Appendix B. 2D Bars problem

B.1 Preprocess (example)

In a 2D bars problem, the number of DOFs per node is 2 (1 = horizontal x-component and 2 = vertical y-component). Consider the example 2D bars structure depicted in Figure B1. It contains 5 nodes (indexed from 1 to 5), and 5 elements or bars (indexed from (1) to (5)). Which index is assigned to each node or bar does not matter, as long as we stick to a defined configuration throughout the whole problem. Bars (1), (3) and (4) are made of a material with elastic modulus E_1 and cross-section area A_1, while bars (2) and (5) are made of another material with E_2 and A_2. Furthermore, these bars are subjected to an initial stress σ_0.

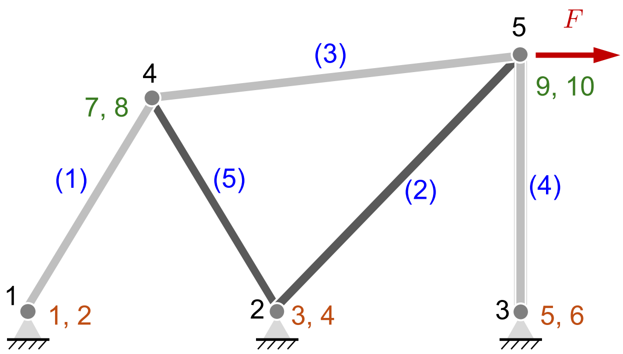

Figure B1. Example 2D bars structure.

In this example, this is how the mesh matrices would be defined:

$$[\mathbf{x}] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \end{bmatrix}; \quad [\mathbf{T}^{\mathrm{n}}] = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 4 & 5 \\ 3 & 5 \\ 2 & 4 \end{bmatrix}; \quad [\mathbf{m}] = \begin{bmatrix} E_1 & A_1 & 0 \\ E_2 & A_2 & \sigma_0 \end{bmatrix}; \quad \{\mathbf{T}^{\mathrm{m}}\} = \begin{Bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \end{Bmatrix}$$

Notice that we have two different material properties, so we only added two rows in the $[\mathbf{m}]$ matrix. We arbitrarily defined the first column to be the elastic modulus and the second column to be the cross-section area. The vector $\{\mathbf{T}^{\mathrm{m}}\}$ determines which material is assigned to each bar. Since $n_{\mathrm{i}} = 2$ in this case we can define the DOFs connectivities matrix as:

$$[\mathbf{T}^{\mathrm{d}}] = \begin{bmatrix} 1 & 2 & 7 & 8 \\ 3 & 4 & 9 & 10 \\ 7 & 8 & 9 & 10 \\ 5 & 6 & 9 & 10 \\ 3 & 4 & 7 & 8 \end{bmatrix}$$

In the same example, we identify the DOFs of nodes 1, 2 and 3 as prescribed (both the x and y-components of the displacement). This results in a total of 6 DOFs prescribed:

$$[\mathbf{p}] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

There is only one point force acting on the horizontal DOF of node 5, so:

$$[\mathbf{F}] = \begin{bmatrix} 5 & 1 & F \end{bmatrix}$$

B.2 Element stiffness function

Inputs:

- Element index: e
- Nodal coordinates matrix: $[\mathbf{x}]$
- Nodal connectivities matrix: $[\mathbf{T}^{\mathrm{n}}]$
- Material properties matrix: $[\mathbf{m}]$
- Material connectivities vector: $\{\mathbf{T}^{\mathrm{m}}\}$

Outputs:

- Element stiffness matrix: $[\mathbf{K}^{*}]$

Algorithm:

1. Retrieve element's vertices coordinates:
   $$[\mathbf{x}^{\mathrm{el}}] = [\mathbf{x}(\mathbf{T}^{\mathrm{n}}(e,:),:)]$$
2. Get bar length and orientation:
   $$l = \sqrt{\left(\mathbf{x}^{\mathrm{el}}(2,1) - \mathbf{x}^{\mathrm{el}}(1,1)\right)^2 + \left(\mathbf{x}^{\mathrm{el}}(2,2) - \mathbf{x}^{\mathrm{el}}(1,2)\right)^2}$$
   $$c = \frac{\mathbf{x}^{\mathrm{el}}(2,1) - \mathbf{x}^{\mathrm{el}}(1,1)}{l}$$
   $$s = \frac{\mathbf{x}^{\mathrm{el}}(2,2) - \mathbf{x}^{\mathrm{el}}(1,2)}{l}$$
3. Retrieve element's material properties:
   $$E = \mathbf{m}(\mathbf{T}^{\mathrm{m}}(e),1)$$
   $$A = \mathbf{m}(\mathbf{T}^{\mathrm{m}}(e),2)$$
4. Compute element stiffness matrix:
   $$[\mathbf{K}^{*}] = \frac{EA}{l}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

B.3 Element force function

Inputs:

- Element index: e
- Nodal coordinates matrix: $[\mathbf{x}]$
- Nodal connectivities matrix: $[\mathbf{T}^{\mathrm{n}}]$
- Material properties matrix: $[\mathbf{m}]$
- Material connectivities vector: $\{\mathbf{T}^{\mathrm{m}}\}$

Outputs:

- Element force vector: $\{\mathbf{f}^{*}\}$

Algorithm:

1. Retrieve element's vertices coordinates:
   $$[\mathbf{x}^{\mathrm{el}}] = [\mathbf{x}(\mathbf{T}^{\mathrm{n}}(e,:),:)]$$
2. Get bar length and orientation:
   $$l = \sqrt{\left(\mathbf{x}^{\mathrm{el}}(2,1) - \mathbf{x}^{\mathrm{el}}(1,1)\right)^2 + \left(\mathbf{x}^{\mathrm{el}}(2,2) - \mathbf{x}^{\mathrm{el}}(1,2)\right)^2}$$
   $$c = \frac{\mathbf{x}^{\mathrm{el}}(2,1) - \mathbf{x}^{\mathrm{el}}(1,1)}{l}$$

$s = \frac{\mathbf{x}^{\mathrm{el}}(2,2) - \mathbf{x}^{\mathrm{el}}(1,2)}{l}$

3. Compute the rotation matrix:

$$[\mathbf{R}] = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

4. Retrieve element's material properties:

$\sigma_0 = \mathbf{m}(\mathbf{T}^{\mathrm{m}}(e), 3)$

$A = \mathbf{m}(\mathbf{T}^{\mathrm{m}}(e), 2)$

5. Compute element stiffness matrix:

$$\{\mathbf{f}^*\} = -\sigma_0 A [\mathbf{R}]^{\mathrm{T}} \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

B.4 Element stress function

Inputs:

- Element index: e
- Nodal coordinates matrix: $[\mathbf{x}]$
- Nodal connectivities matrix: $[\mathbf{T}^{\mathrm{n}}]$
- Material properties matrix: $[\mathbf{m}]$
- Material connectivities vector: $\{\mathbf{T}^{\mathrm{m}}\}$
- Element displacements: $\{\mathbf{u}^{\mathrm{el}}\}$

Outputs:

- Element stress: σ

Algorithm:

1. Retrieve element's vertices coordinates:

$[\mathbf{x}^{\mathrm{el}}] = [\mathbf{x}(\mathbf{T}^{\mathrm{n}}(e,:),:)]$

2. Get bar length and orientation:

$$l = \sqrt{\left(\mathbf{x}^{\mathrm{el}}(2,1) - \mathbf{x}^{\mathrm{el}}(1,1)\right)^2 + \left(\mathbf{x}^{\mathrm{el}}(2,2) - \mathbf{x}^{\mathrm{el}}(1,2)\right)^2}$$

$c = \frac{\mathbf{x}^{\mathrm{el}}(2,1) - \mathbf{x}^{\mathrm{el}}(1,1)}{l}$

$s = \frac{\mathbf{x}^{\mathrm{el}}(2,2) - \mathbf{x}^{\mathrm{el}}(1,2)}{l}$

3. Compute the rotation matrix:

$$[\mathbf{R}] = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

4. Retrieve element's material properties:

$\sigma_0 = \mathbf{m}(\mathbf{T}^{\mathrm{m}}(e), 3)$

$E = \mathbf{m}(\mathbf{T}^{\mathrm{m}}(e), 1)$

5. Compute element strain:

$\varepsilon = \frac{1}{l}[-1 \quad 0 \quad 1 \quad 0][\mathbf{R}]\{\mathbf{u}^{\mathrm{el}}\}$

6. Compute element stress:

$\sigma = E\varepsilon + \sigma_0$