SEQUOIA

International School

ADDITIONAL MATHEMATICS

(MODULE 1)

Name : ______________________________

Class : ______________________________

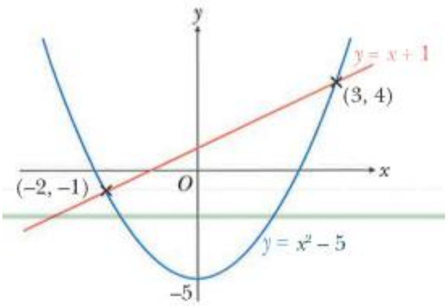
Contents

3 Factors and Polynomials

Simultaneous equations and quadratics

1.1 Simultaneous equations (one linear and one non-linear)

In this section you will learn how to solve simultaneous equations where one equation is linear and the second equation is not linear.



The diagram shows the graphs of $y = x + 1$ and $y = x^2 - 5$.

The coordinates of the points of intersection of the two graphs are $(-2, -1)$ and $(3, 4)$.

We say that $x = -2$ and $x = 3$, $y = 4$ are the solutions of the simultaneous equations $y = x + 1$ and $y = x^2 - 5$.

The solutions can also be found algebraically:

$y = x + 1$ ----(1)

$y = x^2 - 5$ ----(2)

Substitute for y from (1) into (2):

$$\begin{aligned} x + 1 &= x^2 - 5 && \text{rearrange} \\ x^2 - x - 6 &= 0 && \text{factorise} \\ (x + 2)(x - 3) &= 0 \end{aligned}$$

$$x = -2 \text{ or } x = 3$$

Substituting $x = -2$ into (1) gives $y = -2 + 1 = -1$.

Substituting $x = 3$ into (1) gives $y = 3 + 1 = 4$.

The solutions are: $x = -2$, $y = -1$ and $x = 3, y = 4$

Worked Example 1

Solve the simultaneous equations.

$2x + 2y = 7$

$x^2 - 4y^2 = 8,$

Answers

$2x + 2y = 7$ --- (1)

$x^2 - 4y^2 = 8$ --- (2)

From (1), $x = \frac{7-2y}{2}$,

Substitute for x in (2):

$$\left(\frac{7-2y}{2}\right)^2 - 4y^2 = 8$$

expand brackets

$$\frac{49 - 28y + 4y^2}{4} - 4y^2 = 8$$

multiply both sides by 4

$$49 - 28y + 4y^2 - 16y^2 = 32$$

rearrange

$$12y^2 + 28y - 17 = 0$$

factorise

$$(6y + 17)(2y - 1) = 0$$

$$y = -2\tfrac{5}{6} \text{ or } y = \tfrac{1}{2}$$

Substituting $y = -2\frac{5}{6}$ into (1) gives $x = 6\frac{1}{3}$

Substituting $y = \frac{1}{2}$ into (1) gives $x = 3$.

The solutions are: $x = 6\frac{1}{3}$, $y = -2\frac{5}{6}$ and $x = 3$, $y = \frac{1}{2}$.

Exercise 1.1

Basic Level

Solve the following simultaneous equations.

1. $y = x^2$

   $y = x + 6$

Answer:

$x = 3, y = 9$ and $x = -2, y = 4$

2. $y = x - 6$

   $x^2 + xy = 8$

Answer:

$x = -1, y = -7$ and $x = 4, y = -2$

3. $y = x - 1$

$x^2 + y^2 = 25$

Answer:

$x = -3, y = -4$ and $x = 4, y = 3$

4. $xy = 4$

$y = 2x + 2$

Answer:

$x = 1, y = 4$ and $x = -2, y = -2$

5. $x^2 - xy = 0$

$x + y = 1$

Answer:

$x = 0.5, y = 0.5$ and $x = 0, y = 1$

6. $3y = 4x - 5$

$x^2 + 3xy = 10$

Answer:

$x = -1, y = -3$ and $x = 2, y = 1$

7. $2x + y = 7$

$xy = 6$

Answer:

$x = 1.5, y = 4$ and $x = 2, y = 3$

8. $x - y = 2$

$2x^2 - 3y^2 = 15$

Answer:

$x = 3, y = 1$ and $x = 9, y = 7$

9. $x + 2y = 7$

$x^2 + y^2 = 10$

Answer:

$x = 1.8, y = 2.6$ and $x = 1, y = 3$

10. $y = 2x$

$x^2 + xy = 3$

Answer:

$x = -1, y = 2$ and $x = 1, y = 2$

11. $xy = 2$

$x + y = 3$

Answer:

$x = 1, y = 2$ and $x = 2, y = 1$

12. $y^2 = 4x$

$2x + y = 4$

Answer:

$x = 1, y = 2$ and $x = 4, y = -4$

13. $x + 3y = 0$

$2x^2 + 3y = 1$

Answer:

$x = 1, y = -\frac{1}{3}$ and $x = -\frac{1}{2},\ y = \frac{1}{6}$

14. $x + y = 4$

$x^2 + y^2 = 10$

Answer:

$x = 3, y = 1$ and $x = 1, y = 3$

15. $y = 3x$

$2y^2 - xy = 15$

Answer:

$x = -1, y = -3$ and $x = 1, y = 3$

16. $x - 2y = 1$

$4y^2 - 3x^2 = 1$

Answer:

$x = 0, y = -0.5$ and $x = -1, y = -1$

17. $3 + x + xy = 0$

$2x + 5y = 8$

Answer:

$x = -1, y = 2$ and $x = 7\frac{1}{2}, y = -1.4$

18. $xy = 12$

$(x-1)(y+2) = 15$

Answer:

$x = -1.5, y = -8$ and $x = 4, y = 3$

19. Calculate the coordinates of the points where the line $y = 1 - 2x$ cuts the curve $x^2 + y^2 = 2$.
Answer: $(-0.2,\ 1.4)$ and $(1, -1)$

20. The sum of two numbers x and y is 11.
The product of the two numbers is 21.25
a Write down two equations in x and y.
Answer: $x + y = 11, xy = 21.25$
b Solve your equations to find the possible values of x and y.
Answer: $x = 2.5, y = 8.5$ and $x = 8.5, y = 2.5$

21. The sum of the areas of two squares is 818 cm^2.
The sum of perimeters is 160 cm.
Find the lengths of the sides of the squares.
Answer: 17 cm and 23 cm

22. The line $y = 2 - 2x$ cuts the curve $3x^2 - y^2 = 3$ at the points A and B.
Find the length of the line AB.
Answer: $6\sqrt{5}$ or 13.4 to 3 sf

23. The line $2x + 5y = 1$ meets the curve $x^2 + 5xy - 4y^2 + 10 = 0$ at the points A and B. Find the coordinates of the midpoint of AB.
Answer: $(0.5, 0)$

24. The line $y = x - 10$ intersects the curve $x^2 + y^2 + 4x + 6y - 40 = 0$ at the points A and B. Find the length of the line AB.
Answer: $5\sqrt{2}$ or 7.07 to 3 sf

25. The straight-line $y = 2x - 2$ intersects the curve $x^2 - y = 5$ at the points A and B. Given that A lies below the x-axis and the point P lies on AB such that $AP: PB = 3: 1$, find the coordinates of P.
Answer: $(2, 2)$

26. The line $x – 2y = 2$ intersects the curve $x + y^2 = 10$ at two points A and B.
Find the equation of the perpendicular bisector of the line AB.
Answer: $y = -2x - 1$

1.2 Maximum and minimum values of a quadratic function

<u>Worked Example 2</u>

$f(x) = x^2 - 3x - 4 \quad x \in \mathbb{R}$

a) Find the axes crossing points for the graph of $y = f(x)$

b) Sketch the graph of $y = f(x)$ and use the symmetry of the curve to find the coordinates of the minimum point.

c) State the range of the function $f(x)$.

Answers

$y = x^2 - 3x - 4$

When $x = 0, y = -4$.

When $y = 0$,

$$x^2 - 3x - 4 = 0$$
$$(x + 1)(x - 4) = 0$$
$$x = -1 \text{ or } x = 4$$

Axes crossing points are:

$(0, -4), (-1, 0)$ and $(4, 0)$.

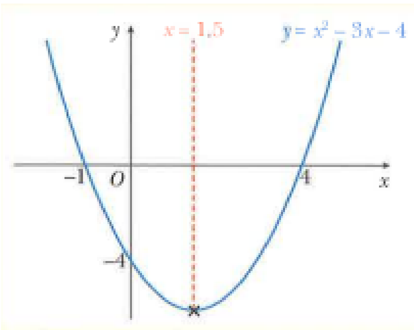
c) The line of symmetry cuts the x-axis midway between -1 and 4.

So the line of symmetry is $x = 1.5$

When $x = 1.5$,

$y = (1.5)^2 - 3(1.5) - 4.$

$y = -6.25$



c) The range is $f(x) \geq -6.25$.

Completing the square

If you expand the expressions $(x+d)^2$ and $(x-d)^2$ you obtain the results:

$$(x+d)^2 = x^2 + 2dx + d^2 \text{ and } (x-d)^2 = x^2 - 2dx + d^2$$

Rearranging these give the following important results:

$$x^2 + 2dx = (x+d)^2 - d^2$$

$$x^2 - 2dx = (x-d)^2 - d^2$$

This is known as **completing the square**.

To complete the square for $x^2 + 8x$:

$$8 \div 2 = 4$$

$$x^2 + 8x = (x+4)^2 - 4^2$$

$$x^2 + 8x = (x+4)^2 - 16$$

To complete the square for $x^2 + 10x$:

$$10 \div 2 = 5$$

$$x^2 + 10x - 3 = (x+5)^2 - 5^2 - 3$$

$$x^2 + 10x - 3 = (x+5)^2 - 28$$

To complete the square for $2x^2 - 8x + 14$ you must first take a factor of 2 out of the expression:

$$2x^2 - 8x + 14 = 2[x^2 - 4x + 7]$$

$$4 \div 2 = 2$$

$$x^2 - 4x + 7 = (x-2)^2 - 2^2 + 7$$

$$x^2 - 4x + 7 = (x-2)^2 + 3$$

So, $2x^2 - 8x + 6 = 2[(x-2)^2 + 3]$

$$= 2[(x-2)^2 + 6$$

You can also use an algebraic method for completing the square, as shown in the following example.

Worked Example 3

Express $2x^2 - 4x + 5$ in the form $p(x-q)^2 + r$, where p, q and r are constants to be found.

Answers

$2x^2 - 4x + 5 = p(x-q)^2 + r$

Expanding the brackets and simplifying gives:

$2x^2 - 4x + 5 = px^2 - 2pq + pq^2 + r$

Comparing coefficients of x^2, coefficients of x and the constant gives:

$2 = p$ ----- (1) $\quad -4 = -2pq$ ----- (2) $\quad 5 = pq^2 + r$ ----- (3)

Substituting $p = 2$ in equation (2) gives $q = 1$.

Substituting $p = 2$ and $q = 1$ in equation (3) gives $r = 3$.

$2x^2 - 4x + 5 = 2(x-1)^2 + 3$.

Completing the square for a quadratic expression or function enables you to:

- write down the maximum or minimum value of the expression
- write down the coordinates of the maximum or minimum point of the function
- sketch the graph of the function
- write down the line of symmetry of the function
- state the range of the function.

In **Worked Example 3** you found that:

$$2x^2 - 4x + 5 = 2(x-1)^2 + 3$$

This part of the expression is a square so it will always be ≥ 0. The smallest value it can be is 0. This occurs when $x = 1$.

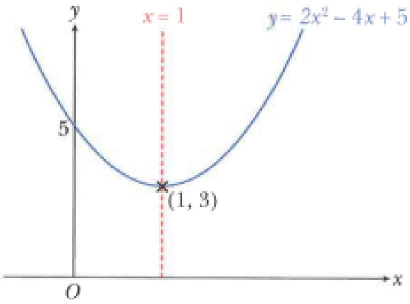
The minimum value of the expression is $2 \times 0 + 3 = 3$ and this minimum occurs when $x = 1$.

So the function $y = 2x^2 - 4x + 5$ will have a minimum point at the point $(1,3)$.

When $x = 0, y = 5$.

The graph of $y = 2x^2 - 4x + 5$ can now be sketched:

The line of symmetry is $x = 1$.



The range is $y \geq 3$.

The general rule is:

For a quadratic function $f(x) = ax^2 + bx + c$ that is written in the form $f(x) = a(x - h)^2 + k$,

i. if $a > 0$, the minimum point is (h, k)
ii. if $a < 0$, the maximum point is (h, k)

Worked Example 4

$f(x) = 2 + 8x - 2x^2 \quad x \in \mathbb{R}$

a) Find the value of a, the value of b and the value of c for which $f(x) = a - b(x + c)^2$.
b) Write down the coordinates of the maximum point on the curve $y = f(x)$.
c) Sketch the graph of $y = f(x)$, showing the coordinates of the points where the graph intersects the x and y-axes.
d) State the range of the function $f(x)$.

Answers

a) $2 + 8x - 2x^2 = a - b(x + c)^2$
$2 + 8x - 2x^2 = a - b(x^2 + 2cx + c^2)$
$2 + 8x - 2x^2 = a - bx^2 - 2bcx - bc^2$

Comparing coefficients of x^2, coefficients of x and the constant gives:

$-2 = -b$ ----- (1) $\qquad 8 = -2bc$ ----- (2) $\qquad 2 = a - bc^2$ ----- (3)

Substituting $b = 2$ in equation (2) gives $c = -2$.
Substituting $b = 2$ and $c = -2$ in equation (3) gives $a = 10$.
So $a = 10, b = 2$ and $c = -2$.

b) $y = 10 - 2(x - 2)^2$

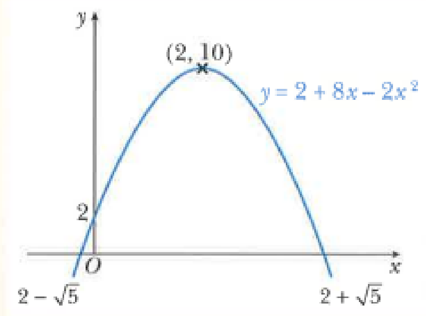
This part of the expression is a square so it will always be ≥ 0. The smallest value it can be is 0. This occurs when $x = 2$.

The maximum value of the expression is $10 - 2 \times 0 = 10$,

and this maximum occurs when $x = 2$.

So the function $y = 2 + 8x - 2x^2$ will have maximum point at the point $(2,10)$.

$y = 2 + 8x - 2x^2$
When $x = 0, y = 2$.
When $y = 0$,



Axes crossing points are: (0, 2), (2 - $\sqrt{5}$, 0) and (2 + $\sqrt{5}$, 0)

d) The range is $f(x) \leq 10$.

Exercise 1.2

Basic Level

1. Use the symmetry of each quadratic function to find the maximum or minimum points.

   Sketch each graph, showing all axis crossing points.

   a) $y = x^2 - 5x - 6$

   b) $y = x^2 - x - 20$

c) $y = x^2 + 4x - 21$

d) $y = x^2 + 3x - 28$

e) $y = x^2 + 4x + 1$

f) $y = 15 + 2x - x^2$

2. Express each of the following in the form $(x-m)^2+n$

a) x^2-8x

Answer: $(x-4)^2-16$

b) x^2-10x

Answer: $(x-5)^2-25$

c) x^2-5x

Answer: $(x-2.5)^2-6.25$

d) $x^2 - 3x$

Answer: $(x - 1.5)^2 - 2.25$

e) $x^2 + 4x$

Answer: $(x - 2)^2 - 4$

f) $x^2 + 7x$

Answer: $(x - 3.5)^2 - 12.25$

g) $x^2 + 9x$

Answer: $(x - 4.5)^2 - 20.25$

h) $x^2 + 3x$

Answer: $(x - 1.5)^2 - 2.25$

3. Express each of the following in the form $(x - m)^2 + n$.

a) $x^2 - 8x + 15$

Answer: $(x - 4)^2 - 1$

b) $x^2 - 10x - 5$

Answer: $(x - 5)^2 - 30$

c) $x^2 - 6x + 2$

Answer: $(x-3)^2 - 7$

d) $x^2 - 3x + 4$

Answer: $(x-1.5)^2 + 1.75$

e) $x^2 + 6x + 5$

Answer: $(x + 3)^2 - 4$

f) $x^2 + 6x + 9$

Answer: $(x + 3)^2 + 0$

g) $x^2 + 4x - 157$

Answer: $(x + 2)^2 - 21$

h) $x^2 + 5x + 6$

Answer: $(x + 2.5)^2 - 0.25$

4. Express each of the following in the form $a(x-p)^2+q$.

a) $2x^2-8x+3$

Answer: $2(x-2)^2-5$

b) $2x^2-12x+1$

Answer: $2(x-3)^2-17$

c) $3x^2 - 12x + 5$

Answer: $3(x-2)^2 - 7$

d) $2x^2 - 3x + 2$

Answer: $2(x-0.75)^2 + 0.875$

e) $2x^2 + 4x + 1$

Answer: $2(x + 1)^2 - 1$

f) $2x^2 + 7x - 3$

Answer: $2(x + 1.75)^2 - 9.125$

g) $2x^2 - 3x + 5$

Answer: $2(x - 0.75)^2 + 3.875$

h) $3x^2 - x + 6$

Answer: $3\left(x - \frac{1}{6}\right)^2 + 5\frac{11}{12}$

5. Express each of the following in the form $m - (x - \mathrm{n})^2$.

a) $6x - x^2$

Answer: $9 - (x - 3)^2$

b) $10x - x^2$

Answer: $25 - (x - 5)^2$

c) $3x - x^2$

Answer: $2.25 - (x - 1.5)^2$

d) $8x - x^2$

Answer: $16 - (x - 4)^2$

6. Express each of the following in the form $a - (x + b)^2$.

a) $5 - 2x - x^2$

Answer: $6 - (x + 1)^2$

b) $8 - 4x - x^2$

Answer: $12 - (x + 2)^2$

c) $10 - 5x - x^2$

Answer: $16.25 - (x - 2.5)^2$

d) $7 + 3x - x^2$

Answer: $9.25 - (x + 1.5)^2$

7. Express each of the following in the form $a - p(x+q)^2$

a) $9 - 6x - 2x^2$

Answer: $13.5 - 2(x + 1.5)^2$

b) $1 - 4x - 2x^2$

Answer: $3 - 2(x + 1)^2$

c) $7 + 8x - 2x^2$

Answer: $15 - 2(x-2)^2$

d) $2 + 5x - 3x^2$

Answer: $4\frac{1}{12} - 3\left(x - \frac{5}{6}\right)^2$

8. a) Express $4x^2 + 2x + 5$ in the form $a(x + b)^2 + c$, where a , b and c are constants.

Answer: $4\left(x + \frac{1}{4}\right)^2 + 4.75$

b) Does the function $y = 4x^2 + 2x + 5$ meet the x-axis? Explain your answer.

Answer: No, since $4.75 > 0$

9. $f(x) = 2x^2 - 8x + 1$

a) Express $2x^2 - 8x + 1$ in the form $a(x+b)^2 + c$, where a and b are integers.

Answer: $2(x-2)^2 - 7$

b) Find the coordinates of the stationary point on the graph of $y = f(x)$.

Answer: $(2, -7)$

10. $f(x) = x^2 - x - 5$ for $x \in \mathbb{R}$. Find the smallest value of $f(x)$ and the corresponding value of x.

Answer: $-5.25, 0.5$

11. $f(x) = 5 - 7x - 2x^2$ for $x \in \mathbb{R}$. Write $f(x)$ in the form $p - 2(x - q)^2$, where p and q are constants to be found.

Answer: $11\frac{1}{8} - 2\left(x + 1\frac{3}{4}\right)^2$

12. $f(x) = 14 + 6x - 2x^2$ for $x \in \mathbb{R}$.

a) Express $14 + 6x - 2x^2$ in the form $a + b(x + c)^2$, where a, b and c are constants.

Answer: $18.5 - 2(x - 1.5)^2$

b) Write down the coordinates of the stationary point on the graph of $y = f(x)$.

Answer: $(1.5, 18.5)$

c) Sketch the graph of $y = f(x)$.

Answer: n-shaped curve, vertex $(1.5, 18.5)$

13. $f(x) = 7 + 5x - x^2$ for $0 \leq x \leq 7.$

a) Express $7 + 5x - x^2$ in the form $a - (x + b)^2$, where a, and b are constants.

Answer: $13.25 - (x - 2.5)^2$

b) Find the coordinates of the turning point of the function *f (x),* stating whether it is a maximum or minimum point.

Answer: (2.5, 13.25), Maximum

14. The function f is such that $f(x) = 2x^2 - 8x + 3$. Write $f(x)$ in the form $2(x + a)^2 + b$, where a, and b are constants to be found.

Answer: $2(x - 2)^2 - 5$

15. $f(x) = 1 + 4x - x^2$ for $x \geq 2$

a) Express $1 + 4x - x^2$ in the form $a - (x + b)^2$, where a and b are constants to be found.

Answer: $5 - (x - 2)^2$

b) Find the coordinates of the turning point of the function $f(x)$, stating whether it is a maximum or minimum point.

Answer: (2,5) Maximum

Intermediate level

1. Express each of the following quadratic expressions in the form $a(x-h)^2+k$, where a, h and k are constants.

   a) $3x^2-4x+1$

   Answer: $3(x-\frac{2}{3})^2-\frac{1}{3}$

   b) $-4x^2+2x-3$

   Answer: $-4\left(x-\frac{1}{4}\right)^2-\frac{11}{4}$

c) $(2x+1)^2 - x$

Answer: $4\left(x+\frac{3}{8}\right)^2+\frac{7}{16}$

d) $2-5x-3x^2$

Answer: $-3\left(x+\frac{5}{6}\right)^2+\frac{49}{12}$

2. Find the exact solutions of each of the following quadratic equations by completing the square.

a) $4x^2 - 4x - 31 = 0$

Answer: $x = \frac{1}{2} - 2\sqrt{2}$ or $x = \frac{1}{2} + 2\sqrt{2}$

b) $44 + 4x - x^2 = 0$

Answer: $x = 2 - 4\sqrt{3}$ or $x = 2 + 4\sqrt{3}$

3. Simplify each of the following quadratic equations to the form $ax^2 + bx + c = 0$. Then, use an appropriate method to solve the equations.

a) $(2x - 1)^2 = x$

Answer: $4x^2 - 5x + 1 = 0, x = \frac{1}{4}$ or $x = 1$

b) $(2 - 3x)^2 = 6x - 2$

Answer: $3x^2 - 6x + 2 = 0, x = 1 - \frac{1}{3}\sqrt{3}$ or $x = 1 + \frac{1}{3}\sqrt{3}$

c) $(1-2x)(x+2)=-2$

Answer: $2x^2+3x-4=0, x=\frac{-3-\sqrt{41}}{4}$ or $x=\frac{-3+\sqrt{41}}{4}$

4. Explain if you can solve the equation $4x^2-6x+1=0$ by factorisation. What is another way of solving it? Solve the equation.

   Answer: It cannot be solved by factorisation. $x=\frac{3-\sqrt{5}}{4}$ or $x=\frac{3+\sqrt{5}}{4}$

5. The quadratic expression $ax^2 - 4x + b$ has a minimum value of 1 when $x = 2$, Find the value of a and b by completing the square.
   Answer: $a = 1, b = 5$

6. Using the method of completing the square, show that
   a) $4x^2 - 20x + 26$ is positive for all real x
   b) $-8 + 4x - x^2$ is negative for all real x

7. It is given that $y = x^2 - 3x - \frac{3}{4}$.

a) Find the coordinates of the turning point and sketch the graph of $y = x^2 - 3x - \frac{3}{4}$.

Answer: $\left(\frac{3}{2}, -3\right)$

b) On the same axes, deduce the graph of $y = \left|x^2 - 3x - \frac{3}{4}\right|$, indicating the turning point of the graph clearly.

c) Do the two graphs have a common line of symmetry? Explain.

8. a) $y = 1 + bx + ax^2$ has a maximum value of 4 when $x = -1$. Find the value of a and b.

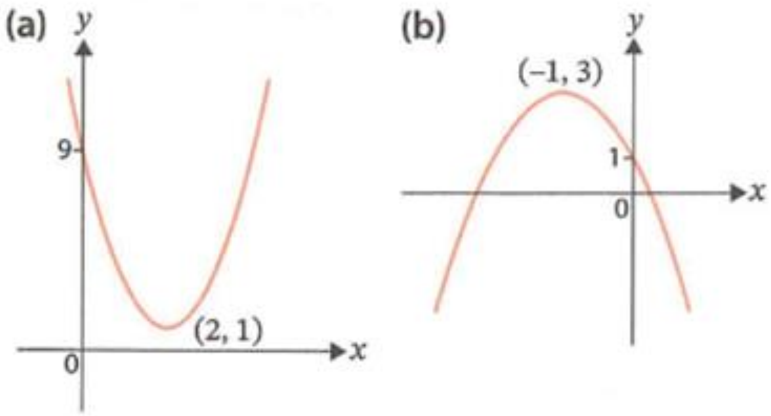
Answer: $a = -3, b = -6$

b) $y = ax^2 + bx + c$ has a minimum value of $-\frac{21}{4}$ when $x = \frac{1}{4}$. When $x = 0, y = -5$.

Find the values of a, b and c.

Answer: $a = 4, b = -2, c = -5$

9. The quadratic graphs are shown below. For each of the graphs, find the quadratic function written in the form $y = a(x-h)^2 + k$, where $a. h$ and k are constants.



a) **Answer:** $y = 2(x-2)^2 + 1$
b) **Answer:** $y = -2(x+1)^2 + 3$

1.3 Graphs of $y = |f(x)|$ where $f(x)$ is quadratic

To sketch the graph of the modulus function $y = |ax^2 + bx + c|$, you must :
First sketch the graph of $y = ax^2 + bx + c$
Reflect in the x-axis the part of the curve $y = ax^2 + bx + c$ that is below the x-axis.

Worked Example 5

Sketch the graph of $y = |x^2 - 2x - 3|$.

Answers

First sketch the graph of $y = x^2 - 2x - 3$.

When $x = 0, y = -3$.

So the y-intercept is -3.

When $y = 0$,

$$x^2 - 2x - 3 = 0$$

$$(x+1)(x+3) = 0$$

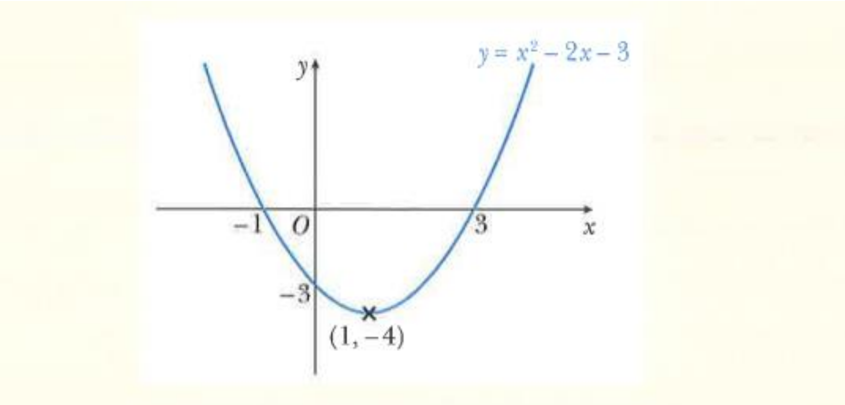
$$x = -1 \text{ or } x = 3.$$

So the x-intercepts are -1 and 3.

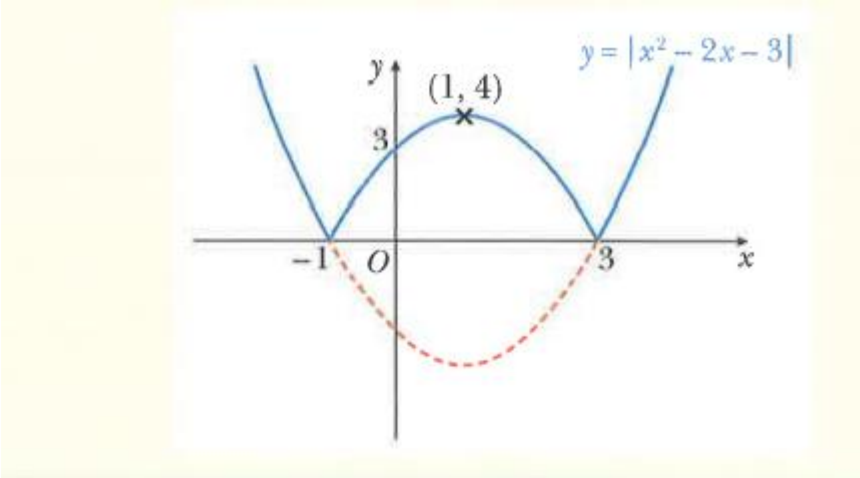
The x-coordinates of the minimum point = $\frac{-1+3}{2}$ = 1.

The y-coordinates of the minimum point $= (1)^2 - 2(1) - 3 = -4$.

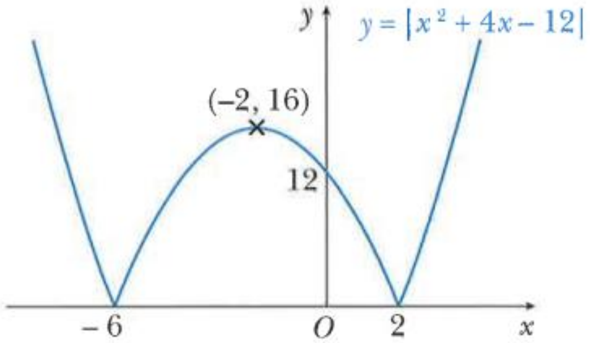
The minimum point is $(1, -4)$.



Now reflect in the x-axis the part of the curve $y = x^2 - 2x - 3$ that is below the x-axis.

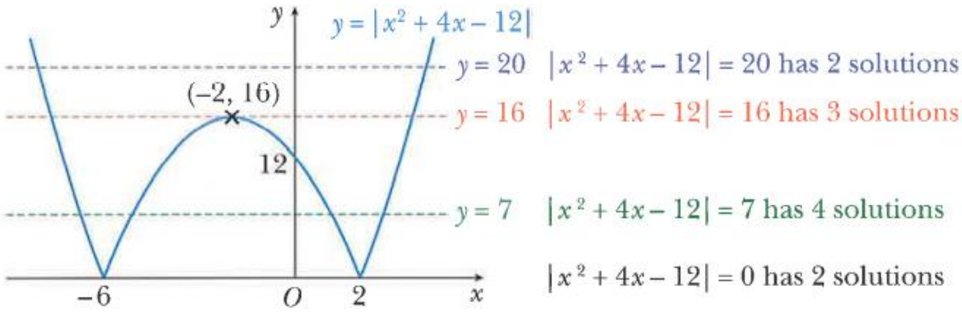


A sketch of the function $y = |x^2 + 4x - 12|$ is shown below.



Now consider using this graph to find the number of solutions of the equation

$|x^2 + 4x - 12| = k$ where $k \geq 0$.



The conditions for the number of solutions of the solutions of the equation $|x^2 + 4x - 12| = k$ are:

Value of k	$k = 0$	$0 < k < 16$	$k = 16$	$k > 16$
Number of solutions	2	4	3	2

Equations involving $|f(x)|$, where $f(x)$ is quadratic, can be solved algebraically:

To solve $|x^2 + 4x - 12| = 16$:

$x^2 + 4x - 12 = 16$ or $x^2 + 4x - 12 = -16$

$x^2 + 4x - 28 = 0$ or $x^2 + 4x + 4 = 0$

$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-28)}}{2(1)}$ or $(x + 2)(x + 2) = 0$

$x = \frac{-4 \pm \sqrt{128}}{2}$ or $x = -2$

$x = -2 \pm 4\sqrt{2}$

$x = 3.66$ or $x = -7.66$ *to 3 sf*

The exact solutions are $x = -2 - 4\sqrt{2}$ or $x = -2$ or $x = -2 + 4\sqrt{2}$.

Exercise 1.3

Basic Level

1. Sketch the graphs of each of the following functions.

   a) $y = |x^2 - 4x + 3\ |$

   b) $y = |x^2 - 2x - 3\ |$

c) $y = |x^2 - 5x + 4\,|$

d) $y = |x^2 - 2x - 8\,|$

e) $y = |2x^2 - 11x - 6\,|$

f) $y = |3x^2 + 5x - 2\,|$

2. $f(x) = 1 - 4x - x^2$

a) Write $f(x)$ in the form $a - (x + b)^2$, where a and b are constants.

Answer: $5 - (x + 2)^2$

b) Sketch the graph of $y = f(x)$.

c) Sketch the graph of $y = |f(x)|$.

3. $f(x) = 2x^2 + x - 3$

a) Write $f(x)$ in the form $a - (x + b)^2$, where a and b are constants.

Answer: $2(x + 0.25)^2 - 3.125$

b) Sketch the graph of $y = f(x)$.

4. a) Find the coordinates of the stationary point on the curve $y = |(x-7)(x+1)$.

   Answer: (3, 16)

   b) Sketch the graph of $y = |(x-7)(x+1)$.

   c) Find the set of values of k for which $|(x-7)(x+1) = k$ has four solutions.

   Answer: $0 < k < 16$

5. a) Find the coordinates of the stationary point on the curve $y = |(x+5)(x+1)|$.
Answer: $(-3, 4)$

b) Find the set of values of k for which $|(x+5)(x+1)| = k$ has two solutions.
Answer: $k > 4$

6. a) Find the coordinates of the stationary point on the curve $y = |(x-8)(x+3)|$.
Answer: $(5.5, 6.25)$

b) Find the value of k for which $|(x-8)(x+3)| = k$ has three solutions.
Answer: 6.257

7. Solve these equations.

a) $|x^2 - 6| = 10$

Answer: $-4, 4$

b) $|x^2 - 2| = 2$

Answer: $-2, 0, 2$

c) $|x^2 - 5x| = 6$

Answer: $-1, 2, 3, 6$

1.4 Quadratic inequalities

Quadratic inequalities can be solved by sketching a graph and considering when the graph is above or below the x-axis.

Worked Example 6

Solve $x^2 - 3x - 4 > 0$.

Answers

Sketch the graph of $y = x^2 - 3x - 4$.

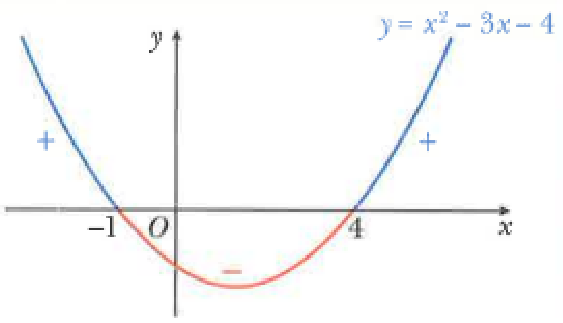
When $y = 0, x^2 - 3x - 4 = 0$

$$(x + 1)(x - 4) = 0$$

$$x = -1 \quad \text{or} \quad x = 4.$$

So the x-axis crossing points are -1 and 4.

For $x^2 - 3x - 4 > 0$ you need to find the range of values of x for which the curve is positive (above the x-axis).



The solution is $x < -1$ and $x > 4$

Worked Example 7

Solve $2x^2 \leq 15 - x$.

Answers

Rearranging: $2x^2 + x - 15 \leq 0$.

Sketch the graph of $y = 2x^2 + x - 15$.

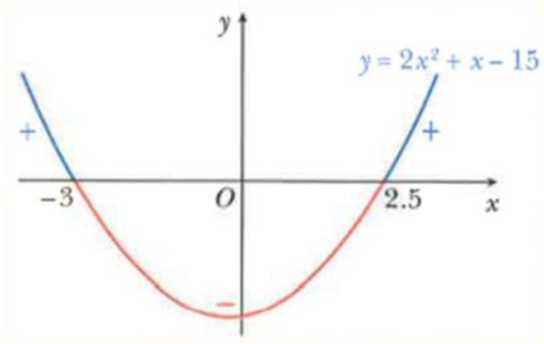
When $y = 0, 2x^2 + x - 15 = 0$

$$(2x - 5)(x + 3) = 0$$

$$x = 2.5 \quad \text{or} \quad x = -3.$$

So the x-axis crossing points are -3 and 2.5.

For $2x^2 + x - 15 \leq 0$ you need to find the range of values of x for which the curve is either zero or negative (below the x-axis).



The solution is $-3 \leq x \leq 2.5$.

Exercise 1.4

Basic Level

1. Solve.
   a) $(x+3)(x-4) > 0$
   Answer: $x < 3, x > 4$

   b) $(x-5)(\ x-1) \leq 0$
   Answer: $1 < x < 5$

   c) $(x-3)(x+7) \geq 0$
   Answer: $x \leq -7, x \geq 3$

   d) $x(x-5) < 0$
   Answer: $0 < x < 5$

e) $(2x + 1)(x - 4) < 0$
Answer: $-0.5 < x < 4$

f) $(3 - x)(x + 1) \geq 0$
Answer: $-1 \leq x \leq 3$

g) $(2x + 3)(x - 5) < 0$
Answer: $-1.5 < x < 5$

h) $(x - 5)^2 \geq 0$
Answer: $-\infty < x < \infty$

i) $(x-3)^2 \leq 0$
Answer: $x = 3$

2. Solve.
   a) $x^2 + 5x - 14 < 0$
   Answer: $-7 < x < 2$

   b) $x^2 + x - 6 \geq 0$
   Answer: $x \leq -3, x \geq 2$

   c) $x^2 - 9x + 20 \leq 0$
   Answer: $4 \leq x \leq 5$

d) $x^2 + 2x - 48 > 0$
Answer: $x < -8, x > 6$

e) $2x^2 - x - 15 \leq 0$
Answer: $-2.5 \leq x \leq 3$

f) $5x^2 + 9x + 4 > 0$
Answer: $x < -1, x > -0.8$

3. Solve.

a) $x^2 < 18 - 3x$

Answer: $-6 < x < 3$

b) $12x < x^2 + 35$

Answer: $x < 5, x > 7$

c) $x(3 - 2x) \leq 1$

Answer: $x \leq 0.5, x \geq 1$

d) $x^2 + 4x < 3(x + 2)$
Answer: $-3 < x < 2$

e) $(x + 3)(1 - x) < x - 1$
Answer: $x < -4, x > 1$

f) $(4x + 3)(3x - 1) < 2x(x - 3)$
Answer: $-0.5 < x < 0.6$

4. Find the set of values of x for which

a) $x^2 - 11x + 24 < 0$ and $2x + 3 < 13$

Answer: $3 < x < 5$

b) $x^2 - 4x \leq 12$ and $4x + 3 > 1$

Answer: $1 < x \leq 6$

c) $x(2x - 1) < 1$ and $7 - 2x < 6$

Answer: $0.5 < x < 1$

d) $x^2 - 3x - 10 < 0$ and $x^2 - 10x + 21 < 0$
Answer: $3 < x < 5$

e) $x^2 + x - 2 > 0$ and $x^2 - 2x - 3 \geq 0$
Answer: $x < -0.2, x \geq 3$

5. Solve.

a) $|x^2 + 2x - 2| < 13$

Answer: $-5 < x < 3$

b) $|x^2 - 8x + 6| < 6$

Answer: $0 < x < 2, 6 < x < 8$

c) $|x^2 - 6x + 4| < 4$

Answer: $0 < x < 2, 4 < x < 6$

1.5 Roots of quadratic equations

The answers to an equation are called the roots of the equation. Consider solving the following three quadratic equations using the formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

$x^2 + 2x - 8 = 0$ $x = \frac{-2 + \sqrt{2^2 - 4 \times 1 \times (-8)}}{2 \times 1}$ $x = \frac{-2 \pm \sqrt{36}}{2}$ $x = 2$ or $x = -4$	$x^2 + 6x + 9 = 0$ $x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 9}}{2 \times 1}$ $x = \frac{-2 \pm \sqrt{0}}{2}$ $x = -1$ or $x = -1$	$x^2 + 2x + 6 = 0$ $x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 6}}{2 \times 1}$ $x = \frac{-2 \pm \sqrt{-20}}{2}$ No solution
2 distinct roots	**2 equal roots**	**0 roots**

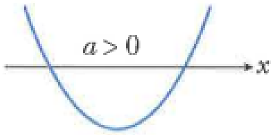
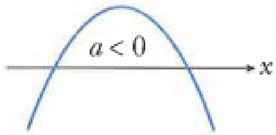
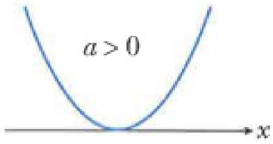
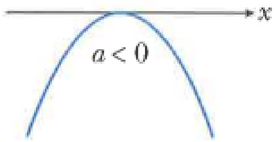
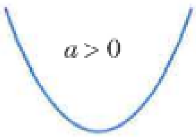
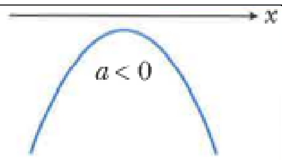
The part of the quadratic formula underneath the square root sign is called the discriminant.

Discriminant $= b^2 - 4ac$

The sign (positive, zero or negative) of the discriminant tells you how many roots there are for a particular quadratic equation.

$b^2 - 4ac$	**Nature of roots**
> 0	2 real distinct roots
$= 0$	2 real equal roots
< 0	0 real roots

There is a connection between the roots of the quadratic equation $ax^2 + bx + c = 0$ and the corresponding curve $y = ax^2 + bx + c$.

$b^2 - 4ac$	**Nature of roots of** $ax^2 + bx + c = 0$	**Shape of curve** $y = ax^2 + bx + c$
> 0	2 real distinct roots	 or  The curve cuts the x-axis at 2 distinct points.
$= 0$	2 real equal roots	 or  The curve touches the x-axis at 1 point.
< 0	0 real roots	 or  The curve is entirely above or entirely below the x-axis.

<u>Worked Example 8</u>

Find the values of k for which $x^2 - 3x + 6 = k(x - 2)$ has two equal roots.

Answers

$$x^2 - 3x + 6 = k(x - 2)$$

$$x^2 - 3x + 6 - kx + 2k = 0$$

$$x^2 - (3 + h)x + 6 + 2k = 0$$

For two equal roots $b^2 - 4ac = 0$.

$$(3 + k)^2 - 4 \times 1(6 + 2k) = 0$$

$$k^2 + 6k + 9 - 24 - 8k = 0$$

$$k^2 - 2k - 15 = 0$$

$$(k + 3)(k - 5) = 0$$

So $k = -3$ or $k = 5$.

Worked Example 9

Find the values of k for which $x^2 + (k-2)x + 4 = 0$ has two distinct roots.

Answers

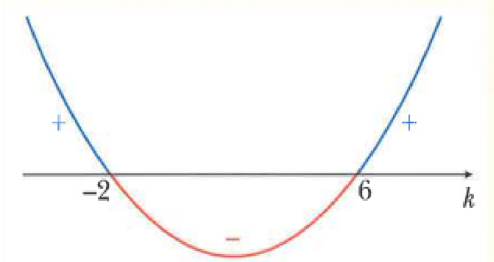
$x^2 + (k-2)x + 4 = 0$

For two distinct roots $b^2 - 4ac > 0$

$$(k-2)^2 - 4 \times 1 \times 4 > 0$$

$$k^2 - 4k + 4 - 16 > 0$$

$$(k+2)(k-6) > 0$$



Critical values are -2 and 6.

So $k < -2$ or $k > 6$.

Exercise 1.5

Basic Level

1. State whether these equations have two distinct roots, two equal roots or no roots.

   a) $x^2 + 4x + 4$
   Answer: two equal roots

   b) $x^2 + 4x - 21$
   Answer: two distinct roots

   c) $x^2 + 9x + 1$
   Answer: two distinct roots

   d) $x^2 - 3x + 15$
   Answer: no roots

e) $x^2 - 6x + 2$
Answer: two distinct roots

f) $4x^2 + 20x + 25$
Answer: two equal roots

g) $3x^2 + 2x + 7$
Answer: no roots

h) $5x^2 - 2x - 9$
Answer: two distinct roots

2. Find the values of k for which $x^2 + kx + 9 = 0$ has two equal roots.

   Answer: $k = \pm 6$

3. Find the values of k for which $kx^2 - 4x + 8 = 0$ has two distinct roots.

   Answer: $k < 0.5$

4. Find the values of k for which $3x^2 + 2x + k = 0$ has no real roots.

   Answer: $k > \frac{1}{3}$

5. Find the values of k for which $(k+1)x^2 + kx - 2k = 0$ has two equal roots.

   Answer: $0, -\frac{8}{9}$

6. Find the values of k for which $kx^2 + 2(k+3)x + k = 0$ has two distinct roots.

   Answer: $k > 1.5$

7. Find the values of k for which $3x^2 - 4x + 5 - k = 0$ has two distinct roots.

   Answer: $k > 3\frac{2}{3}$

8. Find the values of k for which $4x^2 - (k-2)x + 9 = 0$ has two equal roots.

Answer: $k = -10, k = 14$

9. Find the values of k for which $4x^2 + 4(k-2)x + k = 0$ has two equal roots.

Answer: $k = 1, k = 4$

10. Show that the roots of the equation $x^2 + (k - 2)x - 2k = 0$ are real and distinct for all real values of k.

11. Show that the roots of the equation $kx2 + 5x - 2k = 0$ are real and distinct for all real values of k.

Intermediate level

1. Show that the quadratic equation $(t+3)x^2+(2t+5)x+(t+2)=0$ has two distinct real roots for all real values of t except $t=k$. State the value of k.
   Answer: -3

2. Show that the quadratic equation $x^2+(2p+2)x+p^2+2=0$ has no real roots for all real values of p.

3.

a) Prove that the quadratic equation $5x^2 + kx + k - 6 = 0$ has two distinct real roots for all real values of k.

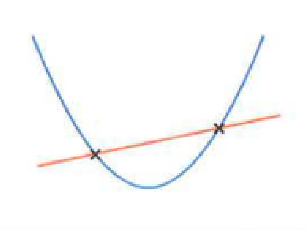
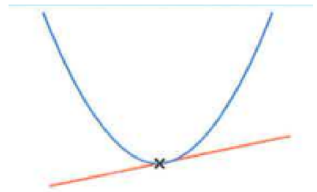
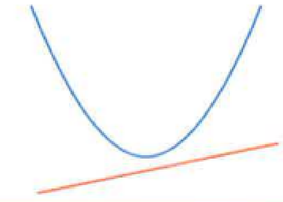
b) Prove that the graph of the quadratic function $y = 5x^2 + kx + k^2$ lies above the x-axis for all real $k \neq 0$.

4. Tom claims that the quadratic equation $ax^2 - 2x = 1 - \frac{a}{2}$ may have two distinct real roots, one repeated real root or no real roots. Do you agree? Explain.
Answer: Yes.

5. Find the smallest positive integer p for which the equation $3x^2 + px + 7 = 0$.
   Answer: 10

1.6 Intersection of a line and a curve

When considering the intersection of a straight line and a parabola, there are three possible situations.

Situation 1	Situation 2	Situation 3
		
2 points of intersection	1 point of intersection	0 point of intersection
The line cuts the curve at two distinct points.	The line touches the curve at one point. This means that the line is a **tangent** to the curve.	The line does not intersect the curve.

You have already learnt that to find the points of intersection of the line $y = x - 6$ with the parabola $y = x^2 - 3x - 4$ you solve the two equations simultaneously.

This would give $x^2 - 3x - 4 = x - 6$

$$x^2 - 4x + 2 = 0.$$

The resulting quadratic equation can then be solved using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The number of points of intersection will depend on the value of $b^2 - 4ac$.
The different situations are given in the table below.

$b^2 - 4ac$	Nature of roots	Line and curve
> 0	2 real distinct roots	2 distinct points of intersection
$= 0$	2 real equal roots	1 point of intersection (line is a tangent)
< 0	0 real roots	no points of intersection

The condition for a quadratic equation to have real roots is $b^2 - 4ac \geq 0$.

Exercise 1.6

Basic Level

1. Find the values of k for which $y = kx + 1$ is a tangent to the curve $y = 2x2 + x + 3$.
   Answer: $k = -3, , k = 5$

2. Find the value of k for which the x-axis is a tangent to the curve
   $y = x^2 + (3 - k)x - (4k + 3)$.
   Answer: $k = -7, k = -3$

3. Find the values of constant c for which the line $y = x + c$ is a tangent to the curve

   $y = 3x + \frac{2}{x}$.

   Answer: $c = \pm 4$

4. Find the set of values of k for which the line $y = 3x + 1$ cuts the curve $y = x2 + kx + 2$ in two distinct points.

   Answer: $k < 1, k > 5$

5. The line $y = 2x + k$ is a tangent to the curve $x^2 + 2xy + 20 = 0$.

   (a) Find the possible values of k.

   Answer: $k = \pm 10$

   (b) For each of these values of k, find the coordinates of the point of contact of the tangent with the curve.

   Answer: $(-2,6), (2,-6)$

6. Find the set of values of k for which the line $y = k - x$ cuts the curve $y = x^2 - 7x + 4$ in two distinct points.
   Answer: $k > -5$

7. Find the values of k for which the line $y = kx - 10$ meets the curve $x^2 + y^2 = 10x$.
   Answer: $k \geq 0.75$

8. Find the set of values of m for which the line $y = mx - 5$ does not meet the curve $y = x^2 - 5x + 4$

   Answer: $-11 < m < 1$

9. The line $y = mx + 6$ is a tangent to the curve $y = x^2 - 4x + 7$. Find the possible values of m.

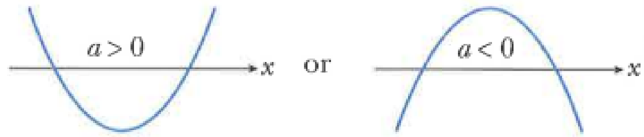
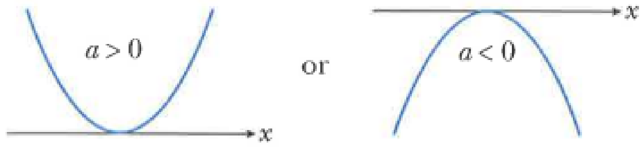
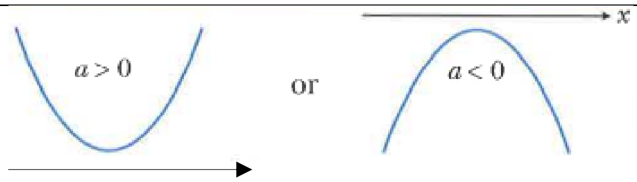
   Answer: $m = -2, m = -6$

Summary

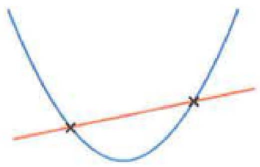
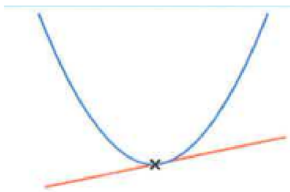
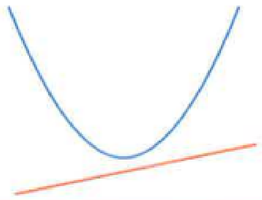
$b^2 - 4ac$	Nature of roots	Line and curve
> 0	2 real distinct roots	2 distinct points of intersection
$= 0$	2 real equal roots	1 point of intersection (line is a tangent)
< 0	0 real roots	no points of intersection

The condition for a quadratic equation to have real roots is $b^2 - 4ac \geq 0$.

Quadratic equation $(ax^2 + bx + c = 0)$ **and corresponding curve** $(y = ax^2 + bx + c)$

$b^2 - 4ac$	**Nature of roots of** $ax^2 + bx + c = 0$	**Shape of curve** $y = ax^2 + bx + c$
> 0	2 real distinct roots	 The curve cuts the x-axis at 2 distinct points.
$= 0$	2 real equal roots	 The curve touches the x-axis at 1 point.
< 0	0 real roots	 The curve is entirely above or entirely below the x-axis.

Quadratic curve and straight line

Situation 1	Situation 2	Situation 3
		
2 points of intersection	1 point of intersection	0 point of intersection
The line cuts the curve at two distinct points.	The line touches the curve at one point. This means that the line is a **tangent** to the curve.	The line does not intersect the curve.

Solving simultaneously the equation of the curve with the equation of the line will give a quadratic equation of the form $ax^2 + bx + c = 0$. The discriminant $b^2 - 4ac$, gives information about the roots of the equation and also about the intersection of the curve with the line.

Examination questions

Worked example

The line $y = 2x - 8$ cuts the curve $2x^2 + y^2 - 5xy + 32 = 0$ at the points A and B.

Find the length of the line AB.

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Answers

$y = 2x - 8$ ------------------------ (1)

$2x^2 + y^2 - 5xy + 32 = 0$----------(2)

Substitute for *y* from (1) in (2):

$2x^2 + (2x - 8)^2 - 5x(2x - 8) + 32 = 0$ expand brackets

$2x^2 + 4x^2 - 32x + 64 - 10x^2 + 40x + 32 = 0$ collect like terms

$-4x^2 + 8x + 96 = 0$ divide by – 4

$x^2 - 2x + 24 = 0$ factorise

$(x - 6)(x + 4) = 0$

$x = 6 \text{ or } x = -4$

When $x = 6$, $y = 2(6) - 8 = 4$.

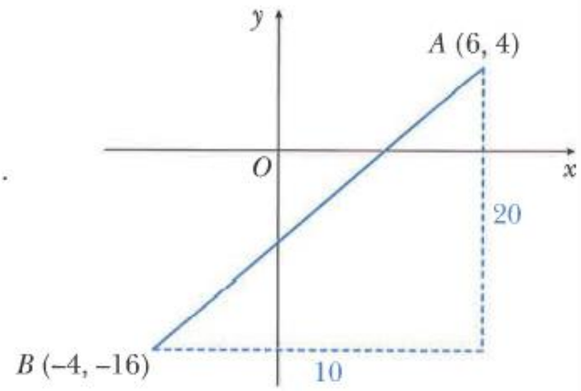
When $x = -4$, $y = 2(-4) - 8 = -16$.

The points of intersection are $A\ (6,4)$ and $B(-4, -16)$.

Using Pythagoras: $AB^2 = 20^2 + 10^2$

$AB^2 = 500$

$AB = 22.4$ to 3sf



1. Find the set of values of k for which the line $y = 2x + k$ cuts the curve $y = x^2 + kx + 5$ at two distinct points. (0606/11_Summer_2012_Q3)
   Answer: $k > 4, k < -4$

2. Solve the simultaneous equations $5x + 3y = 2$ and $\frac{2}{x} - \frac{3}{y} = 1$. (0606/12_Summer_2012_Q4)

   Answer: $x = \frac{1}{5}, y = \frac{1}{3}\ x = 4. y = -6$

3. Find the set of values of k for which the line $y = k - 6x$ is a tangent to the curve $y = x(2x + k)$at two distinct points. (0606/11_Winter_2012_Q2)
Answer: $y = -6, y = 2$ and $x = -6, x = 10$

4. The line $x - 2y = 6$ intersects the curve $x^2 + xy + 10y + 4y^2 = 156$ at the points A and B. Find the length of AB. (0606/12_Winter_2012_Q5)
Answer: $AB = \sqrt{320}\,, 8\sqrt{5}\ or\ 17.9$

5. Find the set of values of m for which the line $y = mx + 2$ does not meet the curve $y = mx^2 + 7x + 11$. (0606/23_Winter_2012_Q5)
Answer: $1 < m < 49$

6. The line $3x + 4y = 15$ cuts the curve $2xy = 9$ at the points A and B. Find the length of AB. (0606/12_Summer_2012_Q5)
Answer: $x = 2, y = \frac{9}{4}$ and $x = 3, y = \frac{3}{2}$ $\quad AB = 1.25$

7. The line $y = 2x - 8$ cuts the curve $2x^2 + y^2 = 5xy + 32 = 0$ at the points *A* and *B*. Find the length of AB. (0606/21_Summer_2012_Q8)
Answer: $22.4 \ or \ \sqrt{500} \ and \ 10\sqrt{5}$

8. Find the set of values of k for which the line $y = k(4x - 3)$does not intersect the curve $y = 4x^2 + 8x - 8$. (0606/11_Summer_2014_Q4)
Answer: $\therefore 3 < k < 4$

9. Find the set of values of k for which the line $y + kx - 2 = 0$ is a tangent to the curve $y = 2x^2 - 9x + 4$. (0606/22_Summer_2014_Q2)
Answer: $k = 5 \text{ and } 13$

10. The line $y = x - 5$ meets the curve $x^2 + y^2 + 2x - 35 = 0$ at the points A and B. Find the exact length of AB. (0606/22_Summer_2014_Q8)
Answer: $\sqrt{72}\ or\ 6\sqrt{2}$

11. The equation of a curve is $y = 2x^2 - 20x + 37$. (0606/23_Summer_2012_Q12)

a) Express y in the form $a(x+b)^2 + c$, where a, b, and c are integers.

Answer: $y = 2(x-5)^2 - 13 \; or \; a = 2, b = -5, c = -13$

b) Write down the coordinates of the stationary point on the curve.

Answer: $(5, -13)$

12. Solve the inequality $4x - 9 > 4x(5 - x)$ (0606/21_Winter_2012_Q1)

Answer: $x < -\frac{1}{2}, x > \frac{9}{2}$

13. It is given that $f(x) = 4 + 8x - x^2$. (0606/21_Winter_2012_Q4)

a) Find the value of a and of b for which $f(x) = a - (x + b)^2$ and hence write down the coordinates of the stationary point of the curve $y = f(x)$

Answer: $a = 20$**,** $a = -4, (4, 20)$

b) On the axes below, sketch the graph of $y = f(x)$, showing the coordinates of the point where your graph intersects with the y-axis.

Answer: Negative quadratic shape, correct position with turning point in first quadrant and 4 marked on y-axis

14. Find the set of values of k for which the line $y = 2x^2 + kx + 2k - 6$ lies above the x-axis for all values of x. (0606/12_Summer_2013_Q4)
Answer: $4 < k < 12$

15. Find the set of values of k for which the curve $y = (k + 1)x^2 - 3x + (k + 1)$ lies below the x-axis. (0606/11_Winter_2013_Q2)
Answer: $k < -\frac{5}{2}$

Review Question

1. Solve $x + 2y = 5,\ \ 2x + y = 2xy$

   Answer: $x = 1, y = 2$ or $x = \frac{5}{2}, y = \frac{5}{4}$

2. Solve $4x^2 - 9(y + 1)^2 = 36,\ \ 2x + 3y = -5$

   Answer: $x = -5, y = \frac{5}{3}$

3. Solve $\frac{2x}{y}+\frac{y}{x}=3,\ 3x-y=2$

   Answer: $x=1, y=1$ or $x=2, y=4$

4. Solve for x and y, $\frac{x^2}{6}-\frac{y}{4}=1,\ x+y=5$

   Answer: $x=3, y=2\ or\ x=-\frac{9}{2}, y=\frac{19}{2}$

5. Find the coordinates of the points of intersection of the line $x + 2y = 10$ and the curve $2y^2 - 7y + x = 0$

   Answer: $\left(5, \frac{5}{2}\right), (6,2)$

6. a) Find the values of c for which the equation $4x^2 - (c + 2)x + c = 1$ has equal roots

   Answer: $2,10$

   b) Find the range of values of k for which curve $y = \frac{16}{x}$ intersects the straight line $y = k - x$ at two distinct points.

   Answer: $k < -8 \text{ or } k > 8$

7. a) Find the set of values of x for which $x(2x-1) > 3$.

Answer: $\left\{x: x < -1 \; or \; x > \frac{3}{2}, x \in \mathbb{R}\right\}$

b) The line $y = 2k - x$ does not intersect the curve $y = (x-1)^2 + k$. Find the range of values of k.

Answer: $k < \frac{3}{4}$

2 Indices and Surds

2.1 Simplifying expressions involving indices

When simplifying expressions involving indices you often need to use more than one of the rules for indices.

In Worked Example 1 and Exercise 2.1 you may assume that all bases are positive real numbers.

WORKED EXAMPLE 1

Simplify $\frac{(4x^2y)^2 \times \sqrt{9x^6y^2}}{(x^3y^2)^{-2}}$.

Answers

$$\frac{(4x^2y)^2 \times \sqrt{9x^6y^2}}{(x^3y^2)^{-2}} = \frac{(4)^2(x^2)^2(y)^2 \times (9)^{\frac{1}{2}}(x^6)^{\frac{1}{2}}(y^2)^{\frac{1}{2}}}{(x^3y^2)^{-2}}$$

$$= \frac{16x^4y^2 \times 3x^3y}{x^{-6}y^{-4}}$$

$$= \frac{48x^7y^3}{x^{-6}y^{-4}}$$

$$= 48x^{13}y^7$$

Exercise 2.1

Basic Level

1. Simplify each of the following.

a) $(x^5)^3$

Answer: x^{15}

b) $x^7 \times x^9$

Answer: x^{16}

c) $x^5 \div x^8$

Answer: x^{-3}

d) $\sqrt{x^{10}}$

Answer: x^5

e) $\sqrt{x^{-4}}$

Answer: x^{-2}

f) $(x^{-3})^5$

Answer: x^{-15}

g) $\sqrt[3]{x^6}$

Answer: x^2

h) $(x^{-1})^2 \times (x^{\frac{1}{2}})^8$

Answer: x^2

i) $3x^2y \times 5x^4y^3$

Answer: $15x^6\, y^{-3}$

j) $\sqrt{25x^6y^{-4}}$

Answer: $5x^3y^{-2}$

k) $(2x^3y^{\frac{3}{2}})^4$

Answer: $16x^{12}y^6$

l) $\sqrt{9x^8y^{-4}} \times \sqrt[3]{8x^6y^{-3}}$

Answer: $6x^6y^{-3}$

2. Simplify each of the following.

a) $\frac{\sqrt[3]{x} \times \sqrt[3]{x^5}}{x^{-2}}$

Answer: x^4

b $\frac{x^2 \times \sqrt{x^5}}{x^{-\frac{1}{2}}}$

Answer: x^5

c $\frac{(\sqrt[3]{x})^2 \times \sqrt{x^6}}{x^{-\frac{1}{3}}}$

Answer: x^4

d $\frac{(3xy)^2 \times \sqrt{x^4 y^6}}{(2x^4 y^3)^2}$

Answer: $\frac{9}{4}x^{-4}y^{-1}$

3. Given that $\frac{(36x^4)^2}{8x^2 \times 3x} = 2^a 3^b x^c$, evaluate a, b and c.

Answer: $a = 1, b = 3, c = 5$

4. Given that $\frac{\sqrt{x^{-1}} \times \sqrt[3]{y^2}}{\sqrt{x^6 y^{-\frac{2}{3}}}} = x^a y^b$, find the value of a and the value of b.

Answer: $a = -\frac{7}{2}, b = 1$

5. Given that $\frac{\sqrt{a^{\frac{4}{5}} b^{-\frac{2}{3}}}}{a^{-\frac{1}{5}} b^{\frac{2}{3}}} = a^x b^y$, find the value of x and the value of y.

Answer: $x = \frac{3}{5}, y = -1$

6. Given that $\frac{(a^x)^2}{b^{5-x}} \times \frac{b^{y-4}}{a^Y} = a^2 b^4$, find the value of x and the value of y.

Answer: $x = 5, y = 8$

7. Simplify $(1 + x)^{\frac{3}{2}} - (1 + x)^{\frac{1}{2}}$.

Answer: $x(1 + x)^{\frac{1}{2}}$

Intermediate level

1.Simplify the algebraic fractions:

a) $\frac{(a^3b^{-3})^2a}{a^{-4}b^2}$

Answer: $a^{11}b^{-8}$

b) $\frac{(3x^{-5}(3x^{-3})^4x^3}{9x\sqrt[3]{x^2}}$

Answer: $3^3x^{-\frac{-47}{3}}$

c) $\dfrac{a^2(a^3b^{-2})^3}{a^{-5}\sqrt{b^2}}$

Answer: $a^{16}b^{-7}$

d) $\dfrac{\sqrt[3]{a}\sqrt[4]{a}\sqrt[5]{a}}{\sqrt[6]{a}\sqrt[7]{a}\,a^{\frac{1}{8}}}$

Answer: $a^{\frac{293}{840}}$

e) $\dfrac{4^{-3}\sqrt{1024}\,x^{\frac{1}{2}}}{256\sqrt{x^3}\,2^{-1}}$

Answer: $2^{-8}x^{-1}$

f) $\dfrac{9x^{-3}\,x^{\frac{2}{3}}(3x^{-1})^{-5}}{81\sqrt[5]{x^4}\sqrt{x^5}}$

Answer: $3^{-7}x^{-\frac{19}{30}}$

2.2 Solving equation involving indices

Consider the equation $2^x = 64$.

The equation has an unknown exponent (or index) and is called an **exponential equation**. You have already learnt how to solve simple exponential equations such as $2^x = 64$. In this section, you will learn how to solve more complicated exponential equations.

Worked Example 2

Solve the equation $2^{3x+1} = 8$.

Answers

$2^{3x+1} = 8$ change 8 to 2^3

$2^{3x+1} = 2^3$ equate the indices

$3x + 1 = 3$ solve for x

$3x = 2$

$x = \frac{2}{3}$

Worked Example 3

Solve the equation $3^{2x-1} \times 9^{x-1} = 243$.

Answers

$$3^{2x-1} \times 9^{x-1} = 243$$ change to base 3

$$3^{2x-1} \times (3^2)^{x-1} = 3^5$$

$$3^{2x-1} \times (3^{2x-2}) = 3^5$$ add the indices on the left hand side

$$3^{4x-3} = 3^5$$ equate the indices

$$4x - 3 = 5$$ solve for x

$$4x = 8$$

$$x = 2$$

Worked Example 4

Solve the simultaneous equation.

$9^x(27^y) \quad = 1$

$4^x \div \sqrt{2})^y \quad = 128$

Answers

$9^x(27^y) \quad = 1$ change to base 3

$4^x \div (\sqrt{2})^y \quad = 128$ change to base 2

$3^{2x} \times 3^{3y} \quad = 3^0$ add indices on the left hand side

$2^{2x} \div 2^{\frac{1}{2}y} \quad = 2^7$ subtract indices on the left hand side

$3^{2x=3y} \quad = 3^0$ equating the indices gives $2x + 3y = 0$

$2^{2x-\frac{1}{2}y} \quad = 2^7$ equating the indices gives $2x - \frac{1}{2}y = 7$

$2x + 3y \quad = 0$ subtract the two equations

$2x - \frac{1}{2}y \quad = 7$

$3\frac{1}{2}y \quad = -7$, so $y = -2$.

Substituting $y = -2$ into $2x + 3y = 0$ gives $x = 3$.

The solution is $x = 3$, $y = -2$.

Worked Example 5

a) Solve the equation $4y^2 + 3y - 1 = 0$.

b) Use your answer to **part a** to solve the equation $4(2^x)^2 + 3(2^x) - 1 = 0$.

Answers

a)

$$4y^2 + 3y - 1 = 0$$

$$(4y - 1)(y + 1) = 0$$

$$4y - 1 = 0 \text{ or } y + 1 = 0$$

$$y = \frac{1}{4} \text{ or } \quad y = -1$$

b)

$$4(2^x)^2 + 3(2^x) - 1 = 0$$

$$2^x = \frac{1}{4} \text{ or } 2^x = -1$$

$$2^x = 2^{-2} \text{ or } 2^x = -1$$

Solving $2^x = 2^{-2}$, gives $x = -2$

There is a solution to $2^x = -1$, since $2^x > 0$ for all real values of x.

The solution is $x = -2$.

Exercise 2.2

Basic Level

1. Solve each of the following equations.

a) $5^{2x} = 5^{7x-1}$

Answer: $\frac{1}{5}$

b) $4^{2x+1} = 4^{3x-2}$

Answer: 3

c) $7^{x^2} = 7^{6-x}$

Answer: $-3 \ or \ 2$

d) $3^{2x^2} = 3^{9x+5}$

Answer: $-\frac{1}{2} \ or \ 5$

2. Solve each of the following equations.

a) $2^{n+1} = 32$

Answer: 4

b) $4^{2n} = 256$

Answer: 2

c) $3^{2n+1} = 27$

Answer: 1

d) $2^{n-1} = \frac{1}{4}$

Answer: -1

e) $5^{n+1} = \frac{1}{125}$
Answer: -4

f) $5^{x^2-16} = 1$
Answer: ± 4

3. Solve each of the following equations.

a) $2^x = 4^3$
Answer: 6

b) $3^{2x-1} = 27^x$
Answer: -1

c) $3^x = 9^{x+5}$
Answer: -10

d) $4^{3x+4} = 8^{4x+12}$
Answer: $-4\frac{2}{3}$

e) $\left(\frac{1}{4}\right)^x = 64$
Answer: -3

f) $4^{5-3x} = \frac{1}{8^{x+1}}$
Answer: $4\frac{1}{3}$

g) $5^{x^2+3} = 25^{2x}$
Answer: $1 \; or \; 3$

h) $3^{x^2-4} = 27^x$
Answer: $-1 \; or \; 4$

4. Solve each of the following equations.

a) $2^{3x} \times 4^{x+1} = 64$

Answer: 0.8

b) $2^{3x+1} \times 8^{x-1} = 128$

Answer: 1.5

c) $(2^{2-x})(4^{2x+3}) = 8$

Answer: $-\frac{5}{3}$

d) $3^{x+1} \times 9^{2-x} = \frac{1}{27}$

Answer: 8

5. Solve each of the following equations.

a) $\frac{27^{2x}}{3^{5-x}} = \frac{3^{2x+1}}{9^{x+3}}$

Answer: 0

b) $\frac{4^{x}}{2^{3-x}} = \frac{2^{3x}}{8^{x-2}}$

Answer: 3

c) $\frac{2^{x+4}}{8^{-x}} = \frac{64}{4^{\frac{1}{2}x}}$

Answer: 0.4

d) $\frac{27^{2x}}{3^{6-x}} = \frac{3^{2x+1}}{9^{x+3}}$

Answer: $\frac{1}{7}$

6. Solve each of the following equations.

a) $3^{2x} \times 2^{x} = \frac{1}{18}$

Answer: -1

b) $2^{2x} \times 5^{x}$=8000

Answer: 3

7. Solve each of the following pairs of simultaneous equations.

a) $4^{x} \div 2^{y} = 16$

$3^{2x} \times 9^{y} = 27$

Answer: $x = \frac{11}{6}, y = -\frac{1}{3}$

b) $125^x \div 5^y = 25$

$2^{3x} \times \left(\frac{1}{8}\right)^{1-y} = 32$

Answer: $x = \frac{7}{6}, y = \frac{3}{2}$

8. a) Solve the equation $2y^2 - 7y - 4 = 0.$

Answer: $-\frac{1}{2}\ or\ 4$

b) Use your answer to **part a** to solve the equation $2(2^x)^2 - 7(2^x) - 4 =$

Answer: 2

9. a) Solve the equation $4y^2 = 15 + 7y$.

Answer: $-\frac{5}{4}$ or 3

b) Use your answer to **part a** to solve the equation $4(9^x) = 15 + 7(3^x)$.

Answer: -1

10. a) Solve the equation $3y = 8 + \frac{3}{y}$.

Answer: $-\frac{1}{3}$ or 3

b) Use your answer to **part a** to solve the equation $3x^{\frac{1}{2}} = 8 + 3x^{-\frac{1}{2}}$

Answer: 9

2.3 Surds

A surd is irrational number of the form $\sqrt{n}$, where n is a positive integer that is not a perfect square.

$\sqrt{2}$, $\sqrt{5}$ and $\sqrt{12}$ are all surds.

$\sqrt{9}$ is not a surd because $\sqrt{9} = \sqrt{3^2} = 3$.

Other examples of surds are $2 + \sqrt{5}$, $\sqrt{7} - \sqrt{2}$ and $\frac{3-\sqrt{2}}{5}$.

When an answer is given using a surd, it is an exact answer.

You can collect like terms together.

$6\sqrt{11} + 3\sqrt{11} = 9\sqrt{11}$ and $5\sqrt{7} + 2\sqrt{7} = 3\sqrt{7}$

<u>Worked Example 6</u>

Simplify $4(5 - \sqrt{3}) - 2(5\sqrt{3} - 1)$.

Answers

$4(5 - \sqrt{3}) - 2(5\sqrt{3} - 1)$ expand the brackets

$= 20 - 4\sqrt{3} - 10\sqrt{3} + 2$ collect like terms

$= 22 - 14\sqrt{3}$

Exercise 2.3

Basic Level

1. Simplify.

a) $3\sqrt{5} + 7\sqrt{5}$

Answer: $10\sqrt{5}$

b) $3\sqrt{10} + 2\sqrt{10}$

Answer: $5\sqrt{10}$

c) $8\sqrt{11} + \sqrt{11}$

Answer: $9\sqrt{11}$

d) $6\sqrt{3} - \sqrt{3}$

Answer: $5\sqrt{3}$

2. $A\ 3\sqrt{5} + 7\sqrt{3}$, $B\ 2\sqrt{5} - 3\sqrt{3}$A $C\ 2\sqrt{3} - \sqrt{5}$

Simplify.

a) $A + B$

Answer: $5\sqrt{5} + 4\sqrt{3}$

b) $A - C$

Answer: $4\sqrt{5} + 5\sqrt{3}$

c) $2A + 3B$

Answer: $12\sqrt{5} + 5\sqrt{3}$

d) $5A + 2B - C$

Answer: $20\sqrt{5} + 27\sqrt{3}$

3. The first 4 terms of a sequence are

$2+3\sqrt{7}$ $\quad 2+5\sqrt{7}$ $\quad 2+7\sqrt{7}$ $\quad 2+9\sqrt{7}$.

a) Write down the $6th$ term of his sequence.

Answer: $2+13\sqrt{7}$

b) Find the sum of the first 5 terms of this sequence.

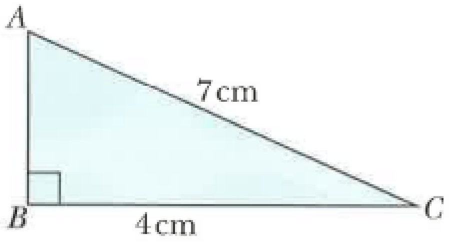
Answer: $10+35\sqrt{7}$

c) Write down an expression for the nth term of this sequence.

Answer: $2+(2n-1)\sqrt{7}$

4. a) Find the exact length of AB.

Answer: $\sqrt{30}$



b) Find the exact perimeter of the triangle.

Answer: $11 + \sqrt{33}$

2.4 Multiplication, division and simplification of surds

You can multiply surds using the rule: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

Worked Example 7

Simplify

a) $\sqrt{3} \times \sqrt{5}$ b) $(\sqrt{8})^2$ c) $2\sqrt{5} \times 3\sqrt{3}$

Answers

a) $\sqrt{3} \times \sqrt{5} = \sqrt{3 \times 5} = \sqrt{15}$

b) $(\sqrt{8})^2 = \sqrt{8} \times \sqrt{8} = \sqrt{64} = 8$ Note: $\sqrt{n} \times \sqrt{n} = n$

c) $2\sqrt{5} \times 3\sqrt{3} = 6\sqrt{15}$

Worked Example 8

Expand and simplify

a) $(4 - \sqrt{3})^2$ b) $(\sqrt{3} + 5\sqrt{2})(\sqrt{2} + \sqrt{3})$

Answers

a) $(4 - \sqrt{3})^2$ square means multiply by itself

$= (4 - \sqrt{3})(4 - \sqrt{3})$ expand the brackets

$= 16 - 4\sqrt{3} - 4\sqrt{3} + 3$ collect like terms

$= 19 - 8\sqrt{3}$

b) $(\sqrt{3} + 5\sqrt{2})(\sqrt{2} + \sqrt{3})$ expand the brackets

$= \sqrt{6} + 3 + 10 + 5\sqrt{6}$ collect like terms

$= 13 + 6\sqrt{6}$

$\sqrt{98}$ can be simplified using the multiplication rule.

$\sqrt{98} = \sqrt{49 \times 2} = \sqrt{49} \times \sqrt{2} = 7\sqrt{2}$

<u>Worked Example 9</u>

Simplify $\sqrt{75} - \sqrt{12}$.

Answers

$\sqrt{75} - \sqrt{12}$	$75 = 25 \times 3$ and $12 = 4 \times 3$
$= \sqrt{25} \times \sqrt{3} - \sqrt{4} \times \sqrt{3}$	$\sqrt{25} = 5$ and $\sqrt{4} = 2$
$= 5 \times \sqrt{3} - 2 \times \sqrt{3}$	collect like terms
$= 3\sqrt{3}$	

You can divide surds using the rule: $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

<u>Worked Example 10</u>

Simplify $\frac{\sqrt{77}}{\sqrt{11}}$

Answers

$\frac{\sqrt{77}}{\sqrt{11}} = \sqrt{\frac{77}{11}} = \sqrt{7}$

Exercise 2.4

Basic Level

1. Simplify.

a) $\sqrt{18} \text{ x } \sqrt{2}$
Answer: 6

b) $\sqrt{2} \text{ x } \sqrt{72}$
Answer: 12

c) $\sqrt{5} \text{ x } \sqrt{6}$
Answer: $\sqrt{30}$

d) $(\sqrt{2})^2$
Answer: 2

e) $(\sqrt{13})^2$
Answer: 13

f) $(\sqrt{5})^3$
Answer: $5\sqrt{5}$

g) $3\sqrt{2} \text{ x } 5\sqrt{3}$
Answer: $15\sqrt{6}$

h) $7\sqrt{5} \text{ x } 2\sqrt{7}$
Answer: $14\sqrt{35}$

2. Simplify.

a) $\frac{\sqrt{112}}{\sqrt{28}}$
Answer: 2

b) $\frac{\sqrt{52}}{\sqrt{26}}$
Answer: $\sqrt{2}$

c) $\frac{\sqrt{12}}{\sqrt{3}}$
Answer: 2

d) $\frac{\sqrt{12}}{\sqrt{108}}$
Answer: $\frac{1}{3}$

e) $\frac{\sqrt{15}}{\sqrt{3}}$
Answer: $\sqrt{5}$

f) $\frac{\sqrt{54}}{\sqrt{6}}$
Answer: 3

g) $\frac{\sqrt{5}}{\sqrt{81}}$
Answer: $\frac{\sqrt{5}}{9}$

h) $\frac{\sqrt{88}}{2\sqrt{11}}$
Answer: $\sqrt{2}$

i) $\frac{9\sqrt{20}}{3\sqrt{5}}$
Answer: 6

3. Simplify.

a) $\sqrt{8}$
Answer: $2\sqrt{2}$

b) $\sqrt{12}$
Answer: $2\sqrt{3}$

c) $\sqrt{20}$
Answer: $2\sqrt{5}$

d) $\sqrt{50}$
Answer: $5\sqrt{2}$

e) $\sqrt{72}$
Answer: $6\sqrt{2}$

f) $\sqrt{18}$
Answer: $3\sqrt{2}$

g) $\sqrt{80}$
Answer: $4\sqrt{5}$

h) $\sqrt{90}$
Answer: $3\sqrt{10}$

i) $\sqrt{63}$
Answer: $3\sqrt{7}$

j) $\sqrt{44}$
Answer: $2\sqrt{11}$

k) $\sqrt{125}$
Answer: $5\sqrt{5}$

l) $\sqrt{117}$
Answer: $3\sqrt{13}$

m) $\sqrt{75}$
Answer: $5\sqrt{3}$

n) $\sqrt{3000}$
Answer: $10\sqrt{30}$

o) $\frac{\sqrt{20}}{2}$
Answer: $\sqrt{5}$

p) $\frac{\sqrt{500}}{5}$
Answer: $2\sqrt{5}$

q) $\sqrt{20}$ x $\sqrt{10}$
Answer: $10\sqrt{2}$

r) $\sqrt{8}$ x $\sqrt{5}$
Answer: $2\sqrt{10}$

s) $\sqrt{245}$ x $\sqrt{5}$
Answer: 35

4. Simplify.

a) $5\sqrt{3}+\sqrt{48}$
Answer: $9\sqrt{3}$

b) $\sqrt{12}+\sqrt{3}$
Answer: $3\sqrt{3}$

c) $\sqrt{20}+3\sqrt{5}$
Answer: $5\sqrt{5}$

d) $\sqrt{75}+2\sqrt{3}$
Answer: $7\sqrt{3}$

e) $\sqrt{32}-2\sqrt{8}$
Answer: 0

f) $\sqrt{125}+\sqrt{80}$
Answer: $9\sqrt{5}$

g) $\sqrt{45}-\sqrt{5}$
Answer: $2\sqrt{5}$

h) $\sqrt{20}-5\sqrt{5}$
Answer: $-3\sqrt{5}$

i) $\sqrt{175}-\sqrt{28}+\sqrt{63}$
Answer: $6\sqrt{7}$

j) $\sqrt{50} + \sqrt{72} - \sqrt{18}$
Answer: $8\sqrt{2}$

k) $\sqrt{200} - 2\sqrt{18} + \sqrt{72}$
Answer: $10\sqrt{2}$

l) $\sqrt{80} + 2\sqrt{20} + 4\sqrt{45}$
Answer: $20\sqrt{5}$

m) $5\sqrt{12} - 3\sqrt{48} + 2\sqrt{75}$
Answer: $8\sqrt{3}$

5. Expand and simplify.

a) $\sqrt{2}(3+\sqrt{2})$
Answer: $2+3\sqrt{2}$

b) $\sqrt{3}(2\sqrt{3}+\sqrt{12})$
Answer: $3\sqrt{3}$

c) $\sqrt{2}(5-2\sqrt{2})$
Answer: $5\sqrt{5}$

d) $\sqrt{3}(\sqrt{27}+5)$
Answer: $7\sqrt{3}$

e) $\sqrt{3}(\sqrt{3}-1)$
Answer: 0

f) $\sqrt{5}(2\sqrt{5}+\sqrt{20})$
Answer: $9\sqrt{5}$

g) $(\sqrt{2}+1)(\sqrt{2}-1)$
Answer: $2\sqrt{5}$

h) $(\sqrt{3}+5)(\sqrt{3}-1)$
Answer: $-3\sqrt{5}$

i) $(2+\sqrt{5})(2\sqrt{5}+1)$
Answer: $6\sqrt{7}$

j) $(3-\sqrt{2})(3+\sqrt{2})$
Answer: $8\sqrt{2}$

k) $(4+\sqrt{3})(4-\sqrt{3})$
Answer: $10\sqrt{2}$

l) $(1+\sqrt{5})(1-\sqrt{5})$
Answer: 0

m) $(\sqrt{7}+\sqrt{5})(\sqrt{7}+2\sqrt{5})$
Answer: $8\sqrt{3}$

6. Expand and simplify.

a) $(2 + \sqrt{5})^2$

Answer: $9 + 4\sqrt{5}$

b) $(5 - \sqrt{3})^2$

Answer: $28 - 10\sqrt{3}$

c) $(4 + 5\sqrt{3})^2$

Answer: $91 + 40\sqrt{3}$

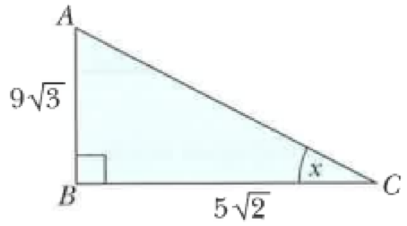
d) $(\sqrt{2} + \sqrt{3})^2$

Answer: $5 + 2\sqrt{6}$

7. A rectangle has sides of length $(2 + \sqrt{8})$cm and $(7 - \sqrt{2})$ cm.
Find the area of the rectangle.
Express your answer in the form $a + b\sqrt{2}$, where a and b are integers.
Answer: $10 + 12\sqrt{2}\ cm^2$

8. a) Find the value of AC^2.

Answer: 293



b) Find the value of $\tan x$.

Write your answer in the form $\frac{a\sqrt{6}}{b}$, where a and b are integers.

Answer: $a = 9, b = 10$

c) Find the area of the triangle.

Write your answer in the form $\frac{p\sqrt{6}}{q}$, where p and q are integers.

Answer: $p = 45, q = 2$

9. A cuboid has a square base.

The sides of the square are of length $(1+\sqrt{2})$cm.

The height of the cuboid is $(5-\sqrt{2})$cm.

Find the volume of the cuboid.

Express your answer in the form $a+b\sqrt{2}$, where a and b are integers.

Answer: $11+7\sqrt{2}\ \mathrm{cm}^2$

2.5 Rationalising the denominator of a fraction

You rationalise the denominator of a fraction when it is a surd.
To rationalise the denominator of a fraction means to turn an irrational denominator into a rational number.

In earlier Exercise, you found that

$$\left(4+2\sqrt{3}\right)\left(4-2\sqrt{3}\right)=(4)^2-8\sqrt{3}-\left(2\sqrt{3}\right)^2=16=12=4$$

This is an example of the product of two irrational numbers $\left(4+2\sqrt{3}\right)$ and $\left(4-2\sqrt{3}\right)$ giving a rational number (4).

$4+2\sqrt{3}$ and $4-2\sqrt{3}$ are called conjugate surds.

The product of two conjugate surds always gives a rational number.

The product of conjugate surds $a+b\sqrt{c}$ and $a-b\sqrt{c}$ is a rational number.
So you can rationalise the denominator of a fraction using these rules:

- For fractions of the form $\frac{1}{\sqrt{a}}$, multiply numerator and denominator by $\sqrt{a}$.
- For fractions of the form $\frac{1}{a+b\sqrt{c}}$, multiply numerator and denominator by $a-b\sqrt{c}$.
- For fractions of the form $\frac{1}{a-b\sqrt{c}}$, multiply numerator and denominator by $a+b\sqrt{c}$.

Worked Example 11

Rationalise the denominators of

a) $\frac{2}{\sqrt{5}}$ b) $\frac{5}{2+\sqrt{3}}$ c) $\frac{\sqrt{7}+3\sqrt{2}}{\sqrt{7}-\sqrt{2}}$

Answers

a) $\frac{2}{\sqrt{5}}$ multiply numerator and denominator by $\sqrt{5}$

$= \frac{2\sqrt{5}}{\sqrt{5}\,\sqrt{5}}$

$= \frac{2\sqrt{5}}{5}$

b) $\frac{5}{2+\sqrt{3}}$ multiply numerator and denominator by $(2-\sqrt{3})$

$= \frac{5(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$ use $(x+y)(x-y) = x^2 - y^2$ to expand the denominator

$= \frac{10-5\sqrt{3}}{(2)^2-(\sqrt{3})^2}$

$= 10 - 5\sqrt{3}$

c) $\frac{\sqrt{7}+3\sqrt{2}}{\sqrt{7}-\sqrt{2}}$ multiply numerator and denominator by $(\sqrt{7}+\sqrt{2})$

$= \frac{(\sqrt{7}+3\sqrt{2})(\sqrt{7}+\sqrt{2})}{(\sqrt{7}-\sqrt{2})(\sqrt{7}+\sqrt{2})}$ use $(x+y)(x-y) = x^2 - y^2$ to expand the denominator

$= \frac{7+\sqrt{14}+3\sqrt{14}+6}{(\sqrt{7})^2-\sqrt{2})^2}$

$= \frac{13+4\sqrt{14}}{15}$

Exercise 2.5

Basic Level

1. Rationalise the denominators.

a) $\frac{1}{\sqrt{5}}$

Answer: $\frac{\sqrt{5}}{5}$

b) $\frac{3}{\sqrt{2}}$

Answer: $\frac{3\sqrt{2}}{2}$

c) $\frac{9}{\sqrt{3}}$

Answer: $3\sqrt{3}$

d) $\frac{4}{\sqrt{5}}$

Answer: $\frac{4\sqrt{5}}{5}$

e) $\frac{12}{\sqrt{3}}$

Answer: $4\sqrt{3}$

f) $\frac{4}{\sqrt{12}}$

Answer: $\frac{2\sqrt{3}}{3}$

g) $\frac{3}{\sqrt{8}}$

Answer: $\frac{3\sqrt{2}}{4}$

h) $\frac{\sqrt{2}}{\sqrt{32}}$

Answer: $\frac{1}{4}$

i) $\frac{\sqrt{3}}{\sqrt{15}}$

Answer: $\frac{\sqrt{5}}{5}$

j) $\frac{5}{2\sqrt{2}}$

Answer: $\frac{5\sqrt{2}}{4}$

k) $\frac{7}{2\sqrt{3}}$

Answer: $\frac{7\sqrt{3}}{6}$

l) $\frac{1+\sqrt{5}}{\sqrt{5}}$

Answer: $\frac{5+\sqrt{5}}{5}$

m) $\frac{3-\sqrt{2}}{\sqrt{2}}$

Answer: $\frac{-2+3\sqrt{2}}{2}$

n) $\frac{14-\sqrt{7}}{\sqrt{7}}$

Answer: $-1+2\sqrt{7}$

2. Rationalise the denominators and simplify.

a) $\frac{1}{1+\sqrt{2}}$

Answer: $-1+\sqrt{2}$

b) $\frac{1}{3+\sqrt{5}}$

Answer: $\frac{3-\sqrt{5}}{4}$

c) $\frac{4}{3-\sqrt{5}}$

Answer: $3+\sqrt{5}$

d) $\frac{5}{2+\sqrt{5}}$

Answer: $-10+5\sqrt{5}$

e) $\frac{2}{2-\sqrt{3}}$

Answer: $4+2\sqrt{3}$

f) $\frac{5}{2\sqrt{3}-3}$

Answer: $\frac{15+10\sqrt{3}}{3}$

g) $\frac{2}{2-\sqrt{3}}$

Answer: $4 + 2\sqrt{3}$

h) $\frac{5}{2\sqrt{3}-3}$

Answer: $\frac{15+10\sqrt{3}}{3}$

i $\frac{1}{2\sqrt{3}-\sqrt{2}}$

Answer: $\frac{2\sqrt{3}+\sqrt{2}}{10}$

j $\frac{8}{\sqrt{7}-\sqrt{5}}$

Answer: $4\sqrt{7} + 4\sqrt{5}$

3. Rationalise the denominators and simplify.

a) $\frac{2-\sqrt{3}}{2+\sqrt{3}}$

Answer: $7-4\sqrt{3}$

b) $\frac{1+\sqrt{2}}{3-\sqrt{2}}$

Answer: $\frac{5+4\sqrt{2}}{7}$

c) $\frac{\sqrt{2}+1}{2\sqrt{2}-1}$

Answer: $\frac{5+3\sqrt{2}}{7}$

d) $\frac{\sqrt{7}-\sqrt{2}}{\sqrt{7}+\sqrt{2}}$

Answer: $\frac{9-2\sqrt{14}}{5}$

e) $\frac{\sqrt{5}+1}{3-\sqrt{5}}$

Answer: $2+\sqrt{5}$

f) $\frac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}$

Answer: $\frac{14-\sqrt{187}}{3}$

g) $\frac{\sqrt{3}-\sqrt{7}}{\sqrt{3}+\sqrt{7}}$

Answer: $\frac{-5+\sqrt{21}}{2}$

h) $\frac{\sqrt{23}-\sqrt{37}}{\sqrt{37}+\sqrt{23}}$

Answer: $\frac{30+\sqrt{851}}{7}$

4. Write as a single fraction.

a) $\frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{3}-1}$

Answer: $\sqrt{3}$

b) $\frac{2}{\sqrt{7}+\sqrt{2}} + \frac{1}{\sqrt{7}-2}$

Answer: $\frac{3\sqrt{7}-\sqrt{2}}{5}$

c) $\frac{2}{4-\sqrt{3}} + \frac{1}{4+\sqrt{3}}$

Answer: $\frac{12+\sqrt{3}}{13}$

5. The area of a rectangle is $(8 + \sqrt{10})$ cm^2.

The length of one side is $(\sqrt{5} + \sqrt{2})$ cm.

Find the length of the other side in the form $a\sqrt{5} + b\sqrt{2}$, where a and b are integers.

Answer: $(2\sqrt{5} - \sqrt{2})\text{cm}$

6. A cuboid has a square base of length $(2+ \sqrt{5})$ cm.

The volume of the cuboid is $(16 + 7\sqrt{5})$ cm^3.

Find the height of the cuboid.

Express your answer in the form $a + b\sqrt{5}$, where a and b are integers.

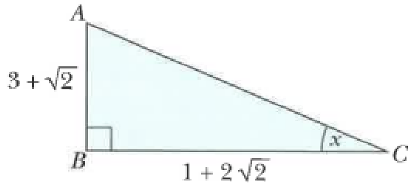
Answer: $(4 - \sqrt{5})cm$

7. A right circular cylinder has a volume of $(25 + 14\sqrt{3})\pi$ cm^3 and a base radius of $(2+\sqrt{3})$cm. Find its height in the form $(a + b\sqrt{3})$ cm, where a and b are integers.

Answer: $(7 - 2\sqrt{3})cm$

9. a) Find the value of $\tan x$. Write your answer in the form $\frac{a+b\sqrt{2}}{c}$, where a, b and c are integers.

Answer: $\frac{1+5\sqrt{2}}{7}$

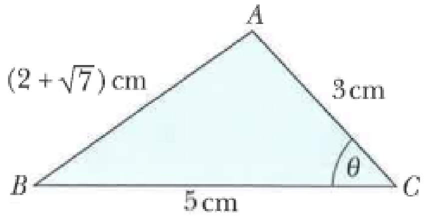


b) Find the area of triangle. Write your answer in the form $\frac{p+q\sqrt{2}}{r}$, where p, q and r are integers.

Answer: $\frac{15+14\sqrt{2}}{2}$

9. Find the value of $\cos\theta$. Write your answer in the form $\frac{a+b\sqrt{7}}{c}$, where a, b and c are integers.

Answer: $\frac{23-4\sqrt{7}}{30}$



2.6 Solving equations involving surds

Worked Example 12

Solve $\sqrt{5}x = \sqrt{2}x + \sqrt{7}$.

Answers

$\sqrt{5}\,x = \sqrt{2}\,x + \sqrt{7}$ collect x's on one side

$\sqrt{5}\,x - \sqrt{2}\,x = \sqrt{7}$ factorise

$x(\sqrt{5} - \sqrt{2}) = \sqrt{7}$ divide both sides by $(\sqrt{5} - \sqrt{2})$

$x = \frac{\sqrt{7}}{\sqrt{5}-\sqrt{2}}$ multiply numerator and denominator by $(\sqrt{5} + \sqrt{2})$

$x = \frac{\sqrt{7}(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$ use $(x + y)(x - y) = x^2 - y^2$ to expand the denominator

$x = \frac{\sqrt{35}+\sqrt{14}}{(\sqrt{5})^2-(\sqrt{2})^2}$

$x = \frac{\sqrt{35}+\sqrt{14}}{5-2}$

$x = \frac{\sqrt{35}+\sqrt{14}}{3}$

Worked Example 13

Solve the simultaneous equations.

$3x + 5y = 20$

$\sqrt{2}\,x - 5\sqrt{2}\,y = 8$

Answers

$3x + 5y = 20$ multiply the first equation by $\sqrt{2}$

$\sqrt{2}\,x - 5\sqrt{2}\,y = 8$

$3\sqrt{2}\,x + 5\sqrt{2}\,y = 20$ add the two equations to eliminate y

$\sqrt{2}\,x - 5\sqrt{2}\,y = 8$

$4\sqrt{2}\,x = 20\sqrt{2} + 8$ divide both sides by $4\sqrt{2}$

$x = \frac{20\sqrt{2}+8}{4\sqrt{2}}$ multiply numerator and denominator by $\sqrt{2}$

$x = \frac{(20\sqrt{2}+8)\times\sqrt{2}}{4\sqrt{2}\times\sqrt{2}}$

$x = \frac{40+8\sqrt{2}}{8}$

$x = 5 + \sqrt{2}$

Substituting $x = 5 + \sqrt{2}$ in the first equation gives

$3(5 + \sqrt{2}) + 5y = 20$

$15 + 3\sqrt{2} + 5y = 20$

$5y = 5 - 3\sqrt{2}$

$y = \frac{5-3\sqrt{2}}{5}$

So the solution is $x = 5 + \sqrt{2}, y = \frac{5-3\sqrt{2}}{5}$.

Worked Example 14

Solve $\sqrt{x} = 2x - 6$.

Answers

$$\sqrt{x} = 2x - 6 \quad \text{square both sides}$$
$$x = (2x - 6)(2x - 6) \quad \text{expand the brackets}$$
$$x = 4x^2 - 24x + 36 \quad \text{collect like terms}$$
$$4x^2 - 25x + 36 = 0 \quad \text{factorise}$$
$$(4x - 9)(x - 4) = 0$$
$$x = \frac{9}{4} \text{ or } x = 4$$

Check $x = \frac{9}{4}$ in the original equation:

$\sqrt{\frac{9}{4}} = \frac{3}{2}$ and $2 \times \frac{9}{4} - 6 = -\frac{3}{2}$, so $x = \frac{9}{4}$ is not a valid solution of the original equation.

Check $x = 4$ in the original equation:

$\sqrt{4} = 2$ and $2 \times 4 - 6 = 2$, so $x = 4$ is a solution of the original equation.

So the final answer is $x = 4$.

Worked Example 15

Solve $\sqrt{3x + 4} - \sqrt{2x + 1} = 1$

Answers

$$\sqrt{3x + 4} - \sqrt{2x + 1} = 1 \quad \text{isolate one of the square roots}$$
$$\sqrt{3x + 4} = 1 + \sqrt{2x + 1} \quad \text{square both sides}$$
$$3x + 4 = (1 + \sqrt{2x + 1})(1 + \sqrt{2x + 1}) \quad \text{expand the brackets}$$
$$3x + 4 = 1 + 2\sqrt{2x + 1} + 2x + 1 \quad \text{isolate the square root and collect like terms}$$
$$x + 2 = 2\sqrt{2x + 1} \quad \text{square both sides}$$
$$(x + 2)^2 = 4(2x + 1) \quad \text{expand the brackets}$$
$$x^2 + 4x + 4 = 8x + 4 \quad \text{collect like terms}$$
$$x^2 - 4x = 0 \quad \text{factorise}$$
$$x(x - 4) = 0$$
$$x = 0 \text{ or } x = 4$$

Check $x = 0$ in the original equation:

$\sqrt{3 \times 0 + 4} - \sqrt{2 \times 0 + 1} = \sqrt{4} - \sqrt{1} = 1$, so $x = 0$ is a solution of the original equation.

Check $x = 4$ in the original equation:

$\sqrt{3 \times 4 + 4} - \sqrt{2 \times 4 + 1} = \sqrt{16} - \sqrt{9} = 1$, so $x = 4$ is also a solution of the original equation.

So the final answer is $x = 0$ or $x = 4$.

Exercise 2.6

Basic Level

1. Solve these equations.

a) $\sqrt{12}x - \sqrt{5}x = \sqrt{3}$

Answer: $\frac{6+\sqrt{15}}{7}$

b) $\sqrt{10}x = \sqrt{5}\,x + \sqrt{2}$

Answer: $\frac{\sqrt{10}+2\sqrt{5}}{5}$

c) $\sqrt{17}x = 2\sqrt{3}x + \sqrt{5}$

Answer: $\frac{\sqrt{85}+2\sqrt{15}}{5}$

2. Solve these simultaneous equations.

a) $3x - y = 5\sqrt{2}$

$2x + y = 5$

Answer: $1 + \sqrt{2}, 3 - 2\sqrt{2}$

b) $x + y = 5$

$\sqrt{6}x + 2y = 12$

Answer: $2 + \sqrt{6}, 3 - \sqrt{6}$

c) $3x + y = 6$

$4x + 3y = 8 - 5\sqrt{5}$

Answer: $2 + \sqrt{5}, -3\sqrt{5}$

d) $2x + y = 11$

$5x - 3y = 11\sqrt{7}$

Answer: $3 + \sqrt{7}, 5 - 2\sqrt{7}$

e) $x + \sqrt{2}y = 5 + 4\sqrt{2}$

$x + y = 8$

Answer: $3 - \sqrt{2}, 5 + \sqrt{2}$

3. Solve the equations.

a) $10\sqrt{x} - 4 = 7\sqrt{x} + 6$

Answer: $\frac{100}{9}$

b) $6\sqrt{x} + 4 = 9\sqrt{x} + 3$

Answer: $\frac{1}{9}$

c) $\sqrt{2x-1}-3=0$
Answer: 5

d) $\sqrt{2x-3}=6$
Answer: 19.5

e) $\sqrt{5x-1}= \sqrt{x+7}$
Answer: 2

f) $3+\sqrt{5x+6}=12$
Answer: 15

g) $\sqrt{5x-8}=2x$
Answer: $2\sqrt{2}$

h) $12-\sqrt{x+5}=7$
Answer: 20

4. Solve these equations.

a) $\sqrt{2x-1}=x$
Answer: 1

b) $\sqrt{x+6}=x$
Answer: 3

c) $\sqrt{2x+3}-x=0$
Answer: 3

d) $\sqrt{10-2x}+x=1$
Answer: -3

e) $\sqrt{x+15} = x+3$
Answer: 1

f) $\sqrt{x+4} + 2 = x$
Answer: 5

g) $\sqrt{x+5} + 1 = x$
Answer: 4

h) $\sqrt{x} = 2x - 6$
Answer: 4

i) $\sqrt{3x+1} - x - 1 = 0$
Answer: $0\ or\ 1$

j) $2x + 3 - \sqrt{20x+9} = 0$
Answer: $0\ or\ 2$

6. The roots of equation $x^2 - 2\sqrt{6}x + 5 = 0$ are p and q, where $p > q$.

   Write $\frac{p}{q}$ in the form $\frac{a+b\sqrt{6}}{c}$, where a, b and c are integers.

   Answer: $\frac{7+2\sqrt{6}}{5}$

7. Find the positive root of the equation $(4-\sqrt{2})x^2 - (1-2\sqrt{2})x - 1 = 0$.

   Write your answer in the form $\frac{a+b\sqrt{2}}{c}$, where a, b and c are integers.

   Answer: $\frac{2+\sqrt{2}}{2}$

Intermediate level

Solve the irrational equations:

1. $2\sqrt{x+5} = x+2$
   Answer: $x = 4$

2. $4 + \sqrt{x-4} = x$
   Answer: $x_1 = 4;\ x_2 = 8$

3. $x - 2\sqrt{x-6} = 6$
   Answer: $x_1 = 8;\ x_2 = 10$

4. $\sqrt{4-x} = 3 - \sqrt{5+x}$
Answer: $x_1 = -5;\ x_2 = 4$

5. $5 + 2\sqrt{x-2} = x$
Answer: $x = 11$

6. $2\sqrt{x-1} + \sqrt{4x-1} = \sqrt{3}$
Answer: $x = 1$

7. $2+\sqrt{x^2-4x+4}=x$
Answer: $x \in \langle 2:\infty)$

8. $\sqrt{x+5}+\sqrt{2-x}=0$
Answer: $\emptyset$

9. $2\sqrt{x+2}+\sqrt{2-4x}=\sqrt{10}$
Answer: $x_1=-2;\ x_2=\frac{1}{2}$

10. $5 - \sqrt{x^2 + 10x + 5} = \sqrt{x^2}$

Answer: $x = 1$

11. $\sqrt{12 - x} - \sqrt{4 - x} = 2$

Answer: $x = 3$

Examination Questions

1. A cuboid has a square base of side $(2+\sqrt{3})$cm and a volume of $(16+9\sqrt{3})$ cm$^{3.}$ Without a calculator, find the height of the cuboid in the form $(a+b\sqrt{3})$cm, where a and b are integers. [4] (0606/23_Summer_2012_Q2)

Answer: $4-\sqrt{3}$

2. Solve the equation $\frac{36^{2y-5}}{6^{3y}} = \frac{6^{2y-1}}{216^{y+6}}$. (0606/23_Summer_2012_Q5)

Answer: $y = -4.5$

3. a) (i) Show that $3\sqrt{5} - 2\sqrt{2}$ is a square root of $53 - 12\sqrt{10}$.

Answer: $\left(3\sqrt{5} - 2\sqrt{2}\right)^2 = 45 - 12\sqrt{10} + 8 = 53 - 12\sqrt{10}$

(ii) State the other square root of $53 - 12\sqrt{10}$.

Answer: $\left(-3\sqrt{5} + 2\sqrt{2}\right)$

b) Express $\frac{6\sqrt{3}+7\sqrt{2}}{4\sqrt{3}+5\sqrt{2}}$ in the form of $a + b\sqrt{6}$, where a and b are integers to be found.

(0606/11_Winter_2012_Q7)

Answer: $\frac{6\sqrt{3}+7\sqrt{2}}{4\sqrt{3}+5\sqrt{2}} \times \frac{4\sqrt{3}-5\sqrt{2}}{4\sqrt{3}-5\sqrt{2}} = -1 + \sqrt{6}$

4. a) Given that $\frac{2^{x-3}}{8^{2y-3}} = 16^{x-y}$, show that $3x + 2y = 6$.

b) Given also that $\frac{5^y}{125^{x-2}} = 25$, find the value of x and of y. (0606/22_Winter_2012_Q6)

Answer: $x = \frac{14}{9}$ and $y = \frac{2}{3}$

5. Without using a calculator, simplify $\frac{(3\sqrt{3}-1)^2}{2\sqrt{3}-3}$, giving your answer in the form $\frac{a\sqrt{3}+b}{3}$,where a and b are integers. (0606/23_Winter_2012_Q3)

Answer: $\frac{38+\sqrt{3}+48}{3}$ $or\ a = 38, b = 48$

6. Express $\frac{y\times(4x^3)^2}{\sqrt{8y^3}}$ in the form $2^a \times x^b \times y^c$, where a, b and c are constants. (0606/22_Summer_2013_Q2)

Answer: $2^{\frac{5}{2}} \times x^6 \times y^{-\frac{1}{2}}$ $or\ a = \frac{5}{2}, b = 6, c = -\frac{1}{2}$

7. Express $\frac{(4\sqrt{5}-2)^2}{\sqrt{5}-1}$ in the form $p\sqrt{5}+q$, where p and q are integers.
(0606/21_Winter_2013_Q2)
Answer: $p = 17, q = 1$

8. Solve the simultaneous equations. (0606/21_Winter_2013_Q5)

$$\frac{4^x}{256^y} = 1024$$
$$3^{2x} \times 9^y = 243$$

Answer: $x = 3 \; or \; y = -0.5$

9. Without using a calculator, express $\frac{(2+\sqrt{5})^2}{\sqrt{5}-1}$ in the form $a+b\sqrt{5}$, where a and b are constants to be found. (0606/22_Summer_2014_Q1)

Answer: $\frac{29}{4}+\frac{13}{4}\sqrt{5}$

10. a) Show that $\left(2\sqrt{2}+4\right)^{2}-8\left(2\sqrt{2}+3\right)=0$

b) Solve the equation $\left(2\sqrt{2}+3\right)x^{2}-\left(2\sqrt{2}+4\right)x+2=0$, giving your answer in the form $a+b\sqrt{2}$, where a and b are integers. (0606/23_Summer_2014_Q5)

Answer: $2-\sqrt{2}$

11. Using the substitution $y = 5^x$ show that the equation $5^{2x+1} - 5^{x+1} + 2 = 2(5^x)$ can be written in the form $ay^2 + by + 2 = 0$ where a and b are constants to be found. (0606/11_Winter_2014_Q4)
Answer: $5y^2 - 7y + 2 = 0$

12. Solve the following simultaneous equations. (0606/13_Winter_2014_Q10)

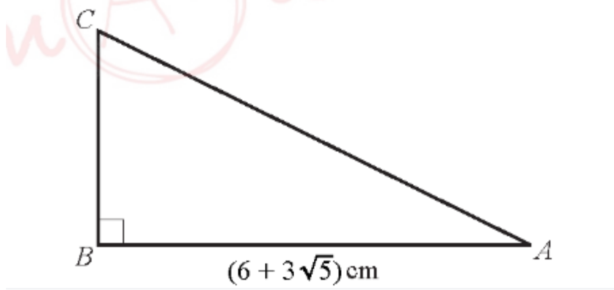
$$\frac{5^x}{25^{3y-2}} = 1$$

$$\frac{3^x}{27^{y-1}} = 81$$

Answer: $x = 6, y = \frac{5}{3}$

13. Integers a and b are such that $\left(a + 3\sqrt{5}\right)^2 + a - b\sqrt{5} = 51$. Find the possible values of a and the corresponding values of b. [6] (0606/21_Winter_2014_Q9)
Answer: $a = -3,2$, $b = -18, 12$

14. Do not use a calculator in this question. (0606/22_Summer_2015_Q3)



The diagram shows the right angled ABC, where $AB = (6 + 3\sqrt{5})$cm and angle $B = 90°$. The area of this triangle is $\left(\frac{36+15\sqrt{5}}{2}\right)$cm^2.

Find the length of the side BC in the form $(a + b\sqrt{5})$cm, where a and b are integers.

Answer: $1 + 2\sqrt{5}$

15. a) Solve the equation $16^{3x-1} = 8^{x+2}$.

Answer: $x = \frac{10}{9}$

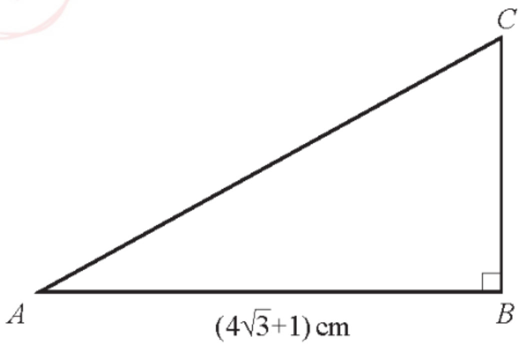
b) Given that $\frac{\left(a^{\frac{1}{3}}b^{-\frac{1}{2}}\right)^3}{a^{-\frac{2}{3}}b^{\frac{1}{2}}} = a^p b^q$, find the value of each of the constants p and q.

(0606/11_Summer_2016_Q2)

Answer: $p = \frac{5}{3}, q = -2$

Review Questions

1. You are not allowed to use a calculator in this question. (0606/13_Winter_2015_Q4)



The diagram shows triangle ABC with side $AB = \left(4\sqrt{3}+1\right)$cm. Angle B is a right angle.

It is given that the area of this triangle is $\frac{47}{2}cm^2$.

Find the length of the side BC in the form $\left(a\sqrt{3}+b\right)cm$, where a and b are integers. [3]

Answer: $BC = 4\sqrt{3} - 1$

2. Using the substitution $u = 2^x$, find the values of x such that $2^{2x+2} = 5(2^x) - 1$. (0606/12_Summer_2012_Q2)

Answer: $x = -2, 0$

3. **You must no use a calculator in this question**. (0606/12_Summer_2012_Q6)

a) Express $\frac{8}{\sqrt{3}+1}$ in the form $a\left(\sqrt{3}-1\right)$, where a is an integer.

An equilateral triangle has sides of length $\frac{8}{\sqrt{3}+1}$.

Answer: $4\left(\sqrt{3}-1\right)$

b) Show that the height of the triangle is $6-2\sqrt{3}$.

Answer: $h=6-2\sqrt{3}$

c) Hence, otherwise, find the area of the triangle in the form $p\sqrt{3}-1$, where p and q are integers.

Answer: $=16\sqrt{3}-24$

4. **Do not use a calculator in this question**. (0606/12_Summer_2016_Q4)
Find the positive value of x for which $(4+\sqrt{5})x^2+(2-\sqrt{5})x-1=0$, giving your answer in the form $\frac{a+\sqrt{5}}{b}$, where a and b are integers.

Answer: $\frac{7+\sqrt{5}}{22}$

5. **Do not use a calculator in this question.** (0606/21_Summer_2016_Q5)

a) Express $\frac{\sqrt{8}}{\sqrt{7}-\sqrt{5}}$ in the form $\sqrt{a}+\sqrt{b}$, where a and b are integers.

Answer: $\sqrt{14}+\sqrt{10}$

b) Given that $28+p\sqrt{3}=\left(q+2\sqrt{3}\right)^2$, where p and q are integers, find the values of p and of q.

Answer: $q=4,-4, p=16,-16$

Factors and polynomials

3.1 Adding, subtracting and multiplying polynomials

A polynomial is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x^1 + a_0$$

where:

- x is a variable
- n is a non-negative integer
- the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are constants
- a_n is called the leading coefficient and $a_n \neq 0$
- a_0 is called the constant term.

The highest power of x in the polynomial is called the degree of the polynomial.

You already know the special names for polynomials of degree 1,2 and 3. These are shown in the table below together with the special name for a polynomial of degree 4.

Polynomial expression	Degree	Name
$ax + b, \quad a \neq 0$	1	linear
$ax^2 + bx + c, \quad a \neq 0$	2	quadratic
$ax^3 + bx^2 + cx + d, \quad a \neq 0$	3	cubic
$ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$	4	quartic

The next example is a recap on how to add, subtract and multiply polynomials.

Worked Example 1

If $P(x) = 2x^3 - 6x^2 - 5$ and $Q(x) = x^3 + 2x - 1$, find an expression for

a) $P(x) + Q(x)$ b) $P(x) - Q(x)$ c) $2Q(x)$ d) $P(x)Q(x)$

Answers

a) $P(x) + Q(x) = 2x^3 - 6x^2 - 5 + x^3 + 2x - 1$

$= 3x^3 - 6x^2 + 2x - 6$

b) $P(x) - Q(x) = (2x^3 - 6x^2 - 5) - (x^3 + 2x - 1)$

$= 2x^3 - 6x^2 - 5 - x^3 - 2x + 1$

$= x^3 - 6x^2 - 2x - 4$

c) $2Q(x) = 2(x^3 + 2x - 1)$

$= 2x^3 + 4x - 2$

d) $P(x)Q(x) = (2x^3 - 6x^2 - 5)(x^3 + 2x - 1)$

$= 2x^3(x^3 + 2x - 1) - 6x^2(x^3 + 2x - 1) - 5(x^3 + 2x - 1)$

$= 2x^6 + 4x^4 - 2x^3 - 6x^5 - 12x^3 + 6x^2 - 5x^3 - 10x + 5$

$= 2x^6 - 6x^5 + 4x^4 - 19x^3 + 6x^2 - 10x + 5$

Exercise 3.1

Basic Level

1. If $P(x) = 3x^4 + 2x^2 - 1$ and $Q(x) = 2x^3 + x^2 + 1$, find an expression for

a) $P(x) + Q(x)$.

Answers: $3x^4 + 2x^3 + 3x^2$

b) $3P(x) + Q(x)$

Answers: $9x^4 + 2x^3 + 7x^2 - 2$

c) $P(x) – 2Q(x)$

Answers: $3x^4 - 4x^3 - 3$

d) $P(x)Q(x)$

Answers: $6x^7 + 3x^6 + 4x^5 + 5x^4 - 2x^3 + x^2 - 1$

2. Find the following products.

a) $(2x - 1)(4x^3 + x + 2)$

Answers: $8x^4 - 4x^3 + 2x^2 + 3x - 2$

b) $(x^3 + 2x^2 - 1)(3x + 2)$

Answers: $3x^4 + 8x^3 + 4x^2 - 3x - 2$

c) $(3x^2 + 2x - 5)(x^3 + x^2 + 4)$

Answers: $3x^5 + 5x^4 - 3x^3 + 7x^2 + 8x - 20$

d) $(x+2)^2(3x^3+x-1)$

Answers: $3x^5+12x^4+13x^3+3x^2-4$

e) $(x^2-5x+2)^2$

Answers: $x^4-10x^3+29x^2-20x+4$

f) $(3x-1)^3$

Answers: $27x^3-27x^2+9x-1$

3. Simplify each of the following.

a) $(2x-3)(x+2)+(x+1)(x-1)$

Answers: $3x^2+x-7$

b) $(3x+1)(x^2+5x+2)-(x^2-4x+2)(x+3)$

Answers: $2x^3+17x^2+21x-4$

c) $(2x^3+x-1)(x^2+3x-4)-(x+2)(x^3-x^2+5x+2)$

Answers: $2x^5+5x^4-8x^3-x^2-19x$

4. If $f(x) = 2x^2 - x - 4$ and $g(x) = x^2 + 5x + 2$, find an expression for

a) $f(x) + xg(x)$

Answers: $x^3 + 7x^2 + x - 4$

b) $[f(x)]^2$

Answers: $4x^4 - 4x^3 - 15x^2 + 8x + 16$

3.2 Division of polynomials

To be able to divide a polynomial by another polynomial you first need to remember how to do long division with numbers.

The steps for calculating $5508 \div 17$ are:

$$
\begin{array}{rl}
\;324 & \\
17\overline{)\,5508} & \text{Divide 55 by 17} \\
\underline{51} & 3 \times 17 = 51 \\
40 & 55 - 51 = 4\text{, bring down the 0 from the next column} \\
\underline{34} & \text{Divide 40 by } 17, 2 \times 17 = 34 \\
68 & 40 - 34 = 6\text{, bring down the 8 from the next column} \\
\underline{68} & \text{Divide 68 by } 17, 4 \times 17 = 68 \\
0 & 68 - 68 = 0
\end{array}
$$

So $\underbrace{5508}_{\text{dividend}} \div \underbrace{17}_{\text{divisor}} = \underbrace{324}_{\text{quotient}}$

The same process can be applied to the division of polynomials.

Worked Example 2

Divide $x^3 - 5x^2 + 8x - 4$ by $x - 2$.

Answers

Step 1:

$$\begin{array}{r@{\,}l}
 & x^2 \\ \cline{2-2}
x-2 \big) & x^3 - 5x^2 + 8x - 4 \\
 & x^3 - 2x^2 \\ \cline{2-2}
 & -3x^2 + 8x
\end{array}$$

divide the first term of the polynomial by x, $x^3 \div x = x^2$
multiply $(x-2)$ by x^2, $x^2(x-2) = x^3 - 2x^2$
subtract, $(x^3 - 5x^2) - (x^3 - 2x^2) = -3x^2$
bring down the $8x$ from the next column

Step 2: Repeat the process

$$\begin{array}{r@{\,}l}
 & x^2 - 3x \\ \cline{2-2}
x-2 \big) & x^3 - 5x^2 + 8x - 4 \\
 & x^3 - 2x^2 \\ \cline{2-2}
 & -3x^2 + 8x \\
 & -3x^2 + 6x \\ \cline{2-2}
 & 2x - 4
\end{array}$$

divide $-3x^2$ by x, $-3x^2 \div x = -3x$
multiply $(x-2)$ by $-3x$, $-3x(x-2) = -3x^2 + 6x$
subtract, $(-3x^2 + 8x) - (-3x^2 + 6x) = 2x$
bring down the -4 from the next column

Step 3: Repeat the process

$$\begin{array}{r@{\,}l}
 & x^2 - 3x + 2 \\ \cline{2-2}
x-2 \big) & x^3 - 5x^2 + 8x - 4 \\
 & x^3 - 2x^2 \\ \cline{2-2}
 & -3x^2 + 8x \\
 & -3x^2 + 6x \\ \cline{2-2}
 & 2x - 4 \\
 & 2x - 4 \\ \cline{2-2}
 & 0
\end{array}$$

divide $2x$ by x, $2x \div x = 2$
multiply $(x-2)$ by 2, $2(x-2) = 2x - 4$
substract, $(2x - 4) - (2x - 4) = 0$

So $(x^3 - 5x^2 + 8x - 4) \div (x - 2) = x^2 - 3x + 2$.

Worked Example 3

Divide $2x^3 - x + 51$ by $x + 3$.

There are no x^2 terms in $2x^3 - x \mp 51$ so we write it as $2x^3 + 0x^2 - x + 51$.

Answers

Step 1:

$$
\begin{array}{r|l}
 & 2x^2 \\
\hline
x+3 & 2x^3 + 0x^2 - x + 51 \\
 & \underline{2x^3 + 6x^2} \\
 & \quad -6x^2 - x
\end{array}
$$

divide the first term of the polynomial by x, $2x^3 \div x = 2x^2$
multiply $(x + 3)$ by $2x^2$, $2x^2(x + 3) = 2x^3 + 6x^2$
subtract, $(2x^3 + 0x^2) - (2x^3 + 6x^2) = -6x^2$
bring down the $-x$ from the next column

Step 2: Repeat the process

$$
\begin{array}{r|l}
 & 2x^2 - 6x \\
\hline
x+3 & 2x^3 + 0x^2 - x + 51 \\
 & \underline{2x^3 + 6x^2} \\
 & \quad -6x^2 - x \\
 & \quad \underline{-6x^2 - 18x} \\
 & \qquad\qquad 17x + 51
\end{array}
$$

divide $-6x^2$ by x, $-6x^2 \div x = -6x$
multiply $(x + 3)$ by $-6x$, $-6x(x + 3) = -6x^2 - 18x$
subtract, $(-6x^2 - x) - (-6x^2 - 18x) = 17x$
bring down the 51 from the next column

Step 3: Repeat the process

$$
\begin{array}{r|l}
 & x^2 - 6x + 17 \\
\hline
x+3 & 2x^3 + 0x^2 - x + 51 \\
 & \underline{2x^3 + 6x^2} \\
 & \quad -6x^2 - x \\
 & \quad \underline{-6x^2 - 18x} \\
 & \qquad\qquad 17x + 51 \\
 & \qquad\qquad \underline{17x + 51} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

divide $17x$ by x, $17x \div x = 17$
multiply $(x + 3)$ by 17, $17(x + 3) = 17x + 51$
substract, $(17x + 51) - (17x + 51) = 0$

So $(2x^3 - x + 51) \div (x + 3) = 2x^2 - 6x + 17$.

Exercise 3.2

Basic Level

1. Simplify each of the following.

a) $(x^3 + 3x^2 - 46x - 48) \div (x + 1)$

Answer: $x^2 + 2x - 48$

b) $(x^3 - x^2 - 3x + 2) \div (x - 2)$

Answer: $x^2 + x - 1$

c) $(x^3 - 20x^2 + 100x - 125) \div (x - 5)$

Answer: $x^2 - 15x + 25$

d) $(x^3 - 3x - 2) \div (x - 2)$

Answer: $x^2 + 2x + 1$

e) $(x^3 - 3x^2 - 33x + 35) \div (x - 7)$

Answer: $x^2 + 4x - 5$

f) $(x^3 + 2x^2 - 9x - 18) \div (x + 2)$

Answer: $x^2 - 9$

2. Simplify each of the following.

a) $(3x^3 + 8x^2 + 3x - 2) \div (x + 2)$

Answer: $3x^2 + 2x - 1$

b) $(6x^3 11x^2 - 3x - 2) \div (3x + 1)$

Answer: $2x^2 + 3x - 2$

c) $(3x^3 - 11x^2 + 20) \div (x - 2)$

Answer: $3x^2 - 5x - 10$

d) $(3x^3 - 21x^2 + 4x - 28) \div (x - 7)$

Answer: $3x^2 + 4$

3. Simplify

a) $\frac{3x^3-3x^2-4x+4}{x-1}$

Answer: $3x^2 - 4$

b) $\frac{2x^3+9x^2+25}{x+5}$

Answer: $2x^2 - x + 5$

c) $\frac{3x^3-50x+8}{3x^2+12x-2}$

Answer: $x-4$

d) $\frac{x^3-14x-15}{x^2-3x-5}$

Answer: $x+3$

4. a) Divide $x^4 - 1$ by $(x + 1)$

Answer: $x^3 - x^2 + x - 1$

b) Divide $x^3 - 8$ by $(x - 2)$

Answer: $x^2 + 2x + 4$

3.3 The factor theorem

In **Worked example 2** you found that $x - 2$ divided exactly into $(x^3 - 5x^2 + 8x - 4)$.

$$(x^3 - 5x^2 + 8x - 4) \div (x - 2) = x^2 - 3x + 2$$

This can also be written as:

$$(x^3 - 5x^2 + 8x - 4) = (x - 2)(x^2 - 3x + 2)$$

If a polynomial $P(x)$ is divided exactly by a linear factor $x - c$ to give the polynomial $Q(x)$, then

$P(x) = (x - c)\ Q(x)$

Substituting $x = c$ into this formula gives $P(c) = 0$.

Hence:

If for a polynomial $P(x)$, $P(c) = 0$ then $x - c$ is a factor of $P(x)$.

This is known as the factor theorem

For example, when $x = 2$,

$4x^3 - 8x^2 - x + 2 = 4(2)^3 - 8\ (2)^2 - 2 + 2 = 32 - 32 - 2 + 2 = 0.$

Therefore $x - 2$ is a factor of $4x^3 - 8x^2 - x + 2$.

The factor theorem can be extended to:

If for a polynomial $P(x)$, $P\left(\frac{b}{a}\right) = 0$ then $x - c$ is a factor of $P(x)$.

For example, when $x = \frac{1}{2}$,

$$4x^3 - 2x^2 + 8x - 4 = 4\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{2}\right) - 4 = \frac{1}{2} - \frac{1}{2} + 4 - 4 = 0$$

Therefore $2x - 1$ is a factor of $4x^3 - 2x^2 + 8x - 4$.

Worked Example 4

Show that $x-3$ is a factor of $x^3-6x^2+11x-6$ by

a) algebraic division

b) the factor theorem

Answers

a) Divide $x^3-6x^2+11x-6$ by $x-3$

$$
\begin{array}{r l}
 & \;\; x^2-3x+2 \\ \cline{2-2}
x-3 & \big)\; x^3-6x^2+11x-6 \\
 & \;\; \underline{x^3-3x^2} \\
 & \qquad -3x^2+11x \\
 & \qquad \underline{-3x^2+\;9x} \\
 & \qquad\qquad\qquad 2x-6 \\
 & \qquad\qquad\qquad \underline{2x-6} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

The remainder $=0$, so $x-3$ is a factor of $x^3-6x^2+11x-6$

b) Let $f(x)=x^3-6x^2+11x-6$ if $f(3)=0$, then $x-3$ is a factor

$$
\begin{aligned}
f(3) &= (3)^3-6(3)^2+11(3)-6 \\
&= 27-54+33-6 \\
&= 0
\end{aligned}
$$

So $x-3$ is a factor of $x^3-6x^2+11x-6$.

Worked Example 5

$2x^2 + x - 1$ is a factor of $2x^3 - x^2 + ax + b$.

Find the value of a the value of b.

Answers

Let $f(x) = 2x^3 - x^2 + ax + b$.

If $2x^2 + x - 1 = (2x - 1)(x + 1)$ is a factor of $f(x)$, then $2x - 1$ and $x + 1$ are also factors of $f(x)$.

Using the factor theorem $f\left(\frac{1}{2}\right) = 0$ and $f(-1) = 0$.

$f\left(\frac{1}{2}\right) = 0$ gives $2\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 + a\left(\frac{1}{2}\right) + b = 0$

$$\frac{1}{4} - \frac{1}{4} + \frac{a}{2} + b = 0$$

$$a = -2b \quad \text{----------------------(1)}$$

$f(-1) = 0$ gives $2(-1)^3 - (-1)^2 + a(-1) + b = 0$

$$-2 - 1 - a + b = 0$$

$$a = b - 3 \text{ ----------------------(2)}$$

$(2) = (1)$ gives $b - 3 = -2b$

$$3b = 3$$

$$b = 1$$

Substituting in (2) gives $a = -2$.

So $a = -2, b = 1$.

Exercise 3.3

Basic Level

1. Use the factor theorem to show:

a) $x - 4$ is a factor of $x^3 - 3x^2 - 6x + 8$

b) $x + 1$ is a factor of $x^3 - 3x - 2$

c) $x - 2$ is a factor of $5x^3 - 17x^2 + 28$

d) $3x + 1$ is a factor of $6x^3 + 11x^2 - 3x - 2$

2. Find the value of a in each of the following.

a) $x + 1$ is a factor of $6x^3 + 27x^2 + ax + 8$

Answer: 29

b) $x + 7$ is a factor of $x^3 - 5x^2 - 6x + a$

Answer: 546

c) $2x + 3$ is a factor of $4x^3 + ax^2 + 29x + 30$

Answer: 12

3. $x - 2$ is a factor of $x^3 + ax^2 + bx - 4$. Express b in terms of a.

Answer: $b = -2 - 2a$

4. Find the value of a and the value of b in each of the following.

a) $x^2 + 3x + 10$ is a factor of $x^3 + ax^2 + bx + 30$

Answer: $a = 0, b = -19$

b) $2x^2 - 11x + 5$ is a factor of $ax^3 - 17x^2 + bx - 15$

Answer: $a = 2, b = 38$

c) $4x^2 - 4x - 15$ is a factor of $4x^3 + ax^2 + bx + 30$

Answer: $a = -12, b = -7$

5. It is given that $x^2 - 5x + 6$ and $x^3 - 6x^2 11x + a$ have a common factor.

Find the possible value of a.

Answer: $-66,18$

6. $x - 2$ is a common factor of $3x^3 - (a - b)x - 8$ and $x^3 - (a + b)x + 30$.

Find the value of a and the value of b.

Answer: $a = 13.5, b = 5.5$

7. $x-3$ and $2x-1$ are factors of $2x^3-px^2-2qx+q$

a) Find the value of p and the value of q

Answer: $p=1$

b) Explain why $x+3$ is also a factor of the expression

Answer: $q=9$

8. $x + a$ is a factor of $x^3 + 8x^2 + 4ax - 3a$

a) Show that $a^3 - 4a^2 + 3a = 0$

b) Find the possible values of a.

Answer: $0,1,3$

3.4 Cubic expressions and equations

Consider factorizing $x^3 - 5x^2 + 8x - 4$ completely.

In **Worked example 2** you found that $(x^3 - 5x^2 + 8x - 4) \div (x - 2) = x^2 - 3x + 2$

This can be rewritten as: $x^3 - 5x^2 + 8x - 4 = (x - 2)(x^2 - 3x + 2)$

Factorising completely gives: $x^3 - 5x^2 + 8x - 4 = (x - 2)(x - 2)(x - 1)$

Hence if you know one factor of a cubic expression it is possible to then factorise the expression completely. The next example illustrates three different methods for doing this.

Worked Example 6

Factorise $x^3 - 3x^2 - 13x + 15$ completely.

Answers

Let $f(x) = x^3 - 3x^2 - 13x + 15$.

The positive and negative factors of 15 are $\pm 1, \pm 3, \pm 5$ and ± 15.

$f(1) = (1)^3 - 3 \times (1)^2 - 13 \times (1) + 15 = 0$

So $x - 1$ is a factor of $f(x)$.

The other factors can be found by any of the following methods.

Method 1 (by trial and error)

$f(x) = x^3 - 3x^2 - 13x + 15$

$f(1) = (1)^3 - 3 \times (1)^2 - 13 \times (1) + 15 = 0$

So $x - 1$ is a factor of $f(x)$.

$f(-3) = (-3)^3 - 3 \times (-3)^2 - 13 \times (-3) + 15 = 0$

So $x + 3$ is a factor of $f(x)$.

$f(5) = (5)^3 - 3 \times (5)^2 - 13 \times (5) + 15 = 0$

So $x - 5$ is a factor of $f(x)$.

Hence $f(x) = (x - 1)(x - 5)(x + 3)$

Method 2 (by long division)

$$
\begin{array}{rl}
 & \;x^2 - 2x - 15 \\
x - 1 & \overline{)\; x^3 - 3x^2 - 13x + 15} \\
 & \underline{\;\;x^3 - x^2 \qquad\qquad\quad} \\
 & \qquad -2x^2 - 13x \\
 & \qquad \underline{-2x^2 + 2x \qquad} \\
 & \qquad\qquad\quad -15x + 15 \\
 & \qquad\qquad\quad \underline{-15x + 15} \\
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

$$
\begin{aligned}
f(x) &= (x-1)(x^2 - 2x - 15) \\
&= (x-1)(x-5)(x+3)
\end{aligned}
$$

Method 3 (by equating coefficients)

Since $x - 1$ is a factor, $x^3 - 3x^2 - 13x + 15$ can be written as:

$$x^3 - 3x^2 - 13x + 15 = (x-1)(ax^2 + bx + c)$$

Coefficient of x^3 is 1, so $a = 1$
since $1 \times 1 = 1$

Constant term is 15, so $c = -15$
since $-1 \times -15 = 15$

$$x^3 - 3x^2 - 13x + 15 = (x-1)(x^2 + bx - 15) \qquad \text{expand and collect like terms}$$

$$x^3 - 3x^2 - 13x + 15 = x^3 + (b-1)x^2 + (-b-15)x + 15$$

Equating coefficients of x^2: $b - 1 = -3$

$$b = -2$$

$$
\begin{aligned}
f(x) &= (x-1)(x^2 - 2x - 15) \\
&= (x-1)(x-5)(x+3)
\end{aligned}
$$

<u>Worked Example 7</u>

Solve $2x^3 - 3x^2 - 18x - 8 = 0$.

Answers

Let $f(x) = 2x^3 - 3x^2 - 18x - 8$.

The positive and negative factors of 8 are $\pm 1, \pm 2, \pm 4$ and ± 8.

$f(-2) = 2(-2)^3 - 3 \times (-2)^2 - 18 \times (-2) - 8 = 0$

So $x + 2$ is a factor of $f(x)$.

$$2x^3 - 3x^2 - 18x - 8 = (x + 2)(ax^2 + bx + c)$$

Coefficient of x^3 is 2, so $a = 2$ since $1 \times 2 = 2$

Constant term is -8, so $c = -4$ since $2 \times -4 = -8$

$2x^3 - 3x^2 - 18x - 8 = (x + 2)(2x^2 + bx - 4)$ expand and collect like terms

$2x^3 - 3x^2 - 18x - 8 = 2x^3 + (b + 4)x^2 + (2b - 4)x - 8$

Equating coefficients of x^2: $b + 4 = -3$

$b = -7$

$f(x) = (x + 2)(2x^2 - 7x - 4)$

$= (x + 2)(2x - 1)(x - 4)$

Hence, $(x + 2)(2x + 1)(x - 4) = 0$

So, $x = -2$ or $x = -\frac{1}{2}$ or $x = 4$.

Worked Example 8

Solve $2x^3 + 7x^2 - 2x - 1 = 0$.

Answers

Let $f(x) = 2x^3 + 7x^2 - 2x - 1$.

The positive and negative factors of -1 are ± 1.

$f(-1) = 2(-1)^3 + 7 \times (-1)^2 - 2 \times (-1) - 1 \neq 0$

$f(1) = 2(1)^3 + 7 \times (1)^2 - 2 \times (1) - 1 \neq 0$

So $x - 1$ and $x + 1$ are not factors of $f(x)$.

By inspection $f\left(\frac{1}{2}\right) = 2(\frac{1}{2})^3 + 7 \times (\frac{1}{2})^2 - 2 \times \left(\frac{1}{2}\right) - 1 = 0$.

So $2x - 1$ is a factor of:

$$2x^3 + 7x^2 - 2x - 1 = (2x - 1)(ax^2 + bx + c)$$

Coefficient of x^3 is 2, so $a = 1$ since $2 \times 1 = 2$

Constant term is -1, so $c = 1$ since $-1 \times 1 = -1$

$$2x^3 + 7x^2 - 2x - 1 = (2x - 1)(x^2 + bx + 1)$$

$$2x^3 + 7x^2 - 2x - 1 = 2x^3 + (2b - 1)x^2 + (2 - b)x - 1$$

Equating coefficients of x^2: $2b - 1 = 7$

$$b = 4$$

So $2x^3 + 7x^2 - 2x - 1 = (2x - 1)(x^2 + 4x + 1)$

$$x = \frac{1}{2} \text{ or } x = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$x = \frac{1}{2} \text{ or } x = \frac{-4 \pm 2\sqrt{3}}{2}$$

$$x = \frac{1}{2} \text{ or } x = -2 + \sqrt{3} \text{ or } x = -2 - \sqrt{3}$$

Not all cubic expressions can be factorized into 3 linear factors.

Consider the cubic expression $x^3 + x^2 - 36$

Let $f(x) = x^3 + x^2 - 36$

$$f(3) = (3)^3 + (3)^2 - 36 = 0$$

So $x - 3$ is a factor of $f(x)$

$$x^3 + x^2 - 36 = (x-3)(ax^2 + bx + c)$$

Coefficient of x^3 is 1, so $a = 1$
since $1 \times 1 = 1$

Constant term is -36, so $c = 12$
since $-3 \times 12 = -36$

$$x^3 + x^2 - 36 = (x-3)(x^2 + bx + 12)$$

$$x^3 + x^2 - 36 = x^3 + (b-3)x^2 + (12 - 3b)x - 36$$

Equating coefficients of x^2 : $b - 3 = 1$

$$b = 4$$

So $x^3 + x^2 - 36 = (x-3)(x^2 + 4x + 12)$

(Note: $x^2 + 4x + 12$ cannot be factorized into two further linear factors, since the discriminant < 0.)

Exercise 3.4

Basic Level

1. a) Show that $x-1$ is a factor of $2x^3 - x^2 - 2x + 1$

b) Hence factorize $2x^3 - x^2 - 2x + 1$ completely

Answer: $(2x-1)(x+1)(x-1)$

2. Factorize these cubic expressions completely.

a) $x^3 + 2x^2 - 3x - 10$

Answer: $(x-2)(x^2+4x+5)$

b) $x^3 + 4x^2 - 4x - 16$

Answer: $(x+4)(x+2)(x-2)$

c) $2x^3 - 9x^2 - 18x$

Answer: $x(2x+3)(x-6)$

d) $x^3 - 8x^2 + 5x + 14$

Answer: $(x - 7)(x + 1)(x - 2)$

e) $2x^3 - 13x^2 + 17x + 12$

Answer: $(2x + 1)(x - 3)(x - 4)$

f) $3x^3 + 2x^2 - 19x + 6$

Answer: $(x-2)(x+3)(3x-1)$

g) $4x^3 - 8x^2 - x + 2$

Answer: $(2x-1)(2x+1)(x-2)$

h) $2x^3 + 3x^2 - 32x + 15$

Answer: $(2x - 1)(x - 3)(x + 5)$

3. Solve the following equations.

a) $x^3 - 3x^2 - 33x + 35 = 0$

Answer: $-5, 1, 7$

b) $x^3 - 6x^2 + 11x - 6 = 0$

Answer: $1,2,3$

c) $3x^3 + 17x^2 + 18x - 8 = 0$

Answer: $-4, -2, \frac{1}{3}$

d) $2x^3 + 3x^2 - 17x + 12 = 0$

Answer: $-4, 1, 1.5$

e) $2x^3 - 3x^2 - 11x + 6 = 0$

Answer: $-2, 0.5, 3$

f) $2x^3 + 7x^2 - 5x - 4 = 0$

Answer: $-4, -\frac{1}{2}, 1$

g) $4x^3 + 12x^2 + 5x - 6 = 0$

Answer: $-2, -1.5, 0.5$

h) $2x^3 - 3x^2 - 29x + 60 = 0$

Answer: $-4, 2.5, 3$

4. Solve the following equations. Express roots in the form $a \pm b\sqrt{c}$, where necessary.

a) $x^3 + 5x^2 - 4x - 2 = 0$

Answer: $1, -3 \pm \sqrt{7}$

b) $x^3 + 8x^2 + 12x - 9 = 0$

Answer: $-3, -\frac{5}{2} \pm \frac{1}{2}\sqrt{37}$

c) $x^3 + 2x^2 - 7x - 2 = 0$

Answer: $2, -2 \pm \sqrt{3}$

d) $2x^3 + 3x^2 - 17x + 12 = 0$

Answer: $1.5, -4, 1$

5. Solve the equation $2x^3 + 9x^2 - 14x - 9 = 0$

Express roots in the form $a \pm b\sqrt{c}$, where necessary.

Answer: $-0.5, -2 \pm \sqrt{13}$

6. Solve the equation $x^3 + 8x^2 + 12x = 9$

Write your answers correct to 2 decimal places where necessary.

Answer: $-5.54, -3, 0.54$

7. a) Show that $x - 2$ is a factor of $x^3 - x^2 - x - 2$.

b) Hence show that $x^3 - x^2 - x - 2 = 0$ has only one real root and state the value of this root.

Answer: 2

8. $f(x)$ is a cubic polynomial where the coefficient of x^3 is 1.

Find $f(x)$ when the roots of $f(x) = 0$ are

a) $-2, 1$ and 5

Answer: $x^3 - 4x^2 - 7x + 10$

b) $-5, -2$ and 4

Answer: $x^3 + 3x^2 - 18x - 40$

c) $-3, 0$ and 2

Answer: $x^3 + x^2 - 6x$

9. $f(x)$ is a cubic polynomial where the coefficient of x^3 is 2.

Find $f(x)$ when the roots of $f(x) = 0$ are

a) $-0.5, 2$ and 4

Answer: $2x^3 - 11x^2 + 10x + 8$

b) 0.5, 1 and 2

Answer: $2x^3 - 7x^2 + 7x - 2$

c) $-1.5, 1$ and 5

Answer: $2x^3 - 9x^2 - 8x + 15$

10. $f(x)$ is a cubic polynomial where the coefficient of x^3 is 1.

The roots of $f(x) = 0$ are $-3, 1 \pm \sqrt{2}$ and $1 - \sqrt{2}$

Express $f(x)$ as a cubic polynomial in x with integer coefficients.

Answer: $x^3 + x^2 - 7x - 3$

11. $f(x)$ is a cubic polynomial where the coefficient of x^3 is 2.

The roots of $f(x) = 0$ are $\frac{1}{2}$, $2 + \sqrt{3}$ and $2 - \sqrt{3}$

Express $f(x)$ as a cubic polynomial in x with integer coefficients

Answer: $2x^3 - 9x^2 + 6x - 1$

12. $2x + 3$ is a factor of $2x^4 + (a^2 + 1)x^3 - 3x^2 + (1 - a^3)x + 3$

a) Show that $4a^3 - 9a^2 + 4 = 0$

b) Find the possible values of a

Answer: $2, \frac{1 \pm \sqrt{33}}{8}$

Intermediate Exercise 3.4

1. When the polynomial $ax^2 + bx + c$ is divided by $x + 1$ and $x - 2$, the remainders are 10 and 1 respectively. This polynomial is exactly divisible by $x - 1$. Find the values of a, b, and c.
   Answer: $a = 2, b = -5, c = 3$

2. The quadratic expression $2x^2 + 5x + 3$ is a factor of the polynomial $2x^3 + cx^2 + dx + 3$, where c and d are constants.

a) Find the value of c and d.

Answer: $c = 7, d = 8$

b) Deduce the third factor of the polynomial.

Answer: $x + 1$

3. The expression $mx^3 + nx^2 + 9x - 2$, where m and n are constants, has a factor $x - 1$ and leaves a remainder of -120 when divided by $x + 2$. Find the value of m and n.
Answer: $m = 6, n = -13$

4. The polynomial $P(x) = hx^4 + x^3 + kx^2 - 4x + n$ is exactly divisible by $x + 1, x - 2$, and $2x - 1$.
   a) Find the values of h, k and n.
      Answer: $h = 2, k = -9, n = 4$

   b) Find the remaining factor of $P(x)$.
   Answer: $x + 2$

5. The expression $3x^4 + kx^3 - 4x^2 + nx - 4$, where k and n are constants, is divisible by $x^2 - x - 2$.

   a) Find the values of k and n.

      Answer: $k = -3, n = -2$

   b) Using the values in part a), factorise $3x^4 + kx^3 - 4x^2 + nx - 4$ completely.

      Answer: $(x - 2)(x + 1)(3x^2 + 2)$

3.5 The remainder theorem

Consider $f(x) = 2x^3 - 4x^2 + 7x - 37$

Substituting $x = 3$ in the polynomial gives $f(3) = 2(3)^3 - 4(3)^2 + 7(3) - 37 = 2$

When $2x^3 - 4x^2 + 7x - 37$ is divided by $x - 3$, there is a remainder.

$$
\begin{array}{r@{}r}
 & 2x^2 + 2x + 13 \\
\cline{2-2}
x-3 \big) & 2x^3 - 4x^2 + 7x - 37 \\
 & \underline{2x^3 - 6x^2} \phantom{{}+7x-37} \\
 & 2x^2 + 7x \phantom{{}-37} \\
 & \underline{2x^2 - 6x} \phantom{{}-37} \\
 & 13x - 37 \\
 & \underline{13x - 39} \\
 & 2
\end{array}
$$

The remainder is 2. This is the same value as $f(3)$.

$f(x) = 2x^3 - 4x^2 + 7x - 36$, can be written as

$f(x) = (x - 3)(2x^3 + 2x + 13) + 2$

In general:

If a polynomial $P(x)$ is divided by $x - c$ to give the polynomial

$Q(x)$ and a remainder R, then

$P(x) = (x - c)Q(x) + R.$

Substituting $x = c$ into this formula gives $P(c) = R$

This leads to the remainder theorem:

If a polynomial $P(x)$ is divided by $x - c$, the remainder is $P(c)$.

The Remainder Theorem can be extended to:

If a polynomial $P(x)$ is divided by $ax - b$, the remainder is $P\left(\frac{b}{a}\right)$.

Worked Example 9

Find the remainder when $7x^3 + 6x^2 - 40x + 17$ is divided by $(x+3)$ by using

a) algebraic division b) factor theorem

Answers

a) Divide $7x^3 + 6x^2 - 40x + 17$ by $(x+3)$.

$$
\begin{array}{rl}
 & 7x^2 + 15x + \;\; 5 \\
x+3 \big) & \overline{7x^3 + 6x^2 - 40x + 17} \\
 & \underline{7x^3 + 21x^2} \\
 & \quad -15x^2 - 40x \\
 & \quad \underline{-15x^2 + 45x} \\
 & \qquad\qquad 5x + 17 \\
 & \qquad\qquad \underline{5x + 15} \\
 & \qquad\qquad\qquad 2
\end{array}
$$

The remainder is 2.

b) Let $f(x) = 7x^3 + 6x^2 - 40x + 17$

$$
\begin{aligned}
\text{Remainder} &= f(-3) \\
&= 7(-3)^3 + 6(-3)^2 - 40(-3) + 17 \\
&= -189 + 54 + 120 + 17 \\
&= 2
\end{aligned}
$$

Worked Example 10

$f(x) = 2x^3 + ax^2 - 9x + b$

When $f(x)$ is divided by $x - 1$, the remainder is 1.
When $f(x)$ is divided by $x + 2$, the remainder is 19.
Find the value of a and of b.

Answers

$f(x) = 2x^3 + ax^2 - 9x + b$

When $f(x)$ is divided by $x - 1$, the remainder is 1 means that: $f(1) = 1$.

$$2(1)^3 + a(1)^2 - 9(1) + b = 1$$

$$2 + a - 9 + b = 1$$

$$a + b = 8 \text{ -------------------------(1)}$$

When $f(x)$ is divided by $x + 2$, the remainder is 19 means that:$f(-2) = 19$.

$$2(-2)^3 + a(-2)^2 - 9(-2) + b = 19$$

$$-16 + 4a + 18 + b = 19$$

$$4a + b = 17 \text{ -------------------------(1)}$$

$(2) - (1)$ gives $3a = 9$

$$a = 3$$

Substituting $a = 3$ in equation (2) gives $b = 5$.

$a = 3$ and $b = 5$

Exercise 3.5

Basic Level

1. Find the remainder when

a) $x^3 + 2x^2 - x + 3$ is divided by $x - 1$

Answer: 5

b) $x^3 - 6x^2 + 11x - 7$ is divided by $x - 2$

Answer: -1

c) $x^3 - 3x^2 - 33x + 30$ is divided by $x + 2$

Answer: 76

d) $2x^3 - x^2 - 18x + 11$ is divided by $2x - 1$

Answer: 2

2. a) When $x^3 + x^2 + ax - 2$ is divided by $x - 1$, the remainder is 5.

Find the value of *a.*

Answer: 5

b) When $2x^3 - 6x^2 + 7x + b$ is divided by $x + 2$, the remainder is 3.

Find the value of *b.*

Answer: 57

c) When $2x^3 + x^2 + cx - 10$ is divided by $2x - 1$, the remainder is -4.

Find the value of *c.*

Answer: 11

3. $f(x) = x^3 + ax^2 + bx - 5$

$f(x)$ has a factor of $x - 1$ and leaves a remainder of 3 when divided by $x + 2$.

Find the value of a and b.

Answer: $a = 4, b = 0$

4. $f(x) = x^3 + ax^2 + 11x + b$

$f(x)$ has a factor of $x - 2$ and leaves a remainder of 24 when divided by $x - 5$.

Find the value of a and b.

Answer: $a = -6, b = -6$

5. $f(x) = x^3 - 2x^2 + ax + b$

$f(x)$ has a factor of $x - 3$ and leaves a remainder of 15 when divided by $x + 2$.

a) Find the value of a and b

Answer: $a = -8. b = 15$

b) Solve the equation $f(x) = 0$

Answer: $3, \frac{-1+\sqrt{21}}{2}, \frac{-1-\sqrt{21}}{2}$

6. $f(x) = 4x^3 + 8x^2 + ax + b$

$f(x)$ has a factor of $2x - 1$ and leaves a remainder of 48 when divided by $x - 2$.

a) Find the value of a and b.

Answer: $a = -9, b = 2$

b) Find the remainder when $f(x)$ is divided by $x - 1$.

Answer: 5

7. $f(x) = 2x^3 + (a+1)x^2 - ax + b$

When $f(x)$ is divided by $x - 1$, the remainder is 5.

Show that $a = -4$ and find the value of b.

Answer: $b = 2$

8. $f(x) = ax^3 + bx^2 + 5x - 2$

When $f(x)$ is divided by $x - 1$, the remainder is 6.

When $f(x)$ is divided by $2x + 1$, the remainder is -6.

Find the value of a and of b.

Answer: $a = 6, b = -3$

9. $f(x) = x^3 - 5x^2 + ax + b$

$f(x)$ has a factor of $x - 2$

a) Express b in terms of a.

Answer: $b = 12 - 2a$

b) When $f(x)$ is divided by $x + 1$, the remainder is -9.

Find the value of a and b.

Answer: $a = 5, b = 2$

10. $f(x) = x^3 + ax^2 + bx + c$

The roots of $f(x) = 0$ are $2, 3$, and k.

When $f(x)$ is divided by $x - 1$, the remainder is -8.

a) Find the value of k.

Answer: 5

b) Find the remainder when $f(x)$ is divided by $x + 1$.

Answer: -72

11. $f(x) = 4x^3 + ax^2 + 13x + b$

$f\ (x)$ has a factor of $2x - 1$ and leaves a remainder of 21 when divided by $x - 2$.

a) Find the value of a and of b

Answer: $a = -8, b = -5$

b) Find the remainder when the expression is divided by $x + 1$.

Answer: -30

12. $f(x) = x^3 - 8x^2 + kx - 20$

When $f(x)$ is divided by $x - 1$, the remainder is R.

When $f(x)$ is divided by $x - 2$, the remainder is $4R$.

Find the value of k.

Answer: 32

13. $f(x) = x^3 + 2x^2 - 6x + 9$

When $f(x)$ is divided by $x + a$, the remainder is R.

When $f(x)$ is divided by $x - a$, the remainder is $2R$.

a) Show that $3a^3 - 2a^2 - 18a - 9 = 0$.

b) Solve the equation in part (a) completely.

Answer: $3, \frac{-7+\sqrt{13}}{6}, \frac{-7-\sqrt{13}}{6}$

14. $f(x) = x^3 + 6x^2 + kx - 15$

When $f\ (x)$ is divided by $x - 1$, the remainder is R.

When $f\ (x)$ is divided by $x + 4$, the remainder is $-R$.

a) Find the value of k

Answer: 3

b) Hence find the remainder when the expression is divided by $x + 2$

Answer: -5

15. $P(x) = 5(x-1)(x-2)(x-3) + a(x-1)(x-2) + b(x-1) + c$

It is given that when $P(x)$ is divided by each of $x-1$, $x-2$ and $x-3$ the remainders are $7, 2$ and 1 respectively. Find the values of *a, b*, and *c*.

Answer: $a = 2, b = -5, c = 7$

Summary

The **factor theorem:**

If, for a polynomial $P(x), P(c) = 0$ then $x - c$ is a factor of $P(x)$.

If, for a polynomial $P(x), P\left(\frac{a}{b}\right) = 0$ then $ax - b$ is a factor of $P(x)$.

The **remainder theorem:**

If a polynomial $P(x)$ is divided by $x - c$, the remainder is $P(c)$.

If a polynomial $P(x)$ is divided by $ax - b$, the remainder is $P\left(\frac{b}{a}\right)$.

Intermediate Exercise 3.5

1. The polynomial $x^4 - 3x^2 - x + 3$ leaves a remainder p when divided by $3 - 2x$. Find the value of p.
   Answer: $-\frac{3}{16}$

2. When the expression $6x^2 + 5kx - 33k$, where k is a constant, is divided by $x - k$, the remainder is -22. Find the possible values of k.
   Answer: $1, 2$

3. The expression $x^4 + ax^2 + bx - 1$, where a and b are constants, leaves remainders 3 and 9 when divided by $x + 1$ and $x - 2$ respectively. Find the value of a and of b.
   Answer: $a = 2, b = -3$

4. When the polynomial $P(x) = x^3 + kx - x + 2$, where p is a constant, is divided by $x + 2$, the remainder is double the remainder when $P(x)$ is divided by $x - 1$. Find
   a) the value of k,
      Answer: 4

   b) the remainder when $P(x)$ is divided by $x + 2$.
      Answer: 12

5. The expression $6x^3 - 5x^2 + kx + 1$, where k is a constant, leaves the same remainder when divided by $x - 1$ and $2x - 1$. Find

a) the value of k,

Answer: -3

b) the common remainder.

Answer: -1

Examination Questions

1. The expression $2x^3 + ax^2 + bx - 30$ is divisible by $x + 2$ and leaves a remainder of -35 when divided by $2x - 1$. Find the values of the constants a and b.
   (0606/11_Summer_2012_Q2)
   Answer: $a = 5, b = -13$

2. It is given that $x - 2$ is a factor of $f(x) = x^3 + kx^2 - 8x - 8$.
(0606/22_Summer_2012_Q5)

a) Find the value of the integer k.

Answer: $k = 4$

b) Using your values of k, find the non-integer roots of the equation $f(x) = 0$ in the form $a \pm \sqrt{b}$, where a and b are integers.

Answer: $-3 \pm \sqrt{5}$

3. a) Given that $3x^3 + 5x^2 + px + 8 = (x - 2)(ax^2 + bx + c)$, find the value of each of the integers a, b, c, and p. (0606/13_Winter_2012_Q8)
Answer: $p = -26, a = 3, b = 11, c = -4$

b) Using the values found in part a), factorise completely $3x^3 + 5x^2 + px + 8$.
Answer: $(x - 2)(3x - 1)(x + 4)$

4. a) The remainder when the expression $x^3 + 9x^2 + bx + c$ is divided by $x - 2$ is twice the remainder when the expression is divided by $x - 1$. Show that $c = 24$.
(0606/21_Winter_2012_Q10)
Answer: $c = 4$

b) Given that $x + 8$ is a factor $x^3 + 9x^2 + bx + 24$, shows that the equation $x^3 + 9x^2 + bx + 24 = 0$ has only one real root.

5. It is given that $f(x) = 6x^3 - 5x^2 + ax + b$ has a factor of $x + 2$ and leaves a remainder of 27 when divided by $x - 1$. (0606/12_Summer_2013_Q7)

a) Show that $b = 40$ and find the value of a.

Answer: $a = -14, b = 40$

b) Show that $f(x) = (x + 2)(px^2 + qx + r)$, where p, q and r are integers to be found.

Answer: $f(x) = (x + 2)(6x^2 - 17x + 20)$

c) Hence solve $f(x) = 0$.

Answer: $x = -2$

6. The function $f(x) = x^3 + x^2 + ax + b$ is visible by $x - 3$ and leaves a remainder of 20 when divided by $x + 1$. (0606/21_Summer_2013_Q12)

a) Show that $b = 6$ and find the value of a.

Answer: $a = -14, b = 6$

b) Using your value of a and taking b as 6, find the non-integer roots of the equation $f(x) = 0$ in the form $p \pm \sqrt{q}$,where p and q are integers.

Answer: $-2 \pm \sqrt{6}$

7. The function $f(x) = ax^3 + 4x^2 + bx - 2$, where a and b are constants, is such that $2x - 1$ is a factor. Given that the remainder when $f(x)$is divided $x - 2$ is twice the remainder when $f(x)$ is divided by $x + 1$, find the value of a and b. (0606/11_Winter_2013_Q6)

   Answer: $a = -2, b = \frac{5}{2}$

8. a) Find, in terms of p, the remainder when $x^3 + px^2 + p^2x + 21$ is divided by $x + 3$. (0606/13_Summer_2014_Q3)
Answer: $f(-3) = 9p - 3p^2 - 6$

b) Hence find the set of values of p for which this remainder is negative.
Answer: $p < 1, p > 2$

9. The expression $2x^3 + ax^2 + bx + 12$ has a factor of $x - 4$ and leaves a remainder of -12 when divided by $x - 1$. Find the value of each of the constants a and b. (0606/21_Summer_2014_Q4)
Answer: $a = -3, b = -23$

10. a) Given that $x + 1$ is a factor of $3x^3 - 14x^2 - 7x + d$, show that $d = 10$.
(0606/22_Summer_2014_Q3)

c) Show that $3x^3 - 14x^2 - 7x + 10$ can be written in the form $(x + 1)(ax^2 + bx + c)$, where a, b and c are constants to be found.

Answer: $3x^2 - 17x + 10$

c) Hence solve the equation $3x^3 - 14x^2 - 7x + 10 = 0$.

Answer: $-1, 5, \frac{2}{3}$

11. The expression $f(x) = 3x^3 + 8x^2 - 33 + p$ has a factor of $x - 2$.
(0606/23_Winter_2014_Q1)

a) Show that $p = 10$ and express $f(x)$ a product of a linear factor and a quadratic factor.
Answer: $f(x) = (x - 2)(3x^2 + 14x - 5)$

b) Hence solve the equation $f(x) = 0$.
Answer: $x = 2, -5, \frac{1}{3}$

12. The polynomial $f(x) = ax^3 - 15x^2 + bx - 2$ has a factor of $2x - 1$ and a remainder of 5 when divided by $x - 1$. (0606/11_Summer_2015_Q6)

a) Show that $b = 8$ and find the value of a.
Answer: $b = 8, a = 14.$

b) Using the values of a and b from part a), express $f(x)$ in the form $(2x - 1)g(x)$, where $g(x)$ is a quadratic factor to be found.
Answer: $(2x - 1)(7x^2 - 4x + 2)$

13. a) Show that $x = -2$ is a root of the polynomial equation $15x^3 + 26x^2 - 11x - 6 = 0$. (0606/22_Summer_2015_Q12)
Answer: $-120 + 104 + 22 - 6 = 0$

b) Find the remainder when $15x^3 + 26x^2 - 11x - 6$ is divided by $x - 3$.
Answer: 600

c) Find the value of p and of q such that $15x^3 + 26x^2 - 11x - 6$ is a factor of $15x^4 + px^3 - 37x^2 + qx + 6$.
Answer: $p = 11, q = 5$

14. It is given that $f(x) = 4x^3 - 4x^2 - 15x + 18$. (0606/21_Winter_2015_Q1)

a) Show that $x + 2$ is a factor of $f(x)$.

Answer: $f(-2) = -32 - 16 + 30 + 18 = 0$

b) Hence factorise $f(x)$ completely and solve the equation $f(x) = 0$.

Answer: $x = -2, 1.5$

Review Question

1. Show that expressions $x^3 + (k-2)x^2 + (k-7)x - 4$ has a factor $x+1$ for all values of k. If the expression also has a factor $x+2$, find the values of k and the third factor.
   Answer: $3, x-2$

2. Given that $x^3 + x - 4 = (x^2 + x - 1)(x-1) + Ax + B$ for all values of x, find the values of A and B. Hence or otherwise, find the remainder when $x^3 + x - 4$ is divided $x^2 + x - 1$.
   Answer: $A = 3, B = -5; 3x - 5$

3. Given that $f(x) = x^3 - 7x + 6$, calculate the remainders when $f(x)$ is divided by $4 - x$ and $x + 3$, respectively. Factorise $f(x)$ completely. By using the substitution $y = \frac{1}{x}$, or otherwise, solve the equation $6x^3 - 7x^2 + 1 = 0$.
Answer: $42; 0;\ (x+3)(x-1)(x-2);\ -\frac{1}{3}, \frac{1}{2}, 1$

4. The expression $3(x+2)^5 + (x+k)^2$ leaves a remainder of 7 when divided by $x + 1$. Determine the values of k.
Answer: $-1, 3$

5. If $x + 2$ is a factor of $x^4 + (p - 1)x^2 - p^2$ but not of $x^3 + px - 3x + 10$, find the value of p.
Answer: 6