Art Gallery Problem [Victor Klee, 1973]

- Input : Art Gallery
- Output : Minimum number of guards that can safe-guard or cover the interior walls of the gallery

- The Guggenheim Museum in Bilbao: hard to supervise (Image courtesy: BBC.com)

Example

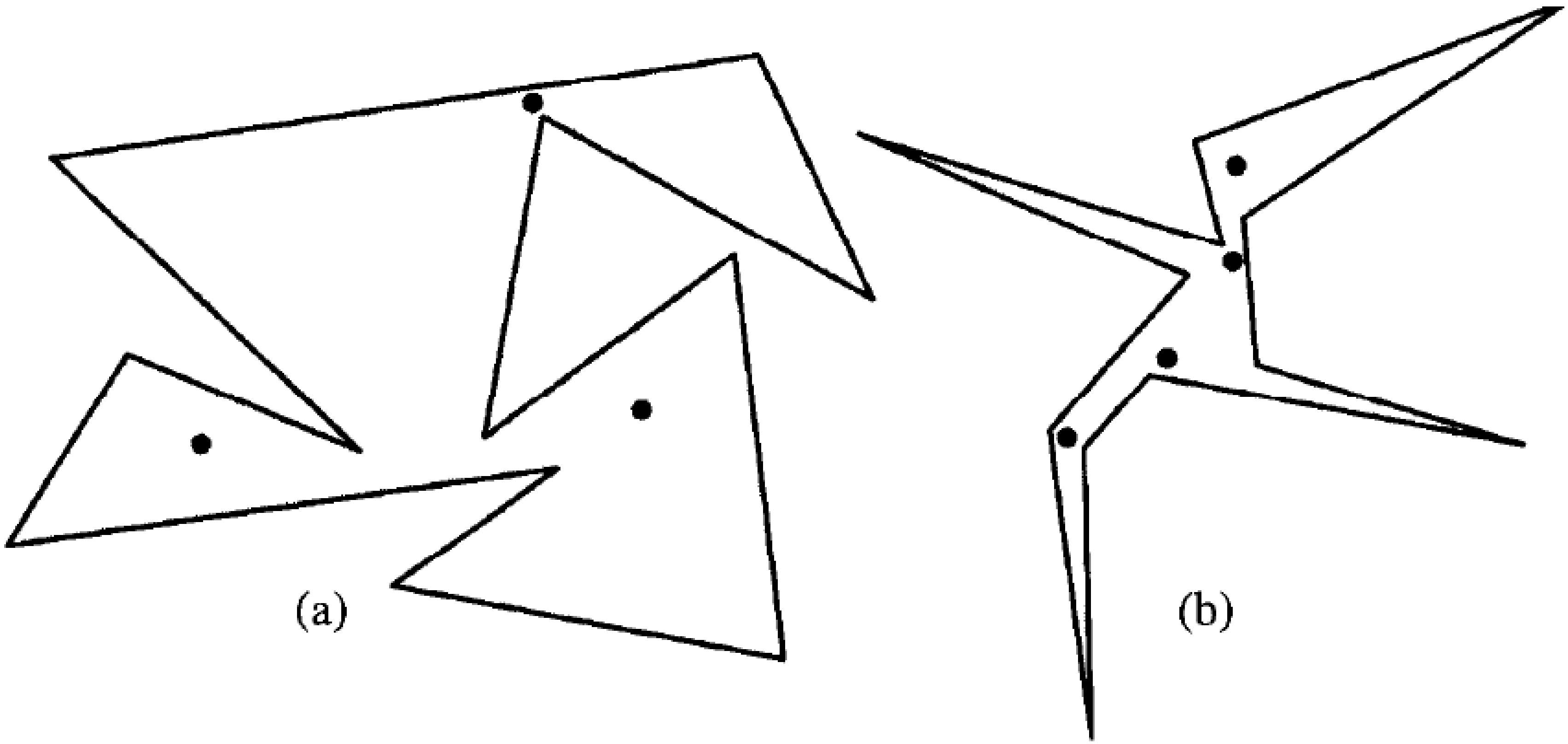
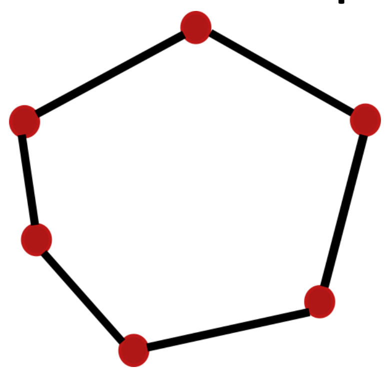
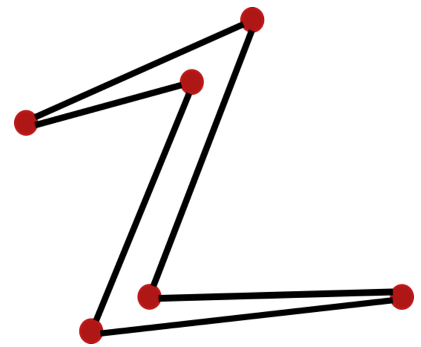


FIGURE 1.3 Two polygons of $n = 12$ vertices: (a) requires 3 guards; (b) requires 4 guards.

Our objective in simple words

- There can be different polygons with *n* vertices

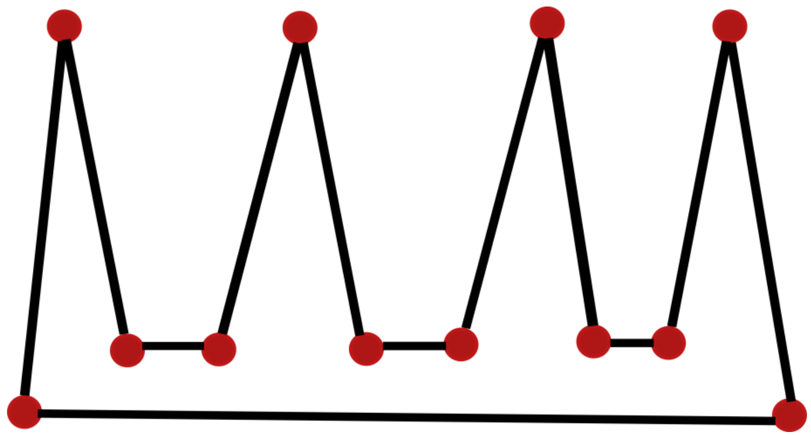
- Find the minimum # of guards for each *P*
- $g(P) = \min_S \{S \text{ covers } P\}$
- Get the maximum # among those
- $G(n) = \max_{P_n} g(P_n)$
- **Max over Min Formulation**

Bounds on $G(n)$

- Certainly at least one guard is always necessary
  - $1 <= G(n)$, hence lower bound of $G(n)$ is 1
- n guards suffice for any P with n vertices
  - stationing point guards near/on every vertex of P
  - $G(n) <= n$, hence upper bound of $G(n)$ is *n or more*

Generalize : $G(n)$ = Floor (n/3)

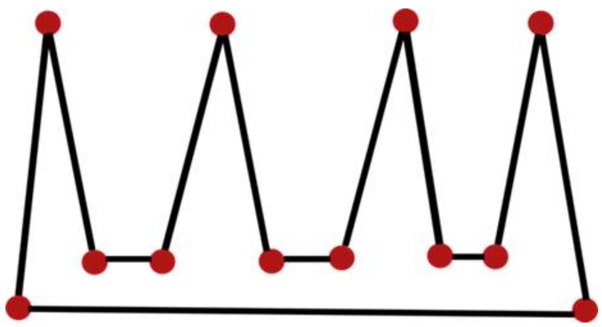
- Consider the comb structure:



- What is the minimum number of guards?
- One guard for each prong?

$G(n)$ = Floor (n/3)

- Proof for ***G*(*n*) = Floor (n/3)**
- First proof was by Chavtal in 1975

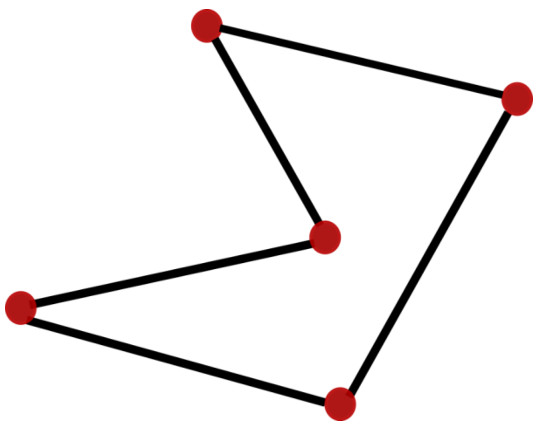


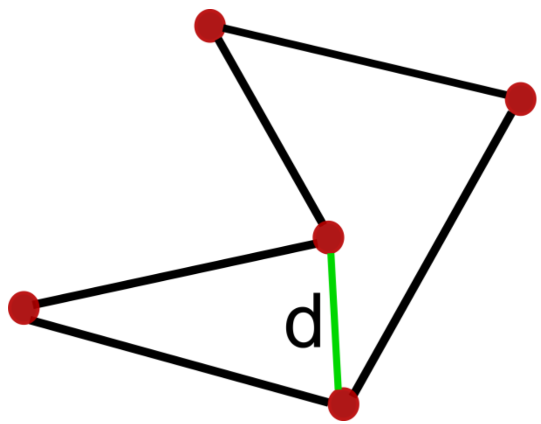
Outline on Chavtal’s proof

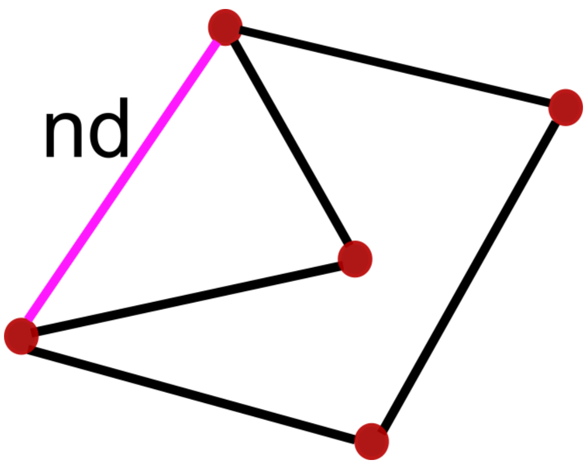
- Mathematical induction
  - By removing a part of polygon & reducing the no. of vertices
  - Applying induction hypotheses
  - Reattach the removed part of the polygon
- Complex proof with many cases

A simpler proof by Fisk in 1978

- Frisk’s proof (1978): Partitioning a polygon to triangles with a diagonal
- What is a diagonal?
- Diagonal of a *P* is a line segment between its vertices which are clearly visible to each other





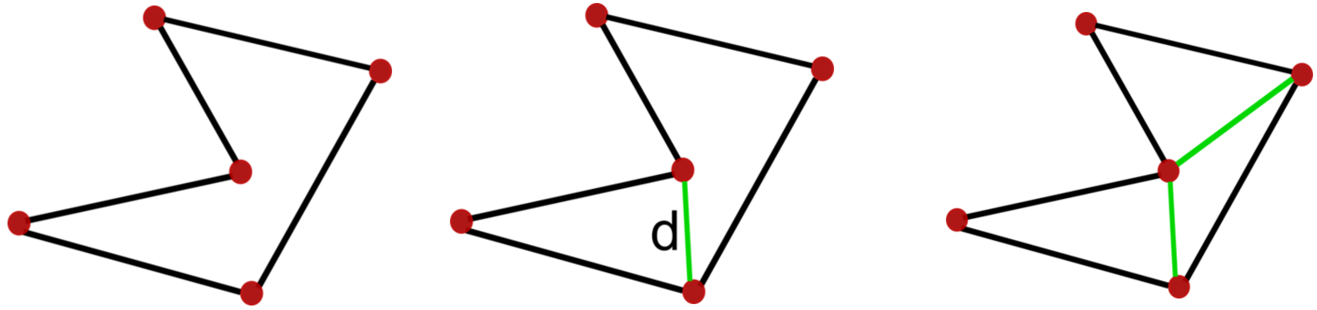


Diagonal of P

- Diagonal of a *P* is a line segment between its vertices a, b which are clearly visible to one another
- Intersection of the closed segment *ab* with ∂P ?
- $ab \cap \partial P = \{a,b\}$
- Open segment ab does not intersect with ∂P
- A diagonal can not make grazing contact with ∂P

Partitioning P into triangles

- Add / Insert non-crossing diagonals
- Non-crossing diagonals are those whose intersection is a subset of their endpoints
  - They share no interior points
- If we add as many non-crossing diagonals in to P as possible, the interior is partitioned in to Δs

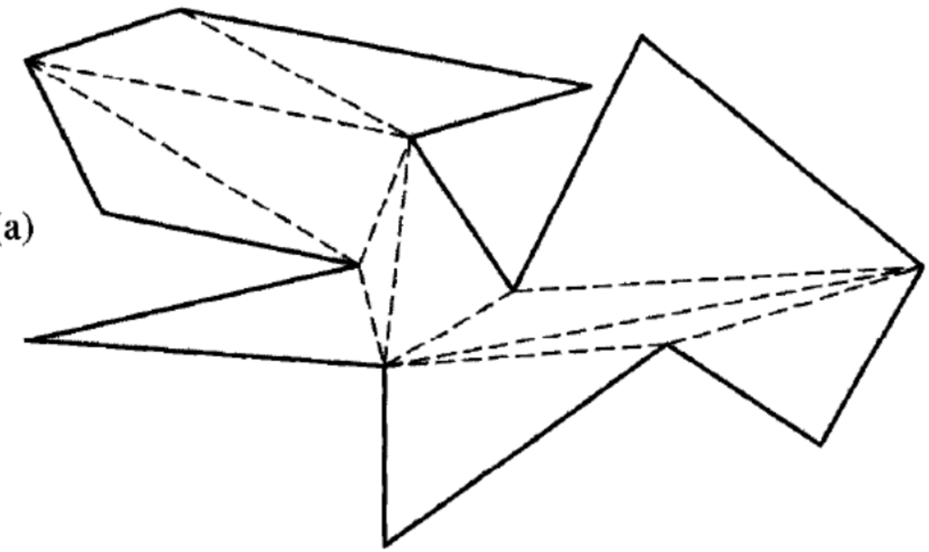


- Such partition is called a **Triangulation of P**

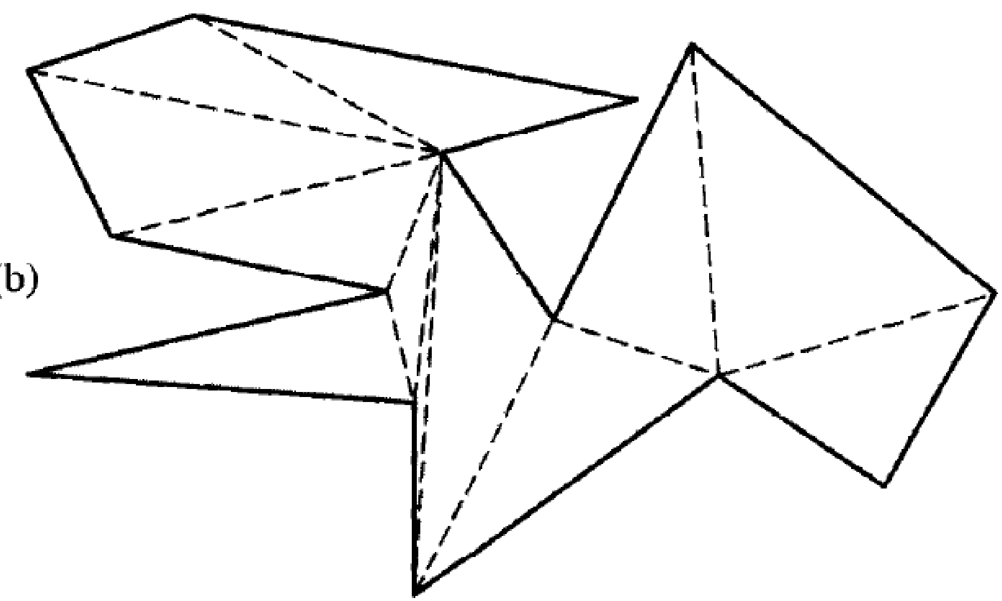
Triangulation(*T*) of A Polygon

- Any rule / order for adding a diagonal?
- Diagonals can be added in any order
- Is *T* unique?
- Exercise: Triangulate a polygon with 10 vertices
- No unique triangulation

(a)



(b)



Get back to the proof by Fisk

- Recall that, we started with Frisk's proof
- Deviated our discussion to what is a diagonal and what is a triangulation
- Now we will get back to the proof by Fisk
- Do u remember what we were trying to prove?

Fisk’s proof for sufficiency of floor (n/3) guards for any P of n vertices

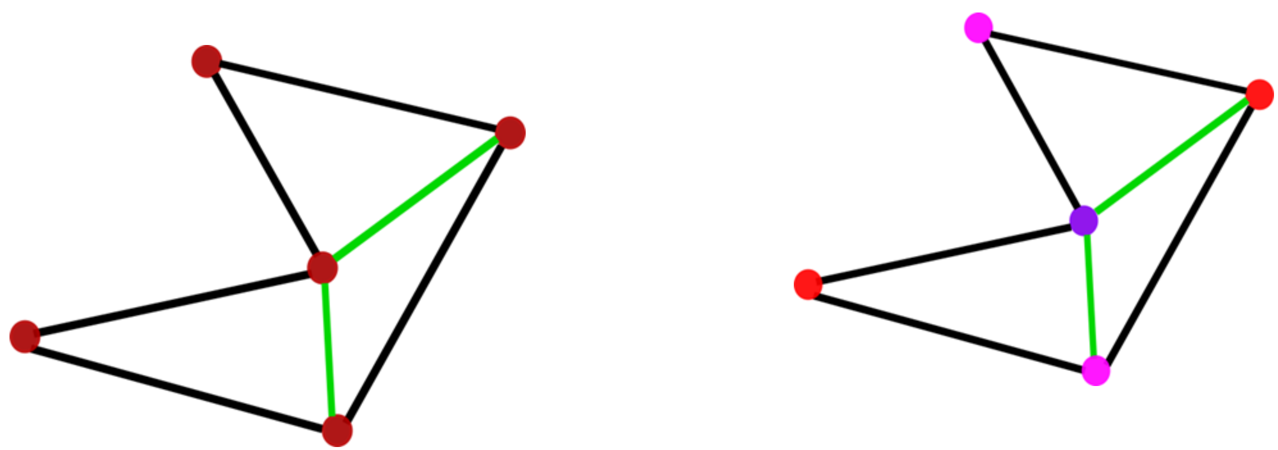
- Two important steps in Fisk’s Proof :
- 1. Triangulate P
- 2. Three-color the triangulation
- What do we mean by three-coloring ?
- On which structure do we apply three-coloring?
- Three-coloring or in general k-coloring is associated with graphs

Our graph : Triangulation of P

- In this case, what is G = (V, E)?
- V = vertices of P
- E = edges of P + diagonals of P
- What is a k-coloring of G?
- k-coloring of G :
  - Assign k colors to the vertices of G
  - No two vertices connected by an edge are assigned the same color

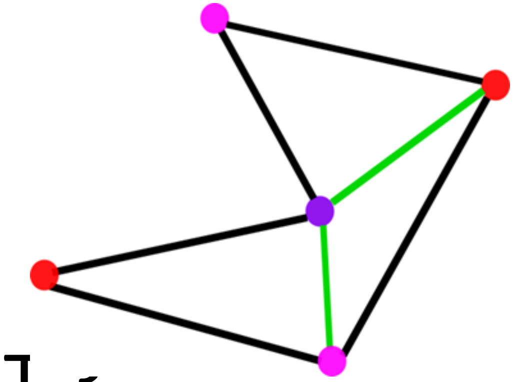
Three-Coloring for Triangulation

- Exercise: Draw P with 10 vertices
- Triangulate P
- 3-color T
- Eg: P with 5 vertices



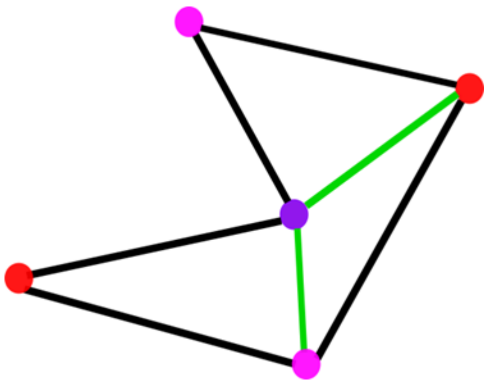
Fisk's proof for sufficiency of floor (*n*/3) guards for any P

- Steps in Fisk's Proof :
- 1. Triangulate P
- 2. Three-color the triangulation
- What do we do after 3-coloring of T ?
- What do we observe after 3-coloring?
- Placing guards on any of the colors covers the whole polygon



Observation after 3-coloring

- How do we argue for the claim *"Placing guards on any of the colors covers the whole polygon"*
- Each Δ must have one of the colors on its three vertices
- Every Δ has a blue color at one of its vertices
- Place guards on every blue colored vertices
- Every Δ has a guard at one its corners now
- Every Δ is covered by one of its guards
- Thus, the entire P is covered/ safe guarded

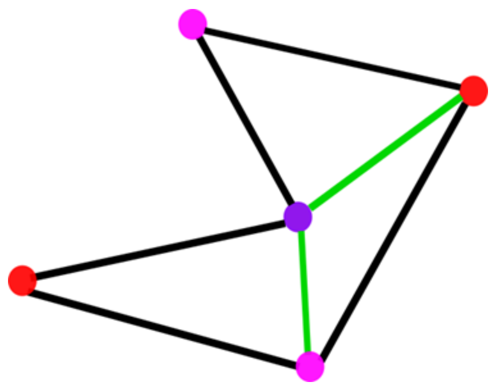


Fisk's proof for sufficiency of floor (n/3) guards for any P

- Steps in Fisk's Proof :
- 1. Triangulate P
- 2. Three-color the triangulation
- 3. Observe that placing guards on any of the colors covers the whole polygon

Fisk's proof for sufficiency of floor (n/3)

- Consider a 3-colored T of P



- If we place guards on every vertex with one color ?
- # of guards we obtain covers P
- It is not necessarily be the minimum or $G(n)$ by its definition

Last step of Fisk's proof

- Suppose there are 22 vertices for P
- What might be # of Red, Blue and Green colored vertices?
- # of Red colored vertices might be 8
- # of Blue colored vertices might be 7
- # of Green colored vertices might be 7
- Do we have a preference over the color we select for placing the guards?
- We place guards either on Blue or Green, as it is less in # compared to Red colored vertices
- # Blue or Green is Floor(n/3)

How to give a mathematical justification/result to support the final step?

- Suppose there are *n* vertices in P
- We can color maximum of *n* vertices
- Three colors have to be split to *n*/3
- **At least one color should not be more than *n*/3**
- Suppose all # Red , # Blue & #Green are greater than *n/3*, what will happen?
- # Red + # Blue + #Green = more than n

At least one color should not be more than *n*/3

- There are *n* vertices for the T of P
- There are 3 colors to color the vertices
- Our requirement: At least one color must not be used more than *n*/3 times
- Since n is an integer, at least one color must not be used more than floor(*n*/3) times
- **Pigeon-hole principle**

Pigeon-hole principle

- If *n* objects to be placed in to *k* pigeon holes, then at least one hole must contain no more than *n*/*k* objects
- If each one of the *k* holes contain more than *n*/*k* objects, the total number of objects would exceed *n*
- In our case *n* is the # of vertices of T of P and *k* is the # of colors used (*k* =3)
- **At least one color must not be used more than floor(*n*/3) times**

Fisk's proof for sufficiency of floor (n/3) guards for any P

- Steps in Fisk's Proof :
- 1. Triangulate P
- 2. Three color the triangulation
- 3. Observe that placing guards on any of the colors covers the whole polygon
- 4. Apply Pigeon-hole principle
- Place guards at vertices with the least frequently used color in the 3-coloring
- This covers the polygon with no more than

  $G(n)$= floor (n/3) guards

Exercise

- Try Fisk’s proof on a polygon of 12 vertices
- With different triangulations of same P, do we get the same # of guards ?

Reference

- J. O Rourke, *Computational Geometry in C*, 2/e, Cambridge University Press, 1998

Thank you