{x}) \geq 0 \text{ for all } i = 1, \dots, m\}. \tag{1}$$

Note that $f_1(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, so this is a redundant constraint in the description of $\mathcal{X}$, however it will simplify the notation later on. It should also be noted that $\mathcal{X}$ is a compact set, as it is a closed and bounded set.

We next consider a homogeneous polynomial f_0 such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$. Note that this is equivalent to having $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \operatorname{cone} \mathcal{X} \setminus \{0\}$, where

$$\begin{aligned} \operatorname{cone} \mathcal{X} &:= \{0\} \cup \{\lambda \mathbf{x} \mid \lambda > 0,\ \mathbf{x} \in \mathcal{X}\} \\ &= \{\mathbf{x} \in \mathbb{R}^n_+ \mid f_i(\mathbf{x}) \geq 0 \text{ for all } i = 1, \dots, m\}. \end{aligned}$$

The aim of this section is to find simple certificates, based on polynomials with nonnegative coefficients, to certify that $f_0(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \operatorname{cone} \mathcal{X}$.

We begin by considering how Theorem 2.3 can be extended for $\mathbf{x}$ being restricted to the non-negative orthant but with the polynomials not being restricted to have even degree.

Theorem 3.1. *Let f_0 to be a homogeneous polynomial such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$, where $\mathcal{X}$ is as given in (1). Then for some $r \in \mathbb{Z}_+$, there exists homogeneous polynomials $g_1, \dots, g_m \in \mathbb{R}[\mathbf{x}]$ such that $g_i(\mathbf{z} \circ \mathbf{z})$ is SOS for all i and $(\mathbf{e}^\mathsf{T}\mathbf{x})^r f_0(\mathbf{x}) \equiv \sum_{i=1}^m f_i(\mathbf{x}) g_i(\mathbf{x})$.*

Proof. We begin by substituting $(\mathbf{z} \circ \mathbf{z})$ for $\mathbf{x}$, and using Theorem 2.3 to consider the homogeneous polynomials $f_i(\mathbf{z} \circ \mathbf{z})$, which have even degree. From this we see that for some $r \in \mathbb{Z}_+$ there exists sets of homogeneous polynomials $\{h_j \mid j \in \mathcal{I}_i\} \subseteq \mathbb{R}[\mathbf{x}]$ for all $i = 1, \dots, m$ such that

$$(\mathbf{e}^\mathsf{T}(\mathbf{z} \circ \mathbf{z}))^r f_0(\mathbf{z} \circ \mathbf{z}) \equiv (\mathbf{z}^\mathsf{T}\mathbf{z})^r f_0(\mathbf{z} \circ \mathbf{z}) \equiv \sum_{i=1}^m f_i(\mathbf{z} \circ \mathbf{z}) \sum_{j \in \mathcal{I}_i} \big(h_j(\mathbf{z})\big)^2.$$

We now note that for all j there exists a unique set of homogenous polynomials

$\{h_{j,\mathbf{t}} \mid \mathbf{t} \in \{0,1\}^n\} \subseteq \mathbb{R}[\mathbf{x}]$ such that $h_j(\mathbf{z}) \equiv \sum_{\mathbf{t}\in\{0,1\}^n} \mathbf{z}^{\mathbf{t}} h_{j,\mathbf{t}}(\mathbf{z}\circ\mathbf{z})$. We then get that

$$(\mathbf{e}^\mathsf{T}(\mathbf{z}\circ\mathbf{z}))^r f_0(\mathbf{z}\circ\mathbf{z}) \equiv \sum_i f_i(\mathbf{z}\circ\mathbf{z}) \sum_{j\in\mathcal{I}_i} \sum_{\mathbf{t}\in\{0,1\}^n} (\mathbf{z}\circ\mathbf{z})^{\mathbf{t}} \big(h_{j,\mathbf{t}}(\mathbf{z}\circ\mathbf{z})\big)^2$$
$$+ \sum_i f_i(\mathbf{z}\circ\mathbf{z}) \sum_{j\in\mathcal{I}_i} \sum_{\substack{\mathbf{s},\mathbf{t}\in\{0,1\}^n: \\ \mathbf{s}\neq\mathbf{t}}} \mathbf{z}^{\mathbf{s}+\mathbf{t}} h_{j,\mathbf{s}}(\mathbf{z}\circ\mathbf{z}) h_{j,\mathbf{t}}(\mathbf{z}\circ\mathbf{z}).$$

We shall call terms of the form $\mathbf{z}^{2\mathbf{m}}$ for some $\mathbf{m} \in \mathbb{Z}^n_+$ even terms and all other terms odd terms. We note that all terms on the left hand side are even. Therefore all odd terms on the right hand side will cancel out, i.e. the odd degree terms have coefficients equal to zero (we actually use the well-known fact that all polynomials that vanish on a set with interior point must be coefficient-wise equal to zero). For the first part of the right hand side we have only even terms and for the second part we have only odd terms. Therefore

$$(\mathbf{e}^\mathsf{T}(\mathbf{z}\circ\mathbf{z}))^r f_0(\mathbf{z}\circ\mathbf{z}) \equiv \sum_i f_i(\mathbf{z}\circ\mathbf{z}) \sum_{\mathbf{t}\in\{0,1\}^n} (\mathbf{z}\circ\mathbf{z})^{\mathbf{t}} \sum_{j\in\mathcal{I}_i} \big(h_{j,\mathbf{t}}(\mathbf{z}\circ\mathbf{z})\big)^2.$$

Now letting $g_i(\mathbf{x}) \equiv \sum_{\mathbf{t}\in\{0,1\}^n} \mathbf{x}^{\mathbf{t}} \sum_{j\in\mathcal{I}_i} \big(h_{j,\mathbf{t}}(\mathbf{x})\big)^2$ for all i we get the required result. In fact, from [ZVP06, Section 4.3], for an arbitrary homogeneous polynomial $g \in \mathbb{R}[\mathbf{x}]$, we have that $g(\mathbf{z}\circ\mathbf{z})$ is SOS if and only if there exists a set of homogeneous SOS polynomials $\{q_{\mathbf{t}} \mid \mathbf{t} \in \{0,1\}^n\} \subseteq \mathbb{R}[\mathbf{x}]$ such that $g(\mathbf{x}) \equiv \sum_{\mathbf{t}\in\{0,1\}^n} \mathbf{x}^{\mathbf{t}} q_{\mathbf{t}}(\mathbf{x})$. □

Remark 3.2. *It is trivial to see in the previous theorem that, without loss of generality, for all i we have either*

$$g_i(\mathbf{x}) \equiv 0 \qquad \textit{or} \qquad \deg(g_i) + \deg(f_i) = r + \deg(f_0).$$

We next consider the following proposition and the corresponding corollary.

Proposition 3.3. *Let f_0 to be a homogeneous polynomial such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$, where $\mathcal{X}$ is as given in (1). Then for any homogeneous polynomial $h \in \mathbb{R}[\mathbf{x}]$ such that $\deg(h) = \deg(f_0)$, there exists $\varepsilon > 0$ such that $f_0(\mathbf{x}) - \varepsilon h(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$.*

Proof. If $h(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{X}$ then the result trivially holds. From now on we consider when this is not the case.

As $\mathcal{X}$ is compact, the minimums and maximums of continuous functions over it are attained. Therefore both $\min_{\mathbf{x}}\{f_0(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X}\}$ and $\max_{\mathbf{x}}\{|h(\mathbf{x})| \mid \mathbf{x} \in \mathcal{X}\}$ are positive and finite.

Now letting ε be the positive scalar given as follows, we get the required result.

$$\varepsilon = \frac{\min_{\mathbf{x}}\{f_0(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X}\}}{2\max_{\mathbf{x}}\{|h(\mathbf{x})| \mid \mathbf{x} \in \mathcal{X}\}}. \qquad □$$

Corollary 3.4. *Let f_0 to be a homogeneous polynomial such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$, where $\mathcal{X}$ is as given in (1). We let*

$$c_i = -\deg(f_i) + \max\{\deg(f_j) \mid j = 0, \dots, m\} \qquad \textit{for all } i = 0, \dots, m,$$

and for all $\varepsilon \in \mathbb{R}$ we define the homogeneous polynomial

$$f_\varepsilon(\mathbf{x}) :\equiv (\mathbf{e}^\mathsf{T}\mathbf{x})^{c_0} f_0(\mathbf{x}) - \varepsilon \sum_{i=1}^{m} (\mathbf{e}^\mathsf{T}\mathbf{x})^{c_i} f_i(\mathbf{x}).$$

Then there exists an $\varepsilon > 0$ such that $f_\varepsilon(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$.

We can now combine the results from this section with the Pólya's Positivstellensatz from Theorem 2.1 to prove our new Positivstellensatz:

Theorem 3.5. *Let f_0 to be a homogeneous polynomial such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$, where $\mathcal{X}$ is as given in (1). Then for some $r \in \mathbb{Z}_+$, there exists homogeneous polynomials $g_1, \dots, g_m \in \mathbb{R}[\mathbf{x}]$ such that all their coefficients are non-negative and $(\mathbf{e}^\mathsf{T}\mathbf{x})^r f_0(\mathbf{x}) \equiv \sum_{i=1}^m f_i(\mathbf{x}) g_i(\mathbf{x})$.*

Proof. Using the notation and results from Corollary 3.4, there exists $\varepsilon > 0$ such that $f_\varepsilon(\mathbf{x})$ is a homogeneous polynomial with $f_\varepsilon(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$. Using Theorem 3.1, this implies that for some $r_1 \in \mathbb{Z}_+$ there exists homogenous polynomials $h_1, \dots, h_m \in \mathbb{R}[\mathbf{x}]$ such that $h_i(\mathbf{z} \circ \mathbf{z})$ is SOS for all i and $(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_1} f_\varepsilon(\mathbf{x}) \equiv \sum_{i=1}^m f_i(\mathbf{x}) h_i(\mathbf{x})$. Therefore

$$(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_1+c_0} f_0(\mathbf{x}) \equiv \sum_{i=1}^{m} f_i(\mathbf{x}) \left(\varepsilon(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_1+c_i} + h_i(\mathbf{x})\right).$$

Furthermore, by Remark 3.2, without loss of generality, for all $i = 1, \dots, m$, either $h_i(\mathbf{x}) \equiv 0$ or

$$\deg(f_i) + \deg(h_i) = r_1 + \deg(f_\varepsilon) = r_1 + \deg(f_i) + c_i.$$

In other words, for all $i = 1, \dots, m$, either $h_i(\mathbf{x}) \equiv 0$ or $\deg(h_i) = r_1 + c_i$. From this it can be seen that $\left(\varepsilon(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_1+c_i} + h_i(\mathbf{x})\right)$ is homogeneous and positive for all $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, and thus, by Theorem 2.1, for some $r_2 \in \mathbb{Z}_+$ there exists homogeneous polynomials $g_1, \dots, g_m \in \mathbb{R}[\mathbf{x}]$, with all their coefficients being non-negative, such that

$$(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_2} \left(\varepsilon(\mathbf{e}^\mathsf{T}\mathbf{x})^{r_1+c_i} + h_i(\mathbf{x})\right) \equiv g_i(\mathbf{x}) \qquad \text{for all } i = 1, \dots, m.$$

Now letting $r = (r_1 + r_2 + c_0) \in \mathbb{Z}_+$, we get the required result. ☐

Remark 3.6. *We briefly recall that searching for polynomials with nonnegative coefficients can be done with linear programming (LP), whilst searching for SOS polynomials* *can be done with a semidefinite programming (SDP). This is the main advantage of our approach as we are able to use the efficient methods for solving LPs, for example the simplex method.*

4 Infinite number of polynomials

We finish this paper by noting that the Positivstellensätze given in Theorems 2.3 and 3.5 can be extended for infinitely many polynomials using the following theorem.

Theorem 4.1. *Consider a set of homogeneous polynomials* $\{f_0\} \cup \{f_i \mid i \in \mathcal{I}\} \subseteq \mathbb{R}[\mathbf{x}]$ *with infinite cardinality. Then* $f_0(\mathbf{x}) > 0$ *for all* $\mathbf{x} \in \bigcap_{i\in\mathcal{I}} f_i^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$ *if and only if there exists a subset* $\mathcal{J} \subseteq \mathcal{I}$ *of finite cardinality such that* $f_0(\mathbf{x}) > 0$ *for all* $\mathbf{x} \in \bigcap_{i\in\mathcal{J}} f_i^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$.

Proof. The "if" part is trivial. We proceed with the "only if" part.

We begin by letting $\mathcal{Z}_\emptyset = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_2 = 1\}$, which is a compact set, and we note that for any $\mathcal{J} \subseteq \mathcal{I}$ we have $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \bigcap_{i\in\mathcal{J}} f_i^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$ if and only if $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{Z}_\emptyset \cap \bigcap_{i\in\mathcal{J}} f_i^{-1}(\mathbb{R}_+)$.

Without loss of generality, we assume that for all $i \in \mathcal{I}$ we have $\deg(f_i) \geq 1$ and

$$1 \geq \max_{\mathbf{x}\in\mathbb{R}^n}\{\|\nabla f_i(\mathbf{x})\|_2 \mid \|\mathbf{x}\|_2 \leq 1\}.$$

Considering the mean value theorem for this implies that for all $\mathbf{x}, \mathbf{y} \in \mathcal{Z}_\emptyset$ and all $i \in \mathcal{I}$, there exists $\theta \in [0, 1]$ such that

$$\begin{aligned} f_i(\mathbf{x}) - f_i(\mathbf{y}) &= (\mathbf{x} - \mathbf{y})^\mathsf{T} \nabla f_i(\theta\mathbf{x}) \\ &\leq \|\mathbf{x} - \mathbf{y}\|_2 \, \|\nabla f_i(\theta\mathbf{x})\|_2 \\ &\leq \|\mathbf{x} - \mathbf{y}\|_2 \max_{\mathbf{x}\in\mathbb{R}^n}\{\|\nabla f_i(\mathbf{x})\|_2 \mid \|\mathbf{x}\|_2 \leq 1\} \\ &\leq \|\mathbf{x} - \mathbf{y}\|_2. \end{aligned} \tag{2}$$

By the same line of reasoning we obtain also that for all $\mathbf{x} \in \mathcal{Z}_\emptyset$ and all $i \in \mathcal{I}$ we have

$$|f_i(\mathbf{x})| \leq 1. \tag{3}$$

For all $j \in \mathcal{I}$ and $\mathcal{J} \subseteq \mathcal{I}$, we define the compact sets

$$\begin{aligned} \mathcal{Y} &= \mathcal{Z}_\emptyset \cap f_0^{-1}(-\mathbb{R}_+), \\ \mathcal{Z}_j &= \mathcal{Z}_\emptyset \cap f_j^{-1}(\mathbb{R}_+), \\ \mathcal{Z}_\mathcal{J} &= \mathcal{Z}_\emptyset \cap \bigcap_{i\in\mathcal{J}} f_i^{-1}(\mathbb{R}_+) = \bigcap_{i\in\mathcal{J}} \mathcal{Z}_i. \end{aligned}$$

We have that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{Z}_\emptyset \cap \bigcap_{i \in \mathcal{J}} f_i^{-1}(\mathbb{R}_+)$ if and only if $\mathcal{Y} \cap \mathcal{Z}_\mathcal{J} = \emptyset$. In particular we have $\mathcal{Y} \cap \mathcal{Z}_\mathcal{I} = \emptyset$.

Next we define the following function from $\mathcal{Z}_\emptyset$ to $\mathbb{R}$

$$\xi(\mathbf{x}) := \sup\{-f_i(\mathbf{x}) \mid i \in \mathcal{I}\} \qquad \text{for all } \mathbf{x} \in \mathcal{Z}_\emptyset.$$

From (2) and (3) we get that this is a continuous function from $\mathcal{Z}_\emptyset$ to $[-1, 1]$. Furthermore, as $\mathcal{Y} \cap \mathcal{Z}_\mathcal{I} = \emptyset$, we have that $\xi(\mathbf{x}) \in (0, 1]$ for all $\mathbf{x} \in \mathcal{Y}$. We now set $\varepsilon = \min_\mathbf{x}\{\xi(\mathbf{x}) \mid \mathbf{x} \in \mathcal{Y}\}$. As $\mathcal{Y}$ is compact, ξ is a continuous function and $\xi(\mathbf{x}) \in (0, 1]$ for all $\mathbf{x} \in \mathcal{Y}$, we get that $\varepsilon \in (0, 1]$.

For any $\mathbf{x} \in \mathcal{Y}$, there exists an $i \in \mathcal{I}$ such that $-f_i(\mathbf{x}) \geq \frac{2}{3}\xi(\mathbf{x}) \geq \frac{2}{3}\varepsilon > 0$. Now, for all $\mathbf{y} \in \mathcal{Z}_\emptyset$ such that $\|\mathbf{x} - \mathbf{y}\|_2 \leq \frac{1}{3}\varepsilon$, we have $f_i(\mathbf{y}) \leq f_i(\mathbf{x}) + \|\mathbf{x} - \mathbf{y}\|_2 \leq -\frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon < 0$ and thus $\mathbf{y} \notin \mathcal{Z}_i$. Keeping this in mind, we consider Algorithm 1.

Algorithm 1 Finding a set $\mathcal{J} \subseteq \mathcal{I}$ of finite cardinality such that $\mathcal{Y} \cap \mathcal{Z}_\mathcal{J} = \emptyset$.

Let $\mathcal{J} \leftarrow \emptyset$.
while $\exists \mathbf{z} \in \mathcal{Y} \cap \mathcal{Z}_\mathcal{J}$ **do**
  Let $i \in \mathcal{I}$ such that $f_i(\mathbf{z}) \leq -\frac{2}{3}\varepsilon$ and let $\mathcal{J} \leftarrow \mathcal{J} \cup \{i\}$.
end while
print $\mathcal{J}$.

We see that no $\mathbf{z}$ in this algorithm can be within a distance of $\frac{1}{3}\varepsilon$ of a previous $\mathbf{z}$. Therefore, as $\mathcal{Y}$ is a bounded subset of $\mathbb{R}^n$, the algorithm finishes within a finite number of iterations. The resultant $\mathcal{J}$ then conforms to the requirements in the theorem. □

Note that Algorithm 1 is purely there to aid the proof and is not meant for use in practice. This is due to the fact that simply checking whether $\mathcal{Y} \cap \mathcal{Z}_\mathcal{J} = \emptyset$ is in general an NP-hard problem.

We will now consider a couple of examples connected to Theorem 4.1.

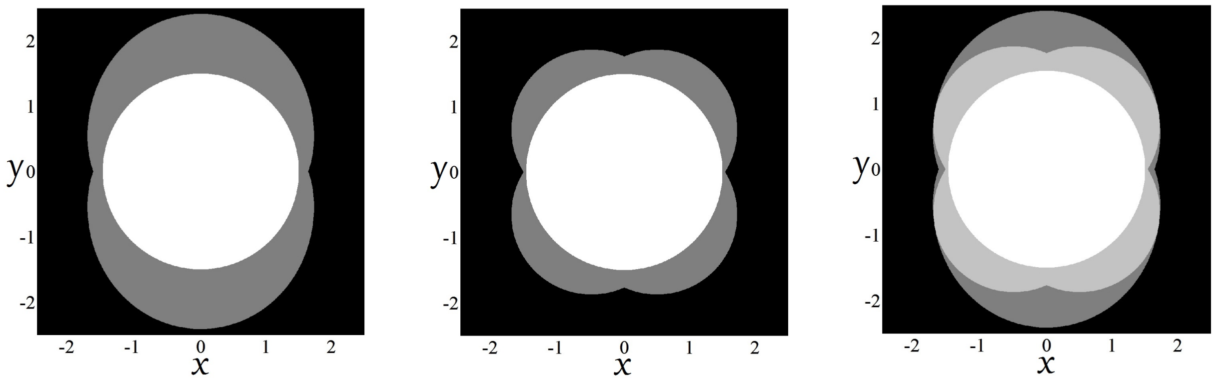
Example 4.2. *Let us consider*

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2_+ \mid (x_1 - a x_2)(x_1 - (a+1)x_2) \geq 0 \ \forall a \in \mathbb{Z}\} = (\mathbb{R}_+ \times \{0\}) \ \cup \ \mathrm{cone}(\mathbb{Z} \times \{1\})$$

and $f_0(\mathbf{x}) \equiv 4x_1^3 - x_1 x_2^2$. *Obviously we have* $f_0(\mathbf{x}) > 0$ *for all* $\mathbf{x} \in \mathcal{X} \setminus \{\mathbf{0}\}$. *We can write* $f_0(\mathbf{x}) \equiv 4(x_1 + x_2)x_1(x_1 - x_2) + 3x_1 x_2^2$. *Therefore* $f_0(\mathbf{x}) > 0$ *for all* $\mathbf{x} \in \mathbb{R}^2_+ \setminus \{\mathbf{0}\}$ *such that* $x_1(x_1 - x_2) \geq 0$, *hence we can keep only one single constraint (the one that corresponds to* $a = 0$*) out of countable many.*

Example 4.3. *Let us consider* $\{f_0\} \cup \{h_\nu \mid \nu \in \mathbb{R}\} \subseteq \mathbb{R}[x, y, z]$ *such that*

$$f_0(x, y, z) = 4x^2 + 4y^2 - 9z^2,$$
$$h_\nu(x, y, z) = \big(x - z\sin(\nu\pi/6)\big)^2 + \big(x - z\cos^3(\nu\pi/6)\big)^2 - 2z^2\cos(\nu\pi/6).$$



(a) The inner white circle represents $\mathcal{Y}$. The black area (and outwards to infinity) represents $\mathcal{Z}_{\mathbb{R}}$.

(b) The inner white circle represents $\mathcal{Y}$. The black area (and outwards to infinity) represents $\mathcal{Z}_{\mathcal{J}}$.

(c) A combination of representations for $\mathcal{Z}_{\mathbb{R}}$, $\mathcal{Z}_{\mathcal{J}}$ and $\mathcal{Y}$.

Figure 1: Representation of Example 4.3.

Note that $f_0(x, y, 0) > 0$ *for all* $(x, y) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$*. From this, and the fact that we are dealing with homogeneous polynomials of degree two, in this example we shall equivalently consider* $\mathcal{Z}_\emptyset = \{(x, y, z) \in \mathbb{R}^3 \mid z = 1\}$*. This makes the visualisations somewhat simpler.*

We then have the following for $\mathcal{I} \subseteq \mathbb{R}$*:*

$$\mathcal{Y} = \{(x, y, 1) \mid f_0(x, y, 1) \leq 0\}, \qquad \mathcal{Z}_{\mathcal{I}} = \{(x, y, 1) \mid h_\nu(x, y, 1) \geq 0 \textit{ for all } \nu \in \mathcal{I}\}.$$

Note that $\mathcal{Z}_{\mathbb{R}}$ *is built from uncountably many constraints and we visualise the sets* $\mathcal{Y}$ *and* $\mathcal{Z}_{\mathbb{R}}$ *in fig. 1a. We have* $\mathcal{Y} \cap \mathcal{Z}_{\mathbb{R}} = \emptyset$ *and thus* $f_0(x, y, z) > 0$ *for all* $(x, y, z) \in \bigcap_{\nu \in \mathbb{R}} h_\nu^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$*.*

If we now consider the set $\mathcal{J} = \{1, 5, 7, 11\}$ *of cardinality four, then visualising* $\mathcal{Y}$ *and* $\mathcal{Z}_{\mathcal{J}}$ *in fig. 1b, we have* $\mathcal{Y} \cap \mathcal{Z}_{\mathcal{J}} = \emptyset$ *and thus* $f_0(x, y, z) > 0$ *for all* $(x, y, z) \in \bigcap_{\nu \in \mathcal{J}} h_\nu^{-1}(\mathbb{R}_+) \setminus \{\mathbf{0}\}$*.*

5 Conclusions

In this paper we proved a generalization of a well-known Positivstellensatz from Pólya [Pól74, HLP88]. The proof of this used the original Positivstellensatz from Pólya, and a Positivstellensatz from Putinar and Vasilescu [PV99a, PV99b]. We showed that for homogeneous polynomials which are positive on the semialgebraic cones defined by homogeneous polynomials and intersected by the non-negative orthant, there exists a Pólya type certificate which can be numerically found by solving an instance of a linear programming problem. We also showed that this can further be extended for infinitely many polynomials in the definition of the semialgebraic cone. An application of these theorems in constructing linear programming based hierarchies for polynomial optimization problems will be presented in the forthcoming paper [DP13].

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