

Tensors for signal and frequency estimation in subspace-based methods: when they are useful?

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Introduction to Singular Spectrum Analysis (SSA)

SSA - family of methods for time series analysis

Problems that can be solved by SSA-related methods:

- Signal extraction
- Frequency estimation
- Smoothing and Noise reduction
- Signal decomposition (Trend and Periodicity extraction)
- Forecasting
- Missing data imputation
- Change in structure detection
- Many others. . .

SSA References

Books:

- J.Elsner and A.Tsonis. Singular Spectrum Analysis: A New Tool in Time Series Analysis, Plenum, 1996.
- N.Golyandina, V.Nekrutrin and A.Zhigljavsky. Analysis of Time Series Structure: SSA and Related Techniques, CRC Press, 2001.
- S.Sanei and H.Hassani. Singular Spectrum Analysis for Biomedical Signals, CRC Press, 2016.
- N.Golyandina, A.Korobeynikov and A.Zhigljavsky. Singular spectrum analysis with R, Springer, 2018.
- N.Golyandina and A.Zhigljavsky. Singular Spectrum Analysis for Time Series, Springer, 2013, 2020 (2nd Edition).

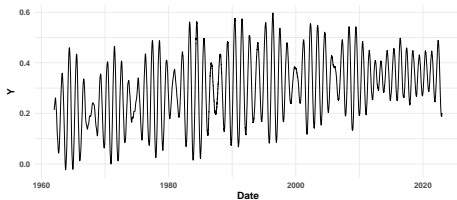
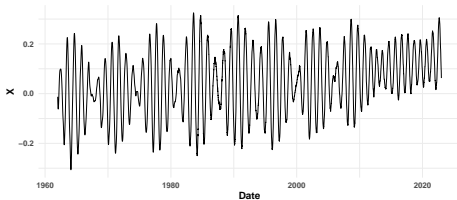
Implementations:

- R Package: Rssa
<https://CRAN.R-project.org/package=Rssa>
- Python Package: PyRssa (Python wrapper over Rssa)
<https://pypi.org/project/pyrssa/>

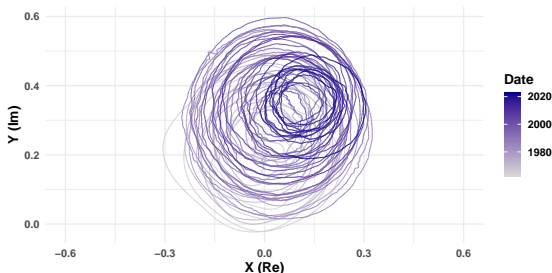
Decomposition and Estimation Example

Data: Coordinates of Earth pole motion [IERS EOP 14 C04]

Raw data:



As complex time series:

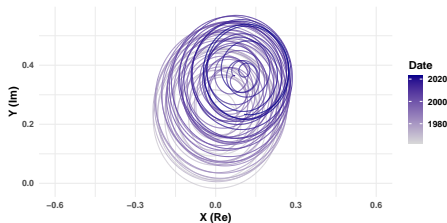


Decomposition and Estimation Example

Decomposition of time series:

- Low-frequency component + high-frequency component
- Signal + noise
- Trend + Seasonality + Noise

Extracted signal:



Signal parameters estimates:

Period (Days)	Change rate
365.41	$-5.5 \cdot 10^{-6}$
433.10	$-2.2 \cdot 10^{-5}$
$\rightarrow \infty$	$2.7 \cdot 10^{-5}$

SSA Algorithm: Embedding

Input: time series $\mathbf{X} = (x_1, x_2, \dots, x_N)$, window length L , signal rank r .

- ① **Embedding.** Constructing the L -Trajectory Hankel matrix $\mathbf{X} \in \mathbb{C}^{L \times K}$ from the series \mathbf{X} , where $K = N - L + 1$:

$$\mathbf{X} = \mathcal{T}^{(L)}(\mathbf{X}) = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_K \\ x_2 & x_3 & x_4 & \dots & x_{K+1} \\ x_3 & x_4 & x_5 & \dots & x_{K+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_L & x_{L+1} & x_{L+2} & \dots & x_N \end{pmatrix}$$

Why not embed into a higher order array (tensor)?

SSA Algorithm: Decomposition, Grouping, Reconstruction

- ② **Decomposition.** Constructing the singular value decomposition (SVD) of the matrix \mathbf{X} : $\mathbf{X} = \sum_{j=1}^{\text{rank } \mathbf{X}} \sqrt{\lambda_j} U_j V_j^H = \sum_{j=1}^{\text{rank } \mathbf{X}} \hat{\mathbf{X}}_j$ where H denotes Hermitian conjugation, U_j and V_j are left and right singular vectors of \mathbf{X} , $\sqrt{\lambda_j}$ — its singular values in descending order.
- ③ **Grouping.** Grouping the terms $\hat{\mathbf{X}}_j$ from the decomposition related to the signal: $\mathbf{S} = \sum_{j=1}^r \hat{\mathbf{X}}_j = \Pi_r \mathbf{X}$, where Π_r is the projector onto the space of matrices with rank not greater than r .
- ④ **Reconstruction.** Applying projection onto the space of Hankel matrices: $\tilde{\mathbf{S}} = \Pi_{\mathcal{H}} \hat{\mathbf{S}}$, and return to the series form: $\tilde{\mathbf{S}} = (\mathcal{T}^{(L)})^{-1} (\tilde{\mathbf{S}})$

Definition

Series X has rank $d < N/2$ in terms of SSA, if the rank of its L -trajectory matrix equals d for any L such that $d \leq \min(L, N - L + 1)$.

If such d exists, then X is called a series of finite rank.

If the signal S is a series of finite rank, then it is generally recommended to use $\text{rank}(S)$ as parameter r in the SSA method

Series rank examples

- rank of S with $s_n = A \exp(\alpha n)$, $\alpha \in \mathbb{C}$, equals 1
- rank of S with $s_n = A \sin(2\pi\omega n + \varphi)$, $0 < \omega < 1/2$, equals 2

Signal Model

What we consider a signal $S = (s_1, s_2, \dots, s_N)$:

- The trajectory matrix $\mathbf{S} = \mathcal{T}^{(L)}(S)$ is rank-deficient (\implies the time series is of some finite rank: $\text{rank}(\mathbf{S}) = r$)
- Any signal S can be represented in the form of a finite sum:

$$s_n = \sum_j p_j(n) \exp(\alpha_j n + i2\pi\omega_j n),$$

where $p_j(n)$ is a polynomial in n

- Real case:

$$s_n = \sum_j p_j(n) \exp(\alpha_j n) \sin(2\pi\omega_j n + \varphi_j),$$

ESPRIT method estimates factors α_j and frequencies ω_j

ESPRIT Algorithm: General Idea

Consider a signal S with elements s_n :

$$s_n = \sum_{j=1}^2 \exp(\alpha_j n + i(2\pi\omega_j n + \varphi_j)) = A_1 z_1^n + A_2 z_2^n$$

where $A_j = \exp(i\varphi_j)$, $z_j = \exp(\alpha_j + i2\pi\omega_j)$

Signal subspace basis is given by

$$\mathbf{M} = \begin{pmatrix} z_1 & z_2 \\ z_1^2 & z_2^2 \\ \vdots & \vdots \\ z_1^L & z_2^L \end{pmatrix} \Rightarrow \overline{\mathbf{M}} = \underline{\mathbf{M}} \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix} \Rightarrow \underline{\mathbf{M}}^- \overline{\mathbf{M}} = \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix}$$

where $\overline{\mathbf{M}}$ denotes \mathbf{M} without the first row, $\underline{\mathbf{M}}$ — without the last row
 $\underline{\mathbf{M}}^-$ denotes the pseudoinverse of $\underline{\mathbf{M}}$

Input: same as in SSA: \mathbf{X} , L , r

① **Embedding.** $\mathbf{X} = \mathcal{T}^{(L)}(\mathbf{X})$

② **Decomposition.** $\mathbf{X} = \sum_{j=1}^{\text{rank } \mathbf{X}} \sqrt{\lambda_j} U_j V_j^H$, $\mathbf{U}_r = [U_1 : U_2 : \dots : U_r]$

③ **Estimation.** Finding eigenvalues z_j of matrix $\underline{\mathbf{U}}_r^- \overline{\mathbf{U}}_r$

From $z_j = \exp(\alpha_j + i2\pi\omega_j)$ parameters α_j and ω_j can be found

Multi-Channel Time Series, MSSA

$$\mathbf{X} = (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(P)}), \quad \mathbf{X}^{(p)} = (x_1^{(p)}, x_2^{(p)}, \dots, x_N^{(p)}) - \text{channels}$$

The only change in the algorithms — Embedding step:

$$\mathbf{X} = \mathcal{T}_{\text{MSSA}}^{(L)}(\mathbf{X}) = [\mathbf{X}^{(1)} : \mathbf{X}^{(2)} : \dots : \mathbf{X}^{(P)}],$$

$$\mathbf{X}^{(p)} = \mathcal{T}^{(L)}(\mathbf{X}^{(p)})$$

When to choose MSSA over SSA for each channel:

- All channels have “similar” structure
- “Supporting” channels with lower noise level

MSSA Trajectory Matrix

$$\mathbf{X} = (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(P)}), \quad \mathbf{X}^{(p)} = (x_1^{(p)}, x_2^{(p)}, \dots, x_N^{(p)})$$

$$\mathbf{X} = \mathcal{T}_{\text{MSSA}}^{(L)}(\mathbf{X}) =$$

$$\left(\underbrace{\begin{pmatrix} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & \dots & x_K^{(1)} \\ x_2^{(1)} & x_3^{(1)} & x_4^{(1)} & \dots & x_{K+1}^{(1)} \\ x_3^{(1)} & x_4^{(1)} & x_5^{(1)} & \dots & x_{K+2}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_L^{(1)} & x_{L+1}^{(1)} & x_{L+2}^{(1)} & \dots & x_N^{(1)} \end{pmatrix}}_{\mathbf{X}^{(1)}} \left| \underbrace{\begin{pmatrix} x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & \dots & x_K^{(2)} \\ x_2^{(2)} & x_3^{(2)} & x_4^{(2)} & \dots & x_{K+1}^{(2)} \\ x_3^{(2)} & x_4^{(2)} & x_5^{(2)} & \dots & x_{K+2}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_L^{(2)} & x_{L+1}^{(2)} & x_{L+2}^{(2)} & \dots & x_N^{(2)} \end{pmatrix}}_{\mathbf{X}^{(2)}} \right| \dots \right)$$

But why limit yourself to matrices?
Matrix is just a 2D tensor

Introducing Tensors to the Algorithm

Basic SSA: Time series $\mathbf{X} \mapsto$ Matrix $\mathbf{X} \mapsto \text{SVD}(\mathbf{X})$
Tensor SSA: Time series $\mathbf{X} \mapsto$ Tensor $\mathcal{X} \mapsto \underbrace{\text{TD}(\mathcal{X})}_{\text{Some Tensor Decomposition}}$

Tensor SVD Extensions:

- Higher-Order SVD (HOSVD)
- Canonical Polyadic Decomposition (CPD)
- T-SVD
- $(L_r, L_r, 1)$ -Decomposition

Reminder: Trajectory Matrix

- Single-Channel:

$$\mathcal{T}^{(L)}(\mathbf{X}) = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_K \\ x_2 & x_3 & x_4 & \dots & x_{K+1} \\ x_3 & x_4 & x_5 & \dots & x_{K+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_L & x_{L+1} & x_{L+2} & \dots & x_N \end{pmatrix}$$

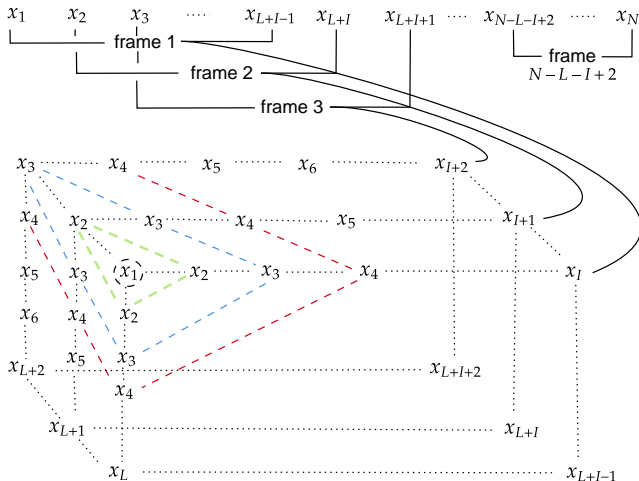
- Multi-Channel:

$$\mathcal{T}_{\text{MSSA}}^{(L)}(\mathbf{X}) = \left(\begin{array}{cccc|cccc|c} x_1^{(1)} & x_2^{(1)} & \dots & x_K^{(1)} & x_1^{(2)} & x_2^{(2)} & \dots & x_K^{(2)} & \dots \\ x_2^{(1)} & x_3^{(1)} & \dots & x_{K+1}^{(1)} & x_2^{(2)} & x_3^{(2)} & \dots & x_{K+1}^{(2)} & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ x_L^{(1)} & x_{L+1}^{(1)} & \dots & x_N^{(1)} & x_L^{(2)} & x_{L+1}^{(2)} & \dots & x_N^{(2)} & \dots \end{array} \right)$$

Mapping Single-Channel Time Series to a Tensor

$$\mathbf{X} = (x_1, x_2, \dots, x_N)$$

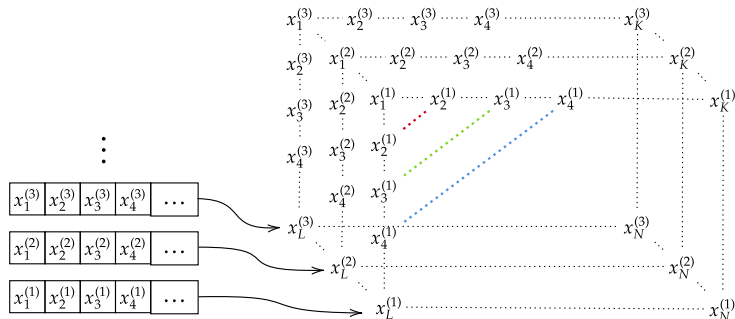
$$\mathbf{X} \mapsto \mathcal{T}_{\text{T-SSA}}^{(I,L)}(\mathbf{X}) = \mathcal{X} \in \mathbb{C}^{I \times L \times K}, K = N - I - L + 2$$



Mapping Multi-Channel Time Series to a Tensor

$$\mathbf{X} = (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(P)}), \mathbf{X}^{(p)} = (x_1^{(p)}, x_2^{(p)}, \dots, x_N^{(p)})$$

$$\mathbf{X} \mapsto \mathcal{T}_{\text{T-MSSA}}^{(L)}(\mathbf{X}) = \mathcal{X} \in \mathbb{C}^{P \times L \times K}, K = N - L + 1$$



Some Tensor Decompositions

Unlike in matrix case, there exist several definitions of tensor ranks and SVD generalizations based on them.

Definition (Tensor rank)

Tensor \mathcal{A} has rank 1, if there exist vectors B , C and D such that $\mathcal{A} = B \circ C \circ D$, where \circ denotes an outer product.

Tensor \mathcal{A} has rank R , if it can be represented as a sum of R rank-1 tensors: $\mathcal{A} = \sum_{i=1}^R \mathcal{B}_i$, $\text{rank}(\mathcal{B}_i) = 1$, and such R is minimal.

The representation of a tensor as a sum of R rank-1 tensors is called **Canonical Polyadic Decomposition (CPD)**.

- Considered for signal extraction in [Kouchaki, Sanei (2013)]
- Does not provide any form of orthogonality of components
- Requires to know the tensor rank in advance
- No connection between signal SSA-rank and the rank of a trajectory tensor

Some tensor decompositions

Definition

n -mode vectors of a tensor \mathcal{A} are vectors obtained from \mathcal{A} by varying the index of the n -th direction and keeping the other indices fixed (analog of rows and columns of a matrix).

n -rank of a tensor \mathcal{A} , denoted by $R_n = \text{rank}_n(\mathcal{A})$, is the dimension of the linear space spanned by the n -mode vectors.

HOSVD: $\mathcal{A} = \sum_{i=1}^{R_1} \sum_{l=1}^{R_2} \sum_{k=1}^{R_3} z_{ilk} U_i^{(1)} \circ U_l^{(2)} \circ U_k^{(3)}$

- Considered for signal parameter estimation in [Papy et al. (2005)], [Papy et al. (2009)]
- Provides orthogonality of components
- Does not require any prior knowledge about tensor n -ranks
- There is a proven connection between number of components and signal SSA-rank

Higher-Order SVD. Higher-Order Orthogonal Iterations

$$\text{SVD}(\mathbf{X}) = \sum_{j=1}^{\text{rank}(\mathbf{X})} \sqrt{\lambda_j} U_j V_j^H$$

$$\text{HOSVD}(\mathcal{X}) = \sum_{i=1}^{\text{rank}_1(\mathcal{X})} \sum_{l=1}^{\text{rank}_2(\mathcal{X})} \sum_{k=1}^{\text{rank}_3(\mathcal{X})} z_{ilk} U_i^{(1)} \circ U_l^{(2)} \circ U_k^{(3)}$$

- $\tilde{\mathbf{X}} = \sum_{j=1}^R \dots \Rightarrow \left\| \mathbf{X} - \tilde{\mathbf{X}} \right\|_F = \min_{\text{rank}(\hat{\mathbf{X}}) \leq R} \left\| \mathbf{X} - \hat{\mathbf{X}} \right\|_F$
- $\tilde{\mathcal{X}} = \sum_{i=1}^{R_1} \sum_{l=1}^{R_2} \sum_{k=1}^{R_3} \dots \Rightarrow \left\| \mathcal{X} - \tilde{\mathcal{X}} \right\|_F \geq \min_{\text{rank}_m(\hat{\mathcal{X}}) \leq R_m} \left\| \mathcal{X} - \hat{\mathcal{X}} \right\|_F$

Truncation of SVD is optimal, but truncation of HOSVD is not

Iterative algorithm for finding optimal approximation – HOOI

T-SSA, T-MSSA and T-ESPRIT with HOSVD

Input: time series X , window length: (I, L) for single-channel or L for multi-channel, signal ranks (r_1, r_2, r_3) , d — estimation dimension for HO-ESPRIT.

① Embedding.	Single-channel	$X \mapsto \mathcal{T}_{\text{T-SSA}}^{(I,L)}(X) = \mathcal{X}$
	Multi-channel	$X \mapsto \mathcal{T}_{\text{T-MSSA}}^{(L)}(X) = \mathcal{X}$

② **Decomposition & Approximation.** Using (r_1, r_2, r_3)
 $\mathcal{X} \mapsto \text{Trunc}(\text{HOSVD}(\mathcal{X})) = \tilde{\mathcal{S}}$ or $\mathcal{X} \mapsto \text{HOOI}(\mathcal{X}) = \tilde{\mathcal{S}}$

③ **Reconstruction or Estimation.**

- **Reconstruction.** $S = \mathcal{T}^{-1} \left(\Pi_{\mathcal{H}_T}(\tilde{\mathcal{S}}) \right)$, $\Pi_{\mathcal{H}_T}$ — projector onto the space of Hankel tensors
- **Estimation.** Finding eigenvalues z_j of matrix $\underline{U}^{-} \overline{U}$, where $\underline{U} = \underline{U}_d = \left[U_1^{(d)} : U_2^{(d)} : \dots : U_{r_d}^{(d)} \right]$. From $z_j = \exp(\alpha_j + i2\pi\omega_j)$ factors α_j and frequencies ω_j of the signal can be found

Single-Channel Series Comparison

$$x_n = e^{\alpha_1 n} e^{2\pi i \omega_1 n} + e^{\alpha_2 n} e^{2\pi i \omega_2 n} + \zeta_n$$

ζ_n — Complex white gaussian noise, $D(\zeta_n) = 0.04^2$, $\omega_1 = 0.2$, $\omega_2 = 0.22$, $\alpha_1 = \alpha_2 = 0$ (same results for $\alpha_1 = \alpha_2 < 0$ and $\alpha_1 < \alpha_2 < 0$).

RMSE results with respect to parameters choice:

- Parameters estimation

	Best	Worst	Mean
T-ESPRIT	0.0012	0.0033	0.0016
ESPRIT	0.0013	0.0140	0.0031

T-method is better

- Signal extraction

	Best	Worst	Mean
T-SSA	0.019	0.024	0.020
SSA	0.018	0.031	0.021

T-method is worse with optimal parameter choice but better on average

Multi-Channel Series Comparison

$$x_n^{(m)} = a_1^{(m)} e^{2\pi i \omega_1 n} + a_2^{(m)} e^{2\pi i \omega_2 n} + \zeta_n^{(m)},$$

$\zeta_n^{(m)}$ — Complex white gaussian noise, $D(\zeta_n^{(m)}) = 0.2^2$,

$\omega_1 = 0.2, \omega_2 = 0.22$

RMSE results with respect to parameters choice:

- Parameters estimation

	Best	Worst	Mean
T-M-ESPRIT	5.33e-04	9.99e-04	6.16e-04
M-ESPRIT	5.36e-04	1.80e-03	7.13e-04

T-method is better

- Signal extraction

	Best	Worst	Mean
T-MSSA	0.065	0.068	0.066
MSSA	0.066	0.152	0.086

T-method is better

Dstack Modifications

Possible problem for ESPRIT: components with close frequencies can mix into one in the presence of noise. Solution: using Dstack mapping.

Consider $\mathbf{X} = (x_1, x_2, \dots, x_N)$, $M = \lfloor N/D \rfloor$, then

$$\text{Dstack}_D(\mathbf{X}) = \mathcal{D}_D(\mathbf{X}) = \left[\begin{array}{c|c|c|c} x_1 & x_2 & \dots & x_D \\ x_{D+1} & x_{D+2} & \dots & x_{2D} \\ x_{2D+1} & x_{2D+2} & \dots & x_{3D} \\ \vdots & \vdots & \dots & \vdots \\ x_{(M-1)D+1} & x_{(M-1)D+2} & \dots & x_{MD} \end{array} \right]$$

$\underbrace{\hspace{10em}}_{\mathbf{X}_D^{(1)}}$

$\underbrace{\hspace{10em}}_{\mathbf{X}_D^{(2)}}$

$\underbrace{\hspace{10em}}_{\mathbf{X}_D^{(D)}}$

Dstack-SSA	$\mathbf{X} \mapsto \mathbf{X} = \mathcal{T}_{\text{MSSA}}^{(L)}(\mathcal{D}_D(\mathbf{X}))$
Dstack-T-SSA	$\mathbf{X} \mapsto \mathcal{X} = \mathcal{T}_{\text{T-MSSA}}^{(L)}(\mathcal{D}_D(\mathbf{X}))$

Undersampling: $\omega \mapsto \hat{\omega} = D\omega \implies |\omega| \leq \frac{1}{2D}$ is required

Single-Channel Series Comparison with Dstack

$$x_n = \cos(2\pi\omega_1 n) + \cos(2\pi\omega_2 n) + \xi_n$$

$\omega_1 = 0.02$, $\omega_2 = 0.0205$, ξ_n — white gaussian noise, $D(\xi_n) = \sigma^2$

RMSE results with respect to parameters choice:

- Parameters estimation, $\sigma = 0.2$

	Best	Worst	Mean
ESPRIT	4.98e-05	5.66e-05	5.32e-05
Dstack-ESPRIT	5.30e-05	7.63e-05	6.24e-05
T-Dstack-ESPRIT	5.40e-05	6.24e-05	5.86e-05

T-method
is worse

- Parameters estimation, $\sigma = 0.6$

	Best	Worst	Mean
ESPRIT	0.101	0.138	0.115
Dstack-ESPRIT	0.009	0.011	0.010
T-Dstack-ESPRIT	0.003	0.005	0.004

T-method is better

Single-Channel Series Comparison with Dstack

$$x_n = \cos(2\pi\omega_1 n) + \cos(2\pi\omega_2 n) + \xi_n$$

$\omega_1 = 0.02$, $\omega_2 = 0.0205$, ξ_n — white gaussian noise, $D(\xi_n) = \sigma^2$

RMSE results with respect to parameters choice:

- Signal extraction, $\sigma = 0.2$

	Best	Worst	Mean
SSA	0.020	0.021	0.021
Dstack-SSA	0.052	0.066	0.057
T-Dstack-SSA	0.050	0.052	0.051

T-method is worse

- Signal extraction, $\sigma = 0.6$

	Best	Worst	Mean
SSA	0.086	0.095	0.090
Dstack-SSA	0.175	0.208	0.189
T-Dstack-SSA	0.175	0.179	0.178

T-method is worse

Results:

- Tensor-based methods are generally less dependent on the parameters choice, and has lower minimal and maximal RMSEs except for single-channel signal extraction case, where basic SSA has lower minimal RMSE
- Mean and median RMSE are close for both methods in the single-channel case but in the multi-channel case tensor methods have advantage
- Tensor methods with applying Dstack mapping can be useful for estimation of parameters in presence of strong noise

Future Work:

- Trying other tensor decompositions (CPD, T-SVD)
- Implementation of tensor modifications for other SSA-based methods
- ...

$$\mathcal{X} \in \mathbb{C}^{I \times L \times K}, \mathbf{X} \in \mathbb{C}^{\hat{L} \times \hat{K}}, I < L < K, \hat{L} < \hat{K}, \\ I + L + K = N + 2, \hat{L} + \hat{K} = N + 1$$

- **SVD**(\mathbf{X}):
 $O(\hat{L}^2 \hat{K})$,
or $O(r \hat{L} \hat{K})$ if only need r -rank approximation,
or $O(rN \log(N))$ if \mathbf{X} is Hankel
- **HOSVD**(\mathcal{X}):
 $O(ILKN)$,
or $O(ILK(r_1 + r_2 + r_3))$ if only need (r_1, r_2, r_3) -rank approximation,
or $O((r_1 + r_2 + r_3)I(L + K) \log(L + K))$ if \mathcal{X} is Hankel