HW04 (Due 10.28)

1 Problem 4.1

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

1. $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad \therefore\ AB \neq BA$

2. $(A+B)^2 = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

   $A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 + 2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}$

   $\therefore\ (A+B)^2 \neq A^2 + 2AB + B^2$

3. $(AB)^2 = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

   $A^2B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

   $\therefore\ (AB)^2 \neq A^2B^2$

2 Problem 4.2

Suppose for a matrix A we have $A^2 = A$

1. Simplify into $sA + tI$, $s, t \in \mathbb{R}$

   $A^3 + 2A^2 - A - I = A(A^2) + 2(A) - A - I = A(A) + A - I = (A^2) + A - I = (A) + A - I = 2A - I$

   $A^2 + 3A + 4I = (A) + 3A + 4I = 4A + 4I$

2. Show that $I + 2A$ is invertible

   Suppose $(I + 2A)^{-1} = sA + tI$, $s, t \in \mathbb{R}$ $\quad$ $A^2 = A$, so all terms with A^k $(k \geq 2)$ will be downgrade to A

   By definition, $(sA + tI)(I + 2A) = I$

   $sAI + 2sA^2 + tI^2 + 2tIA = s(A) + 2s(A) + t(I) + 2t(A) = (s + 2s + 2t)A + t(I) = (3s + 2t)A + t(I)$

   Thus, $t = 1$, $3s + 2t = 0 \to s = -\frac{2}{3}$

   Hence, $I + 2A$ is invertible

3 Problem 4.3

1. $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & & \\ -1 & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$

3. $\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & & 1 & & \\ -1 & & & 1 & \\ -1 & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} = \begin{bmatrix} & & & & \\ & 1 & & & \\ & 2 & 1 & & \\ & 3 & 3 & 1 & \\ & 4 & 6 & 4 & 1 \end{bmatrix}$

4. $\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}^k$; Observe the following

for $k = 2$, $\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}$ (all rows climb up one row)

for $k = 3$, $\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix} = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix}$ (all rows climb up one row)

for $k = 4$, $\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ (all rows climb up one row)

for $k = 5$, $\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix} = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}$ (all rows climb up one row)

Notice, $k = 5$, back to starting position. Why? 4x4 matrix, climb up 4 times obviously reset.

Thus, $\forall\ k\ ,\ n \in \mathbb{N}$,

$k = 4n$, $\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}^k = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$; $k = 4n + 1$, $\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}^k = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}$

$k = 4n + 2$, $\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}^k = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}$; $k = 4n + 3$, $\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}^k = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix}$

4 Problem 4.4

1. $X_{13} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ & & 0 & \\ & & 0 & \\ & & 0 & \end{bmatrix}$; $X_{32} = \begin{bmatrix} & 0 & & \\ & 0 & & \\ 0 & 1 & 0 & 0 \\ & 0 & & \end{bmatrix}$; $X_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & & \\ & 0 & & \\ & 0 & & \end{bmatrix}$

2. $X_{13}X_{32} = (\vec{e}_1\vec{e}_3{}^T)(\vec{e}_3\vec{e}_2{}^T) = \vec{e}_1(\vec{e}_3{}^T\vec{e}_3)\vec{e}_2{}^T = \vec{e}_1 \cdot 1 \cdot \vec{e}_2{}^T = \vec{e}_1\vec{e}_2{}^T = X_{12}$

   $X_{32}X_{13} = (\vec{e}_3\vec{e}_2{}^T)(\vec{e}_1\vec{e}_3{}^T) = \vec{e}_3(\vec{e}_2{}^T\vec{e}_1)\vec{e}_3{}^T = \vec{e}_3 \cdot 0 \cdot \vec{e}_3{}^T = O_{4\times 4}$

3. Verify $X_{ij}^2 = O$ when $i \neq j$; $(\vec{e}_i\vec{e}_j{}^T)(\vec{e}_i\vec{e}_j{}^T) = \vec{e}_i(\vec{e}_j{}^T\vec{e}_i)\vec{e}_j{}^T = \vec{e}_i \cdot 0 \cdot \vec{e}_j{}^T = O$

4. For any two square matrices A, B, show that $AB = BA \;\Leftrightarrow\; (A-I)(B-I) = (B-I)(A-I)$

   " $\Rightarrow$ " : Suppose, $AB = BA$

   $(A-I)(B-I) = (AB) - AI - IB + I^2 = (BA) - IA - BI + I^2 = (B-I)(A-I)$

   " $\Leftarrow$ " : Suppose, $(A-I)(B-I) = (B-I)(A-I)$

   $(A-I)(B-I) - (B-I)(A-I) = (AB - AI - IB + I^2) - (BA - BI - IA + I^2) = AB - BA = 0$

   $\therefore AB = BA$

5 Problem 4.5

Let $J = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & \end{bmatrix}$

1. Going from A to JA: J would be shifting things up

2. Going from A to AJ: J would be shifting things right

3. Given the Pascal's symmetric matrix $P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$, calculate $PJ + J^{\,T}P$

$$PJ + J^{\,T}P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & \end{bmatrix} + \begin{bmatrix} & & & \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} & 1 & 1 & 1 \\ & 1 & 2 & 3 \\ & 1 & 3 & 6 \\ & 1 & 4 & 10 \end{bmatrix} + \begin{bmatrix} & & & \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

Explanation: Pascal's symmetric matrix P comes from this algorithm, $P = \begin{bmatrix} * & \times\,1 & 1 & 1 \\ 1 \rightarrow & 2 & & \\ 1 & & & \\ 1 & & & \end{bmatrix}$

or P [shift right] $(p_{21} \to p_{22})$ + [shift down] P $(p_{12} \to p_{22})$, which is exactly what $PJ + J^{\,T}P$ does.

4. Calculate J, J^2, J^3, J^4

$$J = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & \end{bmatrix} \quad ; \quad J^2 = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & \end{bmatrix}\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & \end{bmatrix} = \begin{bmatrix} & & 1 & \\ & & & 1 \\ & & & \\ & & & \end{bmatrix}$$

$$J^3 = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & \end{bmatrix}\begin{bmatrix} & & 1 & \\ & & & 1 \\ & & & \\ & & & \end{bmatrix} = \begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ & & & \end{bmatrix} \quad ; \quad J^4 = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & \end{bmatrix}\begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ & & & \end{bmatrix} = O$$

5. Show that $I - J$ has inverse $I + J + J^2 + J^3$

$$\begin{aligned}(I - J)(I + J + J^2 + J^3) &= I^2 + IJ + IJ^2 + IJ^3 - JI - J^2 - J^3 - J^4 \\ &= I + J + J^2 + J^3 - J - J^2 - J^3 - O = I \quad (Q.E.D)\end{aligned}$$

6. Calculate $(J + I)^2, (J + I)^3, (J + I)^4$

$$(J + I)^2 = J^2 + JI + IJ + I^2 = J^2 + 2J + I = \begin{bmatrix} 1 & 2 & 1 & \\ & 1 & 2 & 1 \\ & & 1 & 2 \\ & & & 1 \end{bmatrix}$$

$$(J + I)^3 = (J + I)(J^2 + 2J + I) = J^3 + 2J^2 + JI + IJ^2 + 2IJ + I^2$$

$$= J^3 + 3J^2 + 3J + I = \begin{bmatrix} 1 & 3 & 3 & 1 \\ & 1 & 3 & 3 \\ & & 1 & 3 \\ & & & 1 \end{bmatrix}$$

$$(J + I)^4 = (J + I)(J^3 + 3J^2 + 3J + I) = J^4 + 3J^3 + 3J^2 + JI + IJ^3 + 3IJ^2 + 3IJ + I^2$$

$$= J^4 + 4J^3 + 6J^2 + 4J + I = \begin{bmatrix} 1 & 4 & 6 & 4 \\ & 1 & 4 & 6 \\ & & 1 & 4 \\ & & & 1 \end{bmatrix}$$

Proof by Mathematical Induction. $(J + I)^k = \begin{bmatrix} 1 & k & \frac{k(k-1)}{2} & \frac{k(k-1)(k-2)}{6} \\ & 1 & k & \frac{k(k-1)}{2} \\ & & 1 & k \\ & & & 1 \end{bmatrix}$

7. Describe the set of all matrices that commutes with J.

$$AJ = JA \quad \leftrightarrow \quad A_{i,j-1} = A_{i+1,j} \quad \rightarrow \quad A_{i,j} = A_{i-1,j+1} \quad \rightarrow \quad A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \ddots & \vdots \\ 0 & 0 & a_1 & \ddots & a_3 \\ \vdots & \ddots & \ddots & \ddots & a_2 \\ 0 & 0 & \cdots & 0 & a_1 \end{bmatrix}$$

Thus, $A = a_1 I + a_2 J + a_3 J^3 + ... + a_n J^n - 1$

6 Problem 4.6

1. Find A^{-1}

$$\left[\begin{array}{cccc|cccc} 2 & -1 & & & 1 & & & \\ -1 & 2 & -1 & & & 1 & & \\ & -1 & 2 & -1 & & & 1 & \\ & & -1 & 2 & & & & 1 \end{array}\right] \quad (R_2 \to R_2 + \tfrac{1}{2} R_1) \quad \left[\begin{array}{cccc|cccc} 2 & -1 & & & 1 & & & \\ & \frac{3}{2} & -1 & & \frac{1}{2} & 1 & & \\ & -1 & 2 & -1 & & & 1 & \\ & & -1 & 2 & & & & 1 \end{array}\right]$$

$$(R_3 \to R_3 + \tfrac{2}{3} R_2) \quad \left[\begin{array}{cccc|cccc} 2 & -1 & & & 1 & & & \\ & \frac{3}{2} & -1 & & \frac{1}{2} & 1 & & \\ & & \frac{4}{3} & -1 & \frac{1}{3} & \frac{2}{3} & 1 & \\ & & -1 & 2 & & & & 1 \end{array}\right]$$

$$(R_4 \to R_4 + \tfrac{3}{4} R_3) \quad \left[\begin{array}{cccc|cccc} 2 & -1 & & & 1 & & & \\ & \frac{3}{2} & -1 & & \frac{1}{2} & 1 & & \\ & & \frac{4}{3} & -1 & \frac{1}{3} & \frac{2}{3} & 1 & \\ & & & \frac{5}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{array}\right]$$

$$(R_4 \to \tfrac{4}{5} R_4) \quad \left[\begin{array}{cccc|cccc} 2 & -1 & & & 1 & & & \\ & \frac{3}{2} & -1 & & \frac{1}{2} & 1 & & \\ & & \frac{4}{3} & -1 & \frac{1}{3} & \frac{2}{3} & 1 & \\ & & & 1 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{array}\right]$$

$$(R_3 \to R_3 + R_4) \quad \left[\begin{array}{cccc|cccc} 2 & -1 & & & 1 & & & \\ & \frac{3}{2} & -1 & & \frac{1}{2} & 1 & & \\ & & \frac{4}{3} & & \frac{8}{15} & \frac{16}{15} & \frac{8}{5} & \frac{4}{5} \\ & & & 1 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{array}\right]$$

$$(R_3 \to \tfrac{3}{4} R_3) \quad \left[\begin{array}{cccc|cccc} 2 & -1 & & & 1 & & & \\ & \frac{3}{2} & -1 & & \frac{1}{2} & 1 & & \\ & & 1 & & \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ & & & 1 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{array}\right]$$

$$(R_2 \to R_2 + R_3) \quad \left[\begin{array}{cccc|cccc} 2 & -1 & & & 1 & & & \\ & \frac{3}{2} & & & \frac{9}{10} & \frac{9}{5} & \frac{6}{5} & \frac{3}{5} \\ & & 1 & & \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ & & & 1 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{array}\right]$$

$$(R_2 \to \tfrac{2}{3} R_2) \quad \left[\begin{array}{cccc|cccc} 2 & -1 & & & 1 & & & \\ & 1 & & & \frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ & & 1 & & \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ & & & 1 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{array}\right]$$

$$(R_1 \to R_1 + R_2) \quad \left[\begin{array}{cccc|cccc} 2 & & & & \frac{8}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ & 1 & & & \frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ & & 1 & & \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ & & & 1 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{array}\right]$$

$$(R_1 \to \tfrac{1}{2} R_1) \quad \left[\begin{array}{cccc|cccc} 1 & & & & \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ & 1 & & & \frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ & & 1 & & \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ & & & 1 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{array}\right]$$

2. Find A^{-1} (妈呀！Gaussian Elimination 好费手)

$A = E^{-1}U \;\to\; U^{-1}E = A^{-1}$

$$IA = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$E_1 A = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$E_2 E_1 A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$E_3 E_2 E_1 A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}$$

Thus, $E = E_3 E_2 E_1 = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $U^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$ $\quad [\, U \mid I \,] \to [\, I \mid U^{-1} \,]$

Hence, $A^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$$A_2^{-1} - A_1^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{bmatrix} = \frac{1}{5}(5A_2^{-1} - 5A_1^{-1}) = \frac{1}{5} \begin{bmatrix} 16 & 12 & 8 & 4 \\ 12 & 9 & 6 & 3 \\ 8 & 6 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \text{(rank one)}$$

3. Find P^{-1}

$$P = \begin{bmatrix} & 1 & & \\ & & & 1 \\ 1 & & & \\ & & 1 & \end{bmatrix} \text{; Notice, P is a Permutation matrix i.e. it swaps rows/columns}$$

Intuitively, for example: $P_{13}A \;:\; R_1 \leftrightarrow R_3$ and $P_{13}P_{13}A$ swaps back to the starting position

Obviously, same for columns swap.

Since each 2 rows/columns pair up. Thus, the inverse of P is always itself.

4. Find P^{-1}

$$P = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \text{; (4.6.3) the inverse of P is always itself.}$$

5. Find M^{-1}

$$M = \begin{bmatrix} 1 & 2 & & \\ 3 & 4 & & \\ & & 1 & 2 \\ & & 3 & 4 \end{bmatrix} = \begin{bmatrix} A & \\ & A \end{bmatrix} \text{; Find } A^{-1} : \left[\begin{array}{cc|cc} 1 & 2 & 1 & \\ 3 & 4 & & 1 \end{array}\right]$$

$$\underset{(R_2 \to R_2 - 3R_1)}{} \left[\begin{array}{cc|cc} 1 & 2 & 1 & \\ & -2 & -3 & 1 \end{array}\right] \underset{(R_2 \to -\frac{1}{2}R_2)}{} \left[\begin{array}{cc|cc} 1 & 2 & 1 & \\ & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right] \overset{(R_1 \to R_1 - 2R_2)}{} \left[\begin{array}{cc|cc} 1 & & -2 & 1 \\ & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$$

$$\text{Thus, } M^{-1} = \begin{bmatrix} -2 & 1 & & \\ \frac{3}{2} & -\frac{1}{2} & & \\ & & -2 & 1 \\ & & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

6. Find A^{-1}

$A = E^{-1}DU \;\to\; U^{-1}D^{-1}E = A^{-1}$

$$IA = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$E_1A = \begin{bmatrix} 1 & & \\ \frac{1}{3} & 1 & \\ \frac{1}{3} & & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ & \frac{8}{3} & -\frac{4}{3} \\ & -\frac{4}{3} & \frac{8}{3} \end{bmatrix}$$

$$E_2E_1A = \begin{bmatrix} 1 & & \\ & 1 & \\ & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ & \frac{8}{3} & -\frac{4}{3} \\ & -\frac{4}{3} & \frac{8}{3} \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ & \frac{8}{3} & -\frac{4}{3} \\ & & 2 \end{bmatrix}$$

Thus, $E = E_2E_1 = \begin{bmatrix} 1 & & \\ \frac{1}{3} & 1 & \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$, $DU = \begin{bmatrix} 3 & -1 & -1 \\ & \frac{8}{3} & -\frac{4}{3} \\ & & 2 \end{bmatrix} = \begin{bmatrix} 3 & & \\ & \frac{8}{3} & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ & 1 & -\frac{1}{2} \\ & & 1 \end{bmatrix}$,

$D^{-1} = \begin{bmatrix} \frac{1}{3} & & \\ & \frac{3}{8} & \\ & & \frac{1}{2} \end{bmatrix}$, $U^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{6} \\ & 1 & \frac{1}{2} \\ & & 1 \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ & 1 & \frac{1}{2} \\ & & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & & \\ & \frac{3}{8} & \\ & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & & \\ \frac{1}{3} & 1 & \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

7. Find A^{-1}

$$\left[\begin{array}{cccc|cccc} 1 & -a & & & 1 & & & \\ & 1 & -b & & & 1 & & \\ & & 1 & -c & & & 1 & \\ & & & 1 & & & & 1 \end{array}\right] \xrightarrow{(R_3 \to R_3 + cR_4)} \left[\begin{array}{cccc|cccc} 1 & -a & & & 1 & & & \\ & 1 & -b & & & 1 & & \\ & & 1 & & & & 1 & c \\ & & & 1 & & & & 1 \end{array}\right]$$

$$\xrightarrow{(R_2 \to R_2 + bR_3)} \left[\begin{array}{cccc|cccc} 1 & -a & & & 1 & & & \\ & 1 & & & & 1 & b & bc \\ & & 1 & & & & 1 & c \\ & & & 1 & & & & 1 \end{array}\right]$$

$$\xrightarrow{(R_1 \to R_1 + aR_2)} \left[\begin{array}{cccc|cccc} 1 & & & & 1 & a & ab & abc \\ & 1 & & & & 1 & b & bc \\ & & 1 & & & & 1 & c \\ & & & 1 & & & & 1 \end{array}\right]$$

7 Problem 4.7 - find B satisfies

1. $\forall A_{3\times 3}$, $AB = 4A$

   $AB - 4A = O \;\to\; A(B - 4I) = O \;\to\; B = 4I$

2. $\forall A_{3\times 3}$, $BA = 4A$

   $BA - 4A = O \;\to\; (B - 4I)A = O \;\to\; B = 4I$

3. $\forall A_{3\times 3}$, $BA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \;\to\; B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

4. $\forall A_{3\times 3}$, $AB = A \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \;\to\; B = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

5. $B^2 \neq O$, $B^3 = O \to B = \begin{bmatrix} & 1 & \\ & & 1 \\ & & \end{bmatrix}$ or any $B = J_n^{\,n-2}$

6. $B\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}B$

Suppose $B = \begin{bmatrix}a & b\\c & d\end{bmatrix}$

$B\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}B \ \leftrightarrow \ \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} \ \leftrightarrow \ \begin{bmatrix}a+b & a+b\\c+d & c+d\end{bmatrix} = \begin{bmatrix}a+c & b+d\\a+c & b+d\end{bmatrix}$

Thus, $\begin{cases} a+b = a+c \\ a+b = b+d \\ c+d = a+c \\ c+d = b+d \end{cases} \rightarrow \begin{cases} b = c \\ a = d \\ d = a \\ c = b \end{cases}$

Hence, $B = \begin{bmatrix}a & b\\b & a\end{bmatrix}$