Problem 8.1

1. X, Y, Z subset so $X \cup Y$ set

so $|X \cup Y \cup Z| = |X \cup Y| + |Z| - |(X \cup Y) \cap Z|$

$= |X| + |Y| + |Z| - |X \wedge Y| - |X \cap Z \cup Y \cap Z|$

$= |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|$

2. Let $X = \{(x,0) \mid x \in \mathbb{R}\}$, $Y = \{(0,x) \mid x \in \mathbb{R}\}$, $Z = \{(x,x) \mid x \in \mathbb{R}\}$

$\dim(X+Y+Z) = \dim(\mathbb{R}^2) = 2$

$\dim(X) + \dim(Y) + \dim(Z) - \dim(X \wedge Y) - \dim(X \cap Z) - \dim(Y \cap Z) + \dim(X \cap Y \cap Z)$

$1 + 1 + 1 - 0 - 0 - 0 + 0 = 3$

$\therefore \dim(X+Y+Z) \neq \dim(X) + \dim(Y) + \dim(Z) - \dim(X \cap Y) - \dim(X \wedge Z) - \dim(Y \cap Z) + \dim(X \cap Y \cap Z)$

3. $\dim(X+Y+Z) = \dim(X+Y) + \dim(Z) + \dim((X+Y) \cap Z)$

$= \dim(X) + \dim(Y) - \dim(Z) - \dim(X \cap Y) + \dim((X+Y) \cap Z)$

$\forall \vec{v} \in (X \cap Z) + (Y \cap Z)$, $\exists \vec{x_0} \in X \cap Z, \vec{y_0} \in Y \cap Z$ s.t. $\vec{v} = \vec{x_0} + \vec{y_0}$

since $\vec{x_0} \in X$, $\vec{y_0} \in Y$, $\vec{v} \in (X+Y)$

since $\vec{x_0} \in Z$, $\vec{y_0} \in Z$, $\vec{v} \in Z$

so $\vec{v} \in (X+Y) \cap Z$

so $(X \cap Z) + (Y \cap Z) \subseteq (X+Y) \cap Z$

so $\dim((X \cap Z) + (Y \cap Z)) = \dim(X \cap Z) + \dim(Y \cap Z) - \dim(X \cap Y \cap Z) \leq \dim((X+Y) \cap Z)$

$\therefore \dim(X) + \dim(Y) + \dim(Z) - \dim(X \wedge Y) - \dim(X \cap Z) - \dim(Y \cap Z) + \dim(X \cap Y \cap Z) \geq \dim(X) + \dim(Y) + \dim(Z) - \dim(X \cap Y) + \dim((X+Y) \cap Z) = \dim(X+Y+Z)$

Problem 8.2

1. $\mathrm{Ran}(B) = \{B\vec{x} \mid \vec{x} \in \mathbb{R}^d\}$

so $\mathrm{Ran}(B)$ is all the linear combinations of columns of B

since V is equal to all the linear combinations of its basis, and $\mathrm{Ran}(B)$ is all the linear combinations of columns of B, which are the basis for V, $\mathrm{Ran}(B) = V$

$\mathrm{Ker}(B^T) = (\mathrm{Ran}(B))^\perp$

since $\mathrm{Ran}(B) = V$

$\mathrm{Ker}(B^T) = V^\perp$

Since the columns of B form a basis, they must be linearly independent, so $\mathrm{Ker}(B) = \vec{0}$

2. $\mathrm{Ran}(C) = \{C\vec{x} \mid \vec{x} \in \mathbb{R}^d\}$

so $\mathrm{Ran}(C)$ is all the linear combinations of columns of C

since W is equal to all the linear combinations of its basis, and $\mathrm{Ran}(C)$ is all the linear combinations of columns of C, which are the basis for W, $\mathrm{Ran}(C) = W$

$\mathrm{Ker}(C^T) = (\mathrm{Ran}(C))^\perp$

since $\mathrm{Ran}(C) = W$

$\mathrm{Ker}(C^T) = W^\perp$

Since the columns of C form a basis, they must be linearly independent, so $\mathrm{Ker}(C) = \vec{0}$

3. Since $\mathrm{Ker}(C) = \vec{0}$, only $C\vec{0} = \vec{0}$

so only if $B^T\vec{x} = \vec{0}$ will $CB^T\vec{x} = \vec{0}$

Anything times $\vec{0} = \vec{0}$, so if $B^T\vec{x} = \vec{0}$, $CB^T\vec{x} = \vec{0}$

Since $\mathrm{Ker}(B) = \vec{0}$, only $B\vec{0} = \vec{0}$

so only if $C^T\vec{x} = \vec{0}$ will $BC^T\vec{x} = \vec{0}$

Anything times $\vec{0} = \vec{0}$, so if $C^T\vec{x} = \vec{0}$, $BC^T\vec{x} = \vec{0}$

4. $\mathrm{Ker}(CB^T) = \mathrm{Ker}(B^T)$ from p3

$= V^{\perp}$

$\operatorname{Ker}(((CB^T)^T)^T) = \operatorname{Ker}(BC^T)$

$= \operatorname{Ker}(C^T)$ from p.3

$= W^{\perp}$

$\operatorname{Ran}(CB^T) = (\operatorname{Ker}((CB^T)^T))^{\perp}$

$= \operatorname{Ker}(BC^T)^{\perp}$

$= (W^{\perp})^{\perp}$

$= W$

$\operatorname{Ran}((CB^T)^T) = \operatorname{Ker}(CB^T)^{\perp}$

$= (V^{\perp})^{\perp}$

$= V$

Problem 8.3

1. $\left[\begin{array}{ccc|c} 1&2&2&5\\ 2&2&3&5\\ 3&4&5&9 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1&2&2&5\\ 0&-2&-1&-5\\ 3&4&5&9 \end{array}\right] \xrightarrow{R_3-3R_1} \left[\begin{array}{ccc|c} 1&2&2&5\\ 0&-2&-1&-5\\ 0&-2&-1&-6 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1&2&2&5\\ 0&-2&-1&-5\\ 0&0&0&-1 \end{array}\right]$ Pivot in augmented portion, no solution

$\begin{bmatrix} A^T \\ b^T \end{bmatrix} \vec{y} = \begin{bmatrix} \vec{0} \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1&2&3&0\\ 2&2&4&0\\ 2&3&5&0\\ 5&5&9&1 \end{bmatrix} \to \begin{bmatrix} 1&2&3&0\\ 0&-2&-2&0\\ 0&-1&-1&0\\ 0&-5&-6&1 \end{bmatrix} \to \begin{bmatrix} 1&2&3&0\\ 0&-2&-2&0\\ 0&-5&-6&1\\ 0&0&0&0 \end{bmatrix} \to \begin{bmatrix} 1&2&3&0\\ 0&1&1&0\\ 0&-5&-6&1\\ 0&0&0&0 \end{bmatrix} \to \begin{bmatrix} 1&2&3&0\\ 0&1&1&0\\ 0&0&-1&1\\ 0&0&0&0 \end{bmatrix} \to \begin{bmatrix} 1&2&0&3\\ 0&1&0&1\\ 0&0&1&-1\\ 0&0&0&0 \end{bmatrix} \to \begin{bmatrix} 1&0&0&1\\ 0&1&0&1\\ 0&0&1&-1\\ 0&0&0&0 \end{bmatrix}$

$\vec{y} = \begin{bmatrix} 1\\1\\-1 \end{bmatrix}$

So $\vec{y}$ exists, so there is a solution to Problem 2.

So Problem 2 is solvable, but 1 is not

2. $A\vec{x} = \vec{b}$

$\vec{y}^T A \vec{x} = \vec{y}^T \vec{b}$

$(A^T \vec{y})^T \vec{x} = 1$

$\vec{0}^T \vec{x} = 1$

$\therefore 0 = 1$

3. $\vec{b} \in \operatorname{Ran}(A)$

$\therefore \exists \vec{x}$ s.t. $A\vec{x} = \vec{b}$

so there is a solution to $A\vec{x} = \vec{b}$

$\vec{b} \in \operatorname{Ran}(A)$

$\operatorname{Ker}(A^T) = (\operatorname{Ran}(A))^{\perp}$

so $\forall \vec{b} \in \operatorname{Ran}(A), \vec{y} \in \operatorname{Ker}(A^T)$

$\vec{b}^T \vec{y} = \vec{b} \cdot \vec{y} = 0$

but $\vec{b}^T \vec{y} = 1$, contradiction, so there does not exist $\vec{y}$ such that Problem two has a solution

4. If $\vec{b} \notin \operatorname{Ran}(A)$, then there does not exist $\vec{x}$ such that $A\vec{x} = \vec{b}$ by definition, so Problem 1 has no solution.

In addition, we know that there then must be a pivot in the augmented column of RREF($[A\ \vec{b}]$), so the last non-zero row is $[0 \ldots 0\ 1]$

This means that a linear combination of the rows of A resulted in $\vec{0}^T$, and the same lin. comb. of the rows of $\vec{b}$ led to 1.

So $\vec{A}^T \vec{y} = \vec{0}$, and $\vec{b}^T \vec{y} = 1$

So Problem 1 has no solution while Problem 2 does

Problem 8.4

1. Let $A = \begin{bmatrix} 1&0\\ 1&1\\ 0&1\\ 1&0 \end{bmatrix}$

$V = \operatorname{Ran}(A)$

$V^{\perp} = \operatorname{Ker}(A^T) = \operatorname{Ker}\left(\begin{bmatrix} 1&1&0&1\\ 0&1&1&0 \end{bmatrix}\right)$

$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

so basis for $V^\perp$ is $\begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $V^\perp = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\right)$

2. $\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ & 1 & 4 & 0 & 5 \\ & & & 1 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 4 & 0 & 5 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

basis is $\begin{bmatrix} 2 \\ 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \\ 6 \\ -1 \end{bmatrix}$

3. $\text{Ker}([a\ b\ c])$

$[a\ b\ c] \longrightarrow [1\ \tfrac{b}{a}\ \tfrac{c}{a}] \longrightarrow \begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & \frac{b}{a} & \frac{c}{a} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

basis of orthogonal complement: $\begin{bmatrix} \frac{b}{a} \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{c}{a} \\ 0 \\ -1 \end{bmatrix}$

4. Since there are 2 linearly independent rows, A has rank 2, so $\dim \text{Ran}(A) = 2$

so basis for $\text{Ran}(A) = ([\,\text{[illegible]}\,], [\,\text{[illegible]}\,])$, because these two vectors are linearly independent

basis $\text{Ran}(A^T) = ([\,\text{[illegible]}\,], [\,\text{[illegible]}\,])$, because those two vectors are linearly independent and $\dim \text{Ran}(A^T)$ also equals 2

$[\text{[illegible]}] \xrightarrow{R_7 - R_2,\ R_? - R_1} [\text{[illegible]}] \xrightarrow{R_1 \leftrightarrow R_2} [\text{[illegible]}] \xrightarrow{\frac{1}{r} R_1} [\text{[illegible]}] \xrightarrow{R_2 - pR_1} [\text{[illegible]}] \xrightarrow{\text{[illegible]}\, R_2} [\text{[illegible]}] \xrightarrow{R_1 - R_2} [\text{[illegible]}]$

so basis of $\text{Ker}(A) = ([\text{[illegible]}], [\text{[illegible]}], [\text{[illegible]}], [\text{[illegible]}], [\text{[illegible]}], [\text{[illegible]}])$

$\text{Ker}(A^T)$ is every linear combination on the rows that would result in 0, so basis $\text{Ker}(A^T) = ([\text{[illegible]}], [\text{[illegible]}], [\text{[illegible]}], [\text{[illegible]}], [\text{[illegible]}], [\text{[illegible]}])$

5. $\text{RREF}(A) = A$ because all the nonzero rows are already independent, and the zero rows are at the bottom

Similarly, $\text{RREF}(B) = B$

This means that the non-zero rows of A form a basis for $\text{Ran}(A^T)$, and non-zero rows of B form basis for $\text{Ran}(B^T)$

Because row operations preserve the row space of a matrix, any set of bases for a row space can always be reduced into an RREF thats preserves the row relations

Since RREF is unique, if two row spaces are the same, they must have the same RREF. Otherwise, the same row space could have two RREFs.

So $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}$

So $F = G$

Problem 8.5

1. The rank is 3, because there are 3 lin. ind. cols: $[\text{[illegible]}], [\text{[illegible]}], [\text{[illegible]}]$
2. $X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}$
3. $Y = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$
4. $M - XY = [\text{[illegible]}] - \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

$= M - \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} & & 0 & & & \\ & & 0 & & & \\ & & 0 & & & \\ 0 & 0 & 1 & 0 & 0 & 0 \\ & & 0 & & & \\ & & 0 & & & \\ & & 0 & & & \end{bmatrix}$

which has one pivot, so is rank 1

5. $M = [\text{[illegible]}] + [\text{[illegible]}] + [\text{[illegible]}]$