Tensors for signal and frequency estimation in subspace-based methods: when they are useful?

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Introduction to Singular Spectrum Analysis (SSA)

Problems that can be solved by SSA-related methods:

- Signal extraction
- Frequency estimation
- Smoothing and Noise reduction
- Signal decomposition (Trend and Periodicity extraction)
- Forecasting
- Missing data imputation
- Change in structure detection
- Many others. . .

SSA Materials

Books:

- J.Elsner and A.Tsonis. Singular Spectrum Analysis: A New Tool in Time Series Analysis, Plenum, 1996.
- N.Golyandina, V.Nekrutrin and A.Zhigljavsky. Analysis of Time Series Structure: SSA and Related Techniques, CRC Press, 2001.
- S.Sanei and H.Hassani. Singular Spectrum Analysis for Biomedical Signals, CRC Press, 2016.
- N.Golyandina, A.Korobeynikov and A.Zhigljavsky. Singular spectrum analysis with R, Springer, 2018.
- N.Golyandina and A.Zhigljavsky. Singular Spectrum Analysis for Time Series, Springer, 2013, 2020 (2nd Edition).

Implementations:

- R Package: Rssa https://CRAN.R-project.org/package=Rssa
- Python Package: py-ssa-lib (less features) https://pypi.org/project/py-ssa-lib

SSA Decomposition Example

Decomposition of time series:

- Low-frequency component + high-frequency component
- Signal + noise
- \bullet Trend + Seasonality + Noise

example-image.pdf

^{*}Some data*: demonstration of series decomposition with SSA

ESPRIT Frequency Estimation Example

ESPRIT — SSA-related method for parameters estimation

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- *Pole motion data probably*
- Estimate interpretation
- Estimate interpretation

Complex Time Series

Common origins of complex-valued time series:

- Can be constructed from two related features
- Arise as a result of applying the Fourier transform to real data

SSA Algorithm: Embedding

Input: time series $X = (x_1, x_2, \dots, x_N)$, window length L, signal rank r.

Embedding. Constructing the *L-Trajectory* Hankel matrix $\mathbf{X} \in \mathbb{C}^{L \times K}$ from the series X, where K = N - L + 1:

$$\mathbf{X} = \mathcal{T}^{(L)}(\mathsf{X}) = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_K \\ x_2 & x_3 & x_4 & \dots & x_{K+1} \\ x_3 & x_4 & x_5 & \dots & x_{K+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_L & x_{L+1} & x_{L+2} & \dots & x_N \end{pmatrix}$$

SSA Algorithm: Decomposition, Grouping, Reconstruction

- ② **Decomposition**. Constructing the singular value decomposition (SVD) of matrix \mathbf{X} : $\mathbf{X} = \sum_{j=1}^{\mathrm{rank}\,\mathbf{X}} \sqrt{\lambda_j} U_j V_j^{\mathrm{H}} = \sum_{j=1}^{\mathrm{rank}\,\mathbf{X}} \widehat{\mathbf{X}}_j$ where \mathbf{H} denotes Hermitian conjugation, U_j and V_j are left and right singular vectors of \mathbf{X} , $\sqrt{\lambda_j}$ its singular values in descending order.
- **3 Grouping**. Grouping the terms $\widehat{\mathbf{X}}_j$ from the decomposition related to the signal: $\mathbf{S} = \sum_{j=1}^r \widehat{\mathbf{X}}_j = \Pi_r \mathbf{X}$, where Π_r is the projector onto the space of matrices with rank not greater than r.
- **Quantity Reconstruction**. Applying projection onto the space of Hankel matrices: $\widetilde{\mathbf{S}} = \Pi_{\mathcal{H}} \widehat{\mathbf{S}}$, and return to the series form: $\widetilde{\mathsf{S}} = \left(\mathcal{T}^{(L)}\right)^{-1}(\widetilde{\mathbf{S}})$

Series Rank

Definition

Series X has rank d < N/2, if the rank of its L-trajectory matrix equals d for any L such that $d \leq \min(L, N - L + 1)$.

If such d exists, then X is called a series of finite rank.

If the signal S is a series of finite rank, then it is generally recommended to use rank(S) as parameter r in the SSA method

Series rank examples

- rank of S with $s_n = A\sin(2\pi\omega n + \varphi)$, $0 < \omega < 1/2$, equals 2
- rank of S with $s_n = A \exp(\alpha n)$, $\alpha \in \mathbb{C}$, equals 1

Signal Model

What we consider a signal $S = (s_1, s_2, ..., s_N)$:

- The trajectory matrix $\mathbf{S} = \mathcal{T}^{(L)}(\mathsf{S})$ is rank-deficient (\Longrightarrow the time series is of some finite rank: $\mathrm{rank}(\mathsf{S}) = r$)
- Any signal S can be represented in the form of a finite sum:

$$s_n = \sum_j p_j(n) \exp(\alpha_j n + i(2\pi\omega_j n + \varphi_j)),$$

where $p_j(n)$ is a polynomial in n

• Real case:

$$s_n = \sum_j p_j(n) \exp(\alpha_j n) \sin(2\pi\omega_j n + \varphi_j),$$

ESPRIT method estimates damping factors $lpha_j$ and frequencies ω_j

ESPRIT Algorithm: General Idea

$$s_n = \sum_{j=1}^{2} \exp(\alpha_j n + i(2\pi\omega_j n + \varphi_j)) = A_1 z_1^n + A_2 z_2^n$$

where $A_j = \exp(\mathrm{i}\varphi_j)$, $z_j = \exp(\alpha_j + \mathrm{i}2\pi\omega_j)$

Signal subspace basis is given by

$$\mathbf{M} = \begin{pmatrix} z_1 & z_2 \\ z_1^2 & z_2^2 \\ \vdots & \vdots \\ z_1^L & z_2^L \end{pmatrix} \Rightarrow \overline{\mathbf{M}} = \underline{\mathbf{M}} \begin{pmatrix} z_1 \\ & z_2 \end{pmatrix} \Rightarrow \underline{\mathbf{M}}^- \overline{\mathbf{M}} = \begin{pmatrix} z_1 \\ & z_2 \end{pmatrix}$$

where M denotes M without the first row, \underline{M} — without the last \underline{M}^- denotes the pseudoinverse of \underline{M}

ESPRIT Algorithm

Input: same as in SSA: X, L, r

- **1** Embedding. $\mathbf{X} = \mathcal{T}^{(L)}(X)$
- ② Decomposition. $\mathbf{X} = \sum_{j=1}^{\mathrm{rank}\,\mathbf{X}} \sqrt{\lambda_j} U_j V_j^{\mathrm{H}}$, $\mathbf{U}_r = [U_1:U_2:\ldots:U_r]$
- **§** Estimation. Finding eigenvalues z_j of matrix $\underline{\mathbf{U}}_r^- \overline{\mathbf{U}}_r$ From $z_j = \exp(\alpha_j + \mathrm{i} 2\pi \omega_j)$ parameters α_j and ω_j can be found

Multi-Channel Time Series, MSSA

$$\mathbf{X} = \left(\mathbf{X}^{(1)}, \, \mathbf{X}^{(2)}, \, \dots, \, \mathbf{X}^{(P)} \right), \qquad \mathbf{X}^{(p)} = \left(x_1^{(p)}, \, x_2^{(p)}, \, \dots, \, x_N^{(p)} \right) - \text{channels}$$

The only change in the algorithms — Embedding step:

$$\mathbf{X} = \mathcal{T}_{\text{MSSA}}^{(L)}(\mathsf{X}) = \left[\mathbf{X}^{(1)} : \mathbf{X}^{(2)} : \dots : \mathbf{X}^{(P)}\right],$$
$$\mathbf{X}^{(p)} = \mathcal{T}^{(L)}(\mathsf{X}^{(p)})$$

When to chose MSSA over SSA for each channel:

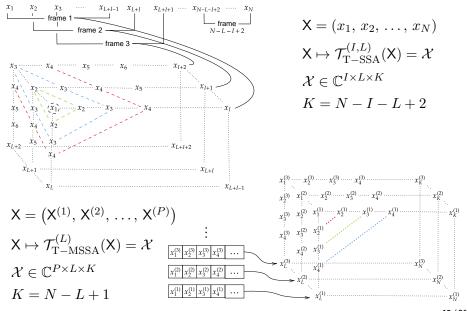
- All channels have "similar" structure
- "Supporting" channels with lower noise level

Intoduction of Tensors

Tensor SVD Extensions:

- Higher-Order SVD (HOSVD)
- Canonical Polyadic Decomposition (CPD)
- T-SVD
- $(L_r, L_r, 1)$ -Decomposition

Mapping Time Series to Tensor



Higher-Order SVD. Higher-Order Orthogonal Iterations

$$SVD(\mathbf{X}) = \sum_{j=1}^{\text{rank}(\mathbf{X})} \sqrt{\lambda_j} U_j V_j^{\text{H}}$$

$$HOSVD(\mathcal{X}) = \sum_{i=1}^{\text{rank}_1(\mathcal{X})} \sum_{l=1}^{\text{rank}_2(\mathcal{X})} \sum_{k=1}^{\text{rank}_3(\mathcal{X})} \mathcal{Z}_{ilk} U_i^{(1)} \circ U_l^{(2)} \circ U_k^{(3)}$$

•
$$\widetilde{\mathbf{X}} = \sum_{j=1}^{R} ... \Rightarrow \left\| \mathbf{X} - \widetilde{\mathbf{X}} \right\|_{F} = \min_{\text{rank}(\widehat{\mathbf{X}}) \leqslant R} \left\| \mathbf{X} - \widehat{\mathbf{X}} \right\|_{F}$$

•
$$\widetilde{\mathcal{X}} = \sum_{i=1}^{R_1} \sum_{l=1}^{R_2} \sum_{k=1}^{R_3} \ldots \Rightarrow \left\| \mathcal{X} - \widetilde{\mathcal{X}} \right\|_F \geqslant \min_{\operatorname{rank}_m(\widehat{\mathcal{X}}) \leqslant R_m} \left\| \mathcal{X} - \widehat{\mathcal{X}} \right\|_F$$

Truncation of SVD is optimal, but truncation of HOSVD is not Iterative algorithm for finding optimal approximation – HOOI

Higher-Order SSA, MSSA, ESPRIT

Input: time series X, window length: (I,L) for single-channel or L for multi-channel, signal ranks $(r_1,\,r_2,\,r_3)$, d — estimation dimension for HO-ESPRIT.

- ② Decomposition & Approximation. Using (r_1, r_2, r_3) $\mathcal{X} \mapsto \operatorname{Trunc}(\operatorname{HOSVD}(\mathcal{X})) = \widetilde{\mathcal{S}} \text{ or } \mathcal{X} \mapsto \operatorname{HOOI}(\mathcal{X}) = \widetilde{\mathcal{S}}$
- Reconstruction or Estimation.
 - Reconstruction. $S = (\mathcal{T})^{-1} \left(\Pi_{\mathcal{H}_T}(\widetilde{\mathcal{S}}) \right)$, $\Pi_{\mathcal{H}_T}$ projector onto the space of Hankel tensors
 - Estimation. Finding eigenvalues z_j of matrix $\underline{\mathbf{U}}^-\overline{\mathbf{U}}$, where $\mathbf{U} = \left[U_1^{(d)}: U_2^{(d)}: \ldots: U_{r_d}^{(d)}\right]$. From $z_j = \exp(\alpha_j + \mathrm{i} 2\pi\omega_j)$ damping factors α_j and frequencies ω_j of the signal can be found

Dtack Modifications

$$X = (x_1, x_2, ..., x_N), M = \lfloor N/D \rfloor$$

$$\operatorname{Dstack}_{D}(\mathsf{X}) = \mathcal{D}_{D}(\mathsf{X}) = \begin{bmatrix} x_{1} & x_{2} & \dots & x_{D} \\ x_{D+1} & x_{D+2} & \dots & x_{2D} \\ x_{2D+1} & x_{2D+2} & \dots & x_{3D} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(M-1)D+1} & x_{D} & x_{D} & x_{D} \end{bmatrix}$$

$$\xrightarrow{\mathsf{Dstack-SSA}} \begin{bmatrix} \mathsf{X} \mapsto \mathbf{X} = \mathcal{T}_{\mathrm{MSSA}}^{(L)} \left(\mathcal{D}_{D}(\mathsf{X}) \right) \\ \mathsf{Dstack-T-SSA} \end{bmatrix} \times \mathcal{X} \mapsto \mathcal{X} = \mathcal{T}_{\mathrm{T-MSSA}}^{(L)} \left(\mathcal{D}_{D}(\mathsf{X}) \right)$$

Undersampling:
$$\omega \mapsto \hat{\omega} = D\omega \implies \max |\omega| \leqslant \frac{1}{2D}$$

Single-Channel Case Comparison, Parameters Estimation

$$x_n = e^{\alpha_1 n} e^{2\pi i \omega_1 n} + e^{\alpha_2 n} e^{2\pi i \omega_2 n} + \zeta_n$$

 ζ_n — Complex white gaussian noise, $D(\zeta_n)=0.04^2$, $\omega_1=0.2$, $\omega_2=0.22$, $\alpha_1=\alpha_2=0$ (same results for $\alpha_1=\alpha_2<0$ and $\alpha_1<\alpha_2<0$).

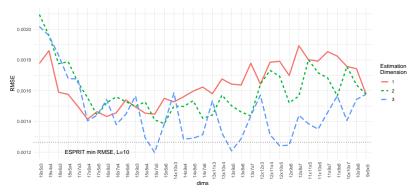


Figure: RMSE of estimates for ω_1 vs window lengths $(I \times L \times K)$

Single-Channel Case Comparison, Signal Extraction

$$x_n = e^{\alpha_1 n} e^{2\pi i \omega_1 n} + e^{\alpha_2 n} e^{2\pi i \omega_2 n} + \zeta_n$$

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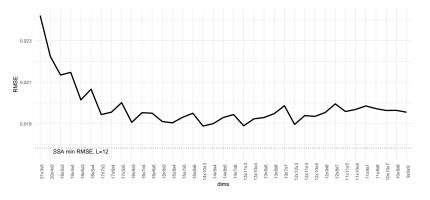


Figure: RMSE of signal estimation vs window lengths $(I \times L \times K)$

Single-Channel Case, Dstack Parameters Estimation

$$x_n=\cos(2\pi\omega_1n)+\cos(2\pi\omega_2n)+\xi_n$$

$$\omega_1=0.02,\,\omega_2=0.0205,\,\xi_n$$
 — white gaussian noise, $\mathrm{D}(\xi_n)=0.2^2$

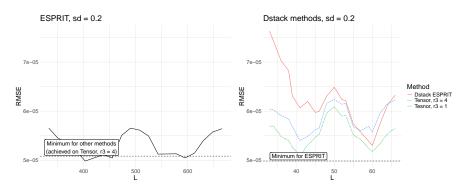


Figure: RMSE of estimates for frequencies by default ESPRIT (left) and Dstack variant (right). Low noise level case.

Single-Channel Case, Dstack Parameters Estimation

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$$\omega_1=0.02,\,\omega_2=0.0205,\,\xi_n$$
 — white gaussian noise, $\mathrm{D}(\xi_n)=0.6^2$

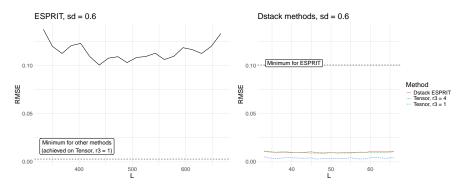


Figure: RMSE of estimates for frequencies by default ESPRIT (left) and Dstack variant (right). High noise level case.

Single-Channel Case, Dstack Signal Extraction

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$$\omega_1=0.02,\,\omega_2=0.0205,\,\xi_n$$
 — white gaussian noise, $\mathrm{D}(\xi_n)=0.2^2$

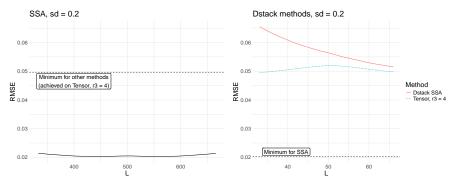


Figure: RMSE of signal estimate by default SSA (left) and Dstack variant (right).

Multi-Channel Case, Parameters Estimation

$$x_n^{(m)}=a_1^{(m)}e^{2\pi\mathrm{i}\omega_1n}+a_2^{(m)}e^{2\pi\mathrm{i}\omega_2n}+\zeta_n^{(m)},$$
 $\zeta_n^{(m)}$ — Complex white gaussian noise, $\mathrm{D}\left(\zeta_n^{(m)}\right)=0.2^2$,
$$\omega_1=0.2,\,\omega_2=0.22$$

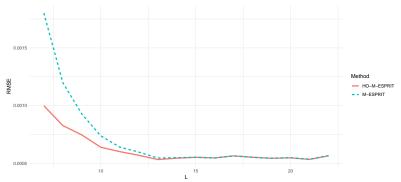


Figure: RMSE of estimates for ω_1 . vs window length L.

Multi-Channel Case, Signal Extraction

$$x_n^{(m)}=a_1^{(m)}e^{2\pi\mathrm{i}\omega_1n}+a_2^{(m)}e^{2\pi\mathrm{i}\omega_2n}+\zeta_n^{(m)},$$
 $\zeta_n^{(m)}$ — Complex white gaussian noise, $\mathrm{D}\left(\zeta_n^{(m)}\right)=0.2^2,$ $\omega_1=0.2,~\omega_2=0.22$

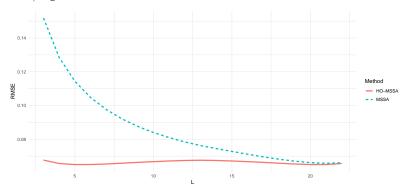


Figure: RMSE of estimates for ω_1 . vs window length L.

Results and Fututre Work

Results:

- Majority of tensor-based methods have lower precision than their matrix-based counterparts for single-channel problems
- Exception is estimation of parameters for components with close frequencies in presence of high-level noise using Dstack
- For multi-channel problems tensor-based methods have generally higher precision but the difference between minimal errors is low

Future Work:

- Trying other tensor decompositions
- Implementation of tensor modifications for other SSA-based methods
- ...

Algorithms Complexities

$$\mathcal{X} \in \mathbb{C}^{I \times L \times K}$$
, $\mathbf{X} \in \mathbb{C}^{\hat{L} \times \hat{K}}$, $I < L < K$, $\hat{L} < \hat{K}$, $I + L + K = N + 2$, $\hat{L} + \hat{K} = N + 1$

- SVD(X): $O(\hat{L}^2\hat{K})$, or $O(r\hat{L}\hat{K})$ if only need r-rank approximation, or $O(rN\log(N))$ if X is Hankel
- HOSVD(\mathcal{X}): O(ILKN), or $O(ILK(r_1+r_2+r_3))$ if only need (r_1,r_2,r_3) -rank approximation, or $O((r_1+r_2+r_3)I(L+K)\log(L+K))$ if \mathcal{X} is Hankel
- HOOI-SSA:

$$O(r_1r_2r_3(I+L+J)),$$

with linear convergence. For precision level of ε :

$$O\left(ILJ(r_1+r_2+r_3)+\frac{1}{\varepsilon}r_1r_2r_3(I+L+J)\right)$$