

# Deep Generative Models

## Lecture 8

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## Recap of previous lecture

- ▶ **Generator:** generative model  $\mathbf{x} = \mathbf{G}(\mathbf{z})$ , which makes generated sample more realistic.
- ▶ **Discriminator:** a classifier  $D(\mathbf{x}) \in [0, 1]$ , which distinguishes real samples from generated samples.

### GAN optimality theorem

The minimax game

$$\min_G \max_D \underbrace{\left[ \mathbb{E}_{\pi(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D(\mathbf{G}(\mathbf{z}))) \right]}_{V(G,D)}$$

has the global optimum  $\pi(\mathbf{x}) = p(\mathbf{x}|\theta)$ , in this case  $D^*(\mathbf{x}) = 0.5$ .

$$\min_G V(G, D^*) = \min_G [2JSD(\pi||p) - \log 4] = -\log 4, \quad \pi(\mathbf{x}) = p(\mathbf{x}|\theta).$$

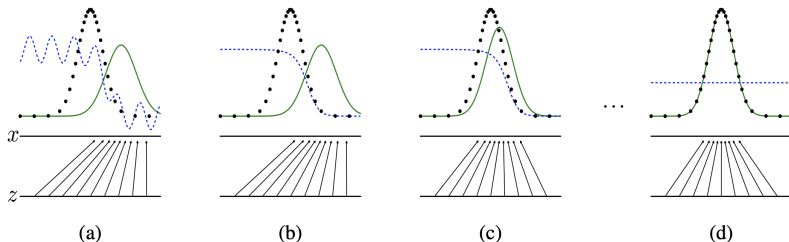
If the generator could be **any** function and the discriminator is **optimal** at every step, then the generator is **guaranteed to converge** to the data distribution.

## Recap of previous lecture

- ▶ Generator updates are made in parameter space, discriminator is not optimal at every step.
- ▶ Generator and discriminator loss keeps oscillating during GAN training.

### Objective

$$\min_{\theta} \max_{\phi} [\mathbb{E}_{\pi(\mathbf{x})} \log D_{\phi}(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D_{\phi}(\mathbf{G}_{\theta}(\mathbf{z})))]$$



# Recap of previous lecture

## Main problems of standard GAN

- ▶ Vanishing gradients (solution: non-saturating GAN);
- ▶ Mode collapse (caused by Jensen-Shannon divergence).

## Standard GAN

$$\min_{\theta} \max_{\phi} [\mathbb{E}_{\pi(\mathbf{x})} \log D_{\phi}(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D_{\phi}(\mathbf{G}_{\theta}(\mathbf{z})))]$$

## Informal theoretical results

The real images distribution  $\pi(\mathbf{x})$  and the generated images distribution  $p(\mathbf{x}|\theta)$  are low-dimensional and have disjoint supports. In this case

$$KL(\pi||p) = KL(p||\pi) = \infty, \quad JSD(\pi||p) = \log 2.$$

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*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*  
*Arjovsky M., Bottou L. Towards Principled Methods for Training Generative Adversarial Networks, 2017*

# Recap of previous lecture

## Wasserstein distance

$$W(\pi, p) = \inf_{\gamma \in \Gamma(\pi, p)} \mathbb{E}_{(\mathbf{x}, \mathbf{y}) \sim \gamma} \|\mathbf{x} - \mathbf{y}\| = \inf_{\gamma \in \Gamma(\pi, p)} \int \|\mathbf{x} - \mathbf{y}\| \gamma(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$$

- ▶  $\gamma(\mathbf{x}, \mathbf{y})$  – transportation plan (the amount of "dirt" that should be transported from point  $\mathbf{x}$  to point  $\mathbf{y}$ ).
- ▶  $\Gamma(\pi, p)$  – the set of all joint distributions  $\gamma(\mathbf{x}, \mathbf{y})$  with marginals  $\pi$  and  $p$  ( $\int \gamma(\mathbf{x}, \mathbf{y}) d\mathbf{x} = p(\mathbf{y})$ ,  $\int \gamma(\mathbf{x}, \mathbf{y}) d\mathbf{y} = \pi(\mathbf{x})$ ).
- ▶  $\gamma(\mathbf{x}, \mathbf{y})$  – the amount,  $\|\mathbf{x} - \mathbf{y}\|$  – the distance.

## Theorem (Kantorovich-Rubinstein duality)

$$W(\pi || p) = \frac{1}{K} \max_{\|f\|_L \leq K} [\mathbb{E}_{\pi(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p(\mathbf{x})} f(\mathbf{x})],$$

where  $\|f\|_L \leq K$  are  $K$ -Lipschitz continuous functions ( $f : \mathcal{X} \rightarrow \mathbb{R}$ ).

# Outline

1. Lipschitzness of Wasserstein GAN critic  
Wasserstein GAN  
WGAN with Gradient Penalty
2. f-divergence minimization
3. Evaluation of likelihood-free models  
Frechet Inception Distance (FID)

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# Wasserstein GAN

## Theorem (Kantorovich-Rubinstein duality)

$$W(\pi||p) = \frac{1}{K} \max_{\|f\|_L \leq K} [\mathbb{E}_{\pi(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p(\mathbf{x})} f(\mathbf{x})],$$

- ▶ Now we have to ensure that  $f$  is  $K$ -Lipschitz continuous.
- ▶ Let  $f_\phi(\mathbf{x})$  be a feedforward neural network parametrized by  $\phi$ .
- ▶ If parameters  $\phi$  lie in a compact set  $\Phi$  then  $f_\phi(\mathbf{x})$  will be  $K$ -Lipschitz continuous function.
- ▶ Let the parameters be clamped to a fixed box  $\Phi \in [-c, c]^d$  (e.x.  $c = 0.01$ ) after each gradient update.

$$\begin{aligned} K \cdot W(\pi||p) &= \max_{\|f\|_L \leq K} [\mathbb{E}_{\pi(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p(\mathbf{x})} f(\mathbf{x})] \geq \\ &\geq \max_{\phi \in \Phi} [\mathbb{E}_{\pi(\mathbf{x})} f_\phi(\mathbf{x}) - \mathbb{E}_{p(\mathbf{x})} f_\phi(\mathbf{x})] \end{aligned}$$



# Wasserstein GAN

## Standard GAN objective

$$\min_{\theta} \max_{\phi} \mathbb{E}_{\pi(\mathbf{x})} \log D_{\phi}(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D_{\phi}(\mathbf{G}_{\theta}(\mathbf{z})))$$

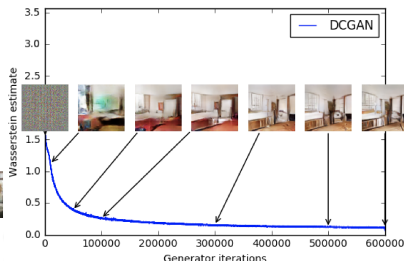
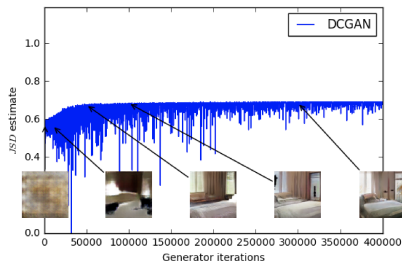
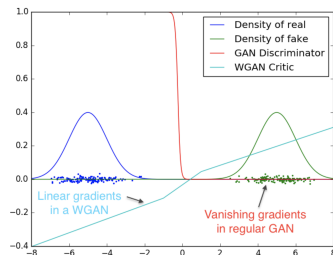
## WGAN objective

$$\min_{\theta} W(\pi || p) \approx \min_{\theta} \max_{\phi \in \Phi} [\mathbb{E}_{\pi(\mathbf{x})} f_{\phi}(\mathbf{x}) - \mathbb{E}_{p(\mathbf{z})} f_{\phi}(\mathbf{G}_{\theta}(\mathbf{z}))].$$

- ▶ Discriminator  $D$  is similar to the function  $f$ , but not the same (it is not a classifier anymore). In the WGAN model, function  $f$  is usually called **critic**.
- ▶ *"Weight clipping is a clearly terrible way to enforce a Lipschitz constraint"*. If the clipping parameter  $c$  is too large, it is hard to train the critic till optimality. If the clipping parameter  $c$  is too small, it could lead to vanishing gradients.

# Wasserstein GAN

- ▶ WGAN has non-zero gradients for disjoint supports.
- ▶  $JSD(\pi||p)$  correlates poorly with the sample quality. Stays constant nearly maximum value  $\log 2 \approx 0.69$ .
- ▶  $W(\pi||p)$  is highly correlated with the sample quality.



# Outline

## 1. Lipschitzness of Wasserstein GAN critic

Wasserstein GAN

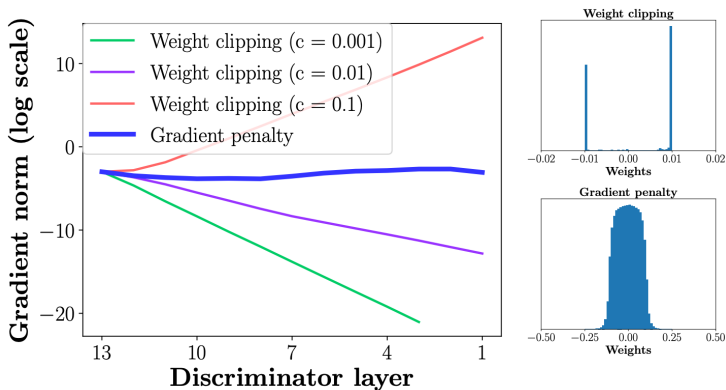
WGAN with Gradient Penalty

## 2. f-divergence minimization

## 3. Evaluation of likelihood-free models

Frechet Inception Distance (FID)

# Wasserstein GAN with Gradient Penalty



## Weight clipping analysis

- ▶ The gradients either grow or decay exponentially.
- ▶ Gradient penalty makes the gradients more stable.

# Wasserstein GAN with Gradient Penalty

## Theorem

Let  $\pi(\mathbf{x})$  and  $p(\mathbf{x})$  be two distributions in  $\mathcal{X}$ , a compact metric space. Let  $\gamma$  be the optimal transportation plan between  $\pi(\mathbf{x})$  and  $p(\mathbf{x})$ . Then

1. there is 1-Lipschitz function  $f^*$  which is the optimal solution of

$$\max_{\|f\|_L \leq 1} \left[ \mathbb{E}_{\pi(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p(\mathbf{x})} f(\mathbf{x}) \right].$$

2. if  $f^*$  is differentiable,  $\gamma(\mathbf{y} = \mathbf{z}) = 0$  and  $\hat{\mathbf{x}}_t = t\mathbf{y} + (1-t)\mathbf{z}$  with  $\mathbf{y} \sim \pi(\mathbf{x})$ ,  $\mathbf{z} \sim p(\mathbf{x}|\boldsymbol{\theta})$ ,  $t \in [0, 1]$  it holds that

$$\mathbb{P}_{(\mathbf{y}, \mathbf{z}) \sim \gamma} \left[ \nabla f^*(\hat{\mathbf{x}}_t) = \frac{\mathbf{z} - \hat{\mathbf{x}}_t}{\|\mathbf{z} - \hat{\mathbf{x}}_t\|} \right] = 1.$$

## Corollary

$f^*$  has gradient norm 1 almost everywhere under  $\pi(\mathbf{x})$  and  $p(\mathbf{x})$ .

# Wasserstein GAN with Gradient Penalty

A differentiable function is 1-Lipschitz if and only if it has gradients with norm at most 1 everywhere.

## Gradient penalty

$$W(\pi||p) = \underbrace{\mathbb{E}_{\pi(\mathbf{x})}f(\mathbf{x}) - \mathbb{E}_{p(\mathbf{x})}f(\mathbf{x})}_{\text{original critic loss}} + \lambda \underbrace{\mathbb{E}_{U[0,1]} \left[ (\|\nabla f(\hat{\mathbf{x}})\|_2 - 1)^2 \right]}_{\text{gradient penalty}},$$

- ▶ Samples  $\hat{\mathbf{x}}_t = t \cdot \mathbf{y} + (1 - t) \cdot \mathbf{z}$  with  $t \in [0, 1]$  are uniformly sampled along straight lines between pairs of points:  $\mathbf{y}$  from the data distribution  $\pi(\mathbf{x})$  and  $\mathbf{z}$  from the generator distribution  $p(\mathbf{x}|\theta)$ .
- ▶ Enforcing the unit gradient norm constraint everywhere is intractable, it turns out to be sufficient to enforce it only along these straight lines.

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# Divergences

- ▶ Forward KL divergence in maximum likelihood estimation.
- ▶ Reverse KL in variational inference.
- ▶ JS divergence in standard GAN.
- ▶ Wasserstein distance in WGAN.

## What is a divergence?

Let  $\mathcal{P}$  be the set of all possible probability distributions. Then  $D : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$  is a divergence if

- ▶  $D(\pi||p) \geq 0$  for all  $\pi, p \in \mathcal{P}$ ;
- ▶  $D(\pi||p) = 0$  if and only if  $\pi \equiv p$ .

## General divergence minimization task

$$\min_p D(\pi||p)$$

## Challenge

We do not know the real distribution  $\pi(\mathbf{x})$ !

# f-divergence family

## f-divergence

$$D_f(\pi||p) = \mathbb{E}_{p(\mathbf{x})} f\left(\frac{\pi(\mathbf{x})}{p(\mathbf{x})}\right) = \int p(\mathbf{x}) f\left(\frac{\pi(\mathbf{x})}{p(\mathbf{x})}\right) d\mathbf{x}.$$

Here  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a convex, lower semicontinuous function satisfying  $f(1) = 0$ .

Name	$D_f(P  Q)$	Generator $f(u)$
Kullback-Leibler	$\int p(x) \log \frac{p(x)}{q(x)} dx$	$u \log u$
Reverse KL	$\int q(x) \log \frac{q(x)}{p(x)} dx$	$-\log u$
Pearson $\chi^2$	$\int \frac{(q(x)-p(x))^2}{p(x)} dx$	$(u-1)^2$
Squared Hellinger	$\int \left(\sqrt{p(x)} - \sqrt{q(x)}\right)^2 dx$	$(\sqrt{u}-1)^2$
Jensen-Shannon	$\frac{1}{2} \int p(x) \log \frac{2p(x)}{p(x)+q(x)} + q(x) \log \frac{2q(x)}{p(x)+q(x)} dx$	$-(u+1) \log \frac{1+u}{2} + u \log u$
GAN	$\int p(x) \log \frac{2p(x)}{p(x)+q(x)} + q(x) \log \frac{2q(x)}{p(x)+q(x)} dx - \log(4)$	$u \log u - (u+1) \log(u+1)$

Nowozin S., Cseke B., Tomioka R. *f*-GAN: Training Generative Neural Samplers using Variational Divergence Minimization, 2016

# f-divergence family

## Fenchel conjugate

$$f^*(t) = \sup_{u \in \text{dom}_f} (ut - f(u)), \quad f(u) = \sup_{t \in \text{dom}_{f^*}} (ut - f^*(t))$$

**Important property:**  $f^{**} = f$  for convex  $f$ .

## f-divergence

$$\begin{aligned} D_f(\pi || p) &= \mathbb{E}_{p(\mathbf{x})} f\left(\frac{\pi(\mathbf{x})}{p(\mathbf{x})}\right) = \int p(\mathbf{x}) f\left(\frac{\pi(\mathbf{x})}{p(\mathbf{x})}\right) d\mathbf{x} = \\ &= \int p(\mathbf{x}) \sup_{t \in \text{dom}_{f^*}} \left(\frac{\pi(\mathbf{x})}{p(\mathbf{x})} t - f^*(t)\right) d\mathbf{x} = \\ &= \int \sup_{t \in \text{dom}_{f^*}} (\pi(\mathbf{x}) t - p(\mathbf{x}) f^*(t)) d\mathbf{x}. \end{aligned}$$

Here we seek value of  $t$ , which gives us maximum value of  $\pi(\mathbf{x})t - p(\mathbf{x})f^*(t)$ , for each data point  $\mathbf{x}$ .

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Nowozin S., Cseke B., Tomioka R. *f*-GAN: Training Generative Neural Samplers using Variational Divergence Minimization, 2016

# f-divergence family

## f-divergence

$$D_f(\pi||p) = \mathbb{E}_{p(\mathbf{x})} f\left(\frac{\pi(\mathbf{x})}{p(\mathbf{x})}\right) = \int p(\mathbf{x}) f\left(\frac{\pi(\mathbf{x})}{p(\mathbf{x})}\right) d\mathbf{x}.$$

## Variational f-divergence estimation

$$\begin{aligned} D_f(\pi||p) &= \int \sup_{t \in \text{dom}_{f^*}} (\pi(\mathbf{x})t - p(\mathbf{x})f^*(t)) d\mathbf{x} \geq \\ &\geq \sup_{T \in \mathcal{T}} \int (\pi(\mathbf{x})T(\mathbf{x}) - p(\mathbf{x})f^*(T(\mathbf{x}))) d\mathbf{x} = \\ &= \sup_{T \in \mathcal{T}} [\mathbb{E}_{\pi} T(\mathbf{x}) - \mathbb{E}_p f^*(T(\mathbf{x}))] \end{aligned}$$

This is a lower bound because of Jensen inequality and restricted class of functions  $\mathcal{T} : \mathcal{X} \rightarrow \mathbb{R}$ .

# f-divergence family

## Variational divergence estimation

$$D_f(\pi || p) \geq \sup_{T \in \mathcal{T}} [\mathbb{E}_{\pi} T(\mathbf{x}) - \mathbb{E}_p f^*(T(\mathbf{x}))]$$

The lower bound is tight for  $T^*(\mathbf{x}) = f' \left( \frac{\pi(\mathbf{x})}{p(\mathbf{x})} \right)$ .

## Example (JSD)

- ▶ Let define function  $f$  and its conjugate  $f^*$

$$f(u) = u \log u - (u + 1) \log(u + 1), \quad f^*(t) = -\log(1 - e^t).$$

- ▶ Let reparametrize  $T(\mathbf{x}) = \log D(\mathbf{x})$ .

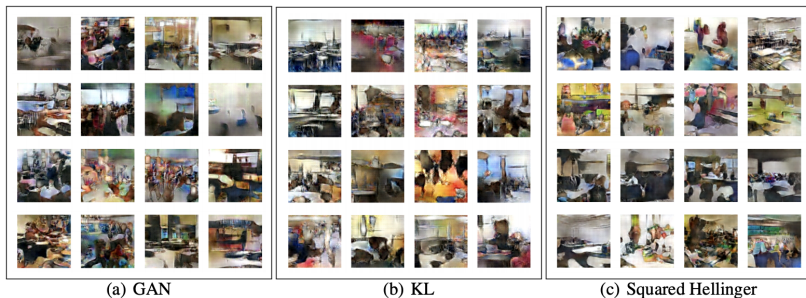
$$\min_G \max_D [\mathbb{E}_{\pi(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D(\mathbf{G}(\mathbf{z})))]$$

# f-divergence family

## Variational divergence estimation

$$D_f(\pi||p) \geq \sup_{T \in \mathcal{T}} [\mathbb{E}_{\pi} T(\mathbf{x}) - \mathbb{E}_p f^*(T(\mathbf{x}))]$$

**Note:** To evaluate lower bound we only need samples from  $\pi(\mathbf{x})$  and  $p(\mathbf{x})$ . Hence, we could fit implicit generative model.



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Nowozin S., Cseke B., Tomioka R. *f*-GAN: Training Generative Neural Samplers using Variational Divergence Minimization, 2016

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# Evaluation of likelihood-free models

How to evaluate generative models?

## Likelihood-based models

- ▶ Split data to train/val/test.
- ▶ Fit model on the train part.
- ▶ Tune hyperparameters on the validation part.
- ▶ Evaluate generalization by reporting likelihoods on the test set.

## Not all models have tractable likelihoods

- ▶ VAE: compare ELBO values.
- ▶ GAN: ???



# Evaluation of likelihood-free models

Let take some pretrained image classification model to get the conditional label distribution  $p(y|\mathbf{x})$  (e.g. ImageNet classifier).

What do we want from samples?

## ► Sharpness



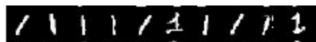
Low sharpness



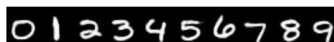
High sharpness

The conditional distribution  $p(y|\mathbf{x})$  should have low entropy (each image  $\mathbf{x}$  should have distinctly recognizable object).

## ► Diversity



Low diversity



High diversity

The marginal distribution  $p(y) = \int p(y|\mathbf{x})p(\mathbf{x})d\mathbf{x}$  should have high entropy (there should be as many classes generated as possible).

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# Frechet Inception Distance (FID)

## Wasserstein metric

$$W_s(\pi, p) = \inf_{\gamma \in \Gamma(\pi, p)} \left( \mathbb{E}_{(\mathbf{x}, \mathbf{y}) \sim \gamma} \|\mathbf{x} - \mathbf{y}\|^s \right)^{1/s}$$

## Theorem

If  $\pi(\mathbf{x}) = \mathcal{N}(\boldsymbol{\mu}_\pi, \boldsymbol{\Sigma}_\pi)$ ,  $p(\mathbf{y}) = \mathcal{N}(\boldsymbol{\mu}_p, \boldsymbol{\Sigma}_p)$ , then

$$W_2(\pi, p) = \|\boldsymbol{\mu}_\pi - \boldsymbol{\mu}_p\|_2^2 + \text{tr} \left[ \boldsymbol{\Sigma}_\pi + \boldsymbol{\Sigma}_p - 2 \left( \boldsymbol{\Sigma}_\pi^{1/2} \boldsymbol{\Sigma}_p \boldsymbol{\Sigma}_\pi^{1/2} \right)^{1/2} \right]$$

## Frechet Inception Distance

$$\text{FID}(\pi, p) = \|\boldsymbol{\mu}_\pi - \boldsymbol{\mu}_p\|_2^2 + \text{tr} \left[ \boldsymbol{\Sigma}_\pi + \boldsymbol{\Sigma}_p - 2 \left( \boldsymbol{\Sigma}_\pi^{1/2} \boldsymbol{\Sigma}_p \boldsymbol{\Sigma}_\pi^{1/2} \right)^{1/2} \right]$$

Representations are the outputs of the intermediate layer from the pretrained classification model.

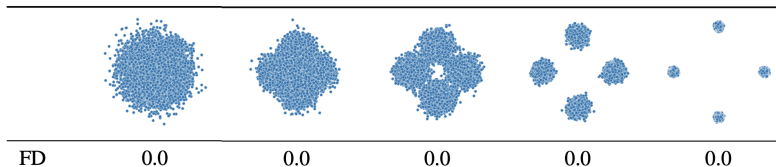
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Heusel M. et al. GANs Trained by a Two Time-Scale Update Rule Converge to a Local Nash Equilibrium, 2017

# Frechet Inception Distance (FID)

$$\text{FID}(\pi, p) = \|\boldsymbol{\mu}_\pi - \boldsymbol{\mu}_p\|_2^2 + \text{tr} \left[ \boldsymbol{\Sigma}_\pi + \boldsymbol{\Sigma}_p - 2 \left( \boldsymbol{\Sigma}_\pi^{1/2} \boldsymbol{\Sigma}_p \boldsymbol{\Sigma}_\pi^{1/2} \right)^{1/2} \right]$$

- ▶ Needs a large sample size for evaluation.
- ▶ Calculation of FID is slow.
- ▶ High dependence on the pretrained classification model.
- ▶ Uses the normality assumption!



## Summary

- ▶ Wasserstein GAN uses Kantorovich-Rubinstein duality for getting Earth Mover distance as model objective.
- ▶ Weight clipping is a terrible way to enforce Lipschitzness. Gradient Penalty adds regularizer to loss that uses necessary conditions for optimal critic.
- ▶ f-divergence family is a unified framework for divergence minimization, which uses variational approximation. Standard GAN is a special case of it.
- ▶ We need measure of quality for implicit models like GANs. One way is to analyze sharpness and diversity of samples.
- ▶ Frechet Inception Distance is the most popular metric for GAN evaluation.