Tensors for signal and frequency estimation in subspace-based methods: when they are useful?

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Introduction to Singular Spectrum Analysis (SSA)

Problems that can be solved by SSA-related methods:

- Signal extraction
- Frequency estimation
- Smoothing and Noise reduction
- Signal decomposition (Trend and Periodicity extraction)
- Forecasting
- Missing data imputation
- Change in structure detection
- Many others...

SSA Materials

Books:

- J.Elsner and A.Tsonis. Singular Spectrum Analysis: A New Tool in Time Series Analysis, Plenum, 1996.
- N.Golyandina, V.Nekrutrin and A.Zhigljavsky. Analysis of Time Series Structure: SSA and Related Techniques, CRC Press, 2001.
- S.Sanei and H.Hassani. Singular Spectrum Analysis for Biomedical Signals, CRC Press, 2016.
- N.Golyandina, A.Korobeynikov and A.Zhigljavsky. Singular spectrum analysis with R, Springer, 2018.
- N.Golyandina and A.Zhigljavsky. Singular Spectrum Analysis for Time Series, Springer, 2013, 2020 (2nd Edition).

Implementations:

- R Package: Rssa
  https://CRAN.R-project.org/package=Rssa
- Python Package: py-ssa-lib (less features)
  https://pypi.org/project/py-ssa-lib

SSA Decomposition Example

Decomposition of time series:

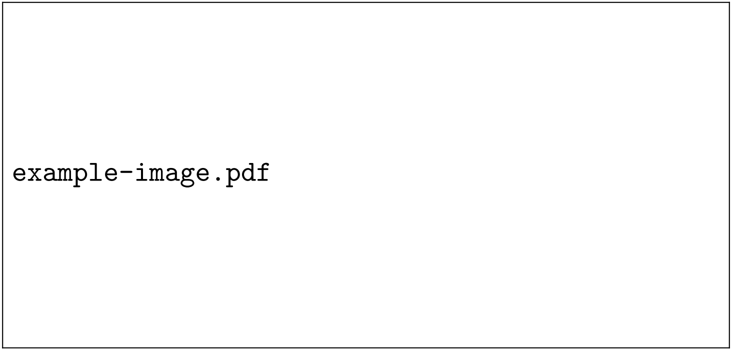
- Low-frequency component + high-frequency component
- Signal + noise
- Trend + Seasonality + Noise

example-image.pdf

Some data: demonstration of series decomposition with SSA

ESPRIT Frequency Estimation Example

ESPRIT — SSA-related method for parameters estimation



Pole motion data probably

1. Estimate — interpretation
2. Estimate — interpretation

Complex Time Series

Common origins of complex-valued time series:

- Can be constructed from two related features
- Arise as a result of applying the Fourier transform to real data

SSA Algorithm: Embedding

Input: time series $\mathsf{X} = (x_1, x_2, \ldots, x_N)$, window length L, signal rank r.

1. **Embedding**. Constructing the L-*Trajectory* Hankel matrix $\mathbf{X} \in \mathbb{C}^{L \times K}$ from the series X, where $K = N - L + 1$:

$$\mathbf{X} = \mathcal{T}^{(L)}(\mathsf{X}) = \begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_K \\ x_2 & x_3 & x_4 & \ldots & x_{K+1} \\ x_3 & x_4 & x_5 & \ldots & x_{K+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_L & x_{L+1} & x_{L+2} & \ldots & x_N \end{pmatrix}$$

SSA Algorithm: Decomposition, Grouping, Reconstruction

2. **Decomposition**. Constructing the singular value decomposition (SVD) of matrix $\mathbf{X}$: $\mathbf{X} = \sum_{j=1}^{\operatorname{rank}\mathbf{X}} \sqrt{\lambda_j} U_j V_j^{\mathrm{H}} = \sum_{j=1}^{\operatorname{rank}\mathbf{X}} \widehat{\mathbf{X}}_j$ where H denotes Hermitian conjugation, U_j and V_j are left and right singular vectors of $\mathbf{X}$, $\sqrt{\lambda_j}$ — its singular values in descending order.

3. **Grouping**. Grouping the terms $\widehat{\mathbf{X}}_j$ from the decomposition related to the signal: $\mathbf{S} = \sum_{j=1}^{r} \widehat{\mathbf{X}}_j = \Pi_r \mathbf{X}$, where Π_r is the projector onto the space of matrices with rank not greater than r.

4. **Reconstruction**. Applying projection onto the space of Hankel matrices: $\widetilde{\mathbf{S}} = \Pi_{\mathcal{H}} \widehat{\mathbf{S}}$, and return to the series form: $\widetilde{\mathsf{S}} = \left(\mathcal{T}^{(L)}\right)^{-1} (\widetilde{\mathbf{S}})$

Series Rank

Definition

Series X has rank $d < N/2$, if the rank of its L-trajectory matrix equals d for any L such that $d \leqslant \min(L, N - L + 1)$.
If such d exists, then X is called a series of finite rank.

If the signal S is a series of finite rank, then it is generally recommended to use $\mathrm{rank}(\mathsf{S})$ as parameter r in the SSA method

Series rank examples

- rank of S with $s_n = A\sin(2\pi\omega n + \varphi)$, $0 < \omega < 1/2$, equals 2
- rank of S with $s_n = A\exp(\alpha n)$, $\alpha \in \mathbb{C}$, equals 1

Signal Model

What we consider a signal $\mathsf{S} = (s_1, s_2, \dots, s_N)$:

- The trajectory matrix $\mathbf{S} = \mathcal{T}^{(L)}(\mathsf{S})$ is rank-deficient ($\Longrightarrow$ the time series is of some finite rank: $\operatorname{rank}(\mathsf{S}) = r$)
- Any signal S can be represented in the form of a finite sum:

$$s_n = \sum_j p_j(n) \exp(\alpha_j n + \mathrm{i}(2\pi\omega_j n + \varphi_j)),$$

  where $p_j(n)$ is a polynomial in n
- Real case:

$$s_n = \sum_j p_j(n) \exp(\alpha_j n) \sin(2\pi\omega_j n + \varphi_j),$$

ESPRIT method estimates damping factors α_j and frequencies ω_j

ESPRIT Algorithm: General Idea

$$s_n = \sum_{j=1}^{2} \exp(\alpha_j n + \mathrm{i}(2\pi\omega_j n + \varphi_j)) = A_1 z_1^n + A_2 z_2^n$$

where $A_j = \exp(\mathrm{i}\varphi_j)$, $z_j = \exp(\alpha_j + \mathrm{i}2\pi\omega_j)$

Signal subspace basis is given by

$$\mathbf{M} = \begin{pmatrix} z_1 & z_2 \\ z_1^2 & z_2^2 \\ \vdots & \vdots \\ z_1^L & z_2^L \end{pmatrix} \Rightarrow \overline{\mathbf{M}} = \underline{\mathbf{M}} \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix} \Rightarrow \underline{\mathbf{M}}^{-}\overline{\mathbf{M}} = \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix}$$

where $\overline{\mathbf{M}}$ denotes $\mathbf{M}$ without the first row, $\underline{\mathbf{M}}$ — without the last
$\underline{\mathbf{M}}^{-}$ denotes the pseudoinverse of $\underline{\mathbf{M}}$

ESPRIT Algorithm

Input: same as in SSA: X, L, r

1. **Embedding**. $\mathbf{X} = \mathcal{T}^{(L)}(\mathsf{X})$
2. **Decomposition**. $\mathbf{X} = \sum_{j=1}^{\operatorname{rank}\mathbf{X}} \sqrt{\lambda_j} U_j V_j^{\mathrm{H}}$, $\mathbf{U}_r = [U_1 : U_2 : \ldots : U_r]$
3. **Estimation**. Finding eigenvalues z_j of matrix $\underline{\mathbf{U}}_r^{-} \overline{\mathbf{U}}_r$

   From $z_j = \exp(\alpha_j + \mathrm{i}2\pi\omega_j)$ parameters α_j and ω_j can be found

Multi-Channel Time Series, MSSA

$\mathsf{X} = \left(\mathsf{X}^{(1)}, \mathsf{X}^{(2)}, \ldots, \mathsf{X}^{(P)}\right), \qquad \mathsf{X}^{(p)} = \left(x_1^{(p)}, x_2^{(p)}, \ldots, x_N^{(p)}\right)$ – channels

The only change in the algorithms — Embedding step:

$$\mathbf{X} = \mathcal{T}^{(L)}_{\mathrm{MSSA}}(\mathsf{X}) = \left[\mathbf{X}^{(1)} : \mathbf{X}^{(2)} : \ldots : \mathbf{X}^{(P)}\right],$$

$$\mathbf{X}^{(p)} = \mathcal{T}^{(L)}(\mathsf{X}^{(p)})$$

When to chose MSSA over SSA for each channel:

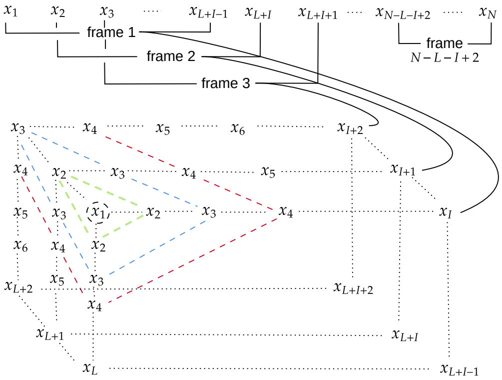
- All channels have "similar" structure
- "Supporting" channels with lower noise level

Intoduction of Tensors

$$\begin{array}{rllll} \text{Basic SSA:} & \text{Time series X} & \mapsto & \text{Matrix } \mathbf{X} & \mapsto & \mathrm{SVD}(\mathbf{X}) \\ \text{Tensor SSA:} & \text{Time series X} & \mapsto & \text{Tensor } \mathcal{X} & \mapsto & \underbrace{\mathrm{TD}}_{\text{Some Tensor Decomposition}}(\mathcal{X}) \end{array}$$

Tensor SVD Extensions:

- Higher-Order SVD (HOSVD)
- Canonical Polyadic Decomposition (CPD)
- T-SVD
- $(L_r, L_r, 1)$-Decomposition



$$\mathsf{X} = (x_1, x_2, \ldots, x_N)$$

$$\mathsf{X} \mapsto \mathcal{T}^{(I,L)}_{\mathrm{T-SSA}}(\mathsf{X}) = \mathcal{X}$$

$$\mathcal{X} \in \mathbb{C}^{I \times L \times K}$$

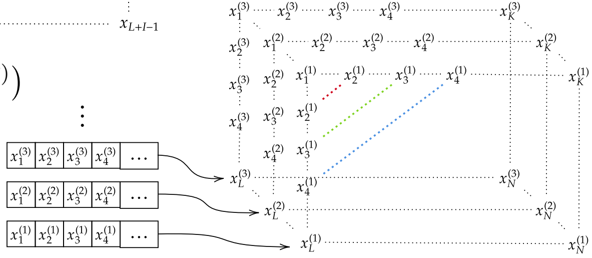
$$K = N - I - L + 2$$

$$\mathsf{X} = \left(\mathsf{X}^{(1)}, \mathsf{X}^{(2)}, \ldots, \mathsf{X}^{(P)}\right)$$

$$\mathsf{X} \mapsto \mathcal{T}^{(L)}_{\mathrm{T-MSSA}}(\mathsf{X}) = \mathcal{X}$$

$$\mathcal{X} \in \mathbb{C}^{P \times L \times K}$$

$$K = N - L + 1$$



Higher-Order SVD. Higher-Order Orthogonal Iterations

$$\mathrm{SVD}(\mathbf{X}) = \sum_{j=1}^{\mathrm{rank}(\mathbf{X})} \sqrt{\lambda_j} U_j V_j^{\mathrm{H}}$$

$$\mathrm{HOSVD}(\mathcal{X}) = \sum_{i=1}^{\mathrm{rank}_1(\mathcal{X})} \sum_{l=1}^{\mathrm{rank}_2(\mathcal{X})} \sum_{k=1}^{\mathrm{rank}_3(\mathcal{X})} \mathcal{Z}_{ilk} U_i^{(1)} \circ U_l^{(2)} \circ U_k^{(3)}$$

- $\widetilde{\mathbf{X}} = \sum_{j=1}^{R} \ldots \Rightarrow \left\| \mathbf{X} - \widetilde{\mathbf{X}} \right\|_F = \min_{\mathrm{rank}(\widehat{\mathbf{X}}) \leqslant R} \left\| \mathbf{X} - \widehat{\mathbf{X}} \right\|_F$
- $\widetilde{\mathcal{X}} = \sum_{i=1}^{R_1} \sum_{l=1}^{R_2} \sum_{k=1}^{R_3} \ldots \Rightarrow \left\| \mathcal{X} - \widetilde{\mathcal{X}} \right\|_F \geqslant \min_{\mathrm{rank}_m(\widehat{\mathcal{X}}) \leqslant R_m} \left\| \mathcal{X} - \widehat{\mathcal{X}} \right\|_F$

Truncation of SVD is optimal, but truncation of HOSVD is not

Iterative algorithm for finding optimal approximation – HOOI

Higher-Order SSA, MSSA, ESPRIT

Input: time series X, window length: (I, L) for single-channel or L for multi-channel, signal ranks (r_1, r_2, r_3), d — estimation dimension for HO-ESPRIT.

1. **Embedding**.

Single-channel	$\mathsf{X} \mapsto \mathcal{T}^{(I,L)}_{\mathrm{T-SSA}}(\mathsf{X}) = \mathcal{X}$
Multi-channel	$\mathsf{X} \mapsto \mathcal{T}^{(L)}_{\mathrm{T-MSSA}}(\mathsf{X}) = \mathcal{X}$

2. **Decomposition & Approximation**. Using (r_1, r_2, r_3)
$\mathcal{X} \mapsto \mathrm{Trunc}(\mathrm{HOSVD}(\mathcal{X})) = \widetilde{\mathcal{S}}$ or $\mathcal{X} \mapsto \mathrm{HOOI}(\mathcal{X}) = \widetilde{\mathcal{S}}$

3. **Reconstruction or Estimation**.
   - **Reconstruction**. $\mathsf{S} = (\mathcal{T})^{-1}\left(\Pi_{\mathcal{H}_{\mathrm{T}}}(\widetilde{\mathcal{S}})\right)$, $\Pi_{\mathcal{H}_{\mathrm{T}}}$ — projector onto the space of Hankel tensors
   - **Estimation**. Finding eigenvalues z_j of matrix $\underline{\mathbf{U}}^{-}\overline{\mathbf{U}}$, where $\mathbf{U} = \left[U_1^{(d)} : U_2^{(d)} : \ldots : U_{r_d}^{(d)}\right]$. From $z_j = \exp(\alpha_j + \mathrm{i}2\pi\omega_j)$ damping factors α_j and frequencies ω_j of the signal can be found

Dtack Modifications

$\mathsf{X} = (x_1, x_2, \ldots, x_N),\ M = \lfloor N/D \rfloor$

$$\mathrm{Dstack}_D(\mathsf{X}) = \mathcal{D}_D(\mathsf{X}) = \left[\begin{array}{c|c|c|c} x_1 & x_2 & \cdots & x_D \\ x_{D+1} & x_{D+2} & \cdots & x_{2D} \\ x_{2D+1} & x_{2D+2} & \cdots & x_{3D} \\ \vdots & \vdots & \cdots & \vdots \\ x_{(M-1)D+1} & x_{(M-1)D+2} & \cdots & x_{MD} \end{array}\right]$$

$$\underbrace{\phantom{x_{(M-1)D+1}}}_{\mathsf{X}_D^{(1)}} \quad \underbrace{\phantom{x_{(M-1)D+2}}}_{\mathsf{X}_D^{(2)}} \quad \cdots \quad \underbrace{\phantom{x_{MD}}}_{\mathsf{X}_D^{(D)}}$$

Dstack-SSA	$\mathsf{X} \mapsto \mathbf{X} = \mathcal{T}_{\mathrm{MSSA}}^{(L)}\left(\mathcal{D}_D(\mathsf{X})\right)$
Dstack-T-SSA	$\mathsf{X} \mapsto \mathcal{X} = \mathcal{T}_{\text{T-MSSA}}^{(L)}\left(\mathcal{D}_D(\mathsf{X})\right)$

Undersampling: $\omega \mapsto \hat{\omega} = D\omega \implies \max|\omega| \leqslant \dfrac{1}{2D}$

$$x_n = e^{\alpha_1 n} e^{2\pi i \omega_1 n} + e^{\alpha_2 n} e^{2\pi i \omega_2 n} + \zeta_n$$

ζ_n — Complex white gaussian noise, $\omega_1 = 0.2$, $\omega_2 = 0.22$, $\alpha_1 = \alpha_2 = 0$ (same results for $\alpha_1 = \alpha_2 < 0$ and $\alpha_1 \leqslant \alpha_2 \leqslant 0$).

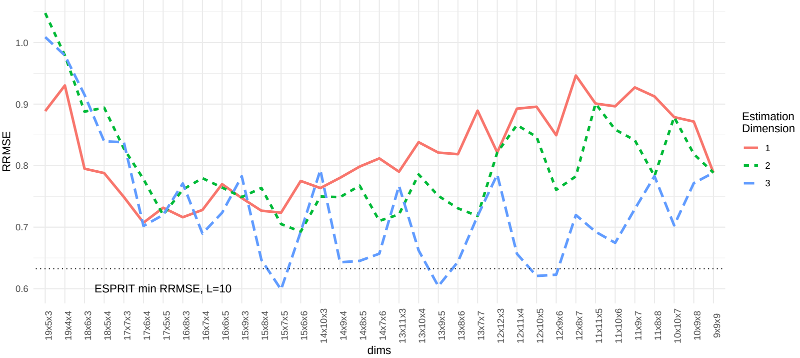


Figure: RRMSE of estimates for ω_1 vs window lengths $(I \times L \times K)$