Probability

Introduction

- **Probability** measures the likelihood of an event occurring, helping us make predictions in uncertain situations.
- Examples: Rolling dice, weather forecasts, medical test accuracy.

Sample Space and Events of an Experiment

- **Experiment**: A process with an uncertain outcome (e.g., flipping a coin).
- **Sample Space (S)**: Set of all possible outcomes.
  - Example: Coin toss → S={H, T}.
- **Event (E)**: A subset of the sample space (e.g., getting heads).
- **Types of Events**:
  - **Simple event**: Contains only one outcome (e.g., rolling a 3 on a die).
  - **Compound event**: Contains multiple outcomes (e.g., rolling an even number).

🔷 **Key Point:** Events are **subsets** of the sample space.

Properties of Probability

- The probability of an event E, denoted as P(E), satisfies:
  1. **0 ≤ P(E) ≤ 1 (**probability is always between 0 and 1).
  2. **P(S) = 1** (something in the sample space must happen).
  3. **For mutually exclusive events** E1,E2,...:

     $$P(E1 \cup E2) = P(E1) + P(E2)$$

     (if two events can't happen together, their probabilities add up).

Key Point: If all outcomes are equally likely,

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total outcomes in } S}$$

Law	Definition	Example (Addition)	Example (Multiplication)
Commutative	Order does not matter	$A + B = B + A$	$A \times B = B \times A$
Associative	Grouping does not matter	$(A + B) + C = A + (B + C)$	$(A \times B) \times C = A \times (B \times C)$
Distributive	Multiplication distributes over addition	Not applicable	$A \times (B + C) = A \times B + A \times C$

Experiments Having Equally Likely Outcomes

- If an experiment has n equally likely outcomes and event E occurs in m ways, then:

  $P(E) = \frac{m}{n}$

- Example: Rolling a die
  - S={1,2,3,4,5,6}
  - Probability of rolling a 3: $P(3) = \frac{1}{6}$.
  - Probability of rolling an even number: $P(\text{even}) = \frac{3}{6} = 0.5$

🔷 **Key Point:** Used when **all outcomes have the same chance** of occurring

Conditional Probability and Independence

- **Conditional Probability**: Probability of event **A** given that event **B** has occurred.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

where:

- $P(A|B)$ is the probability of **A happening given that B has already happened**.
- $P(A \cap B)$ is the probability that **both A and B happen together**.
- $P(B)$ is the probability that **B happens**.
- The condition $P(B)$ > 0 ensures that we are not dividing by zero.

Example:

- Suppose you draw a card from a deck, and we know it is a **red** card (event **B**).
- What is the probability that the card is a **heart** (event **A**)?
- Since half of the red cards are hearts, we calculate : $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

- **Independence**: Two events A and B are independent if the occurrence of one does not affect the probability of the other. Mathematically, this is expressed as:

$$P(A \cap B) = P(A)P(B)$$

  - If this equation holds, then A and B are **independent**.
  - Otherwise, they are **dependent**, meaning that knowing one event affects the likelihood of the other.

  Example:

  - Rolling a die and flipping a coin:
    - Let A be getting a **3** on the die $(P(A) = \frac{1}{6})$.
    - Let B be getting **heads** on the coin $(P(B) = \frac{1}{2})$.
    - Since these events do not influence each other, their joint probability is:

      $P(A \cap B) = P(A)P(B) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

  This confirms **independence**.

🔷**Key Point:** If $P(A \mid B) = P(A)$, then **A** and **B** are **independent**.

Bayes' Theorem

Bayes' Theorem is a fundamental formula in probability that describes how to update our beliefs about an event based on new evidence. It is used extensively in statistics, machine learning, and decision-making.

Formula:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

where:

- $P(A|B)$ = **Posterior Probability** (probability of A given B, i.e., after observing B)
- $P(B|A)$ = **Likelihood** (probability of observing B if A is true)
- $P(A)$ = **Prior Probability** (initial belief about A before observing B)
- $P(B)$ = **Marginal Probability** (total probability of B happening)

Intuition Behind Bayes' Theorem

Bayes' Theorem helps us **update** our beliefs when we get new data. It tells us how likely an event A is **after** considering new evidence B.

💡**Example:** Medical Diagnosis

Suppose a test for a rare disease is **90% accurate**, meaning:

- If you have the disease, the test is positive **90%** of the time.
- If you don't have the disease, the test is positive **5%** of the time (false positives).
- The disease is rare: only **1%** of people have it.

We want to find: **If you test positive, what's the probability that you actually have the disease?**

Using Bayes' Theorem:

- $P(D) = 0.01$ (Prior: Probability of having the disease)
- $P(\neg D) = 0.99$ (Probability of NOT having the disease)
- $P(T|D) = 0.9$ (Likelihood: Probability of testing positive **if you have the disease**)

- $P(T|\neg D) = 0.05$ (False positive rate: Probability of testing positive **if you don't have the disease**)
- $P(T)$ (Total probability of testing positive) is:

  $P(T) = P(T \mid D)P(D) + P(T \mid \neg D)P(\neg D)$

  =(0.9×0.01)+(0.05×0.99)

  =0.009+0.0495

  =0.0585

Now applying Bayes' Theorem:

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)}$$

$$\frac{0.9 \times 0.01}{0.0585} = \frac{0.009}{0.0585} \approx 0.154$$

So even if you test **positive**, the probability of actually having the disease is **only 15.4%**! This happens because false positives are relatively common compared to the actual number of cases.

Why is Bayes' Theorem Important?

- Used in **spam filters** to determine whether an email is spam.
- Used in **medical diagnostics** to refine test results.
- Helps in **machine learning** and **AI** for probabilistic models (e.g., Naïve Bayes classifier).
- Applied in **finance, risk assessment, and decision-making**.

Counting Principles (Combinatorics)

- Used to determine the number of ways events can occur.
- **Multiplication Rule**: If an event can happen in m ways and another in n ways, total ways = $m \times n$.
- **Permutations** (Order Matters):

  $P(n,r) = \frac{n!}{(n-r)!}$

- **Combinations** (Order Doesn't Matter):

  $C(n, r) = \frac{n!}{r!(n-r)!}$

  - Example: Choosing 3 students from a group of 10 → $C(10, 3)$.

🔷 **Key Point:** Use **permutations when order matters and** combinations when **order doesn’t**.
