3.58：要证 schur 补 凹 $\Leftrightarrow \forall z \in R^{n-k}$，$z^T(C-B^TA^{-1}B)z$ 凹

∵ $X \in S_{++}^n$ ∴其任一顺序主子式大于0

∴A任一顺序主子式大于0

$A \in S_{++}$

令 $X_1 = \begin{bmatrix} A_1 & B_1 \\ B_1^T & C_1 \end{bmatrix}$ $X_2 = \begin{bmatrix} A_2 & B_2 \\ B_2^T & C_2 \end{bmatrix}$

即证 $\forall 0<\lambda<1$，$X_1, X_2 \in S_{++}^n$，$\forall z \in R^n$

$z^T(\lambda C_1 + (1-\lambda)C_2 - (\lambda B_1 + (1-\lambda)B_2)^T(\lambda A_1 + (1-\lambda)A_2)^{-1}(\lambda B_1 + (1-\lambda)B_2))z$

$-\lambda z^T C_1 + \lambda z^T B_1^T A_1^{-1} B_1 - (1-\lambda) z^T C_2 - (1-\lambda) z^T B_2^T A_2^{-1} B_2 \geq 0$，

即证 $\forall z$，$z^T B^T A^{-1} B z$ 凸

∵ A正定

∴ $A = D^T D$

$A^{-1} = (D^T D)^{-1} = D^{-1}(D^T)^{-1}$

$z B^T A^{-1} B z$

$= \|(D^{-1})^T B z\|_2$

$\therefore \begin{bmatrix} B^T A^{-1} B & z \\ z^T & t \end{bmatrix} \succeq 0$

①：$\forall z \neq 0$，

$z^T B^T A^{-1} B z$

$= (Bz)^T A^{-1} (Bz)$

∵ $A \in S_{++}^k$

∴ $A^{-1} \in S_{++}^k$，

$z^T B^T A^{-1} B z > 0$，

$B^T A^{-1} B \in S_{++}^{n-k}$

② 易 $f(z, X) = z^T B^T A^{-1} B z$ 在 $R^{n-k} \times S_{++}^n$ 上凸，

只需有 $\forall z$，$f(z, X)$ 在 $S_{++}^n$ 上凸

而 $\mathrm{epi} f = \{(z, X, t) \mid X \succ 0,\ z^T B^T A^{-1} B z \leq t\}$

$= \{(z, X, t) \mid \begin{bmatrix} B^T A^{-1} B & z \\ z^T & t \end{bmatrix} \succeq 0,\ X \succ 0\}$

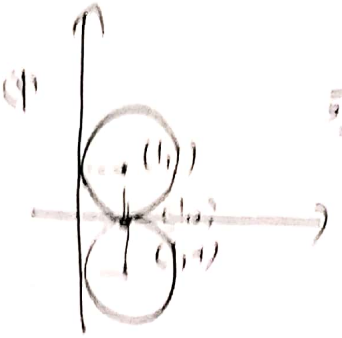
∴有 epi f 为凸集

∴有 $f(z, X)$ 为凸函数

$\forall z$，$z B^T A^{-1} B z$ 是关于 X 凸函数，Schur补为凹函数，得证

~~5.5.6~~
5.2.6

(1)




可行域：~~{(1,0)}~~ {(1,0)}

$x^* = (1,0)$

~~目标函数值~~ {1}

$p = 1$

(2) $\nabla(x_1^2+x_2^2) = (2x_1, 2x_2)$

$\nabla((x_1-1)^2+(x_2-1)^2-1) = (2x_1-2, 2x_2-2)$

$\nabla((x_1-1)^2+(x_2+1)^2-1) = (2x_1-2, 2x_2+2)$

KKT条件：

$(2x_1^*, 2x_2^*) + \lambda_1(2x_1^*-2, 2x_2^*-2) + \lambda_2(2x_1^*-2, 2x_2^*+2) = 0$

即
$$
\begin{cases}
(\lambda_1^*+\lambda_2^*+1)x_1 - (\lambda_1^*+\lambda_2^*) = 0 & \text{①} \\
(\lambda_1^*+\lambda_2^*+1)x_2 - (\lambda_1^*-\lambda_2^*) = 0 & \text{②} \\
(x_1^*-1)^2+(x_2^*-1)^2-1 \le 0 & \text{③} \\
(x_1^*-1)^2+(x_2+1)^2-1 \le 0 & \text{④} \\
\lambda_1^* \ge 0 & \text{⑤} \\
\lambda_2^* \ge 0 & \text{⑥} \\
\lambda_1^*((x_1-1)^2+(x_2+1)^2-1) = 0 & \text{⑦} \\
\lambda_2^*((x_1-1)^2+(x_2+1)^2-1) = 0 & \text{⑧}
\end{cases}
$$

$\lambda_1+\lambda_2 = \lambda_1+\lambda_2+1$

~~$\lambda_1=\lambda_2$~~

~~$x_1=1, x_2=0$~~

~~$\lambda_1=0+\lambda_2=0$~~

~~$\lambda_1=$~~

~~存在 $\lambda_1, \lambda_2$ 使~~

$\because x_2=0$，由② $\lambda_1^*=\lambda_2^*$

$\because$ ③、④、⑦、⑧满足

$\therefore$ 需使①、⑤、⑥即可

$2\lambda_1+1-2\lambda_1=0$ 不存在 $\lambda$

故 $\lambda$

不存在

Week 6

10/26/2021

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5.27: $\|Ax-b\|_2^2 = (Ax-b)^T(Ax-b)$

$$\frac{\partial \|Ax-b\|_2^2}{\partial x} = x^TA^TAx - x^TA^Tb - b^TAx + b^Tb$$

$$= 2A^TAx - 2A^Tb$$

$$\frac{\partial (Gx-h)}{\partial x} = G^T$$

① KKT条件:

$Gx^*-h=0$ -①

$u^* \geq 0$ -② $(u \in R^p)$

$2A^TAx^* - 2A^Tb + G^Tu^* = 0$ -③

$\cancel{2(Ax^*-b)^TAu^*} + G^Tu^* = 0$

$\cancel{2(A^TAx^*-A^Tb)}$

$\cancel{2A^T(Ax^*-b)}\ u^* = 0$

$\cancel{2(Ax-b)^TAu} = 0$

$(\cancel{2x^*A^TA - 2A^Tb} - G^Tu^* = 0$

$(2A^TAx^* - 2A^Tb - G^T)u^* = 0$

②: $2A^TAx = 2A^Tb - G^Tu^*$.

∵ $r(A)=n$, $n \leq m$

∴ $A^TA$ 逆

$$x^* = \frac{1}{2}(A^TA)^{-1}(2A^Tb - G^Tu^*)$$

代入①,有

$$\frac{G}{2}(A^TA)^{-1}(2A^Tb - G^Tu^*) - h = 0$$

$$\frac{G}{2}(A^TA)^{-1}G^Tu^* = h - G(A^TA)^{-1}A^Tb$$

∵ $r(G)=p$, $p \leq n$,

∴ $G(A^TA)^{-1}G^T$ 逆

$$u^* = 2(G(A^TA)^{-1}G^T)^{-1}(h - G(A^TA)^{-1}A^Tb)$$

$$x^* = (A^TA)^{-1}(2A^Tb - G^T(G(A^TA)^{-1}G^T)^{-1}(h - G(A^TA)^{-1}A^Tb))$$

5.28: KKT条件:

$\cancel{Ax^* - b \le 0}$

$$\begin{cases} \begin{bmatrix} 47 \\ 93 \\ 17 \\ 93 \end{bmatrix} + \begin{bmatrix} -4 & -1 & 0 & -6 & 1 \\ -6 & -2 & 3 & -11 & 6 \\ 1 & 7 & -10 & -2 & -1 \\ 3 & 1 & -1 & 12 & -3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} = 0 \quad —① \\ \begin{bmatrix} \lambda_1^* \\ \lambda_2^* \\ \lambda_3^* \\ \lambda_4^* \\ \lambda_5^* \end{bmatrix} \ge 0 \quad —② \\ \begin{bmatrix} -1 & -6 & 13 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} - \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix} \le 0 \quad —③ \\ \begin{bmatrix} \lambda_1^* & \lambda_2^* & \lambda_3^* & \lambda_4^* & \lambda_5^* \end{bmatrix} \left( \begin{bmatrix} -1 & -6 & 13 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} - \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix} \right) = 0 \end{cases}$$

由①,有:

$$\begin{bmatrix} 1 & 1 & 0 & 6 & -1 \\ 0 & 1 & \frac{1}{2} & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 11 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1^* \\ \lambda_2^* \\ \lambda_3^* \\ \lambda_4^* \\ \lambda_5^* \end{bmatrix} = \begin{bmatrix} 47 \\ 24 \\ 16 \\ 77 \end{bmatrix}$$

有:

$$\begin{bmatrix} \lambda_1^* \\ \lambda_2^* \\ \lambda_3^* \\ \lambda_4^* \\ \lambda_5^* \end{bmatrix} = \begin{bmatrix} x_5^* + 3 \\ 2 \\ 2 \\ 7 \\ x_5^* \end{bmatrix}$$

$\because \lambda^* \ge 0$

$\therefore$ 只有 $\lambda_1^*$, $\lambda_2^*$ $\lambda_3^*$ $\lambda_4^*$ 为:

有:

$$\begin{bmatrix} -1 & -6 & 13 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & -1 & -3 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{24} \\ 0 & 0 & 0 & \frac{143}{24} \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ \frac{23}{24} \\ [illegible] \end{bmatrix}$$

向下代回,得

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$