TENSORS FOR SIGNAL AND FREQUENCY ESTIMATION IN SUBSPACE-BASED METHODS: WHEN THEY ARE USEFUL?

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Tensor modifications of singular spectrum analysis for signal extraction and frequency estimation problems in a noisy sum of exponentially modulated sinusoids are reviewed. Modifications using Higher-Order SVD are considered. Numerical comparisons are carried out. It is shown numerically that for, the signal extraction problem, tensor methods generally perform worse than matrix methods for a single-channel series, but can outperform multi-channel SSA for a series system. For frequency estimation, tensor modifications are generally advantageous.

Keywords: time series, signal, frequency estimation, tensor, singular spectrum analysis

1 Introduction

Singular spectrum analysis (SSA) is one of the methods used for time series analysis [1], in which the original time series is transformed into a matrix, called the trajectory matrix, using a given window length L. The singular value decomposition (SVD) of this matrix is then analyzed. When the objective is to estimate the signal and its properties from an observed noisy series, the first r components of the SVD are considered, where r is the rank of the signal trajectory matrix. Based on the selected components, the signal estimation is constructed. A distinctive feature of the method is that it does not require the specification of a signal model. However, SSA can also handle a parametric signal model in the form of a sum of products of polynomials, exponentials and sinusoids. The frequency estimation problem plays a special role. The ESPRIT method uses the estimation of the signal subspace based on the r leading left singular vectors of the trajectory matrix SVD to estimate the frequencies present in the signal. The least squares (LS) version of ESPRIT [2] is also known as Hankel SVD (HSVD), and the total least squares (TLS) version is known as HTLS [3].

A number of works propose tensor modifications of the SSA and ESPRIT methods, where the original series is transformed into a tensor, usually of 3rd order, instead of a matrix [4–6]. One of the common variants of tensor decompositions is Higher-Order SVD (HO-SVD), which generalizes the matrix SVD.

This work aims to compare the performance of matrix and tensor modifications of SSA in solving signal extraction and frequency estimation problems. We will consider the tensor modifications proposed in [6] and [7], which have been adapted for signal extraction.

2 Methods description

2.1 Tensor SSA algorithm layout for signal extraction

The general structure of tensor SSA algorithms based on HO-SVD is as follows (Basic SSA is a special case). Let X be the observed object. The tensor dimensions I, L and K are considered as the window length; some of these dimensions are expressed in terms of the others, or are fixed. The parameters of the algorithm are the values R_1, R_2 and R_3. These are often chosen to be equal to r, but not always.

1. Embedding to the trajectory tensor $\mathbf{X} = \mathcal{T}(\mathsf{X})$.
2. Tensor decomposition $\mathbf{X} = \sum_{i=1}^{I} \sum_{l=1}^{J} \sum_{k=1}^{K} \mathcal{Z}_{ilk} U_i^{(1)} \circ U_l^{(2)} \circ U_k^{(3)}$.
3. Grouping $\hat{\mathbf{X}} = \sum_{i=1}^{R_1} \sum_{l=1}^{R_2} \sum_{k=1}^{R_3} \mathcal{Z}_{ilk} U_i^{(1)} \circ U_l^{(2)} \circ U_k^{(3)}$.
4. Obtaining from $\hat{\mathbf{X}}$ the signal estimate $\hat{\mathsf{X}}$ based on the structure of the trajectory tensor and the operation that is inversed to embedding.

We will further consider two types of input object: single-channel and multi-channel time series.

2.2 Trajectory tensors

Let $\mathsf{X} = (x_1, x_2, \ldots, x_N)$ be a single-channel time series of length N, $x_n \in \mathbb{C}$.

Definition 1. The tensor embedding operator for a single-channel time series with window lengths I and L (then $K = N - I - L + 2$) such that $1 < I, L < N$, $I + L < N + 1$ is a mapping $\mathcal{T}_{I,L}$ that transfers the series X into the tensor $\mathcal{X} \in \mathbb{C}^{I \times L \times K}$ as follows: $\mathcal{X}_{ilk} = x_{i+l+k-2}$, where $i \in \overline{1:I}$, $l \in \overline{1:L}$, $k \in \overline{1:K}$.

Let $\mathsf{X} = \left(\mathsf{X}^{(1)}, \mathsf{X}^{(2)}, \ldots, \mathsf{X}^{(P)}\right)$ be a multi-channel time series consisting of P single-channel series, also called channels.

Definition 2. The tensor embedding operator for a multi-channel time series with window length L (that is, $I = P$, L, $K = N - L + 1$) such that $1 < L < N$ is a mapping $\mathcal{T}_L$ that translates the P-channel time-series X into the tensor $\mathcal{X} \in \mathbb{C}^{P \times L \times K}$ as follows $\mathcal{X}_{plk} = x_{l+k-1}^{(p)}$, where $p \in \overline{1:P}$, $l \in \overline{1:L}$, $k \in \overline{1:K}$.

2.3 Algorithm for signal parameters estimation

Consider the P-channel time series (including the single-channel case $P = 1$) with elements

$$x_n^{(p)} = \sum_{r=1}^{R} a_r^{(p)} e^{\alpha_r n} e^{\mathrm{i}\left(2\pi\omega_r n + \varphi_r^{(p)}\right)},$$

where the model parameters are the amplitudes $a_r^{(p)} \in \mathbb{C}\backslash\{0\}$, phases $\varphi_r^{(p)} \in [0, 2\pi)$, the frequencies $\omega_r \in [0, 1/2]$, and the damping factors $\alpha_r \in \mathbb{R}$. The HO-ESPRIT algorithm that estimates the frequencies and damping factors of a time series is defined as follows. After the embedding step the matrix $\mathbf{U} = \mathbf{U}_d = \left[U_1^{(d)} : U_2^{(d)} : \ldots : U_{R_d}^{(d)}\right]$ for $d \in \{1, 2, 3\}$ is constructed and the following matrix equation

$$\mathbf{U}^{\uparrow} = \mathbf{U}_{\downarrow}\mathbf{Z}$$

is solved with respect to matrix $\mathbf{Z}$, where the up and down arrows placed behind the matrix $\mathbf{U}$ stand for deleting its first and last rows accordingly. The R largest eigenvalues of the matrix $\mathbf{Z}$ are considered to be the estimates of the poles $\lambda_r = e^{\alpha_r + 2\pi \mathrm{i}\omega_r}$, from which the parameters α_r and ω_r can be obtained.

2.4 Dstack modifications

In the paper [7], to improve the speed of the method, it is proposed to transform a single-channel series into a multi-channel series before applying the tensor modification: $x_m^{(d)} = x_{(m-1)D+d}$, where $m \in \overline{1 : (N/D)}$. In that paper only the ESPRIT modification called HTLSDstack is considered, but we will apply this time series transformation for the signal estimation problem as well, and will call the resulting method SSADstack. Tensor modifications are constructed as for a multi-channel series.

3 Comparison of tensor and matrix methods

All numerical comparisons are made using time series that are expressed as sums of sinusoids.

The following methods were compared for single channel time series and signal extraction problem: SSA, HO-SSA, SSADstack, HO-SSADstack with $R_1 = r$ and HO-SSADstack with $R_1 = 1$. It has been shown that, in most cases, the SSA method significantly outperforms other methods in terms of accuracy. When the SSA method is less accurate, the difference is negligible and only occurs for a very narrow range of parameters. This minor disadvantage is therefore not a practical consideration. Of the Dstack methods, SSADstack and HO-SSADstack are the most accurate, with a small difference in accuracy when $R_1 = r$.

For single-channel time series and frequency estimation problem, a signal in the form of two sinusoids with close frequencies was considered. The ESPRIT, HO-ESPRIT, HTLSDstack, HO-HTLSDstack with $R_1 = r$ and HO-HTLSDstack with $R_1 = 1$ methods were compared. It was found that ESPRIT performs more accurately at a low noise level. However, at a medium or high noise level, HO-ESPRIT with optimal parameter selection becomes more accurate. Furthermore, HO-HTLSDstack with $R_1 = 1$ outperforms all methods.

For multi-channel time series, it has been demonstrated that, when all channels are expressed as a sum of sinusoids with equal frequencies, tensor modifications provide more accurate results for both signal extraction and frequency estimation.

4 Conclusions

Numerical comparisons revealed the varying effects of the HO-SVD tensor modifications on different time series problems. For signal extraction from a single-channel time series, the basic matrix method is certainly more accurate. However, for multi-channel time series with an equal set of frequencies across the channels, and for frequency estimation problems, the tensor methods can offer improved accuracy.

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