Problem 4.1

1. $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

They are not the same

2. $(A+B)^2 = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right)^2$

$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}$

They are not the same.

3. $(AB)^2 = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right)^2$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

$A^2B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

They are not the same

Problem 4.2

1. $A^3 + 2A^2 - A - I = A^2A + 2A^2 - A - I$

$= AA + 2A - A - I$

$= A + A - I$

$= 2A - I$

$A^2 + 3A + 4I = A + 3A + 4I$

$= 4A + 4I$

2. $(I + 2A)^{-1} = (sA + tI)$

$(sA + tI)(I + 2A) = I$

$sA + 2sA^2 + tI + 2tA = I$

$2sA^2 + (s+2t)A + (t-1)I = 0$

$2sA + (s+2t)A + (t-1)I = 0$

$(3s+2t)A + (t-1)I = 0$

$t = 1$

$s = -\frac{2}{3}$

$\therefore (I + 2A)^{-1} = -\frac{2}{3}A + I$

Problem 4.3

1. $\underset{P_{13}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \underset{P_{13}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

The first and third rows of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ are swapped twice, resulting in the original matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{-1}}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E_{31}^{-1}}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$

Row 1 is subtracted from Rows 2 and 3

3. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}_{E_{21}^{-1}}\underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ -1 & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}_{E_{31}^{-1}}\underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ -1 & & & 1 & \\ & & & & 1 \end{bmatrix}}_{E_{41}^{-1}}\underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ -1 & & & & 1 \end{bmatrix}}_{E_{51}^{-1}}\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 2 & 1 & & \\ 0 & 3 & 3 & 1 & \\ 0 & 4 & 6 & 4 & 1 \end{bmatrix}$

Row 1 is subtracted from Rows 2, 3, 4, 5

4. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^k$

$k=1$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $\quad \rightarrow k=5$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$k=2$, $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $\quad \vdots$

$k=3$, $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$k=4$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

So if : $1 \equiv k \pmod 4$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$2 \equiv k \pmod 4$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$3 \equiv k \pmod 4$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$0 \equiv k \pmod 4$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Problem 4.4

1. $X_{13} = e_1 e_3^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}[0\,0\,1\,0]$

$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$X_{32} = e_3 e_2^T = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}[0\,1\,0\,0]$

$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$X_{12} = e_1 e_2^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}[0\,1\,0\,0]$

$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. $X_{13}X_{32} = e_1 e_3^T e_3 e_2^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\left([0\,0\,1\,0]\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)[0\,1\,0\,0]$

$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}[1]\,[0\,1\,0\,0]$

$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$X_{32}X_{13} = e_3 e_2^T e_1 e_3^T = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\left([0\,1\,0\,0]\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)[0\,0\,1\,0]$

$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}[0]\,[0\,0\,1\,0]$

$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3. $X_{ij}^2 = e_i e_j^T e_i e_j^T = e_i [0] e_j^T$

$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$= 0$

4. $AB = BA \Rightarrow (A-I)(B-I) = (B-I)(A-I)$

$(A-I)(B-I) = AB - AI - BI + I$

$(B-I)(A-I) = BA - BI - AI + I$

if $AB = BA$

then $(B-I)(A-I) = AB - AI - BI + I = (A-I)(B-I)$

$(A-I)(B-I) = (B-I)(A-I) \Rightarrow AB = BA$

if $(A-I)(B-I) = (B-I)(A-I)$

then $AB - AI - BI + I = BA - BI - AI + I$

so $AB = BA$

Problem 4.5

1. J would be shifting things up

$$J\begin{bmatrix} \vec{a_1}^T \\ \vec{a_2}^T \\ \vec{a_3}^T \\ \vec{a_4}^T \end{bmatrix} = \begin{bmatrix} \vec{a_2}^T \\ \vec{a_3}^T \\ \vec{a_4}^T \\ 0 \end{bmatrix}$$

2. J would be shifting things right

$$[\vec{a_1}\ \vec{a_2}\ \vec{a_3}\ \vec{a_4}]\, J = [0\ \vec{a_1}\ \vec{a_2}\ \vec{a_3}]$$

3. $PJ = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{bmatrix}$ (shift right)

$J^TP = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \end{bmatrix}$ (shift down)

$PJ + J^TP = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$

$P_{i,j} = (P_{i-1,j} + P_{i,j-1})$

So every element in PJ becomes $P_{i,j-1}$

and J^TP becomes $P_{i-1,j}$

So $PJ + J^TP = P_{i-1,j} + P_{i,j-1} = P_{i,j} = P$

except for those elements in the first row and column, resulting in the upper left entry becoming 0

4. $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$J^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (J shifted up/right)

$J^3 = J^2 J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (J^2 shifted up/right)

$J^4 = 0$ (J^3 shifted up/right)

5. $(I-J)(I+J+J^2+J^3)$

$= \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = I$

so $(I-J)^{-1} = (I+J+J^2+J^3)$

6. $(J+I)^2 = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}^2$

$= \begin{bmatrix} 1 & 2 & & \\ & 1 & 2 & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix}$ (row1 → row1 + row2, row2 → row2 + row3, row 3 → row3 + row4)

$(J+I)^3 = \begin{bmatrix} 1 & 3 & & \\ & 1 & 3 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix}$ (row1 → row1 + row2, row2 → row2 + row3, row 3 → row3 + row4)

$(J+I)^4 = \begin{bmatrix} 1 & 4 & & \\ & 1 & 4 & \\ & & 1 & 4 \\ & & & 1 \end{bmatrix}$ (row1 → row1 + row2, row2 → row2 + row3, row 3 → row3 + row4)

$\vdots$

$(J+I)^k = \begin{bmatrix} 1 & k & & \\ & 1 & k & \\ & & 1 & k \\ & & & 1 \end{bmatrix}$

7. $JA = A$ shifted up

$AJ = A$ shifted right

so if $JA = AJ$, then A shifted up must equal A shifted right (A must also be square for the multiplication to be defined)

if $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$, $JA = \begin{bmatrix} e & f & g & h \\ i & j & k & l \\ m & n & o & p \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $AJ = \begin{bmatrix} 0 & a & b & c \\ 0 & e & f & g \\ 0 & i & j & k \\ 0 & m & n & o \end{bmatrix}$

so $e, i, j, m, n, o = 0$, $a = f = k = p$, $b = g = l$, $c = h$

so A is upper diagonal in the form $\begin{bmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix}$

Problem 4.6

1. $\left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\times \frac{1}{2}} \left[\begin{array}{cccc|cccc} 1 & -0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & -0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0 & 0 & 0 & 0.5 \end{array}\right] \xrightarrow{R_2 + \frac{1}{2}R_1} \left[\begin{array}{cccc|cccc} 1 & -0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.75 & -0.5 & 0 & 0.25 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & -0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0 & 0 & 0 & 0.5 \end{array}\right] \xrightarrow{\frac{4}{3}R_2} \left[\begin{array}{cccc|cccc} 1 & -0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & -0.5 & 1 & -0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0 & 0 & 0 & 0.5 \end{array}\right] \xrightarrow{R_3 + \frac{1}{2}R_2} \left[\begin{array}{cccc|cccc} 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & -\frac{1}{2} & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right]$

$$\xrightarrow{R_4+\frac{3}{4}R_3}\left[\begin{array}{cccc|cccc} 1 & -\frac12 & 0 & 0 & \frac12 & 0 & 0 & 0 \\ 0 & 1 & -\frac23 & 0 & \frac13 & \frac23 & 0 & 0 \\ 0 & 0 & \frac23 & -\frac12 & \frac16 & \frac13 & \frac12 & 0 \\ 0 & 0 & 0 & \frac58 & \frac18 & \frac14 & \frac38 & \frac12 \end{array}\right] \xrightarrow{\frac32 R_3}\left[\begin{array}{cccc|cccc} 1 & -\frac12 & 0 & 0 & \frac12 & 0 & 0 & 0 \\ 0 & 1 & -\frac23 & 0 & \frac13 & \frac23 & 0 & 0 \\ 0 & 0 & 1 & -\frac34 & \frac14 & \frac12 & \frac34 & 0 \\ 0 & 0 & 0 & \frac58 & \frac18 & \frac14 & \frac38 & \frac12 \end{array}\right] \xrightarrow{\frac85 R_4}\left[\begin{array}{cccc|cccc} 1 & -\frac12 & 0 & 0 & \frac12 & 0 & 0 & 0 \\ 0 & 1 & -\frac23 & 0 & \frac13 & \frac23 & 0 & 0 \\ 0 & 0 & 1 & -\frac34 & \frac14 & \frac12 & \frac34 & 0 \\ 0 & 0 & 0 & 1 & \frac15 & \frac25 & \frac35 & \frac45 \end{array}\right] \xrightarrow{R_3+\frac34 R_4}\left[\begin{array}{cccc|cccc} 1 & -\frac12 & 0 & 0 & \frac12 & 0 & 0 & 0 \\ 0 & 1 & -\frac23 & 0 & \frac13 & \frac23 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac25 & \frac45 & \frac65 & \frac35 \\ 0 & 0 & 0 & 1 & \frac15 & \frac25 & \frac35 & \frac45 \end{array}\right]$$

$$\xrightarrow{R_2+\frac23 R_3}\left[\begin{array}{cccc|cccc} 1 & -\frac12 & 0 & 0 & \frac12 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac35 & \frac65 & \frac45 & \frac25 \\ 0 & 0 & 1 & 0 & \frac25 & \frac45 & \frac65 & \frac35 \\ 0 & 0 & 0 & 1 & \frac15 & \frac25 & \frac35 & \frac45 \end{array}\right] \xrightarrow{R_1+\frac12 R_2}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac45 & \frac35 & \frac25 & \frac15 \\ 0 & 1 & 0 & 0 & \frac35 & \frac65 & \frac45 & \frac25 \\ 0 & 0 & 1 & 0 & \frac25 & \frac45 & \frac65 & \frac35 \\ 0 & 0 & 0 & 1 & \frac15 & \frac25 & \frac35 & \frac45 \end{array}\right]$$

$$A^{-1}=\frac15\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

2. $$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R_1}\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_2}\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_4+R_3}\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_3+R_4}\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2+R_3}\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1+R_2}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4 & 3 & 2 & 1 \\ 0 & 1 & 0 & 0 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

the difference between the two matrices is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which is rank one

3. $$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \vec{a}^T \\ \vec{b}^T \\ \vec{c}^T \\ \vec{d}^T \end{bmatrix}=\begin{bmatrix} \vec{b}^T \\ \vec{d}^T \\ \vec{a}^T \\ \vec{c}^T \end{bmatrix}$$

so $$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} \vec{b}^T \\ \vec{d}^T \\ \vec{a}^T \\ \vec{c}^T \end{bmatrix}=\begin{bmatrix} \vec{a}^T \\ \vec{b}^T \\ \vec{c}^T \\ \vec{d}^T \end{bmatrix}$$

so $$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{-1}=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

4. $\begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}^{-1}=\begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$ because the last row is moved to the 1st row, 2nd to last to 2nd, etc. which is what needs to be done for $\begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$ to become $\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

5. $$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{array}\right]^{-1}=\left[\begin{array}{c|c} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} & \\ \hline & \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2-3R_1}\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow{-\frac12 R2}\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac32 & -\frac12 \end{array}\right] \xrightarrow{R_1-2R_2}\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac32 & -\frac12 \end{array}\right]$$

so $$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}^{-1}=\begin{bmatrix} -2 & 1 & 0 & 0 \\ \frac32 & -\frac12 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & \frac32 & -\frac12 \end{bmatrix}$$

6. $$\left[\begin{array}{ccc|ccc} 3 & -1 & -1 & 1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1+R_2+R_3}\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R_1}\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 1 & 2 & 1 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_1}\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{array}\right] \xrightarrow{\frac14 R_2}\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac14 & \frac12 & \frac14 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{\frac14 R_3}\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac14 & \frac12 & \frac14 \\ 0 & 0 & 1 & \frac14 & \frac14 & \frac12 \end{array}\right] \xrightarrow{R_1-R_2-R_3}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac12 & \frac14 & \frac14 \\ 0 & 1 & 0 & \frac14 & \frac12 & \frac14 \\ 0 & 0 & 1 & \frac14 & \frac14 & \frac12 \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} \frac12 & \frac14 & \frac14 \\ \frac14 & \frac12 & \frac14 \\ \frac14 & \frac14 & \frac12 \end{bmatrix}$$

7. $$\left[\begin{array}{cccccccc} 1 & -a & & & 1 & & & \\ & 1 & -b & & & 1 & & \\ & & 1 & -c & & & 1 & \\ & & & 1 & & & & 1 \end{array}\right] \xrightarrow{R_3+cR_4}\left[\begin{array}{cccccccc} 1 & -a & & & 1 & & & \\ & 1 & -b & & & 1 & & \\ & & 1 & & & & 1 & c \\ & & & 1 & & & & 1 \end{array}\right] \xrightarrow{R_2+bR_3}\left[\begin{array}{cccccccc} 1 & -a & & & 1 & & & \\ & 1 & & & & 1 & b & bc \\ & & 1 & & & & 1 & c \\ & & & 1 & & & & 1 \end{array}\right] \xrightarrow{R_1+aR_2}\left[\begin{array}{cccccccc} 1 & & & & 1 & a & ab & abc \\ & 1 & & & & 1 & b & bc \\ & & 1 & & & & 1 & c \\ & & & 1 & & & & 1 \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} 1 & a & ab & abc \\ & 1 & b & bc \\ & & 1 & c \\ & & & 1 \end{bmatrix}$$

Problem 4.7

1. $AB=4A$

$A4I=4AI=4A$

$B=4I=\begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix}$

2. $BA=4A$

$BAA^{-1}=4AA^{-1}$

$B=4$

3. $B=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (keeps only the first entry in each column of $A \to$ first row of A)

4. $B=\frac13\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (sums each entry in each row, then multiplies by $\frac13$)

5. A matrix like J in Problem 4.5 that shifts upward every multiplication. Since we want $B^3 = 0$, then we can have $B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

6. B must be square 2×2 for the multiplication to be defined

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$$

so $a+b = a+c \Rightarrow b = c$

$a+b = b+d \Rightarrow a = d$

so B must be in the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, meaning B must be symmetric.