AU 326 Digital Image Process

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HW#: 2

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I. PROBLEM 1

A. Introduction

Prove that both the 2-D continuous and discrete Fourier transforms are linear operations (see Section 2.6.2 for a definition of linearity).

B. Solution

The 2-D continuous Fourier transform is

$$F(\mu, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + vz)} dt dz \tag{1}$$

From eq.(2.6-2) we can check whether the following equation is satisfied:

$$H\left[a_1 f_1(t, z) + a_2 f_2(t, z)\right] = a_1 H\left[f_1(t, z)\right] + a_2 H\left[f_2(t, z)\right] \tag{2}$$

Here H denotes the continuous Fourier transform. Thus we can get

$$\begin{aligned} H\left[a_1 f_1(t, z) + a_2 f_2(t, z)\right] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[a_1 f_1(t, z) + a_2 f_2(t, z)\right] e^{-j2\pi(\mu t + vz)} dt dz \\ &= a_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(t, z) e^{-j2\pi(\mu t + vz)} dt dz + a_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_2(t, z) e^{-j2\pi(\mu t + vz)} dt dz \\ &= a_1 H\left[f_1(t, z)\right] + a_2 H\left[f_2(t, z)\right] \end{aligned} \tag{3}$$

Here H denotes the continuous Fourier transform. From the proof above we can see that the 2-D continuous Fourier transform is a linear operation.

The proof of the discrete Fourier transform is similar.The 2-D discrete Fourier transform is:

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)} \tag{4}$$

Thus

$$\begin{aligned} H\left[a_1 f_1(x, y) + a_2 f_2(x, y)\right] &= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[a_1 f_1(x, y) + a_2 f_2(x, y)\right] e^{-j2\pi(ux/M + vy/N)} \\ &= a_1 \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f_1(x, y) e^{-j2\pi(ux/M + vy/N)} + a_2 \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f_2(x, y) e^{-j2\pi(ux/M + vy/N)} \\ &= a_1 H\left[f_1(x, y)\right] + a_2 H\left[f_2(x, y)\right] \end{aligned} \tag{5}$$

Here H denotes the discrete Fourier transform. The 2-D discrete Fourier transform is a linear operation.

From above we can conclude that both the 2-D continuous and discrete Fourier transforms are linear operations.

II. PROBLEM 2

A. Introduction

Find two images and compute their corresponding spectrum and phase angle, then visualize them. Synthesis new images using one image's spectrum and another image's phase angle, visualize them.

B. Solution

In this part I used Matlab to compute and visualize the corresponding spectrum and phase angle of the two images.

First, I input two images and change them to grey images with the same size. Then I used the Fourier transform to get their corresponding spectrum and phase angle. Then I moved the zero point to the center of the spectrum using the function fftshift( ). Afterwards I exchanged the phase angles of two images and did IFFT to get the processed images. The result can be seen in Figure 1.

C. Code

Here is the code of this problem.

```
%input images
[x,cmap1]=imread('lena.jpg');
[y,cmap2]=imread('image.jpg');
%adjust the size and change to grey images
[~,~,l]=size(x);
if l==3
    x=rgb2gray(x);
end
[m,n,l]=size(y);
if l==3
    y=rgb2gray(y);
end
[a,b]=size(x);
y=imresize(y,[a,b]);
% Fourier transform
xf=fft2(x);
yf=fft2(y);
% the corresponding spectrum and phase angle
xf1=abs(xf);
xf2=angle(xf);
yf1=abs(yf);
yf2=angle(yf);
% move
s1=fftshift(xf);
s11=log(abs(s1)+1);
s2=fftshift(yf);
s22=log(abs(s2)+1);
% exchange the phase angles
xfr=xf1.*cos(yf2)+xf1.*sin(yf2).*1j;
yfr=yf1.*cos(xf2)+yf1.*sin(xf2).*1j;
% IFFT
xr=abs(ifft2(xfr));
yr=abs(ifft2(yfr));
% change to type uint8
xr=uint8(xr);
yr=uint8(yr);
% show
subplot(3,3,1);imshow(x);title('x_the_original_image');
subplot(3,3,4);imshow(y);title('y_the_original_image');
subplot(3,3,2);imshow(s11,[]);title('x_spectrum');
subplot(3,3,3);imshow(xf2,[]);title('x_phase_angle');
subplot(3,3,5);imshow(s22,[]);title('y_spectrum');
subplot(3,3,6);imshow(yf2,[]);title('y_phase_angle');
subplot(3,3,7);imshow(xr,[]);title('x_spectrum_with_y_phase_angle');
subplot(3,3,8);imshow(yr,[]);title('y_spectrum_with_x_phase_angle');
```

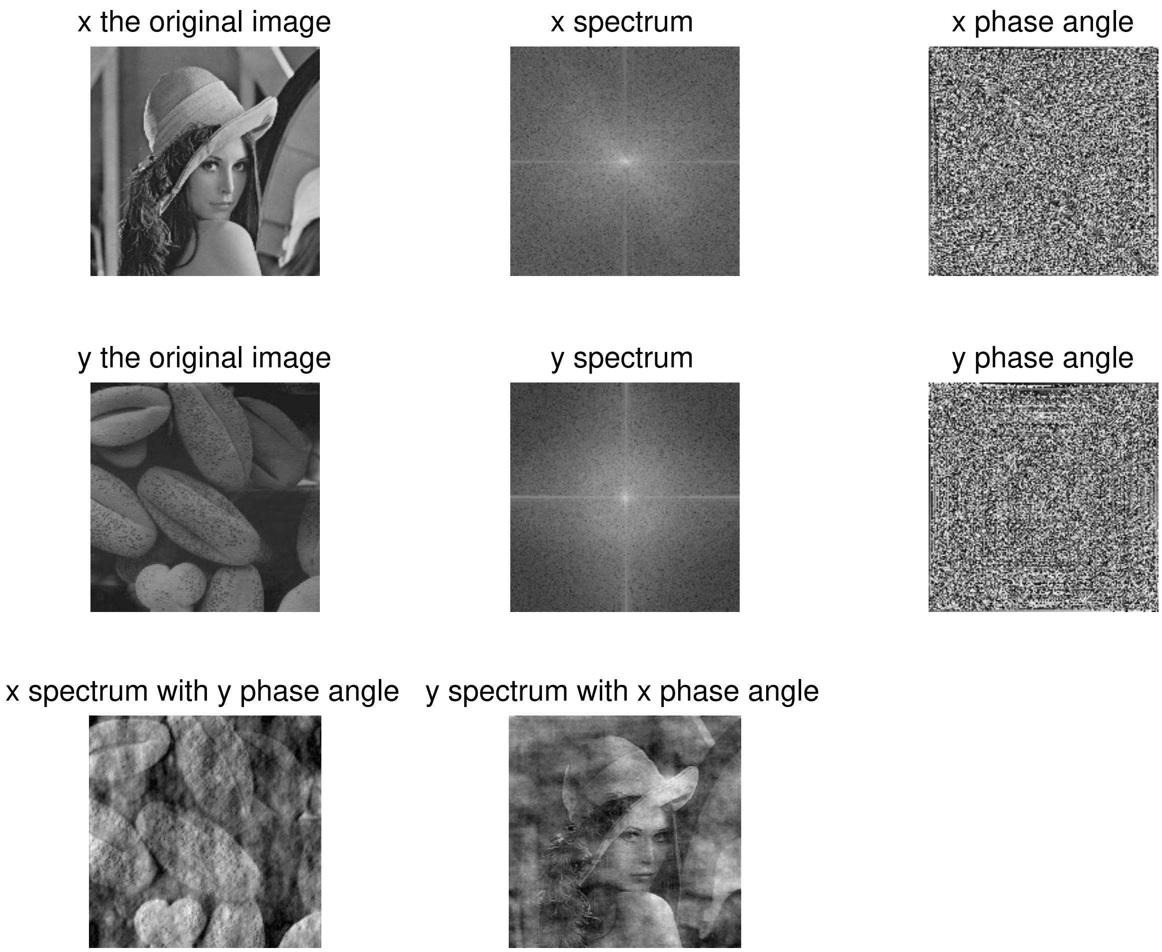

FIG. 1: The result of Problem 2 in Matlab

III. PROBLEM 3

A. Introduction

Show that the saturation component of the complement of a color image cannot be computed from the saturation component of the input image alone.

B. Solution

The functions to transform RGB value to HSI value are

$$H = \begin{cases} \theta, & B \leq G, \\ 360 - \theta, & B > G. \end{cases} \tag{6}$$

$$\theta = \arccos \left\{ \frac{\frac{1}{2}[(R-G)+(R-B)]}{[(R-G)^2+(R-B)(G-B)]^{\frac{1}{2}}} \right\} \tag{7}$$

$$S = 1 - \frac{3}{(R+G+B)}[\min(R,G,B)] \tag{8}$$

$$I = \frac{1}{3}(R+G+B) \tag{9}$$

From the equations above we can see that different colors can have the same saturation value. Consider RGB colors of red (1, 0, 0) and of green (0, 0.59, 0). By calculating we can get their HSI values of (0, 1, 0.33) and (0.33, 1, 0.2). The RGB values of their complemented colors are cyan of (0, 1, 1) and magenta of (1, 0.41, 1). The HSI values are (0.5, 1, 0.66) and (0.83, 0.48, 0.8). So the same saturation component 1 of the original colors corresponds to two different saturation components of 1 and 0.48 of their complemented colors. Thus the conclusion can be drawn that the saturation components of the complement of a color image cannot be computed from the saturation component of the input image alone.

IV. PROBLEM 4

A. Introduction

The arithmetic decoding process is the reverse of the encoding procedure. Decode the message 0.23355 given the following coding model.

B. Solution

According to the arithmetic decoding process, we first divide the [0.0, 1.0) interval by the probabilities in Table I.

Symbol	Probability	Range
a	0.2	[0.0, 0.2)
e	0.3	[0.2, 0.5)
i	0.1	[0.5, 0.6)
o	0.2	[0.6, 0.8)
u	0.1	[0.8, 0.9)
!	0.1	[0.9, 1.0)

TABLE I: The range of each symbol in the first level

The message 0.23355 is between the range of [0.2, 0.5), so the first symbol is e. Then we divide the range of e [0.2, 0.5) by the probabilities similarly in Table II.

Symbol	Probability	Range
a	0.2	[0.20, 0.26)
e	0.3	[0.26, 0.35)
i	0.1	[0.35, 0.38)
o	0.2	[0.38, 0.44)
u	0.1	[0.44, 0.47)
!	0.1	[0.47, 0.50)

TABLE II: The range of each symbol in the second level

The message 0.23355 is between the range of [0.20, 0.26), so the first symbol is a. Then we divide the range of a [0.20, 0.26) by the probabilities similarly in Table III.

Symbol	Probability	Range
a	0.2	[0.200, 0.212)
e	0.3	[0.212, 0.230)
i	0.1	[0.230, 0.236)
o	0.2	[0.236, 0.248)
u	0.1	[0.248, 0.254)
!	0.1	[0.254, 0.260)

TABLE III: The range of each symbol in the third level

The message 0.23355 is between the range of [0.230, 0.236), so the first symbol is i. Then we divide the range of i [0.230, 0.236) by the probabilities similarly in Table IV.

Symbol	Probability	Range
a	0.2	[0.2300, 0.2312)
e	0.3	[0.2312, 0.2330)
i	0.1	[0.2330, 0.2336)
o	0.2	[0.2336, 0.2348)
u	0.1	[0.2348, 0.2354)
!	0.1	[0.2354, 0.2360)

TABLE IV: The range of each symbol in the fourth level

The message 0.23355 is between the range of [0.2330, 0.2336), so the first symbol is i. Then we divide the range of i [0.2330, 0.2336) by the probabilities similarly in Table V.

Symbol	Probability	Range
a	0.2	[0.2330, 0.23312)
e	0.3	[0.23312, 0.23330)
i	0.1	[0.23330, 0.23336)
o	0.2	[0.23336, 0.23348)
u	0.1	[0.23348, 0.23354)
!	0.1	[0.23354, 0.23360)

TABLE V: The range of each symbol in the fifth level

The message 0.23355 is between the range of [0.23354, 0.23360), so the first symbol is !.

To conclude, the message is "eaii!".

V. PROBLEM 5

A. Introduction

Using the Huffman code in the following figure, decode the encoded string 0101000001010111110100.

Original source			Source reduction							
Symbol	Probability	Code	1		2		3		4	
a_2	0.4	1	0.4	1	0.4	1	0.4	1	0.6	0
a_6	0.3	00	0.3	00	0.3	00	0.3	00	0.4	1
a_1	0.1	011	0.1	011	0.2	010	0.3	01		
a_4	0.1	0100	0.1	0100	0.1	011				
a_3	0.06	01010	0.1	0101						
a_5	0.04	01011								

FIG. 2: The Huffman code

B. Solution

1. The first valid code word is 01010, which is the code for symbol a_3.

2. The second valid code word is 00, which is the code for symbol a_6.

3. The third valid code word is 00, which is the code for symbol a_6.

4. The fourth valid code word is 1, which is the code for symbol a_2.

5. The fifth valid code word is 01011, which is the code for symbol a_5.

6. The sixth valid code word is 1, which is the code for symbol a_2.

7. The seventh valid code word is 1, which is the code for symbol a_2.

8. The eighth valid code word is 1, which is the code for symbol a_2.

9. The ninth valid code word is 0100, which is the code for symbol a_4.

In conclusion, the decoded message is $a_3a_6a_6a_2a_5a_2a_2a_2a_4$

VI. PROBLEM 6

A. Introduction

Consider the problem of image blurring caused by uniform acceleration in the x-direction. If the image is at rest at time t = 0 and accelerates with a uniform acceleration x_0(t) = $at^2/2$ for a time T, find the blurring function H(u, v). You may assume that shutter opening and closing times are negligible.

B. Solution

$$\begin{aligned}
H(u,v) &= \int_0^T e^{-j2\pi[ux_o(t)+vy_0(t)]}dt \\
&= \int_0^T e^{-j2\pi[ux_o(t)]}dt \\
&= \int_0^T e^{-j2\pi u\left(\frac{1}{2}at^2\right)}dt \\
&= \int_0^T e^{-j\pi uat^2}dt \\
&= \int_0^T \left[\cos\left(\pi uat^2\right) - j\sin\left(\pi uat^2\right)\right]dt
\end{aligned} \tag{10}$$

VII. PROBLEM 7

A. Introduction

Prove that the Radon transform is a linear operator.

B. Solution

The Radon transform is

$$g(\rho,\theta) = \int_{-\infty-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)dxdy \tag{11}$$

From eq.(2.6-2) we can check whether the following equation is satisfied to prove the Radon transform is a linear operator.

$$H\left[a_1 f_1(t,z) + a_2 f_2(t,z)\right] = a_1 H\left[f_1(t,z)\right] + a_2 H\left[f_2(t,z)\right] \tag{12}$$

Here H denotes the Radon transform. Thus

$$\begin{aligned} H\left[a_1 f_1(t,z) + a_2 f_2(t,z)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[a_1 f_1(x,y) + a_2 f_2(x,y)\right]\delta(x\cos\theta + y\sin\theta - \rho)dxdy \\ &= a_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x,y)\delta(x\cos\theta + y\sin\theta - \rho)dxdy \\ &+ a_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_2(x,y)\delta(x\cos\theta + y\sin\theta - \rho)dxdy \\ &= a_1 H\left[f_1(\rho,\theta)\right] + a_2 H\left[f_2(\rho,\theta)\right] \end{aligned} \tag{13}$$

Here H denotes the Radon transform.

In conclusion, the Radon transform is a linear operator.