Formulas

Sample Mean

- **Sample Mean ($\bar{x}$):** Average of the data values.

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

  Where:

  - n= number of observations
  - xi = each data point

- **Deviations**: Differences from the mean, $xi - \overline{x}$. The sum of deviations is always zero.

Sample Variance and Sample Standard Deviation

- **Sample Variance** (s^2): Measures data spread around the mean

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

- **Sample Standard Deviation (s)**: Square root of the variance, gives spread in original units.

$$s = \sqrt{s^2}$$

Sample Correlation Coefficient

- **Correlation Coefficient (r)**: Measures the strength and direction of a linear relationship between two variables.

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

Chebyshev's Inequality

Mathematically, Chebyshev's inequality states:

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

or equivalently,

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Where

- X is a random variable,
- μ is the mean (expected value) of X,
- σ is the standard deviation of X,
- k is any number greater than 1.

The formula for One-Sided Chebyshev's Inequality

For a random variable X with mean μ and standard deviation σ, the one-sided Chebyshev's inequality states:

$$P(X - \mu \geq k\sigma) \leq \frac{1}{1 + k^2}$$

or

$$P(X - \mu \leq -k\sigma) \leq \frac{1}{1 + k^2}$$

Where:

- X is a random variable,
- μ is the mean of X,
- σ is the standard deviation,
- k is a positive number (i.e., how many standard deviations away from the mean we are considering).

Probability Mass Function (PMF)

A **PMF** assigns probabilities to **specific discrete values**. It must satisfy:

$P(X = x) \geq 0$ for all values of x.

The total probability must sum to 1:

$\sum P(X = x) = 1$

Probability Density Function (PDF)

A **PDF** is used for **continuous random variables**. Instead of assigning probabilities to specific values, it represents a **smooth probability curve**.

$P(a \leq X \leq b) = \int_a^b f(x)\, dx$

Conditional Probability and Independence

- **Conditional Probability**: Probability of event **A** given that event **B** has occurred.

  $$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

  where:

  - $P(A|B)$ is the probability of **A happening given that B has already happened**.
  - $P(A \cap B)$ is the probability that **both A and B happen together**.
  - $P(B)$ is the probability that **B happens**.
  - The condition $P(B)$ > 0 ensures that we are not dividing by zero.

- Law of total probability

  $$P(A) = P(B)P(A|B)$$

  If $B_1, B_2, \ldots, B_n$, is a **partition** of the sample space (i.e., mutually exclusive and collectively exhaustive events), and A is any event, then:

  $P(A) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i)$

- **Independence**: Two events A and B are independent if the occurrence of one does not affect the probability of the other. Mathematically, this is expressed as:

$$P(A \cap B) = P(A)P(B)$$

  - If this equation holds, then A and B are **independent**.
  - Otherwise, they are **dependent**, meaning that knowing one event affects the likelihood of the other.

Bayes' Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

where:

- $P(A|B)$ = **Posterior Probability** (probability of A given B, i.e., after observing B)
- $P(B|A)$ = **Likelihood** (probability of observing B if A is true)
- $P(A)$ = **Prior Probability** (initial belief about A before observing B)
- $P(B)$ = **Marginal Probability** (total probability of B happening)

Counting Principles (Combinatorics)

- **Permutations** (Order Matters):

  $P(n, r) = \frac{n!}{(n-r)!}$

- **Combinations** (Order Doesn't Matter):
  $C(n, r) = \frac{n!}{r!(n-r)!}$

Expectation in Statistics (Expected Value, E[X])

(a) Expectation for a Discrete Random Variable

If X is a **discrete** random variable with values $x_1, x_2, \ldots, x_n$ and probabilities $P(X = x_i)$, then:

$$E[X] = \sum_i x_i P(X = x_i)$$

(b) Expectation for a Continuous Random Variable

If X is **continuous** with probability density function (PDF) $f(x)$, then the expectation is given by:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Properties of Expectation

1. **Linearity of Expectation:**
    - For any two random variables X and Y,

      $$E[aX + bY] = aE[X] + bE[Y]$$

      (where a and b are constants).
    - Example: If $E[X] = 3, E[Y] = 5$, then:

      $$E[2X + 3Y] = 2(3) + 3(5) = 6 + 15 = 21$$

2. **Expectation of a Constant:** $E[c] = c$

   (The expectation of a constant is just the constant itself.)

3. **Expectation of a Sum:**
    - If $X_1, X_2, ..., X_n$ are random variables:

      $$E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i]$$

4. **If X and Y are Independent:** $E[XY] = E[X]E[Y]$

   But if X and Y are **not** independent, this does **not** necessarily hold.

Probability Density Function (PDF) [For Continuous Variables]

The **PDF**, denoted as $f(x)$, represents the likelihood of a random variable taking a specific value.

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

- The area under the curve of the PDF over an interval gives probability.
- Example: The **Normal Distribution** has the famous bell-shaped PDF.

Probability Mass Function (PMF) [For Discrete Variables]

The **PMF**, denoted as $P(X = x)$, gives the probability of a discrete random variable taking exact values.

$$P(X = x) = \text{some probability value}$$

Definition of Variance

The variance of a random variable X, denoted as Var(X) or σ^2, is the **expected squared deviation** from the mean:

$$Var(X) = E[(X - E[X])^2]$$

$$E[X^2] = Var(X) + (E[X])^2$$

$$E[X^2] = Var(X) + \mu^2$$

$$\mu = E[X] \text{ (Since E[X] is the mean)}$$

Variance Formula

Depending on whether X is **discrete** or **continuous**, we compute variance differently.

(a) Variance for Discrete Random Variables

For a discrete random variable X with values $x_1, x_2, \ldots, x_n$ and probabilities $P(X = x_i)$, the variance is:

$$Var(X) = \sum_i P(X = x_i)(x_i - E[X])^2$$

(b) Variance for Continuous Random Variables

For a **continuous** random variable with probability density function (PDF) $f(x)$, variance is:

$$Var(X) = \int_{-\infty}^{\infty} (x - E[X])^2 f(x)\, dx$$

Properties of Variance

1. **Variance of a Constant:**

   $$Var(c) = 0$$

   (A constant does not vary.)

2. **Scaling Property:**

   $$Var(aX) = a^2Var(X)$$

   (Multiplying by a constant a scales the variance by a^2.)

3. **Sum of Independent Random Variables:**

   $$Var(X+Y) = Var(X) + Var(Y)$$

   If X and Y are independent.

4. **Variance and Standard Deviation Relationship:**

   $$\sigma = \sqrt{Var(X)}$$

   Standard deviation (SD) is the **square root of variance**, giving a measure of spread in the same units as the data.

Discrete Distributions

Binomial Distribution

Probability Mass Function (PMF)

The probability of getting exactly k successes in n trials is given by the **binomial formula**:

$$P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$$

Where:

- $P(X = k)$ = Probability of exactly k successes
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ = Number of ways to choose k successes in n trials
- p^k = Probability of getting k **successes**
- k$(1-p)^{n-k}$ = Probability of getting $(n-k)$ **failures**

Binomial Mean and Variance

For $X \sim Binomial(n, p)$:

- **Expected Value (Mean):**

  $E[X] = np$

- **Variance:**

  $Var(X) = np(1-p)$

- **Standard Deviation:**

  $\sigma_X = \sqrt{np(1-p)}$

Bernoulli Random Variable

$X \sim Bernoulli(p)$

1. Probability Mass Function (PMF)

Since X can take only two values (0 or 1), its **PMF** is:

$$P(X = x) = \begin{cases} p, & \text{if } x = 1 \text{ (success)} \\ 1 - p, & \text{if } x = 0 \text{ (failure)} \end{cases}$$

This can be written in compact form using exponents:

$$P(X = x) = p^x (1-p)^{1-x}, \quad x \in \{0, 1\}$$

2. Mean (Expected Value)

The expected value (mean) of a Bernoulli random variable is:

$E[X] = p$

3. Variance

The variance of a Bernoulli random variable is:

$Var(X) = p(1-p)$

Poisson Random Variable

$X \sim Poisson(\lambda)$

Probability Mass Function (PMF)

The probability that a Poisson random variable X takes the value k (i.e., exactly k events occur) is given by:

$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, 3, \ldots$

Where:

- $e \approx 2.718$ (Euler's number)
- λ = **expected number of events per interval**
- $k!$ = k factorial (e.g., 3!=3×2×1)

Mean and Variance

For a Poisson random variable $X \sim Poisson(\lambda)$:

- **Mean (Expectation):** $E[X] = \lambda$
- **Variance:** $Var(X) = \lambda$
- **Standard Deviation:** $\sigma = \lambda$

Continuous Distributions

Uniform Distribution

Probability Density Function (PDF)

Since there are infinitely many possible values, the probability of any **specific** value is **zero**. Instead, we compute probabilities over an interval.

The **PDF** is:

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

- **PDF**:

  $f(x) = \frac{1}{b-a} \; a \leq x \leq b$

- **Mean**:

  $E[X] = \frac{a+b}{2}$

- **Variance**:
  $Var(X) = \frac{(b-a)^2}{12}$

Normal Distribution (Gaussian Distribution)

Probability Density Function (PDF)

For a normal distribution with **mean** μ and **standard deviation** σ, the probability density function (PDF) is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Where:

- $e \approx 2.718$ (Euler's number).
- $\pi \approx 3.1416.$
- σ^2 is the **variance** (square of standard deviation).

Mean: $\mu = 0$.

Standard deviation: $\sigma = 1$

Any normal distribution $N(\mu, \sigma^2)$ can be converted into a **standard normal distribution** using the **Z-score transformation**:

$$Z = \frac{X - \mu}{\sigma}$$
