Discrete vs. Continuous: Distributions Explained

Discrete Distributions

Binomial Distribution

The **binomial distribution** is a **discrete probability distribution** that models the number of **successes** in a fixed number of independent trials, where each trial has only two possible outcomes: **success or failure**.

Conditions for Binomial Distribution

A random variable X follows a **binomial distribution** if:

1. There are **n independent trials**.
2. Each trial has **only two possible outcomes**:
   - **Success** (denoted as 1)
   - **Failure** (denoted as 0)
3. The probability of success is **fixed** and denoted by p, while the probability of failure is $1 - p$.
4. The trials are **independent**, meaning the outcome of one trial does not affect another.

Probability Mass Function (PMF)

The probability of getting exactly k successes in n trials is given by the **binomial formula**:

$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Where:

- $P(X = k)$ = Probability of exactly k successes
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ = Number of ways to choose k successes in n trials
- p^k = Probability of getting k **successes**

- k$(1-p)^{n-k}$ = Probability of getting $(n-k)$ **failures**

📌 **Example:**
A fair coin is tossed
10 times. What is the probability of getting exactly **4 heads**?

- n=10 (trials)
- k=4 (successes = heads)
- p=0.5 (probability of heads)

Using the binomial formula:

$$P(X=4) = \binom{10}{4}(0.5)^4(0.5)^6$$

$$= \frac{10!}{4!(6!)}(0.5)^{10}$$

$$= 210 \times 0.000976 = 0.205$$

So, the probability of exactly **4 heads** in **10 tosses** is **20.5%**.

Binomial Mean and Variance

For $X \sim Binomial(n, p)$:

- **Expected Value (Mean):**

  $$E[X] = np$$

- **Variance:**

  $$Var(X) = np(1-p)$$

- **Standard Deviation:**

  $$\sigma_X = \sqrt{np(1-p)}$$

📌 **Example:** If a basketball player makes 70% of free throws and takes **10 shots**:

- Expected successful shots:

  $$E[X] = 10 \times 0.7 = 7$$

- Variance:

  $$Var(X) = 10 \times 0.7 \times 0.3 = 2.1$$

- Standard Deviation:

  $$\sigma_X = \sqrt{2.1} \approx 1.45$$

So, on average, the player will make **7 shots**, but there is some variation.

Bernoulli Random Variable

A **Bernoulli random variable** is a special type of **discrete random variable** that represents a single trial with only **two possible outcomes**:

- **Success** (usually denoted as 1) with probability p.
- **Failure** (denoted as 0) with probability $1 - p$.

If X is a Bernoulli random variable, we write:

$X \sim Bernoulli(p)$

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1. Probability Mass Function (PMF)

Since X can take only two values (0 or 1), its **PMF** is:

$$P(X = x) = \begin{cases} p, & \text{if } x = 1 \text{ (success)} \\ 1 - p, & \text{if } x = 0 \text{ (failure)} \end{cases}$$

This can be written in compact form using exponents:

$$P(X = x) = p^x (1 - p)^{1-x}, \quad x \in \{0, 1\}$$

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2. Mean (Expected Value)

The expected value (mean) of a Bernoulli random variable is:

$E[X] = p$

This makes sense because if the probability of success is p, we expect X to take value **1** about pp fraction of the time.

📌 **Example:**

If a fair coin is flipped, with **success = heads** (p = 0.5), the expected value is:

E[X]=0.5

which means we expect heads **50% of the time**.

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3. Variance

The variance of a Bernoulli random variable is:

$$Var(X) = p(1 - p)$$

This shows that **as pp gets closer to 0 or 1**, variance decreases (less uncertainty).

📌 **Example:**

For a fair coin flip (p = 0.5):

$$Var(X) = 0.5(1 - 0.5) = 0.25$$

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4. Relationship to Binomial Distribution

The **Bernoulli distribution** is a **special case of the binomial distribution** with **n = 1**:

$$X \sim Binomial(n = 1, p)$$

So, a **single coin flip** (Bernoulli) is just a binomial experiment with one trial.

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5. Examples of Bernoulli Random Variables

✅ **Flipping a coin**: Heads (X = 1), Tails (X = 0).

✅ **Passing an exam**: Pass (X = 1), Fail (X = 0).

✅ **Website Clicks**: User clicks (X = 1), No click (X = 0).

✅ **Product Defects**: Defective (X = 1), Not defective (X = 0).

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6. Summary Table

Feature	Bernoulli Distribution
Definition	A single trial with success (1) or failure (0)
Probability of Success	P(X = 1) = p
Probability of Failure	P(X = 0) = 1 - p
Mean (Expectation)	E[X] = p
Variance	Var(X) = p(1 - p)
Special Case Of	Binomial Distribution (n = 1)

Poisson Random Variable

A **Poisson random variable** represents the number of times an event occurs in a fixed interval of time or space, given that:

1. Events occur **independently** of each other.
2. The average rate (λ) of occurrence is **constant** over time/space.
3. Two events cannot occur at the **exact same instant** (for very small intervals).

A **Poisson-distributed random variable** X is written as:

$$X \sim Poisson(\lambda)$$

Where λ (lambda) is the **mean number of occurrences** in the given interval.

Probability Mass Function (PMF)

The probability that a Poisson random variable X takes the value k (i.e., exactly k events occur) is given by:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, 3, \ldots$$

Where:

- $e \approx 2.718$ (Euler's number)
- λ = **expected number of events per interval**
- $k!$ = k factorial (e.g., 3!=3×2×1)

📌 **Example:**

Suppose calls arrive at a customer service center at an average rate of **3 calls per hour**. What is the probability of receiving **5 calls in an hour**?

Here, $\lambda = 3$ and k=5:

$$P(X = 5) = \frac{e^{-3}3^5}{5!}$$

$$= \frac{0.0498 \times 243}{120}$$

=0.1008

So, there is a **10.08% probability** of receiving exactly 5 calls in an hour.

Mean and Variance

For a Poisson random variable $X \sim Poisson(\lambda)$:

- **Mean (Expectation):** $E[X] = \lambda$
- **Variance:** $Var(X) = \lambda$
- **Standard Deviation:** $\sigma = \lambda$

📌 **Example:**

If a store gets an average of **10 customers per hour**, then:

- Expected customers per hour: E[X]=10
- Variance: Var(X)=10
- Standard deviation: $\sigma = \sqrt{10} \approx 3.16$

When to Use a Poisson Random Variable?

✅ **Counting the number of arrivals or events** over a fixed time or space.

✅ **Examples**:

- Number of **emails** received per hour.
- Number of **customers** entering a shop in a day.
- Number of **machine failures** in a factory per week.
- Number of **earthquakes** in a region per year.

Continuous Distributions

Uniform Distribution

A **uniform distribution** is a **probability distribution** where all outcomes are **equally likely** within a given range. This means that every value in the range has the **same probability** of occurring.

Probability Density Function (PDF)

Since there are infinitely many possible values, the probability of any **specific** value is **zero**. Instead, we compute probabilities over an interval.

The **PDF** is:

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

📌 **Example: Random Number Between 2 and 8**

Let $X \sim U(2, 8)$, meaning X follows a uniform distribution between 2 and 8.

- **PDF**:

  $f(x) = \frac{1}{8-2} = \frac{1}{6}$ for $2 \leq x \leq 8$
- **Mean**:

  $$E[X] = \frac{a+b}{2} = \frac{2+8}{2} = 5$$
- **Variance**:

  $Var(X) = \frac{(b-a)^2}{12} = \frac{(8-2)^2}{12} = \frac{36}{12} = 3$
- **Probability of X being between 4 and 6**:

  $$P(4 \leq X \leq 6) = \int_4^6 \frac{1}{6} dx = \frac{6-4}{6} = \frac{2}{6} = 0.333$$

  So, the probability that X is between 4 and 6 is **33.3%**.

When to Use a Uniform Distribution?

✅ When **all outcomes are equally likely**.

✅ When modeling **random numbers in a fixed range**.

✅ Common in **random sampling, simulation, and cryptography**.

Normal Distribution (Gaussian Distribution)

The **normal distribution**, also called the **Gaussian distribution**, is the most commonly used probability distribution in statistics. It describes **many real-world phenomena**, such as heights, IQ scores, measurement errors, and exam scores.

Properties of a Normal Distribution

✅ **Bell-shaped curve** (Symmetric around the mean).

✅ **Mean = Median = Mode** (all in the center).

✅ **Characterized by two parameters:**

- μ (mean) → **Center** of the distribution.
- σ^2 (variance) → **Spread** of the data.

✅ **Follows the Empirical Rule (68-95-99.7 Rule)** (explained below).

Probability Density Function (PDF)

For a normal distribution with **mean** μ and **standard deviation** σ, the probability density function (PDF) is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Where:

- $e \approx 2.718$ (Euler's number).
- $\pi \approx 3.1416.$
- σ^2 is the **variance** (square of standard deviation).

This function describes the likelihood of X taking a particular value. The **total area under the curve is 1**, meaning it represents a valid probability distribution.

Standard Normal Distribution (Z-Distribution)

A **special case** of the normal distribution is the **standard normal distribution**, which has:

- Mean: $\mu = 0.$
- Standard deviation: $\sigma = 1$

Any normal distribution $N(\mu, \sigma^2)$ can be converted into a **standard normal distribution** using the **Z-score transformation**:

$$Z = \frac{X - \mu}{\sigma}$$

where Z tells us how many standard deviations X is away from the mean.

Empirical Rule (68-95-99.7 Rule)

For a normal distribution:

✅ **68%** of data lies within **1 standard deviation** of the mean ($\mu \pm \sigma$).

✅ **95%** of data lies within **2 standard deviations** ($\mu \pm 2\sigma$).

✅ **99.7%** of data lies within **3 standard deviations** ($\mu \pm 3\sigma$).
