SEREȘ ARTUR

5/S10

$$f(x) = K \cdot e^{-\frac{x^2}{2}} \cdot x^{n-1}, \quad 0 \le x \le \infty$$

a) $K = ?$ pt ca $f(x)$ să fie densitate de repartiție

Pentru ca $f(x)$ să fie densitate de repartiție avem condițiile:

$$\begin{cases} f(x) \geq 0 \\ \int_{-\infty}^{\infty} f(x)\,dx = 1. \end{cases}$$

$$\Rightarrow \int_0^{\infty} K \cdot x^{n-1} \cdot e^{-\frac{x^2}{2}}\,dx = 1.$$

Notăm: $\boxed{\frac{x^2}{2} = u} \Rightarrow x = \sqrt{2u}$

$$\frac{1}{2} \cdot 2x\,dx = du \Rightarrow dx = \frac{du}{x}$$

pt capete: $x = 0 \Rightarrow u = 0$

$x = \infty \Rightarrow u = \infty$

$$K \cdot \int_0^{\infty} x^{n-1} \cdot e^{-u} \cdot \frac{du}{x} = 1$$

$$K \cdot \int_0^{\infty} x^{n-2} \cdot e^{-u}\,du = 1.$$

$$K \cdot \int_0^{\infty} (2u)^{\frac{n-2}{2}} \cdot e^{-u}\,du = 1.$$

$$K \cdot \int_0^{\infty} 2^{\frac{n-2}{2}} \cdot u^{\frac{n-2}{2}} \cdot e^{-u}\,du = 1.$$

$$K \cdot 2^{\frac{n-2}{2}} \int_0^{\infty} u^{\frac{n}{2}-1} \cdot e^{-u}\,du = 1,$$

dar Funcția Gamma: $\Gamma(t) = \int_0^{\infty} x^{t-1} \cdot e^{-x}\,dx$

$$\Rightarrow K \cdot 2^{\frac{n-2}{2}} \cdot \Gamma\left(\frac{n}{2}\right) = 1$$

$$\Rightarrow \boxed{K = \frac{1}{2^{\frac{n}{2}-1} \cdot \Gamma\left(\frac{n}{2}\right)}} \rightarrow \text{constanta căutată!}$$

b) Să se arate că $M(x)=\sqrt{2}\,\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$, și $M(x^2)=n$

$$M(x)=\int_{-\infty}^{\infty} x\cdot f(x)\,dx$$

$$M(x)=\int_{0}^{\infty} k\cdot e^{-\frac{x^2}{2}}\cdot x^n\,dx$$

$$M(x)=k\cdot\int_{0}^{\infty} x^n\cdot e^{-\frac{x^2}{2}}\,dx$$

Notăm $\boxed{\frac{x^2}{2}=u}\Rightarrow x=\sqrt{2u}$

$$\frac{1}{2}\cdot 2x\,dx=du \Rightarrow dx=\frac{du}{x}$$

pt capete: $\begin{cases} x=0\Rightarrow u=0\\ x=\infty\Rightarrow u=\infty\end{cases}$

$$M(x)=k\cdot\int_{0}^{\infty} x^n\cdot e^{-u}\cdot\frac{du}{x}$$

$$M(x)=k\cdot\int_{0}^{\infty} x^{n-1}\cdot e^{-u}\,du$$

$$M(x)=k\cdot\int_{0}^{\infty}(2u)^{\frac{n-1}{2}}\cdot e^{-u}\,du$$

$$M(x)=k\cdot 2^{\frac{n-1}{2}}\cdot\int_{0}^{\infty} u^{\frac{n+1}{2}-1}\cdot e^{-u}\,du;$$

dar $\Gamma(t)=\int_{0}^{\infty} x^{t-1}\cdot e^{-x}\,dx$

$$\Rightarrow M(x)=k\cdot 2^{\frac{n-1}{2}}\cdot\Gamma\left(\frac{n+1}{2}\right),$$

dar $k=\dfrac{1}{2^{\frac{n-2}{2}}\cdot\Gamma\left(\frac{n}{2}\right)}$ → din punctul a)

$$\Rightarrow \boxed{M(x)=\sqrt{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}}$$

$$M(x^2)=\int_{-\infty}^{\infty} x^2\cdot f(x)\,dx$$

$$M(x^2)=\int_{0}^{\infty} x^2\cdot k\cdot e^{-\frac{x^2}{2}}\cdot x^{n-1}\,dx$$

$$M(x^2)=k\cdot\int_{0}^{\infty} x^{n+1}\cdot e^{-\frac{x^2}{2}}\,dx$$

Notăm: $\boxed{\frac{x^2}{2}=u} \Rightarrow x=\sqrt{2u}$

$$\frac{1}{2}\cdot 2x\,dx=du \Rightarrow dx=\frac{du}{x}$$

$$M(x^2)=k\cdot\int_{0}^{\infty} x^{n+1}\cdot e^{-u}\cdot\frac{du}{x}$$

$$M(x^2)=k\int_{0}^{\infty}(2u)^{\frac{n}{2}}\cdot e^{-u}\,du$$

$$M(x^2)=k\cdot 2^{\frac{n}{2}}\cdot\int_{0}^{\infty} u^{\frac{n+2}{2}-1}\cdot e^{-u}\,du$$

$$M(x^2)=k\cdot 2^{\frac{n}{2}}\cdot\Gamma\left(\frac{n+2}{2}\right),$$

dar $k=\dfrac{1}{2^{\frac{n-2}{2}}\cdot\Gamma\left(\frac{n}{2}\right)}$

$$M(x^2)=\frac{1}{2^{\frac{n-2}{2}}\cdot\Gamma\left(\frac{n}{2}\right)}\cdot 2^{\frac{n}{2}}\cdot\Gamma\left(\frac{n+2}{2}\right),$$

dar din Proprietățile Funcției Gamma avem:

$\Gamma(t+1)=t\cdot\Gamma(t)$, adică $\Gamma\left(\frac{n+2}{2}\right)=\frac{n}{2}\cdot\Gamma\left(\frac{n}{2}\right)$

$$\Rightarrow M(x^2)=\frac{1}{2^{\frac{n-2}{2}}\cdot\Gamma\left(\frac{n}{2}\right)}\cdot 2^{\frac{n}{2}}\cdot\frac{n}{2}\cdot\Gamma\left(\frac{n}{2}\right)$$

$$\boxed{M(x^2)=n}$$

SEREȘ ARTUR

4/S10

$$f(x) = \begin{cases} K\left(1+\frac{a}{2}x\right)^{\frac{4}{a^2}-1} \cdot e^{-\frac{2x}{a}}, & -\frac{2}{a} \le x < \infty \\ 0, & \text{în rest} \end{cases}$$

Pentru ca $f(x)$ să fie densitate de repartiție trebuie respectate condițiile:

$$\begin{cases} f(x) \geq 0 \\ \int_{-\infty}^{\infty} f(x)\,dx = 1. \end{cases}$$

Dorim să determinăm valoarea lui K:

$$\int_{-\frac{2}{a}}^{\infty} K\left(1+\frac{a}{2}x\right)^{\frac{4}{a^2}-1} \cdot e^{-\frac{2x}{a}}\,dx = 1$$

$$K\int_{-\frac{2}{a}}^{\infty} \left(1+\frac{a}{2}x\right)^{\frac{4}{a^2}-1} \cdot e^{-\frac{2x}{a}}\,dx = 1.$$

Notăm $\boxed{\frac{a}{2}\cdot x+1 = u} \Rightarrow x = \frac{2(u-1)}{a}$

$$\frac{a}{2}dx = du \Rightarrow dx = \frac{2}{a}du$$

Am făcut această substituție pentru a schimba capetele integralei și a aplica mai apoi funcția Gamma.

$$\left.\begin{array}{l} x = -\frac{2}{a} \Rightarrow u = 0 \\ x = \infty \Rightarrow u = \infty \end{array}\right\} \text{pt capetele integralei}$$

$$K\int_{0}^{\infty} (1+u-1)^{\left(\frac{2}{a}\right)^2-1} \cdot e^{-\left(\frac{2}{a}\cdot\frac{2(u-1)}{a}\right)} \frac{2}{a}\,du = 1$$

$$K\cdot\frac{2}{a}\int_0^{\infty} u^{\left(\frac{2}{a}\right)^2-1} \cdot e^{\boxed{\left(\frac{2}{a}\right)^2(u-1)}}\,du = 1.$$

$\rightarrow$ de aici NU mai știu!

Obs: Dacă în enunț este:

$$K\left(1+\frac{a}{2}x\right)^{\frac{4}{a^2}-1}\cdot e^{-\frac{ax}{2}},\ \text{în loc de } e^{-\frac{2x}{a}},$$

atunci:

$$K\cdot\frac{2}{a}\int_0^\infty u^{\left(\frac{2}{a}\right)^2-1}\cdot e^{u-1}\,du = 1$$

$$\Gamma(t)=\int_0^\infty x^{t-1}\cdot e^{-x}\,dx = 1$$

$$K\cdot\frac{2}{a}\int_0^\infty u^{\left(\frac{a}{2}\right)^2-1}\cdot\frac{e^{u}}{e}\,du = 1.$$

$$\frac{K\cdot 2}{a\cdot e}\cdot\Gamma\left(\left(\frac{a}{2}\right)^2\right) = 1.$$

$$\Rightarrow \boxed{K=\frac{a\cdot e}{2\cdot\Gamma\left(\left(\frac{a}{2}\right)^2\right)}},$$

Nu știu dacă enunțul problemei este corect, cu modificarea scrisă cu roșu iese frumos! Dar ideea de început este bună.