Conkex Optimizatn
Week 4

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[illegible]
软件21

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班级　　姓名　　编号　　科目　　第　页

3.1 (a) ~~$\frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$~~

$\because b > a \quad x = \frac{b-x}{b-a} \cdot a + \frac{x-a}{b-a} \cdot b, \quad \frac{b-x}{b-a} + \frac{x-a}{b-a} = 1$

$\because$ 定义 $f(x)$ 为凸函数，

对 $\forall\, 0 \le \theta \le 1 \quad f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y)$

令 $x = a,\ y = b,\ \theta = \frac{b-x}{b-a}$

$\because a \le x \le b,\ a < b$

$\therefore 0 \le \theta \le 1$

~~$\theta f(a) + (1-\theta) f(b) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$~~

$\therefore f(\theta a + (1-\theta) b) = f(x) \le \theta f(a) + (1-\theta) f(b)$

$= \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$

(b) ~~[illegible]~~

$\because x \in (a, b)$

$\therefore$ 有 $a < x < b$，$x-a > 0$，

$\therefore \exists\, \theta$ 使 $= \frac{b-x}{b-a}$，$x = \theta a + (1-\theta) b$

~~左侧~~

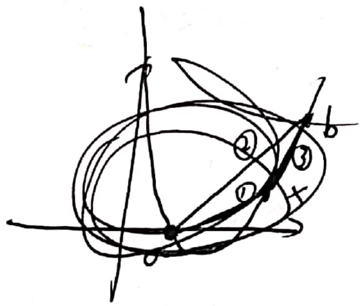
即证：~~$(f(x) - f(a))(b-a) - (f(b) - f(a))(x-a) \le 0$~~

$f(x) = f(\theta a + (1-\theta) b) \le \theta f(a) + (1-\theta) f(b) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$

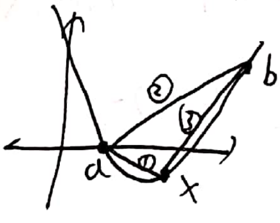
$f(x) - f(a) \le \frac{b-x}{b-a} f(a) - f(a) + \frac{x-a}{b-a} f(b)$

$= \frac{x-a}{b-a} (f(b) - f(a))$

$$\frac{f(x) - f(a)}{x-a} \le \frac{f(b) - f(a)}{b-a}$$



①斜率≤②斜率≤③斜率



~~$f(x) - f(b)$~~

$f(b) - f(x) \ge f(b) - \frac{x-a}{b-a} f(b) - \frac{b-x}{b-a} f(a) = \frac{b-x}{b-a} (f(b) - f(a))$

$\frac{f(b) - f(x)}{b-x} \ge \frac{f(b) - f(a)}{b-a}$

得证

(3) ∵对 $\forall x \in (a,b)$ $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}$，且 f 可微

则 $\lim\limits_{x \to a^+} \frac{f(x)-f(a)}{x-a} = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a) \leq \frac{f(b)-f(a)}{b-a}$

(a右导数)　(导数)

(可微 ⇒ 左导数=右导数=导数)

同理，$\lim\limits_{x \to b^-} \frac{f(x)-f(b)}{x-b} = \lim\limits_{x \to b} \frac{f(x)-f(b)}{x-b} = f'(b) \geq \frac{f(b)-f(a)}{b-a}$

(b左导数)

(4) ~~f(b)=f(~~

∵ f 二次可微，∴ f' 可微，

$\lim\limits_{x \to a^+} \frac{f'(x)-f'(a)}{x-a} = \lim\limits_{x \to a} \frac{f'(x)-f'(a)}{x-a} = f''(a)$

(a右导数)

$\lim\limits_{x \to b^-} \frac{f'(x)-f'(b)}{x-b} = \lim\limits_{x \to b} \frac{f'(x)-f'(b)}{x-b} = f''(b)$

(b左导数)

∴只需证 $\forall a<x<b$，有 $f'(a) \leq f'(x) \leq f'(b)$ 即可

根据~~(结论)~~ $\forall x \in (a,b)$，

f 在 (a,x)，(x,b) 上有定义且为凸函数，

$a<x<b$，

∴ 根据(3)结果，

有 $f'(x) \leq f'(b)$

$f'(a) \leq f'(x)$

(用 x 分别替代(3)中的 a 与 b)

∴ 有 $f''(a) \geq 0$，$f''(b) \geq 0$

3.7: 若f不是常数，则必∃ $x_1, x_2 \in R^n$, $x_1 \neq x_2$, 有 $f(x_1) \neq f(x_2)$，令 $f(x_1)=y_1$, $f(x_2)=y_2$, $y_1<y_2$

若f在R^n有上界N，则对$\forall x \in R^n$, $f(x) \le N$

∵f凸函数，∴ $g_1(t)=f(x_1+t(x_2-x_1))$ 必为凸函数

∵dom f = R^n ∴ $g_1(1)=f(x_2)$, $g_1(0)=f(x_1)$, $g_1(2)=f(2x_2-x_1)$,

∴dom g = R

有 $g_1(1) \le \frac{1}{2}g(0)+\frac{1}{2}g(2)$,

$\frac{1}{2}(g_1(1)-g_1(0)) \le \frac{1}{2}(g_1(2)-g_1(1))$

$g_1(2) \ge g_1(1)+(g_1(1)-g_1(0))$

令 $2x_2-x_1=x_3$,

$f(x_3) \ge y_1+2(y_2-y_1)$

同理，$g_2(t)=f(x_1+t(x_3-x_1))$ 为凸函数.

$g_2(2)=f(2x_3-x_1)=f(x_4)$

$\ge y_1+4(y_2-y_1)$

令 $g_i(t)=f(x_1+t(x_{i+2}-x_1))$,

$g_i(t)$为凸函数

$g_i(2)=f(2x_{i+1}-x_1)$

$=f(x_{i+2})$

$\ge y_1+2^i(y_2-y_1)$

∵ $N-y_1>0$, $y_2-y_1>0$,

∴当 $i>\log_2\frac{N-y_1}{y_2-y_1}$ 时，

$f(x_i)>N$，与$\forall x\in R^n$ $f(x)\le N$矛盾

∴f为常函数

3.32 (a) 对 $\forall x_1, x_2 \in R$, $0<\theta\leq1$

$$fg(\theta x_1+(1-\theta)x_2)$$

$$= f(\theta x_1+(1-\theta)x_2)\, g(\theta x_1+(1-\theta)x_2)$$

$\because \forall x, f(x)\geq 0, g(x)\geq 0,$

$\therefore f(\theta(x_1)+(1-\theta)x_2)$ ~~$f(\theta x_1+(1-\theta)x_2)\leq f(x_2)$~~ $\theta(x_1)+(1-\theta)(x_2)$

$g(\theta x_1+(1-\theta)x_2)$

$\leq (\theta f(x_1)+(1-\theta)f(x_2))$ ~~$g(\theta x_1+(1-\theta)x_2)$~~

$(\theta g(x_1)+(1-\theta)g(x_2))$

~~$\theta f(x_1)+(1-\theta)f(x_2)$~~ $g(x_2)$ $g(x_1)$

$= \theta^2 f(x_1)g(x_1)+(1-\theta)^2 g(x_2)f(x_2)$,记为A

$+\theta(1-\theta)f(x_1)g(x_2)+\theta(1-\theta)f(x_2)g(x_1)$

(b)

$B=\theta fg(x_1)+(1-\theta)fg(x_2)$

$=\theta f(x_1)g(x_1)+(1-\theta)f(x_2)g(x_2)$

$B-A=(\theta-\theta^2)f(x_1)g(x_1)$ $\theta(1-\theta)$

$+\theta(1-\theta)f(x_2)g(x_2)$ $(1-\theta-1)(1-\theta)$

$-\theta(1-\theta)f(x_1)g(x_2)$

$-\theta(1-\theta)f(x_2)g(x_1)$ $(1-\theta-1)(1-\theta)$

$=\theta(1-\theta)(f(x_1)-f(x_2))$

$(g(x_1)-g(x_2))$

$\because f(x)$、$g(x)$都非增/非减

$\therefore B-A\geq 0,$

$f(\theta(x_1)+(1-\theta)x_2)g(\theta(x_1)+(1-\theta)x_2)$

$\leq B$. fg为凸

(6)：对 $\forall x_1, x_2 \in R$ $0 \le \theta \le 1$

$fg(\theta x_1 + (1-\theta) x_2)$

$= f(\theta x_1 + (1-\theta) x_2) g(\theta x_1 + (1-\theta) x_2)$

$\because \forall x, f(x) \ge 0, g(x) \ge 0,$

$f(\theta x_1 + (1-\theta) x_2) g(\theta x_1 + (1-\theta) x_2) \geq (\theta f(x_1) + (1-\theta) f(x_1))(\theta g(x_1) + (1-\theta) g(x_2))$

$= \theta^2 f(x_1) g(x_1) + (1-\theta)^2 g(x_2) f(x_2) + \theta(1-\theta) f(x_1) g(x_2) + \theta(1-\theta) f(x_2) g(x_1)$

（记为A）

$B = \theta fg(x_1) + (1-\theta) fg(x_2)$

$= \theta f(x_1) g(x_1) + (1-\theta) f(x_2) g(x_2)$

$B - A = \theta(1-\theta)(f(x_1) - f(x_2))(g(x_1) - g(x_2))$

$\because$ ~~$f(x_1)$~~ $f(x), g(y)$ 一个增与一个减，$0 < \theta \le 1$

$\therefore B - A \le 0$

$\therefore \forall x_1, x_2 \in R, 0 \le \theta \le 1,$

$fg(\theta x_1 + (1-\theta) x_2) \geq \theta fg(x_1) + (1-\theta) fg(x_2)$，~~凹~~ [illegible] 凹函数

(c) ∵$f(x)$, $g(x)$恒>0 对$\theta\in(0,1)$

$$f/g(\theta x_1+(1-\theta)x_2)=\frac{f(\theta x_1+(1-\theta)x_2)}{g(\theta x_1+(1-\theta)x_2)}=A$$

$$\theta f/g(x_1)+(1-\theta)f/g(x_1)$$

$$=\theta\frac{f(x_1)}{g(x_1)}+(1-\theta)\frac{f(x_2)}{g(x_2)}=B$$

~~∵g为凹函数，则有 $g(\theta x_1+(1-\theta)x_2)\geq\theta g(x_1)+(1-\theta)g(x_2)$~~

f为凸函数，则有 $f(\theta x_1+(1-\theta)x_1)\leq\theta f(x_1)+(1-\theta)f(x_2)$

则有 $A\leq\theta\frac{f(x_1)}{g(\theta x_1+(1-\theta)x_2)}+(1-\theta)\frac{f(x_2)}{g(\theta x_1+(1-\theta)x_2)}=C$

~~∵g为凹函数，则有 $g(\theta x_1+(1-\theta)x_2)\geq\theta g(x_1)+(1-\theta)g(x_2)$~~

$$B-C=\frac{(\theta f(x_1)g(x_2)+(1-\theta)f(x_2)g(x_1))g(\theta x_1+(1-\theta)x_2)-g(x_1)g(x_2)(\theta f(x_1)+(1-\theta)f(x_2))}{g(x_1)g(x_2)g(\theta x_1+(1-\theta)x_2)}$$

$$=\frac{(1-\theta)f(x_2)g(x_1)(g(\theta x_1+(1-\theta)x_2)-g(x_2))-\theta(g(\theta x_1+(1-\theta)x_2)-g(x_1))f(x_1)g(x_2)}{g(x_1)g(x_2)g(\theta x_1+(1-\theta)x_2)}$$

令 $(1-\theta)(g(\theta x_1+(1-\theta)x_2)-g(x_2))-\theta(g(\theta x+(1-\theta)x_2)-g(x_1))=D_1-D_2$

$=g(\theta x_1+(1-\theta)x_2)-\theta g(x_1)-(1-\theta)g(x_2)=P$ ①

∵$g(x)$为凹函数，∴$P\geq0$，

∵$g(x)$非增，∴$D_1\geq0$，$D_2\geq0$ ②

又∵f非减，g非增，$f>0$，$g>0$

∴$f(x_2)g(x_1)>f(x_1)g(x_2)$ ③

由①，②，③

∴$B-C\geq0$，

$A\leq B$，f/g为凹函数

$= QPy$

$x = \frac{1}{\sqrt{\lambda_A}}$

$y = \sqrt{\lambda_A} P x$

$z = \sqrt{\lambda_A} \cdot Q x$,

$x = \frac{1}{\sqrt{\lambda_A}} Q^T z$

minimize $\sum_{i=1}^{n} \frac{\lambda_{Bi}}{\lambda_{Ai}} z_i^2$

s.t. $\|z\| \le 1$

∵ $A_i \in S_{++}^n$, ∴ $\lambda_{Ai} > 0$

当 $B \in S_{++}^n$, $\forall i, \lambda_{Bi} > 0$,

$\sum_{i=1}^{n} \frac{\lambda_{Bi}}{\lambda_{Ai}} z_i^2 \ge 0$, 当 $z=0$时取相等,

$p^* = 0$,

$z^* = 0$, $x^* = 0$

4.21 (c)

∵ $A \in S_{++}^n$,

∴ $A = (A^{\frac{1}{2}})^T A^{\frac{1}{2}}$

令 $y = A^{\frac{1}{2}} x$,

$x = (A^{-\frac{1}{2}})^T \cdot y$

原式 $=$ minimize $y^T \cdot A^{-\frac{1}{2}} B A^{-\frac{1}{2}} y$

s.t. $\|y\| \le 1$

当 $\lambda_{min}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) > 0$ 时, $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ 正定

$\forall y \neq 0$, $y^T (A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) y > 0$,

$$\begin{cases} y^* = 0 \\ x^* = 0 \\ p^* = 0 \end{cases}$$

当 $B \notin S_{++}^n$ 时, $\min(\lambda_{Bi}) \le 0$

$p^* = \min\left(\frac{\lambda_{Bi}}{\lambda_{Ai}}\right)$,

$$z_i^* = \begin{cases} 1 & z_i = \arg\min\left(\frac{\lambda_{Bi}}{\lambda_{Ai}}\right) \text{(只取一个)} \\ 0 \end{cases}$$

$x^* = \frac{1}{\sqrt{\lambda_A}} \cdot Q^T \cdot z^*$

当 $\lambda_{min}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \le 0$ 时,

$p^* = \lambda_{min}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})$

$$y_i = \begin{cases} 0 & \text{others} \end{cases}$$

y^* 为 λ_{min} 对应单位特征向量,

$x^* = A^{-\frac{1}{2}} \cdot y^*$

4.33：(1) 引入变量 x_{n+1}，变为GP：minimize x_{n+1}

∵ $p(x), q(x) > 0$，∴ $x_{n+1} > 0$

s.t: $\frac{p(x)}{x_{n+1}} \leq 1$, $\frac{q(x)}{x_{n+1}} \leq 1$

令 $p(x) = \sum_{k=1}^{m} c_k x_1^{a_{1k}} \cdots x_n^{a_{nk}}$

$q(y) = \sum_{k=1}^{N} f_k x_1^{d_{1k}} \cdots x_n^{d_{nk}}$

令 $\forall k, a_{n+1k} = d_{n+1}k = -1$

取对数：令 $y_i = \ln x_i$，$b_k = \ln c_k$，$e_k = \ln f_k$

原式变为 minimize e^{y_i}

s.t: $\ln \sum_{k=1}^{m} e^{(\sum_{i=1}^{n+1} a_{ik} \cdot y_i) + b_k} \leq 0$

$\ln \sum_{k=1}^{N} e^{\sum_{i=1}^{n+1} d_{ik} \cdot y_i + e_k}$

($\mathrm{dom} f = R_{++}^{n+1}$)，为凸优化问题)

(2) 引入变量 x_{n+1}, x_{n+2}，变为

minimize $e^{x_{n+1}} + e^{x_{n+2}}$

s.t: $\frac{p(x)}{x_{n+1}} \leq 1$, $\frac{q(x)}{x_{n+2}} \leq 1$

令 $p(x) = \sum_{k=1}^{m} c_k x_1^{a_{1k}} \cdots x_n^{a_{nk}}$

$q(y) = \sum_{k=1}^{N} f_k x_1^{d_{1k}} \cdots x_n^{d_{nk}}$

令 $a_{n+1k} = -1$, $a_{n+2k} = 0$

$d_{n+1k} = 0$　$d_{n+2k} = -1$,

取对数，令 $y_i = \ln x_i$, $b_k = \ln c_k$, $e_k = \ln f_k$

问题变为 minimize $e^{e^{y_{n+1}}} + e^{e^{y_{n+2}}}$

s.t: $\ln \sum_{k=1}^{m} e^{\sum_{i=1}^{n+2} a_{ik} \cdot y_i + b_k} \leq 0$

$\ln \sum_{k=1}^{N} e^{\sum_{i=1}^{n+1} d_{ik} \cdot y_i} + e_k \leq 0$

均为凸函数 是凸优化 =)

∵ $p(x) > 0$, $q(x) > 0$，∴ $x_{n+1} > 0$, $x_{n+2} > 0$,

$\mathrm{dom} f = R_{++}^{n+1}$

c) 令 $p(x)=\sum_{k=1}^{m} c_k x_1^{a_{1k}} \cdots x_n^{a_{nk}}$

$q(x)=\sum_{k=1}^{N} f_k x_1^{d_{1k}} \cdots x_n^{d_{nk}}$

$r(x)= h\, x_1^{g_1} \cdots x_n^{g_n}$ $(h>0)$

$\frac{p(x)}{r(x)}=\sum_{k=1}^{m}\frac{c_k}{h} x_1^{a_{1k}-g_1}\cdots x_n^{a_{nk}-g_n}$ 二者仍为正项式

$\frac{q(x)}{r(x)}=\sum_{k=1}^{N}\frac{f_k}{h} x_1^{d_{1k}-g_1}\cdots x_n^{d_{nk}-g_n}$

$\because r(x)>q(x)$

$\therefore \frac{p(x)}{r(x)-q(x)}=\frac{\frac{p(x)}{r(x)}}{1-\frac{q(x)}{r(x)}}$

~~$r(x)>$~~

$r(x)>q(x) \Leftrightarrow \frac{q(y)}{r(x)}<1$

令 $y_i=\ln x_i$, $b_k=\ln\frac{f_k}{h}$, $e_k=\ln\frac{f_k}{h}$

则 $\frac{p(x)}{r(x)}=\sum_{k=1}^{m} e^{\left(\sum_{i=1}^{n}(a_{ik}-g_i)\cdot y_i\right)+b_k}$

$\frac{q(x)}{r(x)}=\sum_{k=1}^{N} e^{\left(\sum_{i=1}^{N}(d_{ik}-g_i)\cdot y_i\right)+e_k}$

取对数，变为：minimize $\ln\sum_{k=1}^{m} e^{\sum_{i=1}^{n}(a_{ik}-g_i)\cdot y_i+b_k} - \ln\left(1-\sum_{k=1}^{N} e^{\sum_{i=1}^{n}(d_{ik}-g_i)y_i+e_k}\right)$

s.t. $\ln\sum_{k=1}^{N} e^{\sum_{i=1}^{n}(d_{ik}-g_i)\cdot y_i+e_k} \leq 0$

(转化优化为 $\ln\frac{p(y)}{r(y)}-\ln\left(1-\frac{q(y)}{r(y)}\right)$

$\ln\frac{p(y)}{r(y)}$ 凸，$\frac{q(y)}{r(y)}$ 凸，$1-\frac{q(y)}{r(y)}$ 凹，

$-\ln x$ 凸且单调递减，

$-\ln\left(1-\frac{q(y)}{r(y)}\right)$ 凸，得优化式凸

均为凸函数，凸优化

4.40 可用t替换 x^TPx，$t \geq x^TPx$

(b) QP

原 = minimize $\frac{1}{2}t + q^Tx + r$

∵ $P \in S_n^+$

∴ $P = Q^TQ$

∴ $t \geq x^TPx$

⇕

$tI - x^TPx \succeq 0$

s.t. ~~$\begin{bmatrix} I & Qx \\ x^TQ^T & tI \end{bmatrix} \succeq 0$~~

~~$\text{diag}(h-Gx) \succeq 0$~~

~~$Ax=B$~~

$$\begin{bmatrix} I & Qx & 0 \\ x^TQ^T & tI & 0 \\ 0 & 0 & \text{diag}(h-Gx) \end{bmatrix} \succeq 0$$

$Ax = b$

(c) QCQP

可用t_i替 x^TP_ix，

∵ P_i半正定，令 $P_i = Q_i^TQ$

~~原 = minimize $\frac{1}{2}t_0 + q_0^Tx + r_0$~~

~~s.t. $\frac{1}{2}t_i + q_i^Tx + r_i \leq 0$~~

~~$Ax = B$~~

$\frac{1}{2}x^TP_ix + q_i^Tx + r_i \leq 0$

⇕

$$\begin{bmatrix} I & Q_ix \\ x^TQ_i^T & -2q_i^Tx-r_i \end{bmatrix} \succeq 0$$

令t使 $t_0 \geq x^TP_0x$

$t_0 \geq x^TPx \Leftrightarrow \begin{bmatrix} I & Q_0x \\ x^TQ_0^T & t_0I \end{bmatrix} \succeq 0$

原 = minimize

$\frac{1}{2}t_0 + q_0^Tx + r_0$

s.t. $\begin{bmatrix} I & Q_0x & & \\ x^TQ_0^T & t_0I & & \\ & & I & Q_ix \\ & & x^TQ_i^T & -2q_i^Tx-r_i \\ & & & & \ddots \end{bmatrix} \succeq 0$

$Ax = b$

SOCP:

~~$\|A_i x+b_i\|_2$~~

~~$= (x^T A_i^T + b_i^T)(A_i x+b_i)$~~

$\|A_i x+b_i\|_2 \le (c_i^T x+d_i)$ SO

$\Updownarrow$

~~$(x^T A_i^T + b_i^T)(A_i x+b_i)$~~

$$\begin{bmatrix} (c_i^T x+d_i)I & A_i x+b_i \\ (A_i x+b_i)^T & c_i^T x+d_i \end{bmatrix} \succeq 0$$

minimize $f^T x$

s.t

$$\begin{bmatrix} (c_1^T x+d_1)I & A_1 x+b_1 & & & & & \\ b_1^T+xA_1^T & c_1^T x+d_1 & & & & & \\ & & (c_2^T x+d_2)I & A_2 x+b_2 & & & \\ & & b_2^T+xA_2^T & c_2^T x+d_2 & & & \\ & & & & \ddots & & \\ & & & & & (c_m^T x+d_m) & A_m x+b_m \\ & & & & & b_m^T+xA_m^T & c_m^T x+d_m \end{bmatrix} \succeq 0$$

$Fx=g$

（$c_i^T x+d_i=0$ 时，上述条件有 $A_i x+b_i = c_i^T x+d_i=0$，矩阵为0矩阵，也半正定，不违反上述条件）

(c) 令 ~~$F(x) = Q^T(x)Q(x)$, $Q(x)$正交~~,

~~$F(x)^{-1} = Q(x)Q^T(x)$~~

令 $t \geq (Ax+b)^T F(x)^{-1}(Ax+b)$

$$t \geq (Ax+b)^T F(x)^{-1}(Ax+b)$$

$(F(x) \succ 0)$

$$\Updownarrow$$

$$\begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & tI \end{bmatrix} \succeq 0$$

~~$(F(x) \succ 0$, 则 $F(x)^{-1}$~~ $\succ 0)$

∴ minimize t

s.t $\begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & tI \end{bmatrix} \succeq 0$