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1. (a) X_i 在u方向的投影为 $t_i u$,

$(x_i - t_i u)^T u = 0$, $t_i = \frac{x_i^T u}{u^T u}$, $\bar{t} = \frac{\sum t_i}{n} = \frac{(x_1 + \cdots + x_n)^T u}{u^T u} = 0$

$t_i u$ 的方差 $\frac{1}{n}\sum_{i=1}^{n} \|t_i u - \bar{t} u\|_2^2 = \frac{1}{n}\sum_{i=1}^{n} t_i^2 (u^T u)$

$= \frac{1}{n}\sum_{i=1}^{n} (u^T u) \cdot \frac{(x_i^T u)^2}{(u^T u)^2}$

$= \sum_{i=1}^{n} \frac{u^T x_i x_i^T u}{n(u^T u)}$

$= \frac{1}{u^T u} \cdot u^T C u$

$\because \|u\|_2^2 = 1$

$\therefore$ 方差 $= u^T C u$

(b) $k=1$

目标函数 $\max_{v_1} \sum_{i=1}^{n} \|x_i - v_1 v_1^T x_i\|_2$

$= \max_{v_1} \sum_{i=1}^{n} (x_i - v_1 v_1^T x_i)^T (x_i - v_1 v_1^T x_i)$

$= \max_{v_1} \sum_{i=1}^{n} x_i^T v_1 v_1^T x_i = \max_{v_1} \sum_{i=1}^{n} (v_1^T x_i)^2$

$\cancel{= \min_{v_1} \sum_{i=1}^{n} (v_1^T x_i)^2}$ $= \max_{v_1} \sum_{i=1}^{n} (v_1 \cdot x_i)^2$

$v_1^T C v_1 = \frac{1}{n}\sum_{i=1}^{n} v_1^T x_i x_i^T v_1$

$= \frac{1}{n}\sum_{i=1}^{n} (x_i^T v_1)^2 = \frac{1}{n}\sum_{i=1}^{n} (v_1 \cdot x_i)^2$

~~n倍方差~~ =

n倍方差 = 目标函数
目标函数最大值 = 最大化方差n倍

2. $\sim W$

$$y = \arg\min_{y'} \sum_{i,j} W_{ij} (y_i' - y_j')^2$$

$W_{ij} = W_{ji}$

该目标函数 $f = \sum_{i,j} W_{ij} (y_i' - y_j')^2$

$$f_i = \sum_j W_{ij} (y_i' - y_j')^2$$

$$= \sum_j W_{ij} (y_i'^2 - 2y_i' y_j' + y_j'^2)$$

$$= y_i'^2 \sum_j W_{ij} - 2y_i' \sum_j W_{ij} y_j' + \sum_j W_{ij} y_j'^2$$

$$f = \sum_{i=1}^n f_i = \underbrace{\sum_{i=1}^n y_i'^2 \cdot D_{ii}}_{y'^T D y'} - 2 \underbrace{\sum_{i=1}^n y_i' \sum_j W_{ij} y_j'}_{y'^T W y'} + \sum_{i=1}^n \sum_j W_{ij} y_j'^2$$

$$= y'^T D y' - 2y'^T W y' + \sum_{i=1}^n \sum_j W_{ij} y_j'^2$$

$$= y'^T L y' - y'^T W y' + \sum_{i=1}^n \sum_{j=1}^n W_{ij} y_j'^2$$

$\Leftarrow$ 证 $\sum_{i=1}^n \sum_{j=1}^n W_{ij} y_j'^2 = y'^T D y'$

$\because \sum_i \sum_j W_{ij} \cdot y_i'^2$

$= \sum_i y_i'^2 \cdot D_{ii}$

$= y'^T D y'$

而对于 $\sum_i \sum_j W_{ij} \cdot y_i'^2$，每个 W_{ij} 只与 $y_i'^2$ 相乘 加权值

对于 $\sum_i \sum_j W_{ij} y_j'^2$，每个 W_{ij} 与 $y_j'^2$ 相乘 加权值

$\because W$ 为对称矩阵：

$\therefore \sum_i \sum_j W_{ij} y_j'^2 = \sum_i \sum_j W_{ji} y_j'^2 = \sum_i \sum_i W_{ij} \cdot y_i'^2$

$= \sum_i \sum_j W_{ij} \cdot y_i'^2 = y'^T D y'$

$\sum_{i=1}^n y_i' \sum_j W_{ij} y_j'$

$= \sum_{i=1}^n y_i' \cdot [\ldots]_{ii}$

$= \sum_{i=1}$

$$\boxed{\begin{aligned} \therefore \text{原式} &= 2(y'^T D y' - y'^T W y') \\ &= 2y'^T L y'，等价 y'^T L y' 最小 \end{aligned}}$$