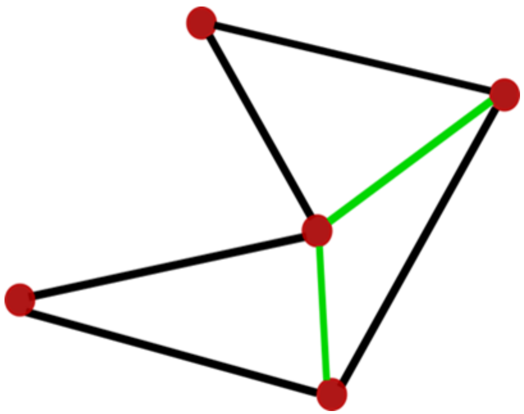
# Triangulation: Theory



- Must every polygon has a triangulation?

- Theorem: **Every polygon *P* of *n* vertices may be partitioned into Δs by the addition of (zero or more) diagonals**

    - We have to prove the existence of a diagonal

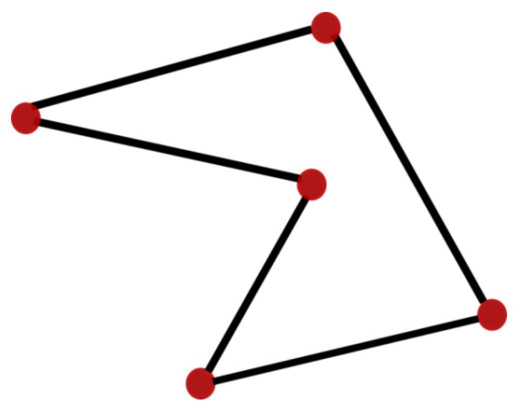
- Lemma 1: **Every polygon of *n* ≥ 4 vertices has a diagonal**

    - For a diagonal to exist, all the points should not be collinear

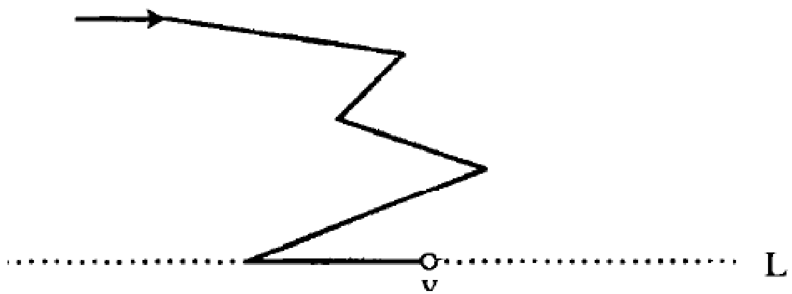
- Lemma 2: **Every polygon *P* must have at least one strictly convex vertex**

# Proof of Lemma2

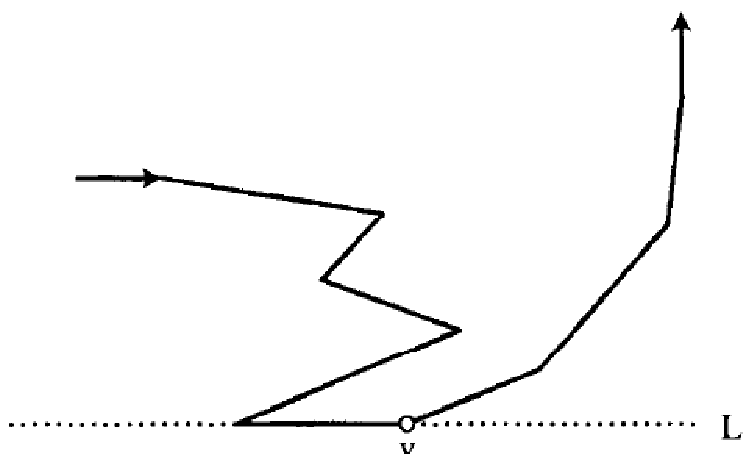
- Lemma 2: **Every polygon *P* must have at least one strictly convex vertex**



# Proof of Lemma 2

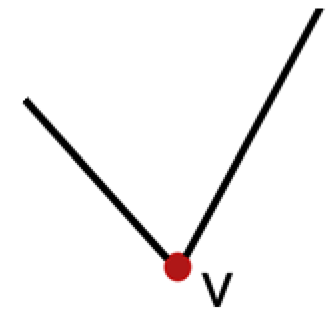


# Proof of Lemma2

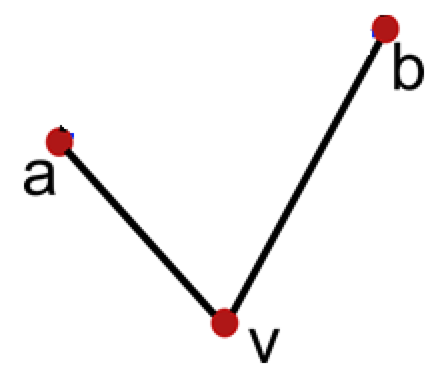


# Proof of Lemma 1

- Lemma 1: **Every polygon of *n* ≥ 4 vertices has a diagonal**
- Let v be a strictly convex vertex



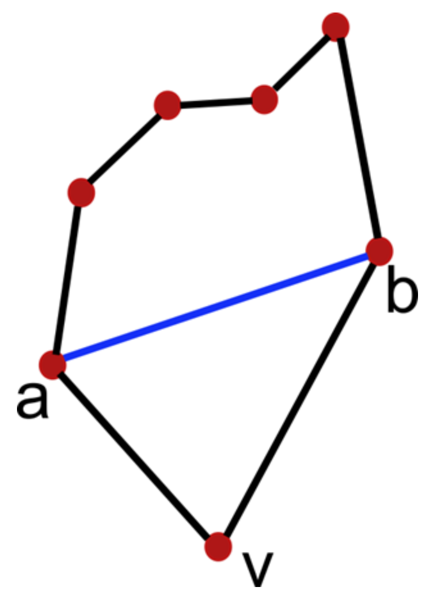

- Let a & b are the vertices adjacent to v
- Case-1: If ab is a diagonal, then the proof is complete




- Draw an example of P in Case 1

# Proof of Lemma 1

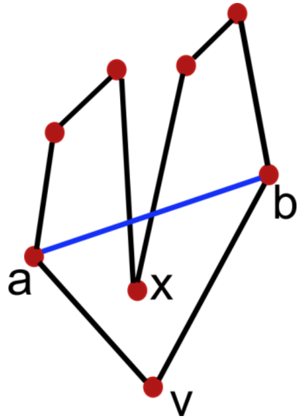
- Case-1: If ab is a diagonal
- An example polygon where ab is a diagonal




- Case-1 : **ab is a diagonal**, **hence trivially proved**

# Proof of Lemma 1

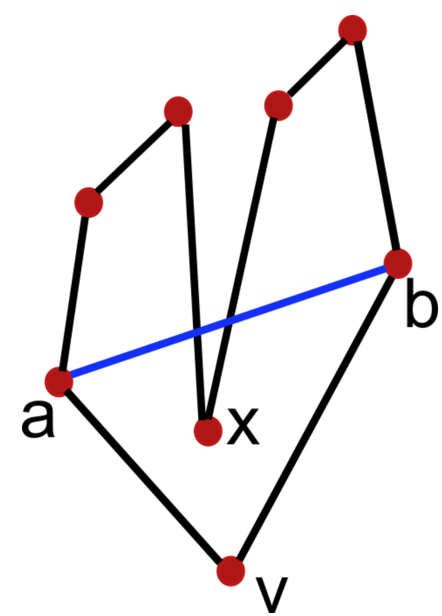
- Will there be a case where ab is not a diagonal?
- Case- 2: ab is not a diagonal
- Draw a polygon of Case 2
- Case 2: (a)Either ab is exterior to P or (b) ab intersects ∂P
- Draw a polygon of Case 2 (a)
- Draw a polygon of Case 2 (b)



# Proof of Lemma 1

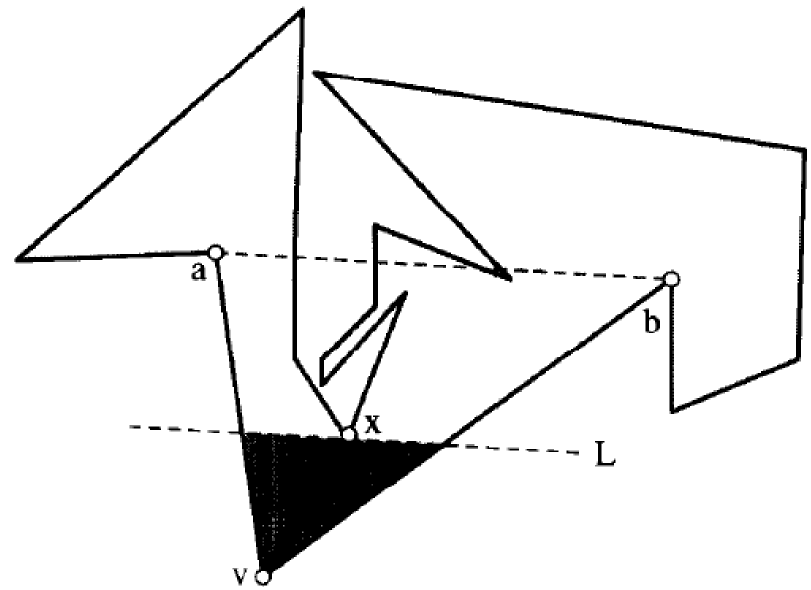
- Case-2: **ab is not a diagonal**
- Since $n > 3$, The closed Δavb contains at least one vertex of P other than a,v,b
- Let x be the vertex of P in Δavb that is closest to v
- Draw a P illustrating a,v,b & x
- x is the first vertex in Δavb hit

  by a line L moving from v to ab



# Proof of Lemma 1

- vx is a diagonal. Why




- The shaded part (interior of Δavb bounded by L that includes v) is empty of points of ∂P
- Hence, vx cannot intersect ∂P except v & x, hence it is a diagonal.
- Recall: Case 2: (a)Either ab is exterior to P or (b) ab intersects ∂P
- **vx is a diagonal** is true in both Case 2(a) and Case 2(b)
- **Both case 1 & 2, ∃ a diagonal, hence Lemma 1 is proved**

# Proof of Theorem

- Theorem: **Every polygon *P* of *n* vertices may be partitioned in to *Δ*s by the addition of (zero or more) diagonals**

- Proof is by induction

  - If *n =3 ,* it is a ***Δ*** , hence trivially proved

  - Let $n \geq 4$, ∃ a diagonal which partitions P to two

  - By induction hypothesis, the claim is true for both the polygons

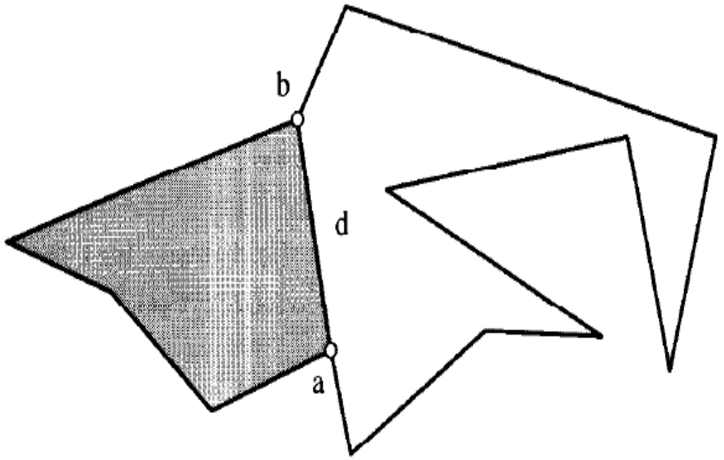
# Properties of Triangulation

- Find out how many diagonals (as a function of n) are there for a triangulation of P with n vertices
- Find out how many triangles (as a function of n) are there for a triangulation of P with n vertices
- How do we prove ?

# Lemma 3 : Number of Diagonals

- Lemma 3: Every triangulation of a polygon P of n vertices uses n-3 diagonals and consists of n-2 triangles
- Proof is by induction
- n=3, both claims are trivially true
- n>=4
- How do we prove?
- Partition P into two polygons $P_1$ and $P_2$ with a diagonal d

# Proof of Lemma 3




- Let d=ab
- # of vertices in $P_1 = n_1$
- # of vertices in $P_2 = n_2$
- $n_1 + n_2 = n+2$ .Why?
- Since a & b are counted in both $P_1$ and $P_2$
- Apply induction hypothesis to $P_1$ and $P_2$
- $(n_1-3) + (n_2-3)+1$
- Why there is a +1 in the previous term?
- +1 corresponds to d, the diagonal added currently

# Proof of Lemma 3

- $(n_1\text{-}3) + (n_2\text{-}3)+1 = (n_1 + n_2)\text{-}6+1$
- $(n_1\text{-}3) + (n_2\text{-}3)+1 = (n+2)\text{-}6+1$
- $(n_1\text{-}3) + (n_2\text{-}3)+1 = n+3\text{-}6$
- $(n_1\text{-}3) + (n_2\text{-}3)+1 = n\text{-}3$ diagonals
- Hence, proved

# Exercise

- Prove the Lemma for the number of triangles

- Lemma 3: Every triangulation of a polygon P of n vertices uses n-3 diagonals and consists of n-2 triangles

# Corollary : Sum of angles

- Corollary : Sum of internal angles of a polygon of n vertices is (n-2) * $180^o$

- Proof

- There are (n-2) Δs in a T of P, by Lemma 3

- Each Δ contributes to $180^o$

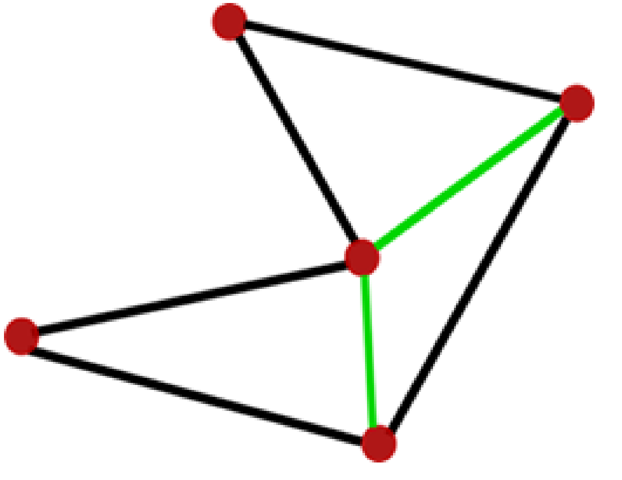
- Hence, proved

# Reading Exercise

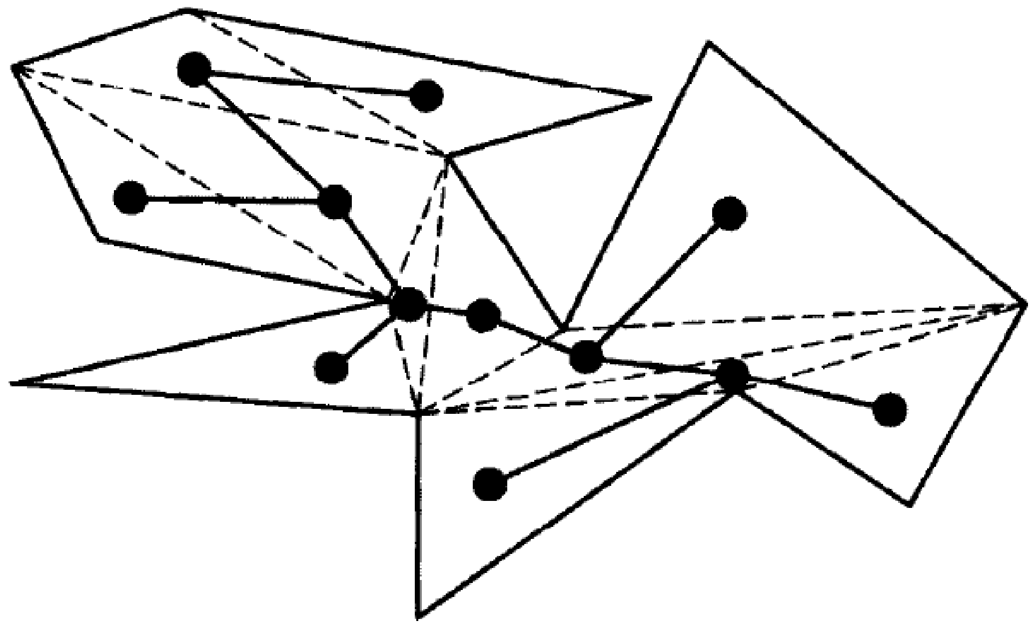
- **Text book: Computational Geometry in C, Joseph O' Rourke -** Chapter 1- Section 1.3 onwards till the end of the chapter
- Area of a polygon (convex/ non-convex)
- Area of a triangle (Cross product, Determinant form) --- Constant time
- Area of quadrilateral (convex/ non-convex)
- Implementation issues :
  - Line segment intersection , Left predicate, collinear predicate etc.
  - Polygon triangulation – two ways

# Dual of a triangulation

- The dual of a T of P is a graph G
- A vertex in G is associated with each triangle of T of P
- An edge between two vertices iff their triangles share a diagonal
- Draw dual of a T of P

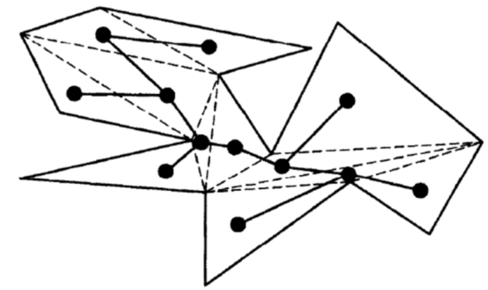


# Dual of a triangulation



- What structure is Dual of a T of P ?
- What is the maximum degree of each vertex of the dual?

# Some other observations on Dual



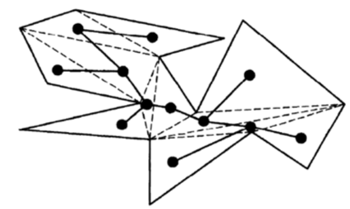
- In a dual, vertices of degree 1 are ?
- Vertices of degree 1 are known as the leaves of the tree
- In a dual, vertices of degree 2 lies on the path of the tree
- In a dual, vertices of degree three are known as the branch points of the tree
- What type of tree is the Dual?
- Binary tree rooted at any vertex of degree one or two

# Ear of a polygon

- Three consecutive vertices of a polygon **a,b,c** form an **ear** of a polygon if **ac** is a diagonal, **b** is the **ear tip**
- Draw a triangulation and mark all ears
- Two ears are nonoverlapping if their triangle interiors are disjoint
- Meisters’s two ears theorem

# Meisters's two ears theorem

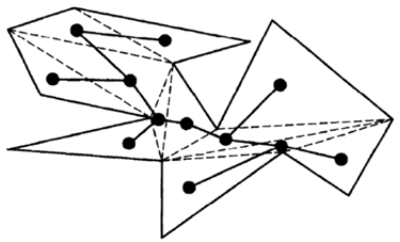
- Theorem: Every polygon of n>=4 vertices has at least two non-overlapping ears.



- Proof
- A leaf node in a dual corresponds to an ear
- A tree of two or more vertices must have at least two leaves
- Why a tree should have two or more leaves?
- Tree should have at least 2 leaves
- n-2 vertices > =2 , since n>=4 by the theorem
- Hence, the theorem is proved

# Theorem : 3-coloring

- Theorem: The triangulation of a Polygon f *n* vertices can be 3-colored.
- **Proof :**
- Induction on *n*
- *n*=3, a triangle can be 3-colored, hence , trivially proved
- n>=4
- We know that T of P has an ear Δabc with ear tip b, by Meisters's two ears theorem
- Form a new polygon P' by cutting the ear
- P' has n-1 vertices
- Apply induction hypotheses to 3-color P'
- Put back the vertex b
- Color it with a different color than the colors used for the vertices a & c.



# Summary : Triangulation - Theory

- Every polygon *P* must have at least one strictly convex vertex
- Every polygon of $n \geq 4$ vertices has a diagonal
- Every polygon *P* of *n* vertices may be partitioned into Δs by the addition of (zero or more) diagonals
- Every triangulation of a polygon P of n vertices uses n-3 diagonals and consists of n-2 triangles
- Sum of internal angles of a polygon of n vertices is $(n-2) * 180^o$
- Dual of a T of P is a tree with each vertex of degree at most three.
- Every polygon of n>=4 vertices has at least two non-overlapping ears.
- The triangulation of a Polygon of *n* vertices can be 3-colored.

# Reference

- J. O Rourke, *Computational Geometry in C*, 2/e, Cambridge University Press, 1998

# Thank you