Appendix C. 3D Bars problem

C.1 Preprocess

The preprocess of a 3D bars problem is practically the same as the one for the 2D case (see Appendix B). The consequence of increasing the dimensions is that now we will have $n_{\mathrm{i}} = 3$ DOFs per node (corresponding to displacements/forces on the x, y and z-directions). Also, since $n_{\mathrm{d}} = 3$ the nodal coordinates matrix will now have 3 columns (corresponding to the x, y and z-coordinates of each node).

For the purposes of this Appendix problem example, let's assume that the bar properties matrix contains the Young's modulus and cross-section area in the first and second columns, and the mass density in the third column, so each row of this matrix looks like:

$$\mathbf{m}(i,:) = [E_i, A_i, \rho_i]$$

While E_i and A_i are required for the structural model, the density ρ_i is included to provide an example of accounting the weight of the structure as an external force. In a general context, this property might not be necessary. Other properties can be added depending on the specific requirements of the problem.

C.2 Element stiffness function

Inputs:

- Element index: e
- Nodal coordinates matrix: $[\mathbf{x}]$
- Nodal connectivities matrix: $[\mathbf{T}^{\mathrm{n}}]$
- Material properties matrix: $[\mathbf{m}]$
- Material connectivities vector: $\{\mathbf{T}^{\mathrm{m}}\}$

Outputs:

- Element stiffness matrix: $[\mathbf{K}^*]$

Algorithm:

1. Retrieve element's vertices coordinates:
   $[\mathbf{x}^{\mathrm{el}}] = [\mathbf{x}(\mathbf{T}^{\mathrm{n}}(e,:),:)]$
2. Get bar length and orientation:
   $\Delta x = \mathbf{x}^{\mathrm{el}}(2,1) - \mathbf{x}^{\mathrm{el}}(1,1)$
   $\Delta y = \mathbf{x}^{\mathrm{el}}(2,2) - \mathbf{x}^{\mathrm{el}}(1,2)$
   $\Delta z = \mathbf{x}^{\mathrm{el}}(2,3) - \mathbf{x}^{\mathrm{el}}(1,3)$
   $l = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$
   $[\mathbf{R}] = \frac{1}{l}\begin{bmatrix} \Delta x & \Delta y & \Delta z & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta x & \Delta y & \Delta z \end{bmatrix}$
3. Retrieve element's material properties:
   $E = \mathbf{m}(\mathbf{T}^{\mathrm{m}}(e),1)$
   $A = \mathbf{m}(\mathbf{T}^{\mathrm{m}}(e),2)$
4. Compute element stiffness matrix:
   $[\mathbf{K}'] = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
   $[\mathbf{K}^*] = [\mathbf{R}]^{\mathrm{T}}[\mathbf{K}'][\mathbf{R}]$

C.3 Element force function

As an example, let's consider the structure is subjected to inertial loads—caused by an acceleration of the whole structure as a rigid body—and must also support its own weight. In such context, we need to take into account the mass of each bar in the structure and transfer the effects to the nodes. The simplest approach for that is a *lumped-mass* model in which the total mass of each bar is distributed equally among its nodes.

Inputs:

- Element index: e
- Nodal coordinates matrix: $[\mathbf{x}]$
- Nodal connectivities matrix: $[\mathbf{T}^{\mathrm{n}}]$
- Material properties matrix: $[\mathbf{m}]$
- Material connectivities vector: $\{\mathbf{T}^{\mathrm{m}}\}$
- Linear acceleration and center of mass of the structure): $\{\dot{\mathbf{V}}\}$ and $\{\mathbf{x}^{\mathrm{cm}}\}$
- Angular velocity and acceleration: $\{\boldsymbol{\Omega}\}$ and $\{\dot{\boldsymbol{\Omega}}\}$
- Coordinates of the center of mass of the structure: $\{\mathbf{x}^{\mathrm{cm}}\}$
- Acceleration of gravity: $\{\mathbf{g}\}$

Outputs:

- Element force vector: $\{\mathbf{f}^*\}$

Algorithm:

1. Retrieve element's vertices coordinates:
   $$[\mathbf{x}^{\mathrm{el}}] = [\mathbf{x}(\mathbf{T}^{\mathrm{n}}(e,:),:)]$$
2. Get bar length:
   $$\Delta x = \mathbf{x}^{\mathrm{el}}(2,1) - \mathbf{x}^{\mathrm{el}}(1,1)$$
   $$\Delta y = \mathbf{x}^{\mathrm{el}}(2,2) - \mathbf{x}^{\mathrm{el}}(1,2)$$
   $$\Delta z = \mathbf{x}^{\mathrm{el}}(2,3) - \mathbf{x}^{\mathrm{el}}(1,3)$$
   $$l = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$
3. Compute relative position of each node with respect to center of mass:
   $$\{\mathbf{r}_1\} = \{\mathbf{x}^{\mathrm{el}}(1,:)\}^{\mathrm{T}} - \{\mathbf{x}^{\mathrm{cm}}\}$$
   $$\{\mathbf{r}_2\} = \{\mathbf{x}^{\mathrm{el}}(2,:)\}^{\mathrm{T}} - \{\mathbf{x}^{\mathrm{cm}}\}$$
4. Compute rigid body acceleration of each node:
   $$\{\mathbf{a}_1\} = \{\dot{\mathbf{V}}\} + \{\dot{\boldsymbol{\Omega}}\} \times \{\mathbf{r}_1\} + \{\boldsymbol{\Omega}\} \times (\{\boldsymbol{\Omega}\} \times \{\mathbf{r}_1\})$$
   $$\{\mathbf{a}_2\} = \{\dot{\mathbf{V}}\} + \{\dot{\boldsymbol{\Omega}}\} \times \{\mathbf{r}_2\} + \{\boldsymbol{\Omega}\} \times (\{\boldsymbol{\Omega}\} \times \{\mathbf{r}_2\})$$
5. Retrieve element's material properties:
   $$\rho = \mathbf{m}(\mathbf{T}^{\mathrm{m}}(e),3)$$
   $$A = \mathbf{m}(\mathbf{T}^{\mathrm{m}}(e),2)$$
6. Compute the element force vector:
   $$\{\mathbf{f}^*\} = \frac{\rho A l}{2}\begin{Bmatrix} \{\mathbf{g}\} - \{\mathbf{a}_1\} \\ \{\mathbf{g}\} - \{\mathbf{a}_2\} \end{Bmatrix} \equiv \frac{\rho A l}{2}\begin{Bmatrix} g(1) - a_1(1) \\ g(2) - a_1(2) \\ g(3) - a_1(3) \\ g(1) - a_2(1) \\ g(2) - a_2(2) \\ g(3) - a_2(3) \end{Bmatrix}$$

C.4 Element stress function

Inputs:

- Element index: e
- Nodal coordinates matrix: $[\mathbf{x}]$
- Nodal connectivities matrix: $[\mathbf{T}^{\mathrm{n}}]$
- Material properties matrix: $[\mathbf{m}]$
- Material connectivities vector: $\{\mathbf{T}^{\mathrm{m}}\}$
- Element displacements: $\{\mathbf{u}^{\mathrm{el}}\}$

Outputs:

- Element stress: σ

Algorithm:

1. Retrieve element's vertices coordinates:
   $$[\mathbf{x}^{\mathrm{el}}] = [\mathbf{x}(\mathbf{T}^{\mathrm{n}}(e,:),:)]$$
2. Get bar length and orientation:
   $$\Delta x = \mathbf{x}^{\mathrm{el}}(2,1) - \mathbf{x}^{\mathrm{el}}(1,1)$$
   $$\Delta y = \mathbf{x}^{\mathrm{el}}(2,2) - \mathbf{x}^{\mathrm{el}}(1,2)$$
   $$\Delta z = \mathbf{x}^{\mathrm{el}}(2,3) - \mathbf{x}^{\mathrm{el}}(1,3)$$
   $$l = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$
   $$[\mathbf{R}] = \frac{1}{l}\begin{bmatrix} \Delta x & \Delta y & \Delta z & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta x & \Delta y & \Delta z \end{bmatrix}$$
3. Retrieve element's material properties:
   $$E = \mathbf{m}(\mathbf{T}^{\mathrm{m}}(e),1)$$
4. Compute element strain:
   $$\varepsilon = \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}[\mathbf{R}]\{\mathbf{u}^{\mathrm{el}}\}$$
5. Compute element stress:
   $$\sigma = E\varepsilon$$