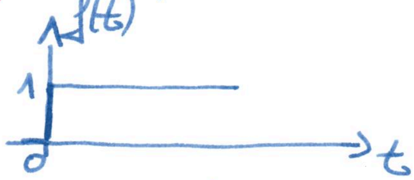
Transformata Laplace

Definiție: În matematică, transformata Laplace este o integrală care convertește o funcție a unei variabile reale t (de obicei t=timp) într-o funcție a unei variabile complexe s (în domeniul frecvenței, cunoscut și sub numele de domeniul s sau planul s). Transformarea are multe aplicații în știință și inginerie, deoarece este un instrument pentru rezolvarea ecuațiilor diferențiale, în special transformă ecuațiile diferențiale în ecuații algebrice.

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt}$$

$\mathcal{L}$ - operator liniar

① $f(t) = \nabla(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$



$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\nabla(t)\} = F(s) = \int_0^{\infty} \nabla(t)\, e^{-st}\, dt = \int_0^{\infty} e^{-st}\, dt = -\frac{1}{s}\int_0^{\infty} -s e^{-st}\, dt =$$

$(e^{-st})' = -s e^{-st}$

$$= -\frac{1}{s}\int_0^{\infty} (e^{-st})'\, dt = -\frac{1}{s} e^{-st}\Big|_0^{\infty}$$

$$= -\frac{1}{s}\Big(\cancel{\lim_{t\to\infty} e^{-st}}^{\,0} - \lim_{t\to 0} e^{-st}\Big)$$

$$= -\frac{1}{s}(0-1) = \frac{1}{s} \Rightarrow$$

$$\boxed{\mathcal{L}\{f(t)\} = \mathcal{L}\{\nabla(t)\} = F(s) = \frac{1}{s}}$$

② $f(t) = e^{-at} \Rightarrow \mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-at} e^{-st}\, dt = \int_0^{\infty} e^{-(s+a)t}\, dt \Rightarrow$

$\left[e^{-(s+a)t}\right]' = -(s+a)e^{-(s+a)t}$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-(s+a)t}\, dt = \frac{-1}{s+a}\int_0^{\infty} -(s+a)e^{-(s+a)t}\, dt = -\frac{1}{s+a}\int_0^{\infty}\left[e^{-(s+a)t}\right]' dt$$

$$= -\frac{1}{s+a} e^{-(s+a)t}\Big|_0^{\infty} = \frac{-1}{s+a}\Big(\cancel{\lim_{t\to\infty} e^{-(s+a)t}}^{\,0} - \lim_{t\to 0} e^{-(s+a)t}\Big) = \frac{1}{s+a} \Rightarrow$$

$$\boxed{\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-at}\} = F(s) = \frac{1}{s+a}}$$

③ $f(t)=f(t-m) \Rightarrow \mathcal{L}\{f(t)\}=F(s)=\int_0^\infty f(t-m)e^{-st}dt=\int_{-m}^{\infty} f(\tau)e^{-s(\tau+m)}d\tau$

notăm: $t-m=\tau \Rightarrow t=\tau+m \Rightarrow dt=d\tau$

$t\to\infty \Rightarrow \tau\to\infty$

$t\to 0 \Rightarrow \tau\to -m$

$$\Rightarrow \mathcal{L}\{f(t)\}=e^{-sm}\int_{-m}^{\infty} f(\tau)e^{-s\tau}d\tau=e^{-sm}\left[\cancel{\int_{-m}^{0} f(\tau)e^{-s\tau}d\tau}+\int_0^\infty f(\tau)e^{-s\tau}d\tau\right]=$$

$$=e^{-sm}\int_0^\infty f(\tau)e^{-s\tau}d\tau=e^{-sm}\mathcal{L}\{f(\tau)\}=e^{-sm}F(s)\Rightarrow$$

$$\boxed{\mathcal{L}\{f(t)\}=\mathcal{L}\{f(t-m)\}=e^{-sm}F(s)}$$

④ $$\mathcal{L}\{f'(t)\}=\int_0^\infty f'(t)e^{-st}dt=f(t)e^{-st}\Big|_0^\infty-\int_0^\infty f(t)(e^{-st})'dt=$$

$$=\underbrace{\lim_{t\to\infty}[f(t)e^{-st}]}_{\cong 0}-\underbrace{\lim_{t\to 0}[f(t)e^{-st}]}_{f(0)}-\int_0^\infty f(t)(-se^{-st})dt=-f(0)+s\underbrace{\int_0^\infty f(t)e^{-st}dt}_{F(s)}$$

$$\Rightarrow \boxed{\mathcal{L}\{f'(t)\}=sF(s)-f(0)}$$

$$\mathcal{L}\{f''(t)\}=\int_0^\infty (f'(t))'e^{-st}dt=f'(t)e^{-st}\Big|_0^\infty-\int_0^\infty f'(t)(e^{-st})'dt=$$

$$=\cancel{\lim_{t\to\infty}[f'(t)e^{-st}]}-\underbrace{\lim_{t\to 0}[f'(t)e^{-st}]}_{f'(0)}+s\underbrace{\int_0^\infty f'(t)e^{-st}dt}_{\mathcal{L}\{f'(t)\}}=s\mathcal{L}\{f'(t)\}-f'(0)=$$

$$=s[sF(s)-f(0)]-f'(0)=s^2F(s)-sf(0)-f'(0)\Rightarrow$$

$$\boxed{\mathcal{L}\{f''(t)\}=s^2F(s)-sf(0)-f'(0)}$$

$$\mathcal{L}\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(0)-\dots-sf^{(n-2)}(0)-f^{(n-1)}(0)\Rightarrow$$

$$\boxed{\mathcal{L}\{f^{(n)}(t)\}=s^nF(s)-\sum_{k=1}^{n}s^{n-k}f^{(k-1)}(0)}$$

⑤ Rezolvați ecuația diferențială $y'(t)+by(t)=1$, $y(0)=y_0=0$

$\mathcal{L}\{y'(t)+by(t)\}=\mathcal{L}\{1\} \Rightarrow$ proprietatea de liniaritate

$\mathcal{L}\{af(t)+bg(t)\}=a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}=aF(s)+bG(s) \Rightarrow$

$\mathcal{L}\{y'(t)\}+b\mathcal{L}\{y(t)\}=\mathcal{L}\{1\} \Rightarrow sY(s)-y(0)+bY(s)=\frac{1}{s} \Rightarrow Y(s)(s+b)=\frac{1}{s}+y(0)$

$\Rightarrow Y(s)(s+b)=\frac{1}{s} \Rightarrow Y(s)=\frac{1}{s(s+b)} \Rightarrow \mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s(s+b)}\right\}=y(t)$

$\frac{1}{s(s+b)}=\frac{A}{s}+\frac{B}{s+b} \Rightarrow A(s+b)+Bs=1 \Rightarrow As+Ab+Bs=1 \Rightarrow$

$s(A+B)+Ab=1 \Rightarrow \begin{cases}A+B=0\\ Ab=1\end{cases} \Rightarrow \boxed{A=\frac{1}{b}} \Rightarrow \boxed{B=-A=-\frac{1}{b}}$

$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s(s+b)}\right\}=\mathcal{L}^{-1}\left\{\frac{1/b}{s}-\frac{1/b}{s+b}\right\} \Rightarrow$

$y(t)=\mathcal{L}^{-1}\left\{\frac{1/b}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{1/b}{s+b}\right\}=\frac{1}{b}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{b}\mathcal{L}^{-1}\left\{\frac{1}{s+b}\right\}=\frac{1}{b}-\frac{1}{b}e^{-bt}$

$\Rightarrow \boxed{y(t)=\frac{1}{b}\left(1-e^{-bt}\right)}$

6) Rezolvați ecuația diferențială $y''(t)-2y'(t)+y(t)=3e^{t}$, $y(0)=1$, $y'(0)=1$

$\mathcal{L}\{y''(t)-2y'(t)+y(t)\}=\mathcal{L}\{3e^{t}\} \Rightarrow$

$\mathcal{L}\{y''(t)\}-2\mathcal{L}\{y'(t)\}+\mathcal{L}\{y(t)\}=3\mathcal{L}\{e^{t}\} \Rightarrow$

$s^2Y(s)-sy(0)-y'(0)-2\left[sY(s)-y(0)\right]+Y(s)=\frac{3}{s-1} \Rightarrow$

$s^2Y(s)-s-1-2sY(s)+2+Y(s)=\frac{3}{s-1} \Rightarrow Y(s)(s^2-2s+1)-s+1=\frac{3}{s-1} \Rightarrow$

$Y(s)(s^2-2s+1)=\frac{3}{s-1}+(s-1)=\frac{3+(s-1)^2}{s-1}=\frac{3+s^2-2s+1}{s-1}=\frac{s^2-2s+4}{s-1} \Rightarrow$

$Y(s)=\frac{s^2-2s+4}{(s-1)^3} \Rightarrow \mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{s^2-2s+4}{(s-1)^3}\right\}=y(t)$

$\frac{s^2-2s+4}{(s-1)^3}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3} \Rightarrow A(s-1)^2+B(s-1)+C=s^2-2s+4$

$A(s^2-2s+1)+B(s-1)+C=s^2-2s+4$

$As^2 - \underline{2As} + A + \underline{Bs} - B + C = s^2 - 2s + 4 \Rightarrow s^2 A + s(-2A + B) + A + C = s^2 - 2s + 4 \Rightarrow$

$\begin{cases} A = 1 \\ -2A + B = -2 \\ A + C = 4 \end{cases} \Rightarrow \boxed{A = 1} \quad -2 + B = -2 \Rightarrow \boxed{B = 0}$

$C = 4 - A \Rightarrow \boxed{C = 3}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{s^2 - 2s + 4}{(s-1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{3}{(s-1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} =$

$= e^t + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\} = e^t + \frac{3}{2}t^2 e^t \Rightarrow \boxed{y(t) = e^t + \frac{3}{2}t^2 e^t}$

$\boxed{\text{Temă}}$: pag 9/3 b) Folosiți transformata Laplace pentru a rezolva următoarea ecuație diferențială:

$\begin{array}{l} y''(t) + 4y(t) = \sin(2t) \\ y(0) = 1,\ y'(0) = 0 \end{array} \Rightarrow \mathcal{L}\{y''(t) + 4y(t)\} = \mathcal{L}\{\sin(2t)\} \Rightarrow$

$\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = \frac{2}{s^2+4} \Rightarrow s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = \frac{2}{s^2+4}$

$Y(s)(s^2 + 4) - s = \frac{2}{s^2+4} \Rightarrow (s^2+4)Y(s) = \frac{2}{s^2+4} + s \Rightarrow (s^2+4)Y(s) = \frac{s^3 + 4s + 2}{s^2+4}$

$\Rightarrow Y(s) = \frac{s^3 + 4s + 2}{(s^2+4)^2} \Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{y(t)\}\} = y(t)$

$\frac{s^3 + 4s + 2}{(s^2+4)^2} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{(s^2+4)^2} \Rightarrow (As + B)(s^2 + 4) + Cs + D = s^3 + 4s + 2$

$\Rightarrow \underline{As^3} + \underline{4As} + \underline{Bs^2} + 4B + \underline{Cs} + D = s^3 + 4s + 2 \Rightarrow As^3 + Bs^2 + s(4A + C) + 4B + D = s^3 + 4s + 2$

$\begin{cases} \boxed{A = 1} \\ \boxed{B = 0} \\ 4A + C = 4 \Rightarrow 4 + C = 4 \Rightarrow \boxed{C = 0} \\ 4B + D = 2 \Rightarrow \boxed{D = 2} \end{cases}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)^2}\right\} = \cos(2t) + \mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)^2}\right\} \Rightarrow$

$y(t) = \cos(2t) + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{2 \cdot 2^3}{(s^2 + 2^2)^2}\right\} \Rightarrow y(t) = \cos(2t) + \frac{1}{8}\left[\sin(2t) - 2t\cos(2t)\right]$

$\mathcal{L}^{-1}\left\{\frac{2a^3}{(s^2+a^2)^2}\right\} = \sin(2t) - 2t\cos(2t)$

$y(t) = \cos(2t) - \frac{t}{4}\cos(2t) + \frac{1}{8}\sin(2t)$

$\boxed{y(t) = \left(1 - \frac{t}{4}\right)\cos(2t) + \frac{1}{8}\sin(2t)}$

Transformata Z

Definiție: În matematică și în domeniul prelucrării semnalelor, transformata Z convertește un semnal în timp discret, care este o secvență de numere reale sau complexe, într-o reprezentare complexă în domeniul frecvenței (domeniul Z sau planul Z). Poate fi considerat ca un echivalent în timp discret al transformării Laplace.

$$\boxed{Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}}$$

Progresie geometrică:

- suma primilor n termeni: - dacă $\boxed{q \neq 1 : S_n = b_1 + b_2 + \dots + b_n = b_1 \frac{q^n - 1}{q - 1}, \forall n \geq 1}$

q = rația progresiei geometrice

b_1 = primul termen

n = numărul termenilor

- dacă $\boxed{q = 1 : S_n = b_1 + b_2 + \dots + b_n = n b_1, \forall n \geq 1}$

① $f(k) = \sigma(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

$$Z\{f(k)\} = Z\{\sigma(k)\} = \sum_{k=0}^{\infty} \sigma(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k} = z^0 + z^{-1} + z^{-2} + \dots + z^{-n}, n \to \infty$$

$$Z\{f(k)\} = 1 + z^{-1} + z^{-2} + \dots + z^{-n} = 1 \cdot \frac{\cancel{(z^{-1})^{n+1}}^{\,0} - 1}{z^{-1} - 1} = \frac{-1}{z^{-1} - 1} = \frac{1}{1 - z^{-1}} \Rightarrow$$

$$\boxed{Z\{f(k)\} = Z\{\sigma(k)\} = \frac{1}{1 - z^{-1}}}$$

② $f(k) = e^{-ak} \Rightarrow Z\{f(k)\} = Z\{e^{-ak}\} = \sum_{k=0}^{\infty} e^{-ak} \cdot z^{-k} = \sum_{k=0}^{\infty} (e^{-a} \cdot z^{-1})^k =$

$$= (e^{-a} z^{-1})^0 + (e^{-a} z^{-1})^1 + (e^{-a} z^{-1})^2 + \dots + (e^{-a} z^{-1})^n, n \to \infty$$

$$Z\{f(k)\} = Z\{e^{-ak}\} = 1 + (e^{-a} z^{-1}) + (e^{-a} z^{-1})^2 + \dots + (e^{-a} z^{-1})^n = 1 \cdot \frac{\cancel{(e^{-a} z^{-1})^{n+1}}^{\,0} - 1}{e^{-a} z^{-1} - 1} =$$

$$= -\frac{1}{e^{-a} z^{-1} - 1} = \frac{1}{1 - e^{-a} z^{-1}} \Rightarrow \boxed{Z\{f(k)\} = Z\{e^{-ak}\} = \frac{1}{1 - e^{-a} z^{-1}}}$$

③ $\mathcal{Z}\{f(k+1)\} = \sum_{k=0}^{\infty} f(k+1) z^{-k} = \sum_{\tau=1}^{\infty} f(\tau) z^{-(\tau-1)} = \sum_{\tau=1}^{\infty} f(\tau) z^{-\tau} \cdot z =$

notăm $k+1=\tau \Rightarrow k=\tau-1$

$k\to\infty \Rightarrow \tau\to\infty$

$k\to 0 \Rightarrow \tau \to 1$

$= z\sum_{\tau=1}^{\infty} f(\tau) z^{-\tau} = z\left[f(1)z^{-1} + f(2)z^{-2} + \dots + f(n)z^{-n}\right],\ n\to\infty$

$\Rightarrow \mathcal{Z}\{f(k+1)\} = z\left[\underbrace{f(0) + f(1)z^{-1} + f(2)z^{-2} + \dots + f(n)z^{-n}}_{F(z)} - f(0)\right] = z\left(F(z) - f(0)\right)$

$\Rightarrow \boxed{\mathcal{Z}\{f(k+1)\} = z\left[F(z) - f(0)\right]}$

$\mathcal{Z}\{f(k+2)\} = \sum_{k=0}^{\infty} f(k+2) z^{-k} = \sum_{\tau=2}^{\infty} f(\tau) z^{-(\tau-2)} = z^{2}\sum_{\tau=2}^{\infty} f(\tau)\cdot z^{-\tau} =$

$k+2=\tau \Rightarrow k=\tau-2$

$= z^{2}\left[f(2)z^{-2} + f(3)z^{-3} + \dots + f(n)z^{-n}\right],\ n\to\infty$

$k\to\infty \Rightarrow \tau\to\infty$

$k\to 0 \Rightarrow \tau\to 2$

$\mathcal{Z}\{f(k+2)\} = z^{2}\left[\underbrace{f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \dots + f(n)z^{-n}}_{F(z)} - f(0) - f(1)z^{-1}\right] \Rightarrow$

$\boxed{\mathcal{Z}\{f(k+2)\} = z^{2}\left[F(z) - f(0) - f(1)z^{-1}\right]}$

$\mathcal{Z}\{f(k+n)\} = z^{n}\left[F(z) - f(0) - f(1)z^{-1} - \dots - f(n-1)z^{-(n-1)}\right] = z^{n}\left[F(z) - \sum_{k=0}^{n-1} f(k) z^{-k}\right]$

$\Rightarrow \boxed{\mathcal{Z}\{f(k+n)\} = z^{n}\left[F(z) - \sum_{k=0}^{n-1} f(k) z^{-k}\right]}$

④ Rezolvați ecuația diferențială folosind transformata $\mathcal{Z}$:

$y(k+1) - a y(k) = 0,\quad y(0) = y_0$

$\mathcal{Z}\{y(k+1) - a y\{k\}\} = \mathcal{Z}\{0\} \Rightarrow \mathcal{Z}\{y(k+1)\} - a\mathcal{Z}\{y(k)\} = 0 \Rightarrow$

$z\left[Y(z) - y(0)\right] - a Y(z) = 0 \Rightarrow z Y(z) - z y_0 - a Y(z) = 0 \Rightarrow Y(z)(z-a) = z y_0 \Rightarrow$

$Y(z) = \frac{z y_0}{z-a} = y_0 \frac{1}{1-az^{-1}} \Rightarrow \mathcal{Z}^{-1}\{\mathcal{Z}\{y(k)\}\} = \mathcal{Z}^{-1}\{Y(z)\} = y(k) \Rightarrow$

$y(k) = \mathcal{Z}^{-1}\left\{y_0 \frac{1}{1-az^{-1}}\right\} = y_0 \mathcal{Z}^{-1}\left\{\frac{1}{1-az^{-1}}\right\} = y_0 a^{k} \Rightarrow \boxed{y(k) = y_0 a^{k}}$

⑤ Rezolvați ecuația diferențială de ordinul 2 folosind transformata Z:

$$y(k+2)-18y(k+1)+32y(k)=0,\ y(0)=0,\ y(1)=2$$

$$\mathcal{Z}\{y(k+2)-18y(k+1)+32y(k)\}=\mathcal{Z}\{0\} \Rightarrow \mathcal{Z}\{y(k+2)\}-18\mathcal{Z}\{y(k+1)\}+32\mathcal{Z}\{y(k)\}=0$$

$$\Rightarrow z^2\left[Y(z)-y(0)-y(1)z^{-1}\right]-18z\left[Y(z)-y(0)\right]+32Y(z)=0 \Rightarrow$$

$$z^2\left[Y(z)-2z^{-1}\right]-18zY(z)+32Y(z)=0 \Rightarrow z^2Y(z)-2z-18zY(z)+32Y(z)=0$$

$$Y(z)(z^2-18z+32)=2z \Rightarrow Y(z)=\frac{2z}{z^2-18z+32}=\frac{2z}{(z-16)(z-2)}=\frac{2z}{z^2(1-16z^{-1})(1-2z^{-1})}$$

$$Y(z)=\frac{2z^{-1}}{(1-2z^{-1})(1-16z^{-1})} \Rightarrow y(k)=\mathcal{Z}^{-1}\{Y(z)\}=\mathcal{Z}^{-1}\{\mathcal{Z}\{y(k)\}\} \Rightarrow$$

$$y(k)=\mathcal{Z}^{-1}\left\{\frac{2z^{-1}}{(1-2z^{-1})(1-16z^{-1})}\right\}$$

$$\frac{2z^{-1}}{(1-2z^{-1})(1-16z^{-1})}=\frac{A}{1-2z^{-1}}+\frac{B}{1-16z^{-1}} \Rightarrow A(1-16z^{-1})+B(1-2z^{-1})=2z^{-1}$$

$$A-16Az^{-1}+B-2Bz^{-1}=2z^{-1} \Rightarrow$$

$$\begin{cases} A+B=0 \\ -16A-2B=2 \end{cases} \Rightarrow \begin{array}{r} A+B=0 \\ -8A-B=1 \\ \hline -7A \quad /=1 \end{array} \Rightarrow \boxed{A=-\frac{1}{7}} \Rightarrow \boxed{B=-A=\frac{1}{7}}$$

$$y(k)=\mathcal{Z}^{-1}\left\{\frac{-1/7}{1-2z^{-1}}\right\}+\mathcal{Z}^{-1}\left\{\frac{1/7}{1-16z^{-1}}\right\}=-\frac{1}{7}\mathcal{Z}^{-1}\left\{\frac{1}{1-2z^{-1}}\right\}+\frac{1}{7}\mathcal{Z}^{-1}\left\{\frac{1}{1-16z^{-1}}\right\} \Rightarrow$$

$$\boxed{y(k)=-\frac{1}{7}2^k+\frac{1}{7}16^k}$$