Analysis of Large Graphs: Link Analysis, PageRank

CS246: Mining Massive Datasets

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New Topic: Graph Data!

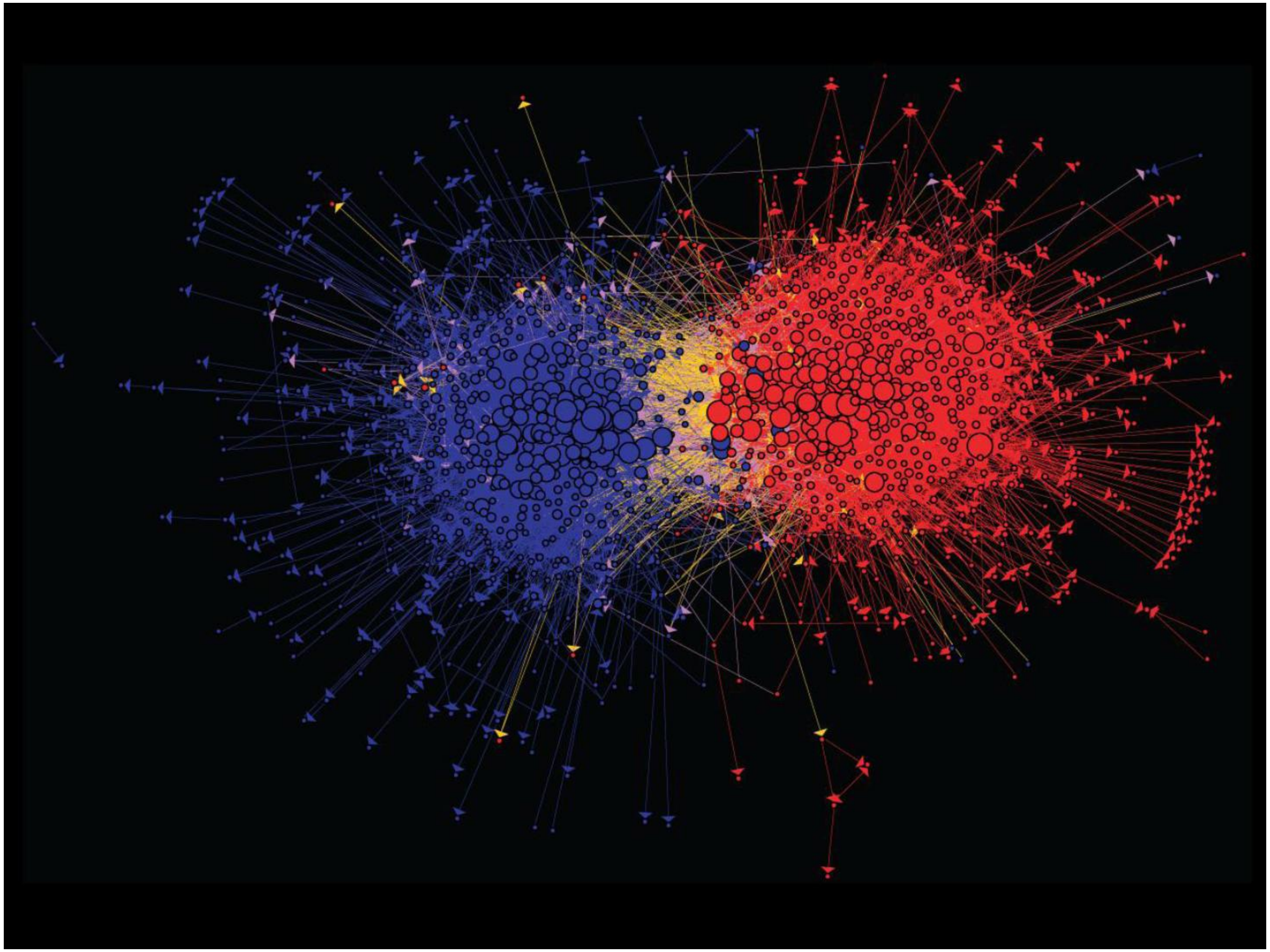
High dim. data	Graph data	Infinite data	Machine learning	Apps
Locality sensitive hashing	**PageRank,** SimRank	Filtering data streams	SVM	Recommender systems
Clustering	Community Detection	Web advertising	Decision Trees	Association Rules
Dimensionality reduction	Spam Detection	Queries on streams	Perceptron, kNN	Duplicate document detection

Graph Data: Social Networks

Facebook social graph

4-degrees of separation [Backstrom-Boldi-Rosa-Ugander-Vigna, 2011]

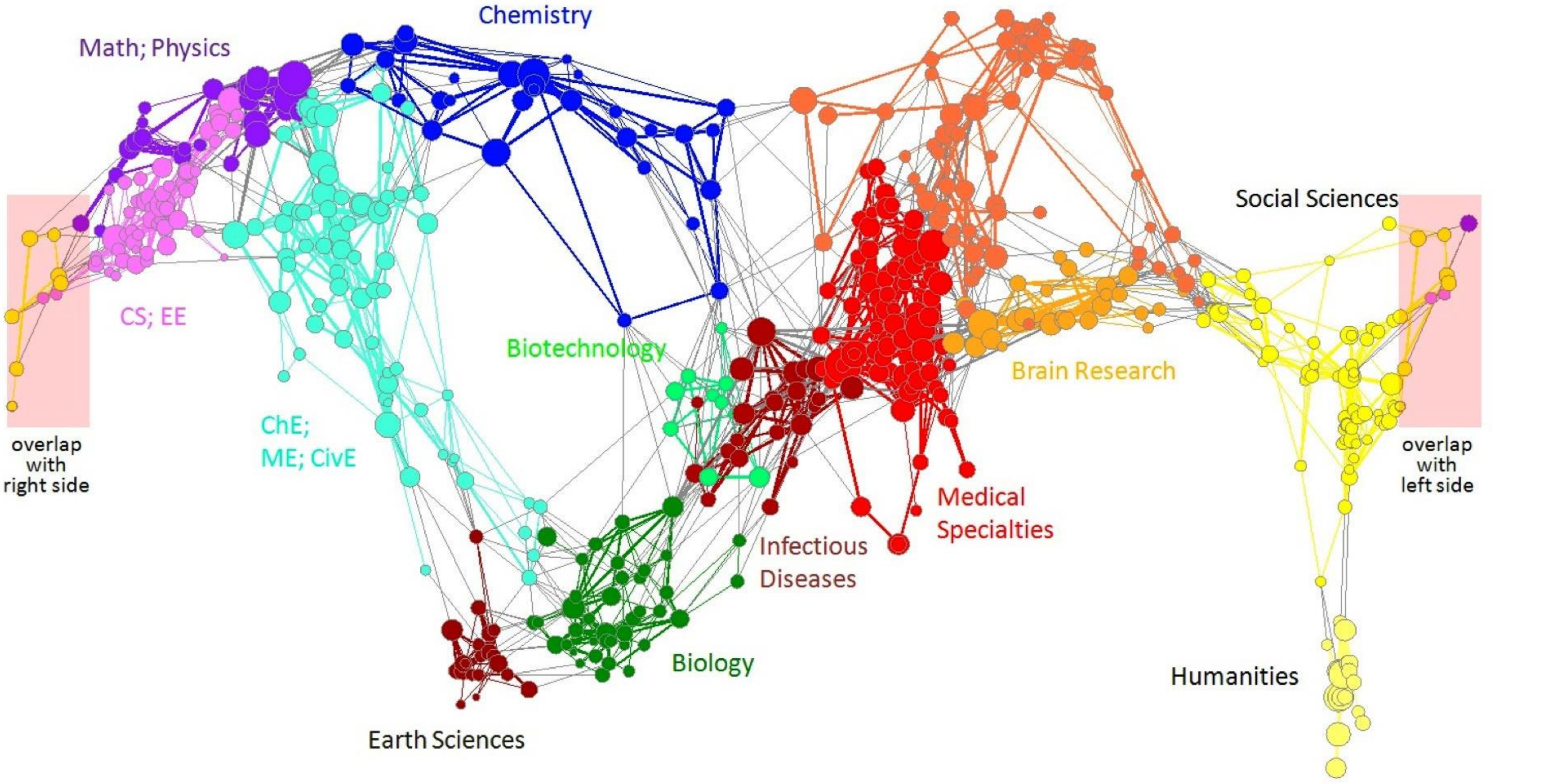
Graph Data: Media Networks



Connections between political blogs

Polarization of the network [Adamic-Glance, 2005]

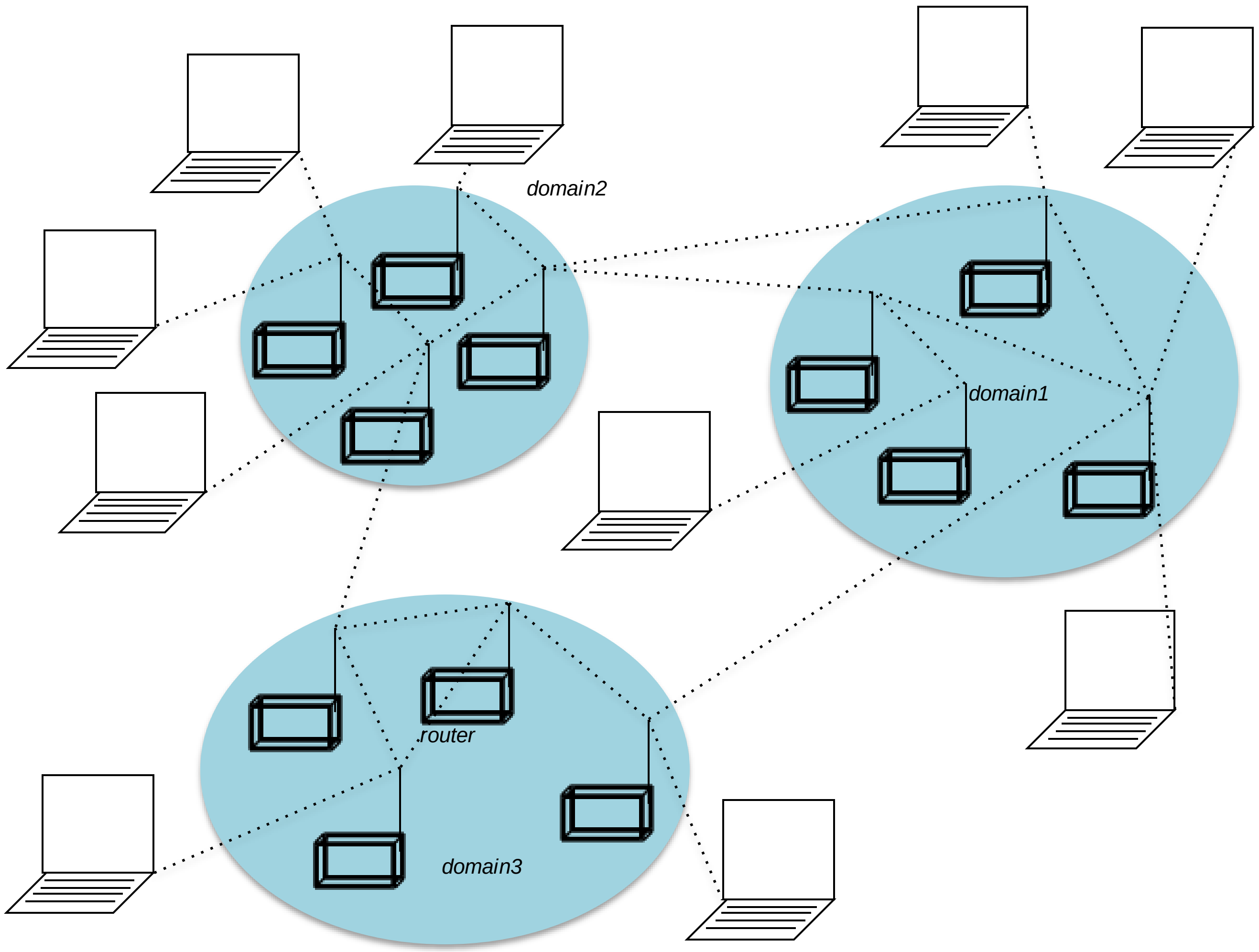
Graph Data: Information Nets



Citation networks and Maps of science

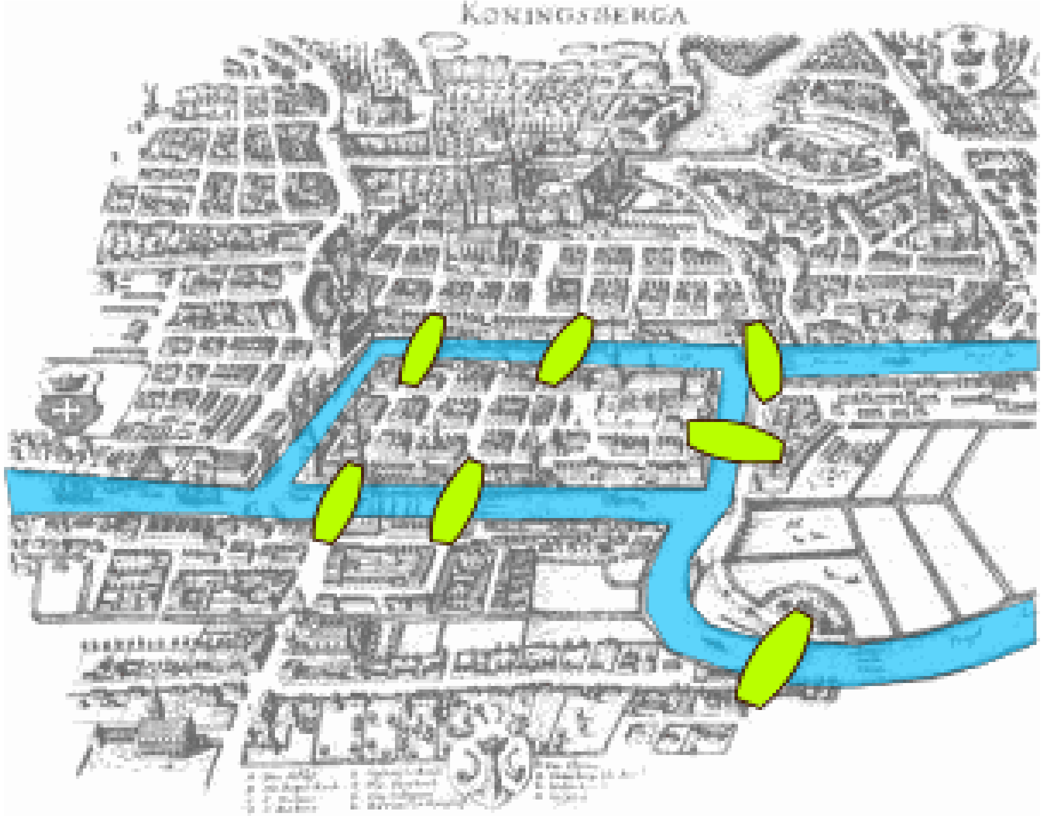
[Börner et al., 2012]

Graph Data: Communication Networks



Internet

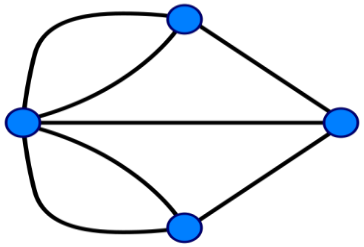
Graph Data: Technological Networks



Seven Bridges of Königsberg

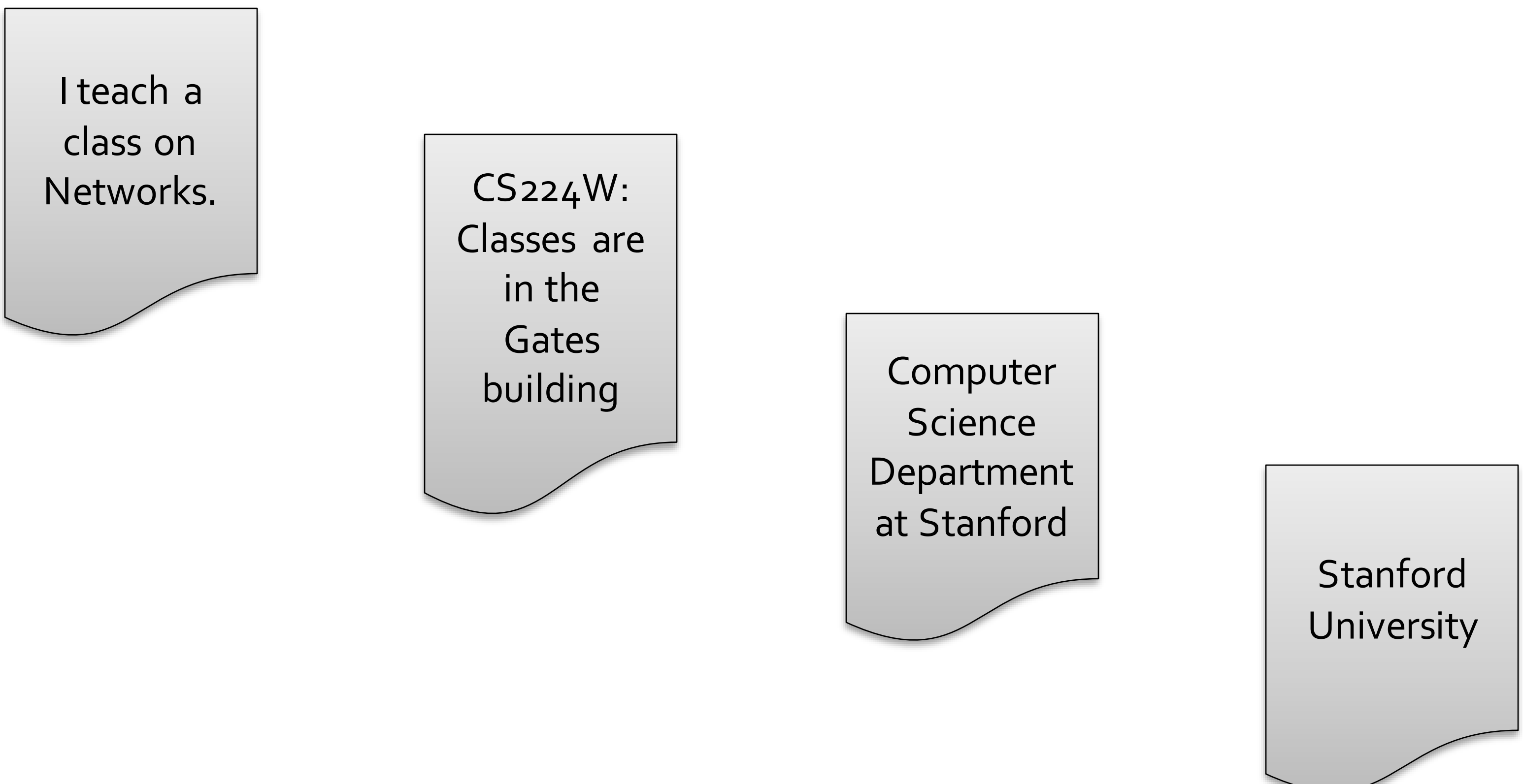
[Euler, 1735]

Return to the starting point by traveling each link of the graph once and only once.



Web as a Graph

- **Web as a directed graph:**
  - **Nodes: Webpages**
  - **Edges: Hyperlinks**



Web as a Graph

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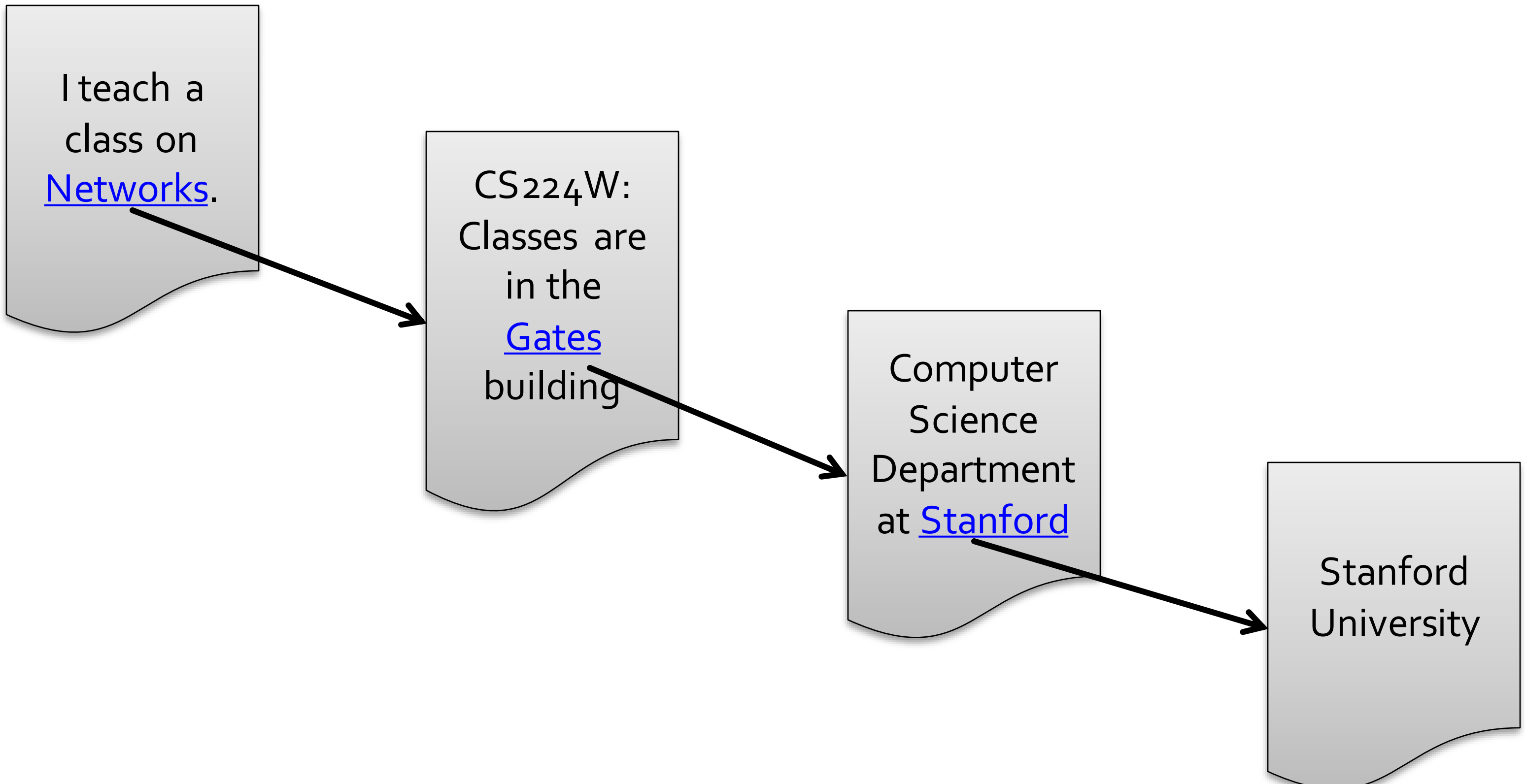
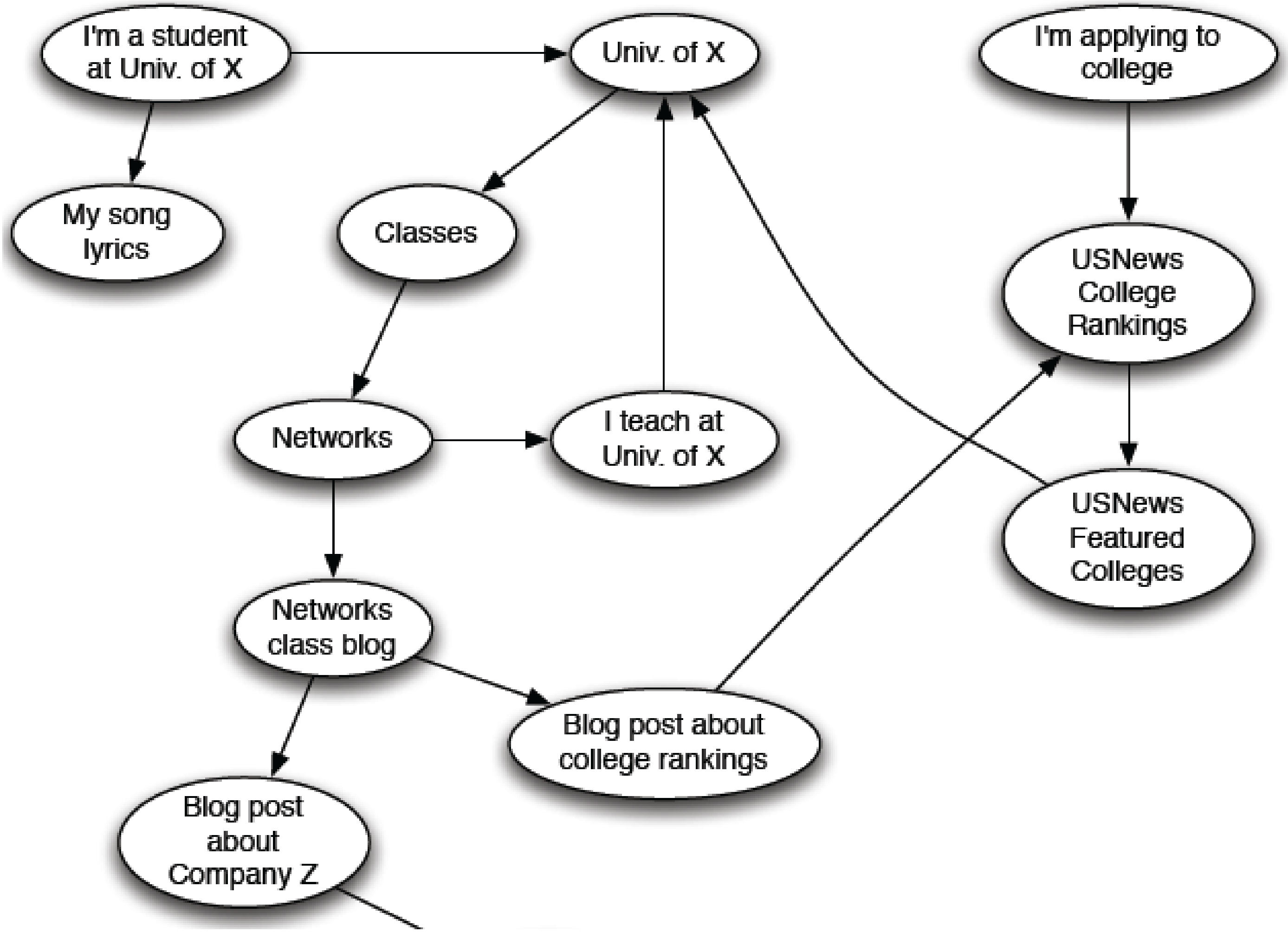


Web as a Directed Graph



Broad Question

- **How to organize the Web?**
- **First try:** Human curated **Web directories**
  - Yahoo, DMOZ, LookSmart
- **Second try: Web Search**
  - **Information Retrieval** investigates: Find relevant docs in a small and trusted set
    - Newspaper articles, Patents, etc.
  - **But:** Web is **huge**, full of untrusted documents, random things, web spam, etc.

Web Search: 2 Challenges

2 challenges of web search:

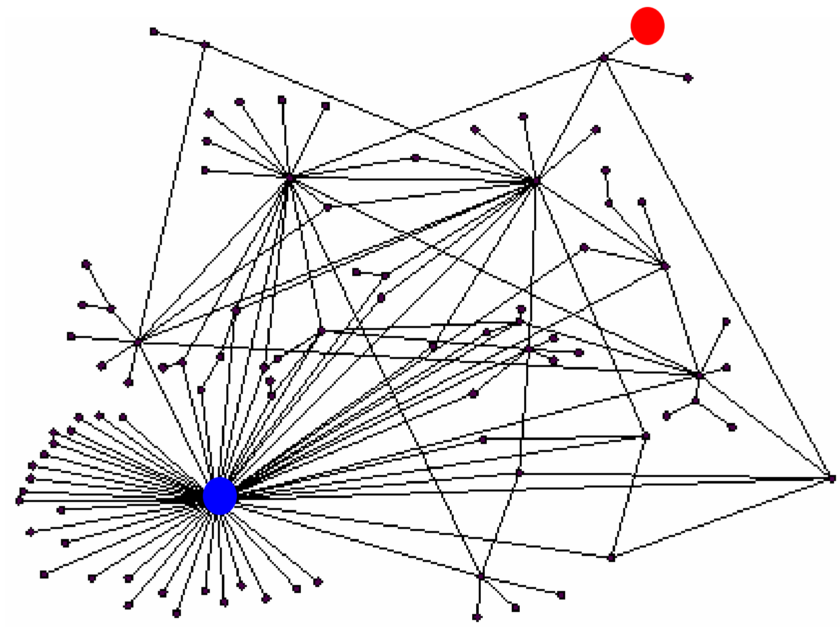
- **(1) Web contains many sources of information Who to "trust"?**
  - **Trick:** Trustworthy pages may point to each other!
- **(2) What is the "best" answer to query "newspaper"?**
  - No single right answer
  - **Trick:** Pages that actually know about newspapers might all be pointing to many newspapers

Ranking Nodes on the Graph

- **All web pages are not equally "important"**

  thispersondoesnotexist.com vs. www.stanford.edu

- There is a large diversity in the web-graph node connectivity.
  Let's rank the pages by the link structure!



Link Analysis Algorithms

- We will cover the following **Link Analysis approaches** for computing **importance** of nodes in a graph:
  - PageRank
  - Topic-Specific (Personalized) PageRank
  - Web Spam Detection Algorithms

PageRank:
The “Flow” Formulation

Links as Votes

- **Idea: Links as votes**
  - **Page is more important if it has more links**
    - In-coming links? Out-going links?
- **Think of in-links as votes:**
  - www.stanford.edu has **millions** in-links
  - thispersondoesnotexist.com has a **few thousands** in-link
- **Are all in-links equal?**
  - **Links from important pages count more**
  - Recursive question!

Intuition – (1)

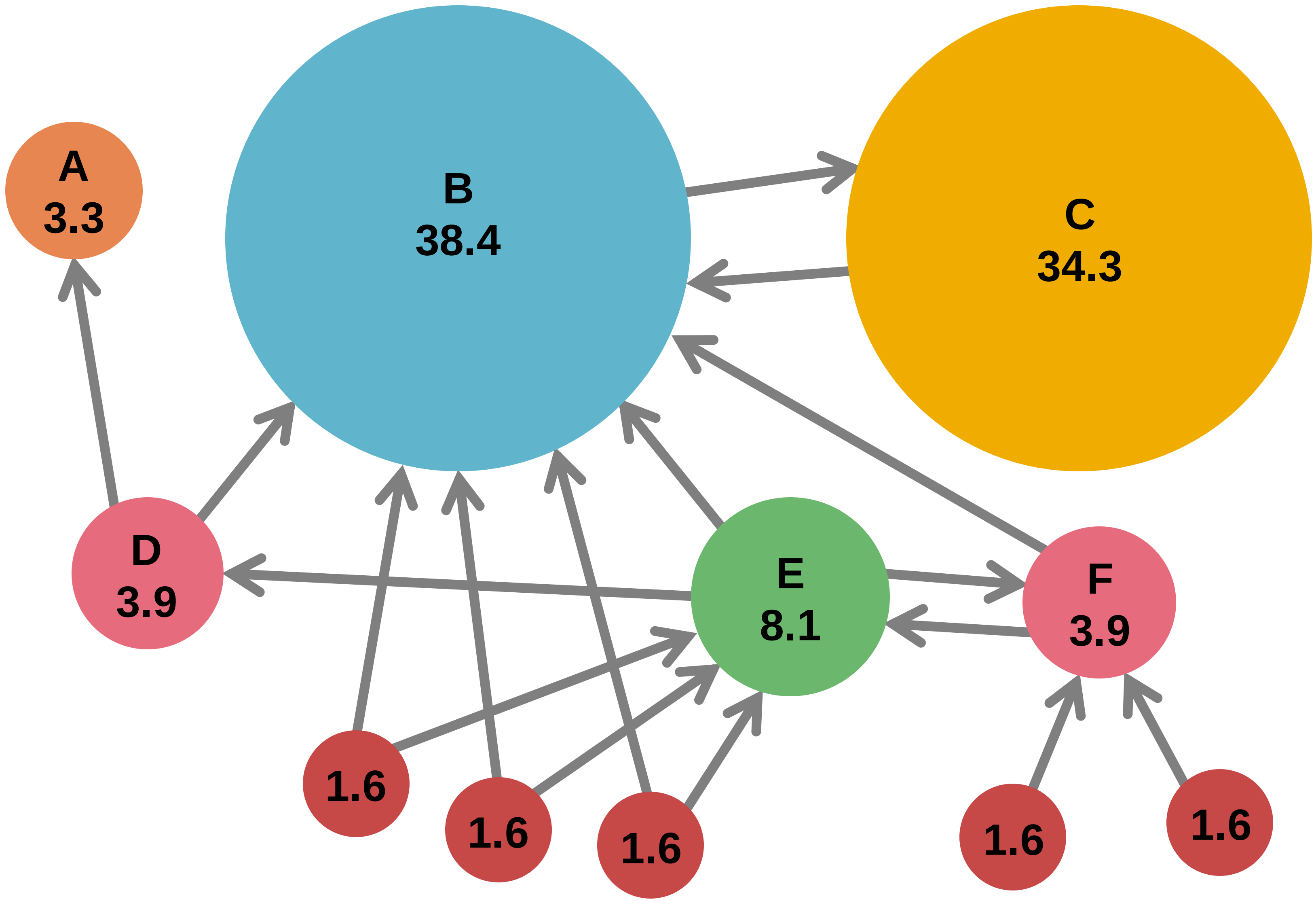
- Web pages are important if people visit them a lot.
- But we can’t watch everybody using the Web.
- A good surrogate for visiting pages is to assume people follow links randomly.
- Leads to *random surfer* model:
  - Start at a random page and follow random out-links repeatedly, from whatever page you are at.
  - *PageRank* = limiting probability of being at a page.

Intuition – (2)

- **Solve the recursive equation:** "importance of a page = its share of the importance of each of its predecessor pages"
  - Equivalent to the random-surfer definition of PageRank

- Technically, *importance* = the principal eigenvector of the transition matrix of the Web
  - A few fix-ups needed

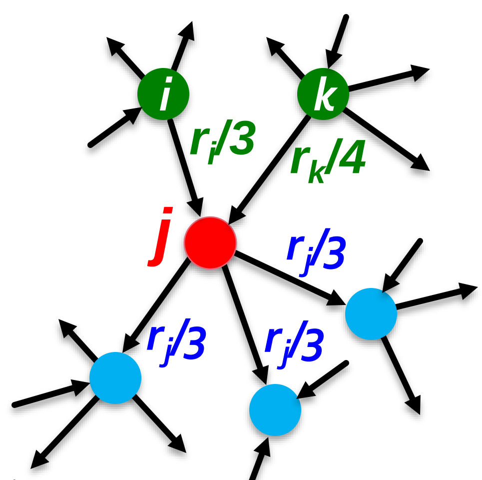
Example: PageRank Scores



Simple Recursive Formulation

- Each link's vote is proportional to the **importance** of its source page
- If page j with importance r_j has n out-links, each link gets r_j / n votes
- Page j's own importance is the sum of the votes on its in-links

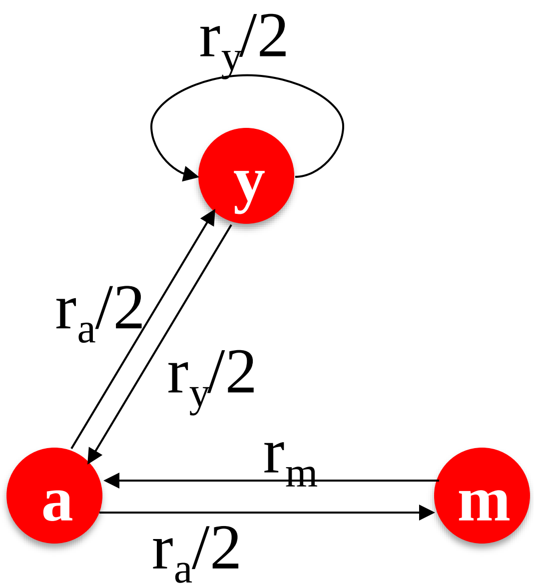
$$r_j = r_i/3 + r_k/4$$



PageRank: The "Flow" Model

- **A "vote" from an important page is worth more**
- **A page is important if it is pointed to by other important pages**
- **Define a "rank" r_j for page j**

The web in 1839



$$r_j = \sum_{i \to j} \frac{r_i}{d_i}$$

d_i … out-degree of node i

"Flow" equations:

$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

r_j are the solutions to the "flow" equation

Solving the Flow Equations

Flow equations:

$r_y = r_y/2 + r_a/2$

$r_a = r_y/2 + r_m$

$r_m = r_a/2$

- **3 equations, 3 unknowns, no constants**
  - No unique solution
  - All solutions equivalent modulo the scale factor
- **Additional constraint forces uniqueness:**
  - $\boldsymbol{r_y + r_a + r_m = 1}$
  - **Solution:** $\boldsymbol{r_y = \frac{2}{5},\ r_a = \frac{2}{5},\ r_m = \frac{1}{5}}$
- **Gaussian elimination method works for small examples, but we need a better method for large web-size graphs**
- **We need a new formulation!**

PageRank: Matrix Formulation

- **Stochastic adjacency matrix $\boldsymbol{M}$**
  - Let page i has d_i out-links
  - If $i \rightarrow j$, then $M_{ji} = \frac{1}{d_i}$ else $M_{ji} = 0$
    - $\boldsymbol{M}$ is a **column stochastic matrix**
      - Columns sum to 1
- **Rank vector $\boldsymbol{r}$:** vector with an entry per page
  - r_i is the importance score of page i
  - $\sum_i r_i = 1$
- **The flow equations can be written**

$$r_j = \sum_{i \rightarrow j} \frac{r_i}{\mathrm{d_i}}$$

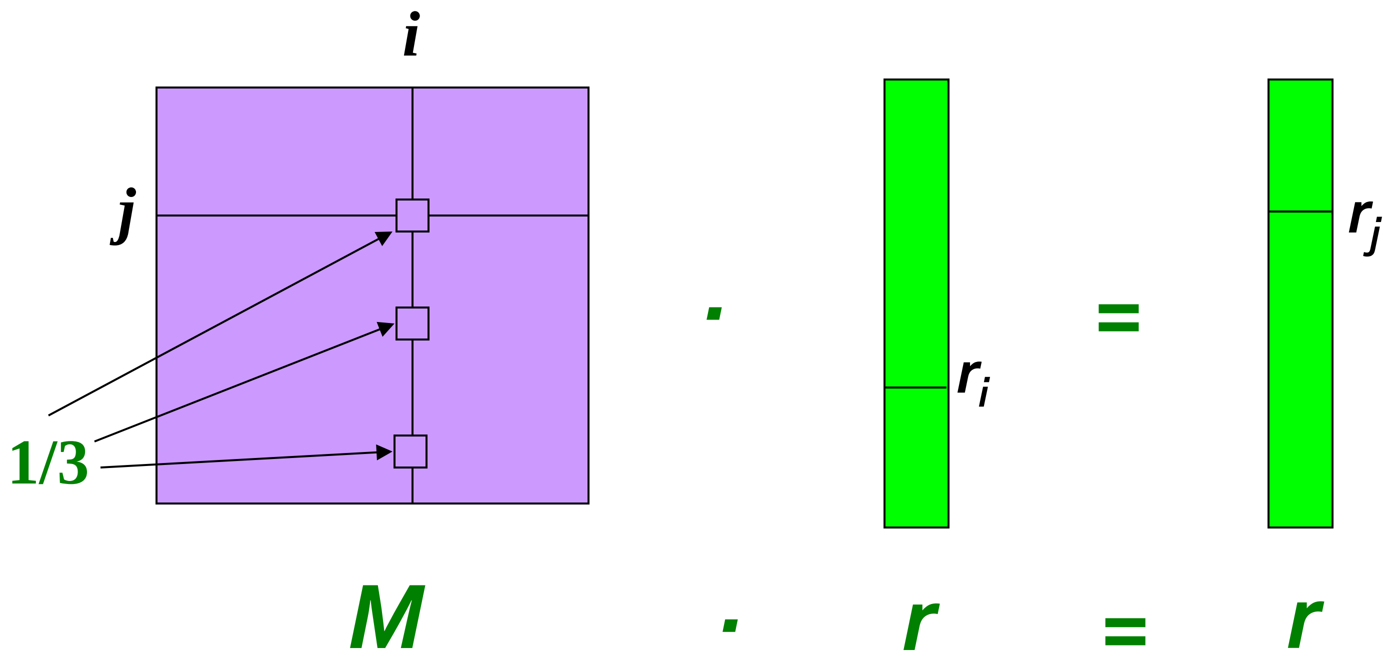
$$\boldsymbol{r} = \boldsymbol{M} \cdot \boldsymbol{r}$$

Example

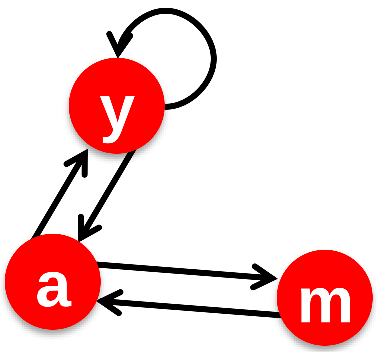
- Remember the flow equation: $r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d}_\mathrm{i}}$
- Flow equation in the matrix form

$$M \cdot r = r$$

  - Suppose page *i* links to 3 pages, including *j*



Example: Flow Equations & M



$$
\begin{array}{c|ccc}
 & r_y & r_a & r_m \\
\hline
r_y & \frac{1}{2} & \frac{1}{2} & 0 \\
r_a & \frac{1}{2} & 0 & 1 \\
r_m & 0 & \frac{1}{2} & 0
\end{array}
$$

$$r = M \cdot r$$

$$r_y = r_y/2 + r_a/2$$

$$r_a = r_y/2 + r_m$$

$$r_m = r_a/2$$

$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix}$$

Eigenvector Formulation

- **The flow equations can be written**

$$\boldsymbol{r} = \boldsymbol{M} \cdot \boldsymbol{r}$$

- So the **rank vector** ***r*** is an **eigenvector** of the stochastic web matrix ***M***
  - In fact, its first or principal eigenvector, with corresponding eigenvalue ***1***
    - Largest eigenvalue of ***M*** is **1** since ***M*** is column stochastic (with non-negative entries)
      - *We know* ***r*** *is unit length and each column of* ***M*** *sums to one, so* $\boldsymbol{Mr} \leq \mathbf{1}$

NOTE: ***x*** is an eigenvector with the corresponding eigenvalue **λ** if:
$$\boldsymbol{Ax} = \boldsymbol{\lambda x}$$

- **We can now efficiently solve for *r*!**
  The method is called Power iteration

Power Iteration Method

- **Given a web graph with *N* nodes, where the nodes are pages and edges are hyperlinks**
- **Power iteration:** a simple iterative scheme
  - Suppose there are *N* web pages
  - Initialize: $\mathbf{r}^{(0)} = [1/N,\ldots.,1/N]^T$
  - Iterate: $\mathbf{r}^{(t+1)} = \mathbf{M} \cdot \mathbf{r}^{(t)}$
  - Stop when $|\mathbf{r}^{(t+1)} - \mathbf{r}^{(t)}|_1 < \varepsilon$

$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d}_\mathrm{i}}$$

d_i …. out-degree of node i

$|\mathbf{x}|_1 = \sum_{1 \le i \le N} |x_i|$ is the $\mathbf{L_1}$ norm
So that **r** is a distribution (sums to 1)

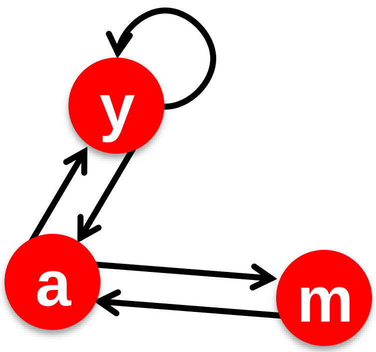
About 50 iterations is sufficient to estimate the limiting solution.

PageRank: How to solve?

- **Power Iteration:**
  - Set $r_j = 1/N$
  - **1:** $r'_j = \sum_{i \to j} \frac{r_i}{d_i}$
  - **2:** $r = r'$
  - Goto **1**
- **Example:**

$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 \\ 1/3 \\ 1/3 \end{matrix}$$

Iteration 0, 1, 2, …



	y	a	m
y	½	½	0
a	½	0	1
m	0	½	0

$r_y = r_y/2 + r_a/2$
$r_a = r_y/2 + r_m$
$r_m = r_a/2$

PageRank: How to solve?

- **Power Iteration:**
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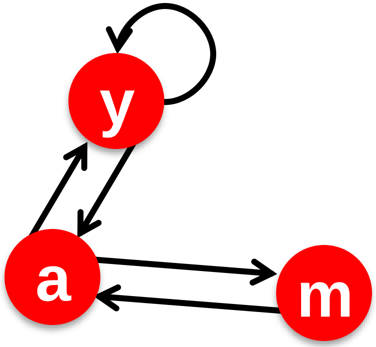


	y	a	m
y	½	½	0
a	½	0	1
m	0	½	0

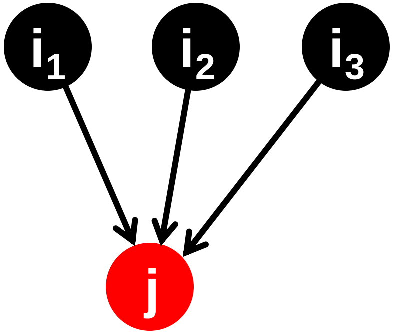
$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{array}{ccccccc} 1/3 & 1/3 & 5/12 & 9/24 & & 6/15 \\ 1/3 & 3/6 & 1/3 & 11/24 & \ldots & 6/15 \\ 1/3 & 1/6 & 3/12 & 1/6 & & 3/15 \end{array}$$

Iteration 0, 1, 2, …

Random Walk Interpretation

- **Imagine a random web surfer:**
  - At any time t, surfer is on some page i
  - At time $t+1$, the surfer follows an out-link from i uniformly at random
  - Ends up on some page j linked from i
  - Process repeats indefinitely
- **Let:**
  - $p(t)$ … vector whose i^{th} coordinate is the prob. that the surfer is at page i at time t
  - So, $p(t)$ is a probability distribution over pages

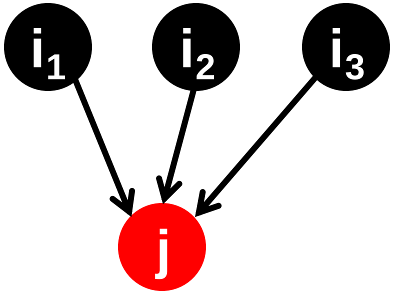


$$r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d_{out}(i)}}$$

The Stationary Distribution

- **Where is the surfer at time *t+1*?**
  - Follows a link uniformly at random

$$\boldsymbol{p}(\boldsymbol{t}+\mathbf{1}) = \boldsymbol{M}\cdot\boldsymbol{p}(\boldsymbol{t})$$



$$p(t+1) = \mathrm{M}\cdot p(t)$$

- Suppose the random walk reaches a state

$$\boldsymbol{p}(\boldsymbol{t}+\mathbf{1}) = \boldsymbol{M}\cdot\boldsymbol{p}(\boldsymbol{t}) = \boldsymbol{p}(\boldsymbol{t})$$

then $\boldsymbol{p}(\boldsymbol{t})$ is **stationary distribution** of a random walk

- **Our original rank vector** $\boldsymbol{r}$ satisfies $\boldsymbol{r} = \boldsymbol{M}\cdot\boldsymbol{r}$
  - **So, $\boldsymbol{r}$ is a stationary distribution for the random walk**

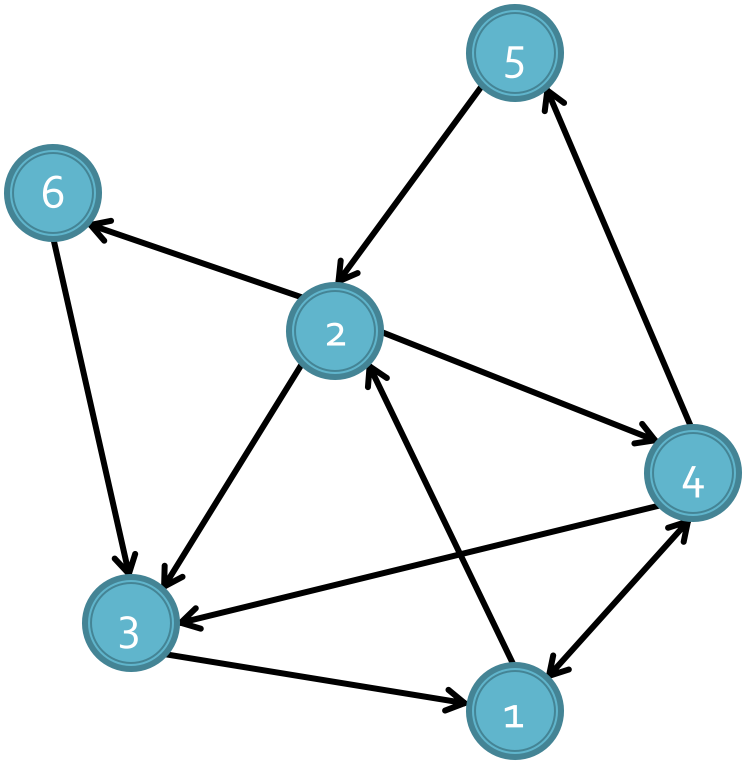
Existence and Uniqueness

- **A central result from the theory of random walks (a.k.a. Markov processes):**

For graphs that satisfy **certain conditions**, the **stationary distribution is unique** and eventually will be reached no matter what is the initial probability distribution at time **t = 0**

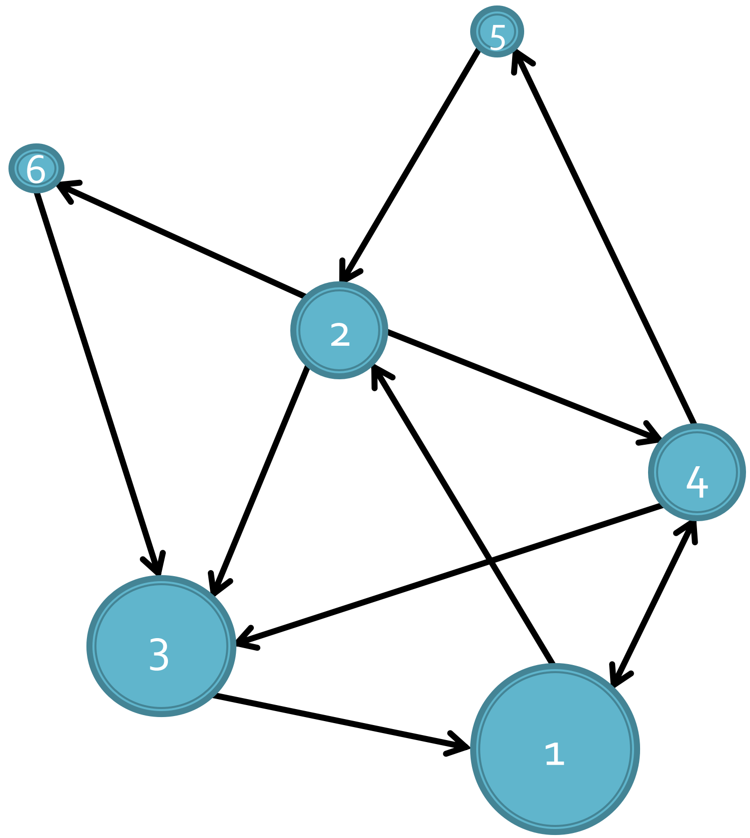
PageRank: test your intuition 1

- Which node has highest PageRank? Second highest?



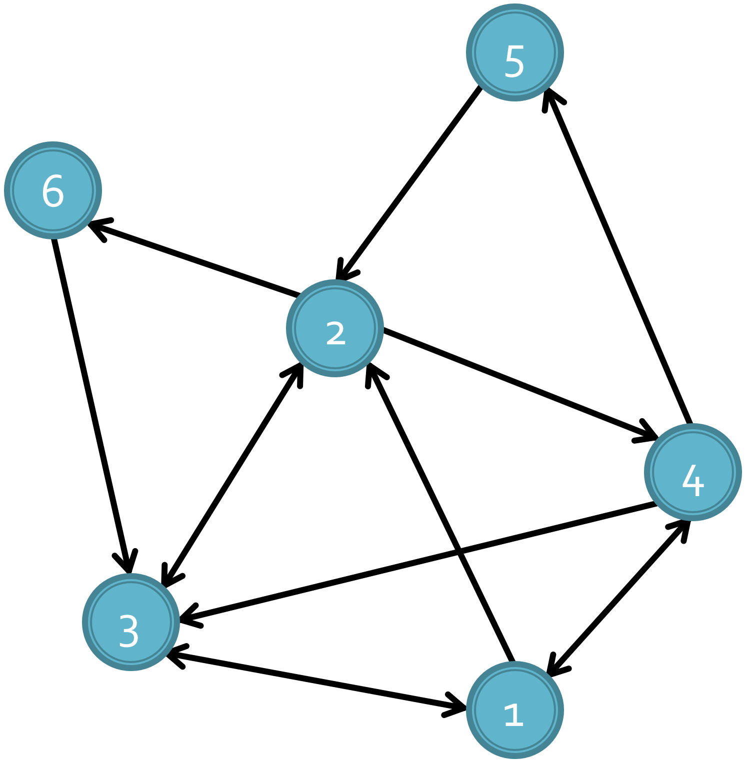
PageRank: test your intuition 1

- Node 1 has the highest PR, followed by Node 3
- Degree ≠ PageRank



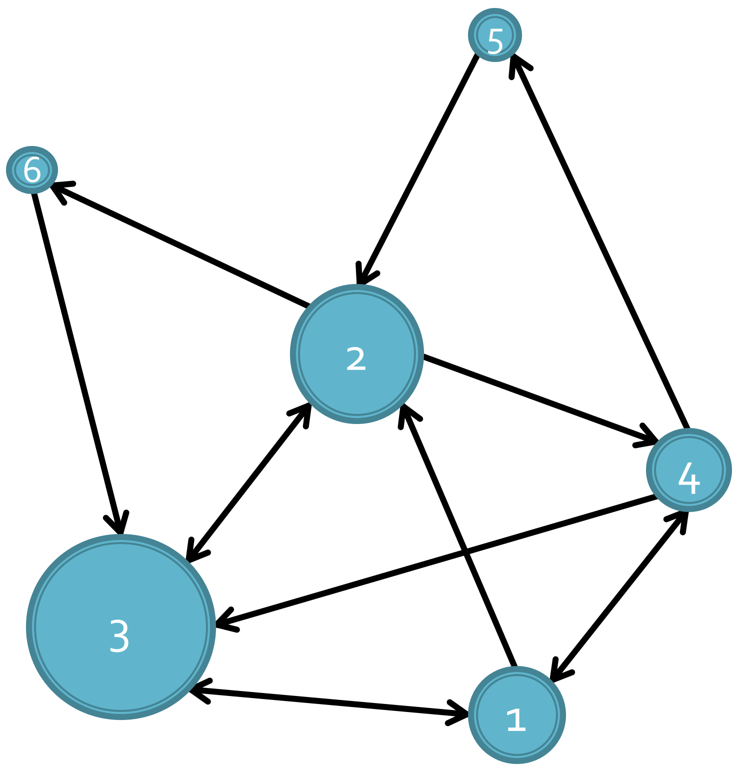
PageRank: test your intuition 2

- Add edge 3 -> 2, 1 -> 3. Now, which node has highest PageRank? Second highest?



PageRank: test your intuition 2

- Node 3 has the highest PR, followed by 2.
- Small changes to graph can change PR!



PageRank:
The Google Formulation

PageRank: Three Questions

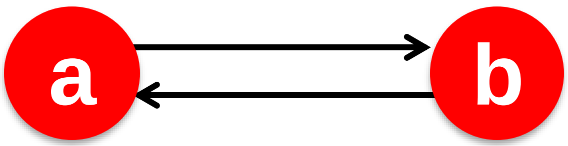
$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d}_\mathrm{i}}$$

or
equivalently

$$r = Mr$$

- **Does this converge?**
- **Does it converge to what we want?**
- **Are results reasonable?**

Does this converge?



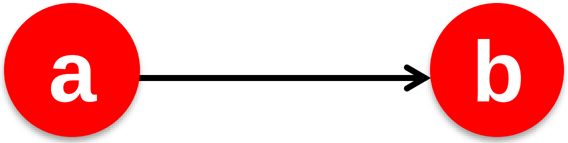
$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d_i}}$$

- **Example:**

r_a	=	1	0	1	0
r_b		0	1	0	1

Iteration 0, 1, 2, …

Does it converge to what we want?



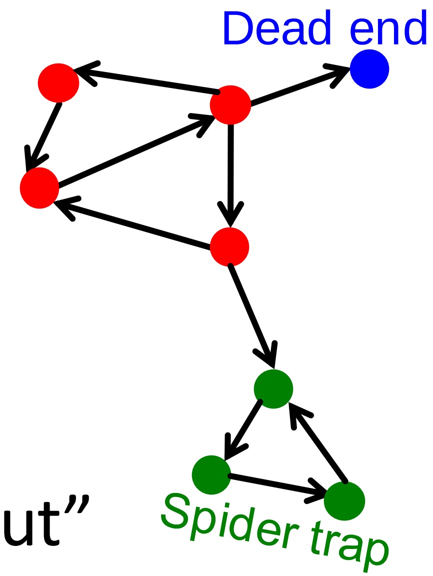
$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d}_i}$$

- **Example:**

r_a	=	1	0	0	0
r_b		0	1	0	0

Iteration 0, 1, 2, …

PageRank: Problems



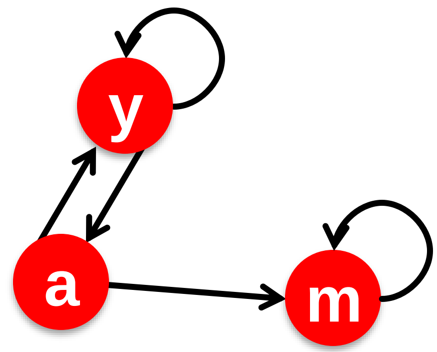
Two problems:

- **(1) Dead ends:** Some pages have no out-links
  - Random walk has "nowhere" to go to
  - Such pages cause importance to "leak out"

- **(2) Spider traps:** (all out-links are within the group)
  - Random walk gets "stuck" in a trap
  - And eventually spider traps absorb all importance

Problem: Spider Traps

- **Power Iteration:**
  - Set $r_j = 1/N$
  - $r_j = \sum_{i \to j} \frac{r_i}{d_i}$
    - And iterate



m is a spider trap

	y	a	m
y	½	½	0
a	½	0	0
m	0	½	1

$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2$$
$$r_m = r_a/2 + r_m$$

- **Example:**

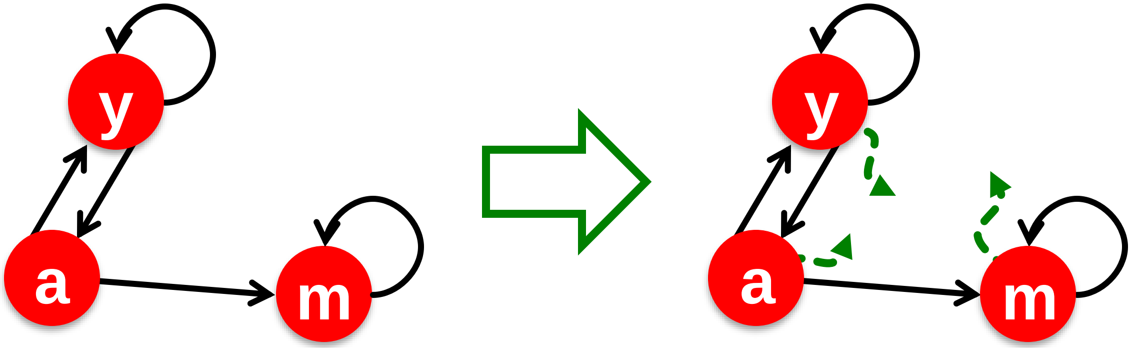
$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 & 2/6 & 3/12 & 5/24 & & 0 \\ 1/3 & 1/6 & 2/12 & 3/24 & \ldots & 0 \\ 1/3 & 3/6 & 7/12 & 16/24 & & 1 \end{matrix}$$

Iteration 0, 1, 2, …

All the PageRank score gets "trapped" in node m.

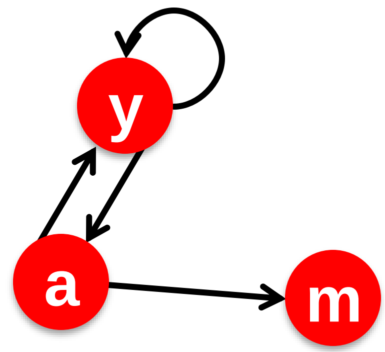
Solution: Teleports!

- **The Google solution for spider traps: At each time step, the random surfer has two options**
  - With prob. β, follow a link at random
  - With prob. **1-**β, jump to some random page
  - β is typically in the range 0.8 to 0.9
- **Surfer will teleport out of spider trap within a few time steps**



Problem: Dead Ends

- **Power Iteration:**
  - Set $r_j = 1/N$
  - $r_j = \sum_{i \to j} \frac{r_i}{d_i}$
    - And iterate



	y	a	m
y	½	½	0
a	½	0	0
m	0	½	0

$r_y = r_y/2 + r_a/2$

$r_a = r_y/2$

$r_m = r_a/2$

- **Example:**

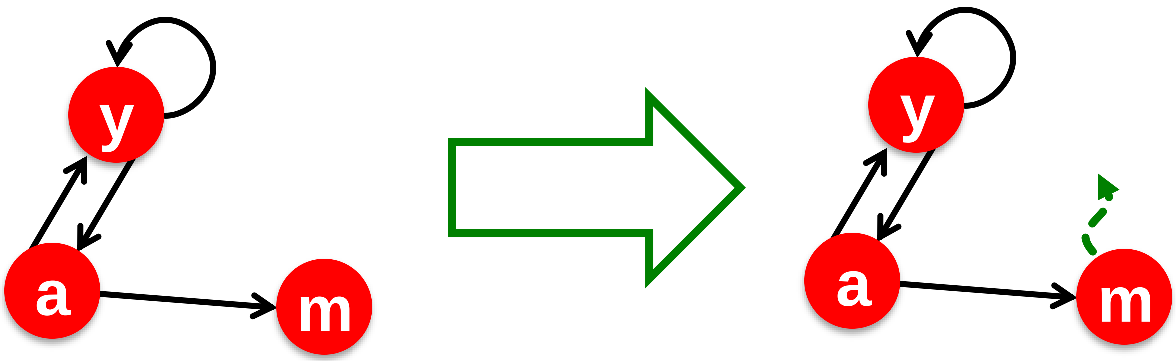
$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 & 2/6 & 3/12 & 5/24 & & 0 \\ 1/3 & 1/6 & 2/12 & 3/24 & \dots & 0 \\ 1/3 & 1/6 & 1/12 & 2/24 & & 0 \end{matrix}$$

Iteration 0, 1, 2, …

Here the PageRank score "leaks" out since the matrix is not stochastic.

Solution: Always Teleport!

- **Teleports:** Follow random teleport links with probability 1.0 from dead-ends
  - Adjust matrix accordingly



	y	a	m
y	½	½	0
a	½	0	0
m	0	½	0

	y	a	m
y	½	½	⅓
a	½	0	⅓
m	0	½	⅓

Why Teleports Solve the Problem?

Why are dead-ends and spider traps a problem and why do teleports solve the problem?

- **Spider-traps** are not a problem, but with traps PageRank scores are **not** what we want
  - **Solution:** Never get stuck in a spider trap by teleporting out of it in a finite number of steps
- **Dead-ends** are a problem
  - The matrix is not column stochastic so our initial assumptions are not met
  - **Solution:** Make matrix column stochastic by always teleporting when there is nowhere else to go

Solution: Random Teleports

- <u>**Google's solution that does it all:**</u>
  At each step, random surfer has two options:
  - With probability β, follow a link at random
  - With probability $1\text{-}\beta$, jump to some random page
- **PageRank equation** [Brin-Page, 98]

$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1-\beta)\frac{1}{N}$$

d_i ... out-degree of node i

This formulation assumes that M has no dead ends. We can either preprocess matrix M to remove all dead ends or explicitly follow random teleport links with probability 1.0 from dead-ends.

The Google Matrix

- **PageRank equation** [Brin-Page, '98]

$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1-\beta)\frac{1}{N}$$

- **The Google Matrix *A*:**

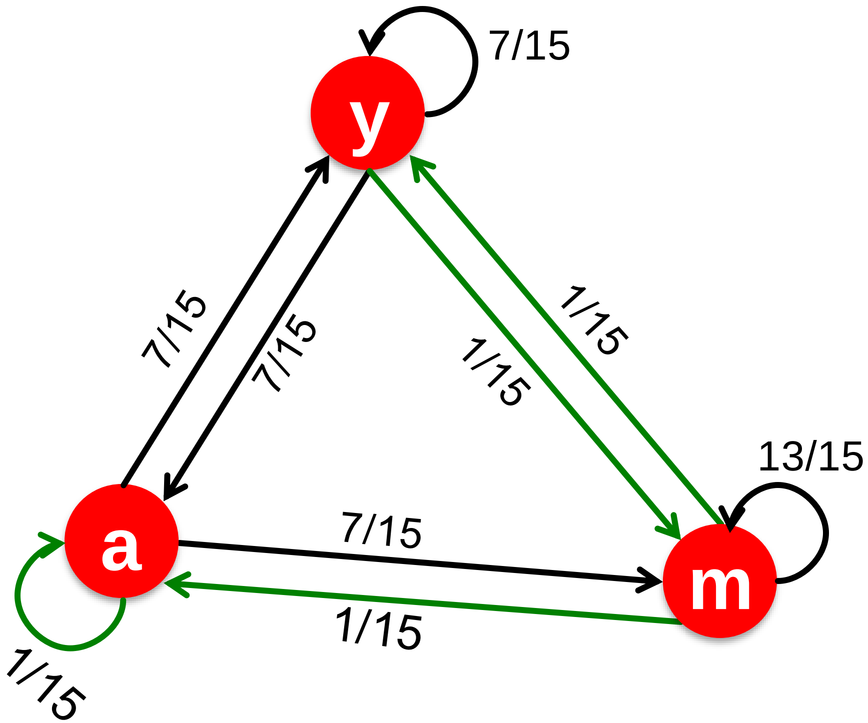
[1/N]$_{NxN}$…N by N matrix where all entries are 1/N

$$A = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$$

- **We have a recursive problem:** $\boldsymbol{r = A \cdot r}$

  And the Power method still works!
- **What is β?**
  - In practice β =*0.8,0.9* (jump every *5* steps on avg.)

Random Teleports ($\beta = 0.8$)



$$0.8 \overset{\mathbf{M}}{\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}} + 0.2 \overset{[1/N]_{NxN}}{\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}}$$

$$\underset{\mathbf{A}}{\begin{array}{c} y \\ a \\ m \end{array} \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}}$$

$$\begin{array}{ccccccccc} y & & 1/3 & 0.33 & 0.24 & 0.26 & & 7/33 \\ a & = & 1/3 & 0.20 & 0.20 & 0.18 & \dots & 5/33 \\ m & & 1/3 & 0.46 & 0.52 & 0.56 & & 21/33 \end{array}$$

How do we actually compute the PageRank?

Computing PageRank

- **Key step is matrix-vector multiplication**
  - $r^{new} = A \cdot r^{old}$
- Easy if we have enough main memory to hold $\mathbf{A}$, $\mathbf{r}^{old}$, $\mathbf{r}^{new}$
- **Say N = 1 billion pages**
  - We need 4 bytes for each entry (say)
  - 2 billion entries for vectors, approx 8GB
  - Matrix **A** has N^2 entries
    - 10^{18} is a large number!

$$\mathbf{A} = \beta \cdot \mathbf{M} + (1-\beta)\,[1/N]_{N \times N}$$

$$A = 0.8 \begin{bmatrix} ½ & ½ & 0 \\ ½ & 0 & 0 \\ 0 & ½ & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}$$

Rearranging the Equation

- $\boldsymbol{r} = \boldsymbol{A} \cdot \boldsymbol{r}$, **where** $\boldsymbol{A_{ji}} = \boldsymbol{\beta}\, \boldsymbol{M_{ji}} + \frac{\boldsymbol{1-\beta}}{\boldsymbol{N}}$
- $r_j = \sum_{\mathrm{i}=1}^{N} A_{ji} \cdot r_i$
- $r_j = \sum_{i=1}^{N} \left[\beta\, M_{ji} + \frac{1-\beta}{N}\right] \cdot r_i$

$$= \sum_{\mathrm{i}=1}^{N} \beta\, M_{ji} \cdot r_i + \frac{1-\beta}{N} \sum_{\mathrm{i}=1}^{N} r_i$$

$$= \sum_{\mathrm{i}=1}^{N} \beta\, M_{ji} \cdot r_i + \frac{1-\beta}{N}$$ since $\sum r_i = 1$

- **So we get:** $\boldsymbol{r} = \boldsymbol{\beta}\, \boldsymbol{M} \cdot \boldsymbol{r} + \left[\frac{\boldsymbol{1-\beta}}{\boldsymbol{N}}\right]_{\boldsymbol{N}}$

Note: Here we assume **M** has no dead-ends

$[x]_N$ … a vector of length N with all entries x

Sparse Matrix Formulation

- We just rearranged the **PageRank equation**

$$\boldsymbol{r} = \boldsymbol{\beta M} \cdot \boldsymbol{r} + \left[\frac{\boldsymbol{1-\beta}}{\boldsymbol{N}}\right]_{\boldsymbol{N}}$$

  - where $[(1\text{-}\beta)/N]_N$ is a vector with all N entries **(1-β)/N**

- ***M*** is a **sparse matrix!** (with no dead-ends)
  - 10 links per node, approx $10N$ entries
- So in each iteration, we need to:
  - Compute $\boldsymbol{r}^{\text{new}} = \beta \boldsymbol{M} \cdot \boldsymbol{r}^{\text{old}}$
  - Add a constant value **(1-β)/N** to each entry in $\boldsymbol{r}^{\text{new}}$
    - **Note if M contains dead-ends then $\sum_j r_j^{new} < 1$ and we also have to renormalize r^{new} so that it sums to 1**

PageRank: The Complete Algorithm

- **Input: Graph G and parameter β**
  - Directed graph G (can have **spider traps** and **dead ends**)
  - Parameter β
- **Output: PageRank vector r^{new}**
  - **Set:** $r_j^{old} = \frac{1}{N}$
  - **repeat until convergence:** $\sum_j \left| r_j^{new} - r_j^{old} \right| < \varepsilon$
    - $\forall j$: $\boldsymbol{r'^{new}_j = \sum_{i \to j} \beta \frac{r_i^{old}}{d_i}}$

      $\boldsymbol{r'^{new}_j = 0}$ if in-degree of $\boldsymbol{j}$ is **0**
    - **Now re-insert the leaked PageRank:**

      $\forall \boldsymbol{j}$: $\boldsymbol{r_j^{new} = r'^{new}_j + \frac{1-S}{N}}$ **where:** $S = \sum_j r'^{new}_j$
    - $\boldsymbol{r^{old} = r^{new}}$

If the graph has no dead-ends then the amount of leaked PageRank is **1-β**. But since we have dead-ends the amount of leaked PageRank may be larger. We have to explicitly account for it by computing **S**.

Sparse Matrix Encoding

- **Encode sparse matrix using only nonzero entries**
  - Space proportional roughly to number of links
  - Say 10N, or 4*10*1 billion = 40GB
  - **Still won't fit in memory, but will fit on disk**

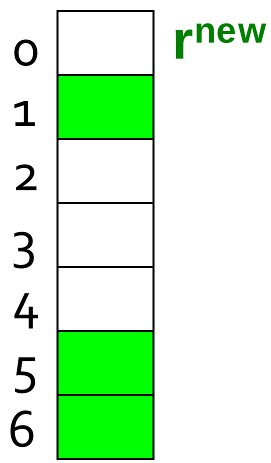
source node	degree	destination nodes
0	3	1, 5, 7
1	5	17, 64, 113, 117, 245
2	2	13, 23

Basic Algorithm: Update Step

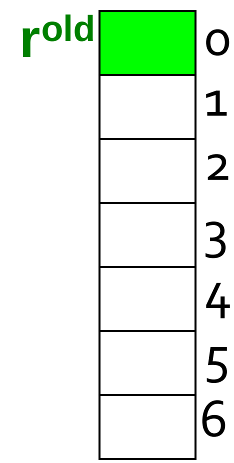
- **Assume enough RAM to fit r^{new} into memory**
  - Store r^{old} and matrix **M** on disk
- **1 step of power-iteration is:**

Initialize all entries of $\mathbf{r^{new}} = \mathbf{(1\text{-}\beta) / N}$
For each page i (of out-degree d_i):
  Read into memory: i, d_i, $dest_1, \ldots, dest_{d_i}$, $r^{old}(i)$
  For $\mathbf{j = 1 \ldots d_i}$
    $\mathbf{r^{new}(dest_j) \mathrel{+}= \beta\, r^{old}(i) / d_i}$

Assuming no dead ends



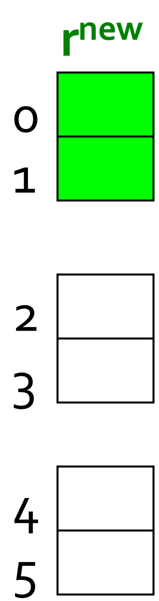
source	degree	destination
0	3	1, 5, 6
1	4	17, 64, 113, 117
2	2	13, 23



Analysis

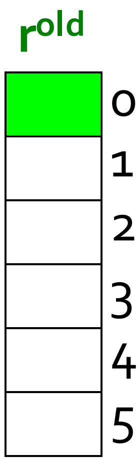
- **Assume enough RAM to fit r^{new} into memory**
  - Store r^{old} and matrix M on disk
- **In each iteration, we have to:**
  - Read r^{old} and M
  - Write r^{new} back to disk
  - **Cost per iteration of Power method:**
    $= 2|\boldsymbol{r}| + |\boldsymbol{M}|$
- **Question:**
  - What if we could not even fit r^{new} in memory?

Block-based Update Algorithm



src	degree	destination
0	4	0, 1, 3, 5
1	2	0, 5
2	2	3, 4

M



- Break r^{new} into k blocks that fit in memory
- Scan M and r^{old} once for each block

Analysis of Block Update

- **Similar to nested-loop join in databases**
  - Break $\boldsymbol{r}^{\text{new}}$ into $\boldsymbol{k}$ blocks that fit in memory
  - Scan $\boldsymbol{M}$ and $\boldsymbol{r}^{\text{old}}$ once for each block
- **Total cost:**
  - $\boldsymbol{k}$ scans of $\boldsymbol{M}$ and $\boldsymbol{r}^{\text{old}}$
  - **Cost per iteration of Power method:**
    $k(|\boldsymbol{M}| + |\boldsymbol{r}|) + |\boldsymbol{r}| = \boldsymbol{k|M| + (k+1)|r|}$
- **Can we do better?**
  - **Hint:** $\boldsymbol{M}$ is much bigger than $\boldsymbol{r}$ (approx 10-20x), so we must avoid reading it $\boldsymbol{k}$ times per iteration

Block-Stripe Update Algorithm

r^{new}

- 0
- 1

src	degree	destination
0	4	0, 1
1	2	0

- 2
- 3

src	degree	destination
0	4	3
2	2	3

- 4
- 5

src	degree	destination
0	4	5
1	2	5
2	2	4

r^{old}

- 0
- 1
- 2
- 3
- 4
- 5

Break *M* into stripes! Each stripe contains only destination nodes in the corresponding block of r^{new}

Block-Stripe Analysis

- Break ***M*** into stripes
  - Each stripe contains only destination nodes in the corresponding block of $\boldsymbol{r}^{\text{new}}$
- Some additional overhead per stripe
  - But it is usually worth it
- **Cost per iteration of Power method:**

=$|\boldsymbol{M}|(\mathbf{1}+\varepsilon) + (\boldsymbol{k}+\mathbf{1})|\boldsymbol{r}|$

where ε is a small number.

Some Problems with PageRank

- **Measures generic popularity of a page**
  - Biased against topic-specific authorities
  - **Solution:** Topic-Specific PageRank (**next**)
- **Uses a single measure of importance**
  - Other models of importance
  - **Solution:** Hubs-and-Authorities
- **Susceptible to Link spam**
  - Artificial link topographies created in order to boost page rank
  - **Solution:** TrustRank

Extras

Why Power Iteration works? (1)

- **Power iteration:**
  A method for finding dominant eigenvector (the vector corresponding to the largest eigenvalue)
  - $\boldsymbol{r}^{(1)} = \boldsymbol{M} \cdot \boldsymbol{r}^{(0)}$
  - $\boldsymbol{r}^{(2)} = \boldsymbol{M} \cdot \boldsymbol{r}^{(1)} = \boldsymbol{M}\left(\boldsymbol{M}\boldsymbol{r}^{(0)}\right) = \boldsymbol{M}^{2} \cdot \boldsymbol{r}^{(0)}$
  - $\boldsymbol{r}^{(3)} = \boldsymbol{M} \cdot \boldsymbol{r}^{(2)} = \boldsymbol{M}\left(\boldsymbol{M}^{2}\boldsymbol{r}^{(0)}\right) = \boldsymbol{M}^{3} \cdot \boldsymbol{r}^{(0)}$
- **Claim:**
  Sequence $\boldsymbol{M} \cdot \boldsymbol{r}^{(0)}, \boldsymbol{M}^{2} \cdot \boldsymbol{r}^{(0)}, \ldots \boldsymbol{M}^{k} \cdot \boldsymbol{r}^{(0)}, \ldots$ approaches the dominant eigenvector of $\boldsymbol{M}$

Why Power Iteration works? (2)

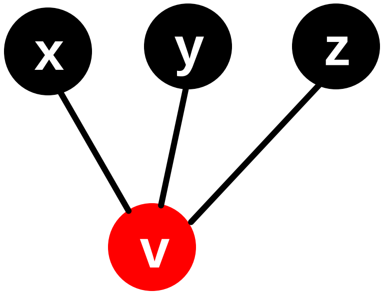
- **Claim:** Sequence $\boldsymbol{M} \cdot \boldsymbol{r}^{(0)}, \boldsymbol{M}^{2} \cdot \boldsymbol{r}^{(0)}, \ldots \boldsymbol{M}^{k} \cdot \boldsymbol{r}^{(0)}, \ldots$ approaches the dominant eigenvector of $\boldsymbol{M}$
- **Proof:**
  - Assume ***M*** has ***n*** linearly independent eigenvectors, $x_1, x_2, \ldots, x_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $\lambda_1 > \lambda_2 > \cdots > \lambda_n$
  - Vectors $x_1, x_2, \ldots, x_n$ form a basis and thus we can write: $r^{(0)} = c_1\, x_1 + c_2\, x_2 + \cdots + c_n\, x_n$
  - $$\begin{aligned} \boldsymbol{M}\boldsymbol{r}^{(0)} &= \boldsymbol{M}(\boldsymbol{c_1}\, \boldsymbol{x_1} + \boldsymbol{c_2}\, \boldsymbol{x_2} + \cdots + \boldsymbol{c_n}\, \boldsymbol{x_n}) \\ &= c_1(Mx_1) + c_2(Mx_2) + \cdots + c_n(Mx_n) \\ &= c_1(\lambda_1 x_1) + c_2(\lambda_2 x_2) + \cdots + c_n(\lambda_n x_n) \end{aligned}$$
  - **Repeated multiplication on both sides produces** $$M^k r^{(0)} = c_1(\lambda_1^k x_1) + c_2(\lambda_2^k x_2) + \cdots + c_n(\lambda_n^k x_n)$$

Why Power Iteration works? (3)

- **Claim:** Sequence $\boldsymbol{M} \cdot \boldsymbol{r}^{(0)}, \boldsymbol{M}^{2} \cdot \boldsymbol{r}^{(0)}, \ldots \boldsymbol{M}^{k} \cdot \boldsymbol{r}^{(0)}, \ldots$ approaches the dominant eigenvector of $\boldsymbol{M}$
- **Proof (continued):**
  - Repeated multiplication on both sides produces
    $M^{k}r^{(0)} = c_1(\lambda_1^k x_1) + c_2(\lambda_2^k x_2) + \cdots + c_n(\lambda_n^k x_n)$
  - $M^{k}r^{(0)} = \lambda_1^k \left[c_1 x_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^k x_n \right]$
  - Since $\lambda_1 > \lambda_2$ then fractions $\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1} \ldots < 1$
    and so $\left(\frac{\lambda_i}{\lambda_1}\right)^k = 0$ as $k \to \infty$ (for all $i = 2 \ldots n$).
  - **Thus:** $\boldsymbol{M}^{k}\boldsymbol{r}^{(0)} \approx \boldsymbol{c}_1(\boldsymbol{\lambda}_1^k \boldsymbol{x}_1)$
    - Note if $c_1 = 0$ then the method won't converge

PageRank for Undirected Graphs

- **Given an <u>undirected</u> graph with *N* nodes, where the nodes are pages and edges are hyperlinks**



- **Claim** [*Existence*]**:** For node v, $r_v = d_v/2m$ is a solution.
- Proof:
  - Iteration step: $\mathbf{r}^{(t+1)} = \mathbf{M} \cdot \mathbf{r}^{(t)}$

$$r_v^{(t+1)} = \frac{r_x^t}{d_x} + \frac{r_y^t}{d_y} + \frac{r_z^t}{d_z}$$

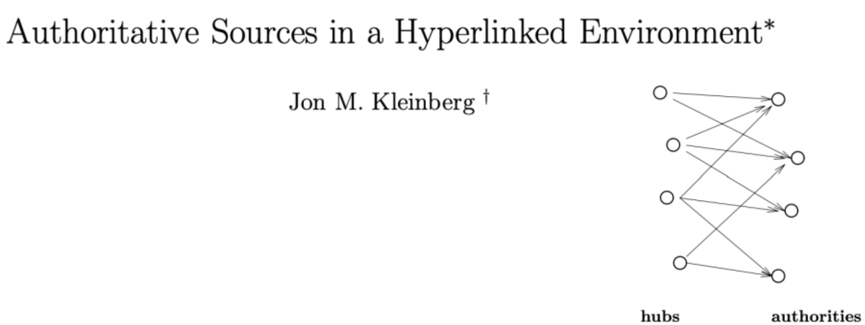
  - Substitute $r_i = d_i/2m$:

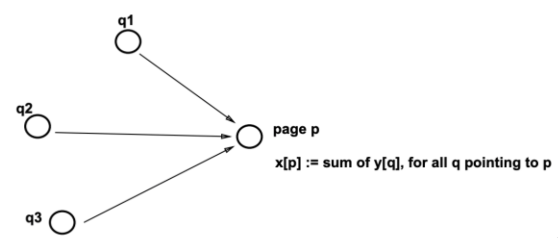
$$r_v^{(t+1)} = \frac{3}{2m}$$

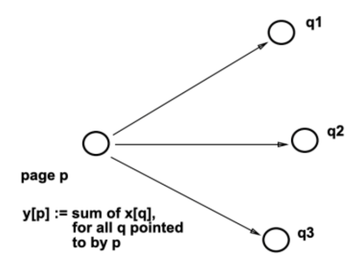
- Done! Uniqueness: exercise! m = #edges

Historical note on Link Analysis

- Classic work: Markov chains, citation analysis
- RankDex patent [Robin Li, '96]
  - Key idea: use backlinks (led to Baidu!)
- HITS Algorithm [Kleinberg, SODA '98]
  - Key idea: iterative scoring!







- PageRank work [Page et al, '98]