TENSORS FOR SIGNAL AND FREQUENCY ESTIMATION IN SUBSPACE-BASED METHODS: WHEN THEY ARE USEFUL?

N.A. Khromov[1], N.E. Golyandina[2]

[1 2] *St. Petersburg State University*
St. Petersburg, Russia
e-mail: [1]hromovn@mail.ru, [2]n.golyandina@spbu.ru

Tensor modifications of singular spectrum analysis for signal extraction and frequency estimation problems in a noisy sum of exponentially modulated sinusoids are reviewed. Modifications using Higher-Order SVD are considered. Numerical comparisons are carried out. It is shown numerically that for the signal extraction problem, tensor methods are inferior to matrix methods in most cases for a single-channel series, but can outperform multichannel SSA for a series system. For frequency estimation, tensor modifications are generally advantageous.

Keywords: time series, signal, frequency estimation, tensor, singular spectrum analysis

1 Introduction

One of the methods for time series analysis is singular spectrum analysis (SSA) [1], in which the original time series is transformed into a matrix, called trajectory matrix, by a given window length L and then the singular value decomposition (SVD) of this matrix is analyzed. If the task is to estimate the signal and its properties from the observed noisy series, the first r components of the SVD are considered, where r is the rank of the signal trajectory matrix. Based on the selected components, the signal estimation is constructed. A distinctive feature of the method is that it does not require the specification of a signal model. However, at the same time SSA allows to handle a parametric model of the signal in the form of a sum of products of polynomials, exponents and sinusoids. A special role is played by the frequency estimation problem. Based on the estimation of the signal subspace using the first r left singular vectors of the trajectory matrix SVD, the ESPRIT method estimates the frequencies present in the signal. The LS version of ESPRIT [2] is also called HSVD, and the TLS version [3] is called HTLS.

A number of works propose tensor modifications of the SSA and ESPRIT methods, where the original series is transformed into a tensor, usually of 3rd order, instead of a matrix. One of the common variants of tensor decompositions is Higher-Order SVD (HO-SVD), which generalizes the matrix SVD.

The purpose of this work is to numerically compare tensor and matrix modifications of SSA for solving signal extraction and frequency estimation problems. We will consider the tensor modifications proposed in [4] and [5], extended for signal extraction.

2 Methods description

2.1 Tensor SSA algorithm layout for signal extraction

The general structure of tensor SSA algorithms based on HO-SVD is as follows (basic SSA is its special case). Let X be the observed object. The tensor dimensions I, L and K are considered as the window length, where some of them are expressed through others or fixed. The parameters of the algorithm are the values R_1, R_2 and R_3. They are frequently chosen to be equal to r, for example, but not always.

1. Embedding $\mathbf{X} = \mathcal{T}(\mathsf{X})$ — trajectory tensor.
2. Decomposition $\mathbf{X} = \sum_{i=1}^{I} \sum_{l=1}^{J} \sum_{k=1}^{K} \mathcal{Z}_{ilk} U_i^{(1)} \circ U_l^{(2)} \circ U_k^{(3)}$.
3. Grouping $\hat{\mathbf{X}} = \sum_{i=1}^{R_1} \sum_{l=1}^{R_2} \sum_{k=1}^{R_3} \mathcal{Z}_{ilk} U_i^{(1)} \circ U_l^{(2)} \circ U_k^{(3)}$.
4. Obtaining from $\hat{\mathbf{X}}$ the signal estimate $\hat{\mathsf{X}}$ based on the structure of the trajectory tensor and the operation inverse to embedding.

We will further consider two variants of input objects: single-channel and multi-channel time series.

2.2 Trajectory tensors

Let $\mathsf{X} = (x_1, x_2, \ldots, x_N)$ be a single-channel time series of length N, $x_n \in \mathbb{C}$.

Definition 1. The tensor embedding operator for a single-channel time series with window lengths I and L such that $1 < I, L < N$, $I + L < N + 1$ is a mapping $\mathcal{T}_{I,L}$ that translates the series X into the tensor $\mathcal{X} \in \mathbb{C}^{I \times L \times K}$ as follows $\mathcal{X}_{ilk} = x_{i+l+k-2}$, where $i \in \overline{1:I}$, $l \in \overline{1:L}$, $k \in \overline{1:K}$, $K = N - I - L + 2$.

Let $\mathsf{X} = (\mathsf{X}^{(1)}, \mathsf{X}^{(2)}, \ldots, \mathsf{X}^{(P)})$ be a multi-channel time series consisting of P single-channel series, also called channels.

Definition 2. The tensor embedding operator for a multi-channel time series with window length L such that $1 < L < N$ is a mapping $\mathcal{T}_L$ that translates the P-channel time-series X into the tensor $\mathcal{X} \in \mathbb{C}^{L \times K \times P}$ $(K = N - L + 1)$ as follows $\mathcal{X}_{lkp} = x_{l+k-1}^{(p)}$, where $l \in \overline{1:L}$, $k \in \overline{1:K}$, $p \in \overline{1:P}$.

2.3 Algorithm for signal parameters estimation.

Consider the P-channel time series (including $P = 1$) with elements

$$x_n^{(p)} = \sum_{r=1}^{R} a_r^{(p)} e^{\alpha_r n} e^{\mathrm{i}\left(2\pi\omega_r n + \varphi_r^{(p)}\right)},$$

where the model parameters are the amplitudes $a_r^{(p)} \in \mathbb{C} \setminus \{0\}$, phases $\varphi_r^{(p)} \in [0, 2\pi)$, frequencies $\omega_r \in [0, 1/2]$, and damping factors $\alpha_r \in \mathbb{R}$. The HO-ESPRIT algorithm that estimates the frequencies and damping factors of a time series is defined as follows. After the embedding step the matrix $\mathbf{U} = \mathbf{U}_d = \left[U_1^{(d)} : U_2^{(d)} : \ldots : U_{R_d}^{(d)}\right]$ for $d \in \{1, 2, 3\}$ is constructed and the following matrix equation

$$\mathbf{U}^{\uparrow} = \mathbf{U}_{\downarrow}\mathbf{Z}$$

is solved with respect to matrix $\mathbf{Z}$, where the up and down arrows placed behind the matrix $\mathbf{U}$ stand for deleting its first and last rows accordingly. The R largest eigenvalues of the matrix $\mathbf{Z}$ are considered to be the estimates of poles $\lambda_r = e^{\alpha_r + 2\pi \mathrm{i} \omega_r}$, from which the parameters α_r and ω_r can be obtained.

2.4 Dstack modifications

In the paper [5], to improve the speed of the method, it is proposed to transform a single-channel series into a multi-channel series before applying the tensor modification: $x_m^{(d)} = x_{(m-1)D+d}$, where $m \in \overline{1 : (N/D)}$. In that paper only the ESPRIT modification called HTLSDstack is considered, but we will apply this time series transformation for the signal estimation problem as well, and will call the resulting method SSADstack. Tensor modifications are constructed as for a multichannel series.

3 Comparison of tensor methods with matrix methods

All numerical comparisons were made for time series expressed as a sum of sinusoids.

The following methods were compared for single channel time series and signal extraction problem: SSA, HO-SSA, SSADstack, HO-SSADstack with $R_3 = r$ and HO-SSADstack with $R_3 = 1$. It was shown that in most cases the SSA method significantly outperforms other methods in terms of accuracy, and when it is less accurate, the difference is insignificant and is present only in a very narrow range of parameters, which makes this small advantage unrealizable in actual practice. Among Dstack methods the most accurate are SSADstack and HO-SSADstack with $R_3 = r$ with small difference in accuracy.

For single-channel time series and frequency estimation problem, a signal in the form of two sinusoids with close frequencies was considered. The ESPRIT, HO-ESPRIT, HTLSDstack, HO-HTLSDstack with $R_3 = r$ and HO-HTLSDstack with $R_3 = 1$ methods were compared. It was obtained that at low noise level ESPRIT performs more accurately, but at medium and high noise level HO-ESPRIT becomes more accurate with optimal parameter selection, and HO-HTLSDstack with $R_3 = 1$ outperforms all methods.

For multi-channel time series, it was shown that in the case when all the channels of the series are expressed as a sum of sinusoids, and the frequencies presented in each channel are equal, tensor modifications give more accurate results, both in the signal extraction problem and in the frequency estimation problem.

4 Conclusion

The performed numerical comparisons showed different effects of the tensor HO-SVD modifications for different time series. For signal extraction of a single-channel time series, the basic matrix method is unambiguously more accurate. For multi-channel time series with equal set of frequencies in the channels and for the frequency estimation problem, the tensor methods can give an advantage in accuracy.

References

[1] Golyandina N.E., Nekrutkin V.V., Zhigljavsky A.A. (2001). *Analysis of Time Series Structure*. Chapman and Hall/CRC: Boca Raton.

[2] Roy R., Paulraj A., Kailath T. (1986). ESPRIT-A subspace rotation approach to estimation of parameters of cisoids in noise. *IEEE Transactions on Acoustics, Speech, and Signal Processing*. Vol. **34**, Num. **5**, pp. 1340-1342.

[3] Roy R., Kailath T. (1989). ESPRIT-estimation of signal parameters via rotational invariance techniques. *IEEE Transactions on Acoustics, Speech, and Signal Processing*. Vol. **37**, Num. **7**, pp. 984-995.

[4] Papy J.M., De Lathauwer L., Van Huffel S. (2005). Exponential data fitting using multilinear algebra: the single-channel and multi-channel case. *Linear Algebra with Applications*. Vol. **12**, Num. **8**, pp. 809-826.

[5] Papy J.M., De Lathauwer L., Van Huffel S. (2009). Exponential data fitting using multilinear algebra: the decimative case. *Journal of Chemometrics*. Vol. **23**, Num. **7-8**, pp. 341-351s.