Convex Optimization
Week 8
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数博21
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5.27: $\|Ax-b\|_2^2 = (Ax-b)^T(Ax-b)$

$= x^TA^TAx - 2x^TA^Tb + b^Tb$

a. $\nabla f_0(x) = 2A^TAx - 2A^Tb$　　$\nabla h(x) = G^T$

KKT条件:

$$\begin{cases} 2A^T(Ax^*-b) + G^Tv^* = 0 \\ Gx^* = h \end{cases}$$

$\because \exists x \in \mathrm{int}(\mathrm{dom} f)$

$\therefore$ 成立slater条件

x^*与v^*同时成立

对偶问题: $L(x,v) = x^TA^TAx - 2x^TA^Tb + b^Tb + (x^TG^Tv - h^Tv)$

$= x^TA^TAx - x^T(2A^Tb - G^Tv) + b^Tb - h^Tv$

$g(v) = \inf_x L(x,v)$

$\frac{\partial L}{\partial x} = 2A^TAx - (2A^Tb - G^Tv)$

$\frac{\partial^2 L}{\partial x^2} = 2A^TA$，正定

$\therefore \frac{\partial L}{\partial x}=0$时，$x^* = (A^TA)^{-1}(A^Tb - \frac{1}{2}G^Tv)$

(A^TA可逆)

$g(v) = L(x^*,v)$

$= -(A^Tb - \frac{1}{2}G^Tv)^T(A^TA)^{-1}(A^Tb - \frac{1}{2}G^Tv)$

$+ b^Tb - h^Tv$

$\frac{\partial g(v)}{\partial v} = G(A^TA)^{-1}(A^Tb - \frac{1}{2}G^Tv) - h^T$

$\frac{\partial^2 g(v)}{\partial v^2} = -\frac{1}{2}G(A^TA)^{-1}G^T$，负定

有极大值

$\therefore \frac{\partial g(v)}{\partial v} = 0$时是最优

$v^* = 2(G(A^TA)^{-1}G^T)^{-1}(G(A^TA)^{-1}A^Tb - h)$

$2G(A^TA)^{-1}A^Tb - 2h^T$
$= G(A^TA)^{-1}G^Tv$

(∵成立slater条件，∴原问题与对偶问题的最优解同时取到)

$x^* = (A^TA)^{-1}(A^Tb - \frac{1}{2}G^Tv^*)$

5.29

$\nabla f_0 = (-6x_1+2,\ 2x_2+2,\ 4x_3+2)$

$\nabla h = (2x_1, 2x_2, 2x_3)$

KKT条件:

$x_1^2 + x_2^2 + x_3^2 = 1$

$(v-3)x_1^2 = 1$

$(v+1)x_2^2 = 1$

$(v+2)x_3^2 = 1$

$$L(x,v) = -3x_1^2 + x_2^2 + 2x_3^2 + 2x_1 + 2x_2 + 2x_3 + v x_1^2 + v x_2^2 + v x_3^2 - v = 0$$

$$\frac{\partial L(x,v)}{\partial x} = (-6x_1 + 2vx_1 + 2,\ 2x_2 + 2vx_2 + 2,\ 4x_3 + 2vx_3 + 2)$$

$$\frac{\partial^2 L(x,v)}{\partial^2 x} = \begin{bmatrix} 2v-6 & 0 & 0 \\ 0 & 2v+2 & 0 \\ 0 & 0 & 2v+4 \end{bmatrix}$$

当 $v>3$ 时，才能取极小值 正定，

此时：$x_1^* = -\frac{1}{v-3}$

$x_2^* = -\frac{1}{v+1}$

$x_3^* = -\frac{1}{v+2}$

$4.04 + 0.97 + 0.20 + 0.17$

$404 + 97 + 20 + 17 = 538$

$$g(v) = \inf_{x \in \mathrm{dom}} L(x,v) = \frac{v-3}{(v-3)^2} + \frac{v+1}{(v+1)^2} + \frac{v+2}{(v+2)^2} - \frac{2}{v-3} - \frac{2}{v+1} - \frac{2}{v+2} - v$$

$$= -\frac{1}{v-3} - \frac{1}{v+1} - \frac{1}{v+2} - v$$

minimize $\frac{1}{v-3} + \frac{1}{v+1} + \frac{1}{v+2} + v$

s.t. $v>3$

当 $\frac{1}{(v-3)^2} + \frac{1}{(v+1)^2} + \frac{1}{(v+2)^2} = 1$ 时 取到最值

即：$3v^4 + 36v + 49$

$x^6 - 14x^4 - 12x^3 + 49x^2 + 84x + 36$

$v^6 - 17v^4 - 12v^3 + 49v^2 + 48v - 11 = 0$

应满足KKT的解 x, v：

① $v = -3.15$ $x = (0.16, 0.47, 0.61)$

② $v = 0.22$, $x = (0.16, -0.82, -0.45)$

③ $v = 1.84$ $x = (0.40, -0.13, -0.26)$

④ $v = 4.04$, $x = (-0.97, -0.20, -0.17)$ 才满足

~~仅4.04满足~~

此时：$x^* = (-0.97, -0.20, -0.17)$

$v^* = 4.04$

$p^* = -5.37$

扫描全能王 创建

② 若 $p^* = d^*$:

则 $~~L(x^*,\lambda^*,u^*)~~ = 1$

$d^* = \max_{\lambda\geq0,u} \inf_{x\in dom f} L(x,\lambda,u)$

（都没划掉 证明有效）

3. $L(x,\lambda,u) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} u_j h_j(x)$ $\lambda_i \geq 0$

$g(\lambda,u) = \inf_{x\in dom f} L(x,\lambda,u)$

①: 若 x^*,λ^*,u^* 为 $L(x,\lambda,u)$ 鞍点，则有

$\forall x \in dom f, \lambda\geq0, u$:

$L(x^*,\lambda,u) \leq L(x^*,\lambda^*,u^*) \leq L(x,\lambda^*,u^*)$

（~~没划掉~~）

（~~没划掉~~ 证明有效）

~~$\therefore x^*,\lambda^*,u^* = \arg\max_{\lambda\geq0,u} \inf_{x\in dom f} L(x,\lambda,u)$~~

~~$\sup_{\lambda\geq0,u} \inf_{x\in dom f} L(x,\lambda,u) = \inf_{x\in dom f} \sup_{\lambda\geq0,u} L(x,\lambda,u)$~~

$\therefore x^*,\lambda^*,u^* = \arg \sup_{\lambda\geq0,u} \inf_{x\in dom f} L(x,\lambda,u)$

此时 λ^*,u^* 为对偶问题最优解，

$d^* = L(x^*,\lambda^*,u^*)$

$\because$ 对 $\forall \lambda\geq0, u$ 有

$L(x^*,\lambda,u) \leq L(x^*,\lambda^*,u^*)$

取 $\lambda=0, u=0$:

$f(x^*) \leq L(x^*,\lambda^*,u^*)$

而 $\because \lambda\geq0$, $f_i(x)\leq0$, $h_j(x)=0$,

$f(x) \geq L(x,\lambda,u)$

$\therefore f(x^*) \geq L(x^*,\lambda^*,u^*)$

若 x^* 不为原问题最优解，则令 x' 为最优解

$f(x') \geq L(x',\lambda^*,u^*)$

而 $L(x',\lambda^*,u^*) \geq L(x^*,\lambda^*,u^*) = f(x^*)$

与 $f(x') < f(x^*)$ 矛盾

$\therefore x^*$ 为原问题最优解 得证

$p^* = f(x^*)$

$d^* = L(x^*,\lambda^*,u^*)$

$\leq f(x^*)$

$\therefore p^* = d^*$

$= L(x^*,\lambda^*,u^*)$

得证

② 若强对偶

则有 x^*,λ^*,u^*，使 $f(x^*) = p$ 为原问题最优值

$g(\lambda^*,u^*) = d$ 为对偶问题最优值，

$p^* = d^*$

则

$f(x^*) = L(x^*,\lambda^*,u^*) = p^* = d^*$

且 $\forall x \in dom f$,

$f(x^*) \leq f(x)$,

$\forall \lambda, u$ 使 $\lambda\geq0$,

$L(x^*,\lambda^*,u^*) \geq L(x^*,\lambda,u)$

$\forall x$,

$L(x^*,\lambda^*,u^*) = f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{j=1}^{p} u_j^* f_j(x^*)$

$L(x,\lambda^*,u^*) = f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{j=1}^{p} u_j^* f_j(x)$

若 $\exists x'$ 使 $L(x',\lambda^*,u^*) < L(x^*,\lambda^*,u^*)$

则 $\arg\inf_{x\in dom f} L(x,\lambda,u)$ 应取 x 而非 x^*,

$\therefore \exists \lambda',u'$

使 $L(x',\lambda',u') = d^* \leq L(x',\lambda^*,u^*) < L(x^*,\lambda^*,u^*) = p^*$,

与 $p^* = d^*$ 矛盾，得证