CS2006D Discrete Structures

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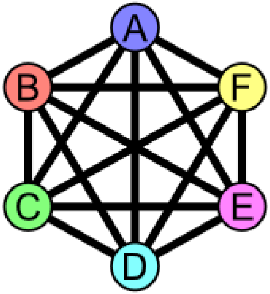
In a group of n peoples, there exist three mutual friends or 3 mutual enemies.

1. Is the above statement true when n=6?
2. What about n=5?



Königsberg Bridge Problem

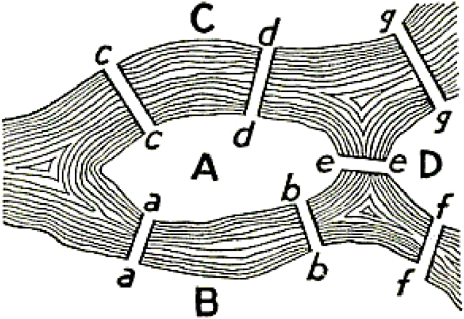
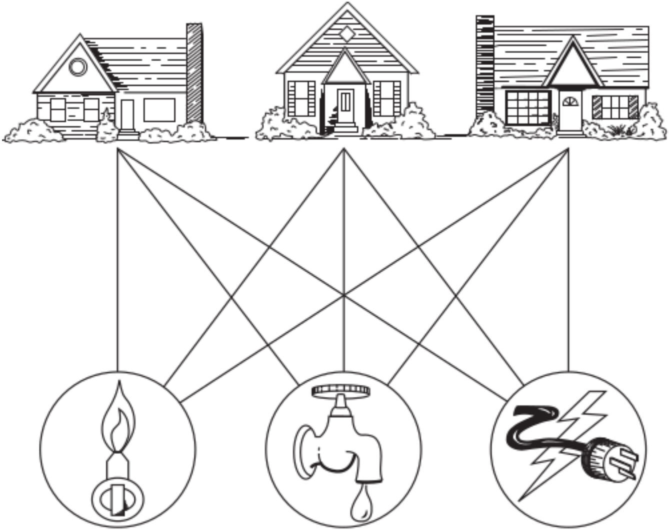


Figure 98. *Geographic Map: The Königsberg Bridges.*

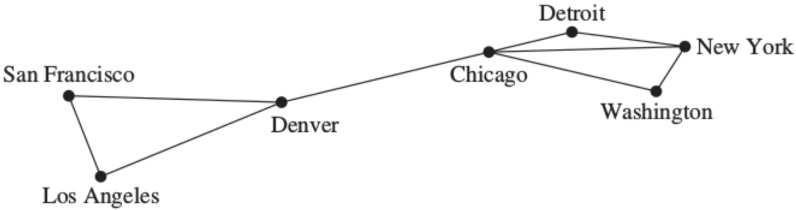
Three Houses and Three Utilities



Coloring the India Map

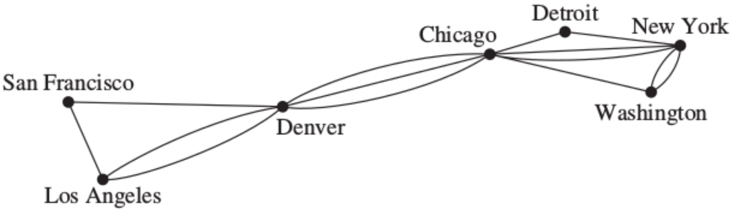
A graph G (V , E) consists of V , a nonempty set of vertices (or nodes) and E, a set of edges

- ⋆ A graph with an infinite vertex set or an infinite number of edges is called an infinite graph
- ⋆ A graph with a finite vertex set and a finite edge set is called a finite graph
- ⋆ A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph



Graphs

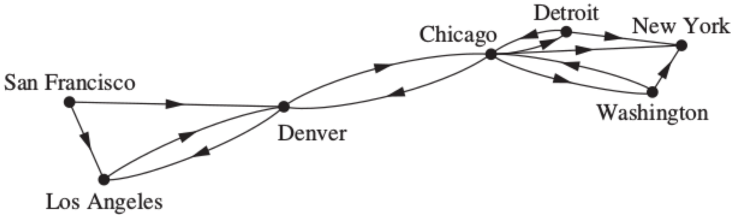
- In a simple graph, each edge is associated to an unordered pair of vertices, and no other edge is associated to this same edge
- Graphs that may have multiple edges connecting the same vertices are called multigraphs
- When there are m different edges associated to the same unordered pair of vertices $\{u, v\}$, we also say that $\{u, v\}$ is an edge of multiplicity m



- Edges that connect a vertex to itself are called loops
- Undirected graphs have no direction on their edges

A directed graph (or digraph) (V , E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E

- ⋆ Each directed edge is associated with an ordered pair of vertices
- ⋆ Directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) are called directed multigraphs



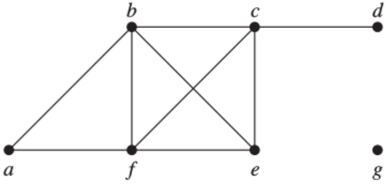
- ⋆ When there are m directed edges, each associated to an ordered pair of vertices (u, v), we say that (u, v) is an edge of **multiplicity** m

- When a directed graph has no loops and has no multiple directed edges, it is called a simple directed graph
- A graph with both directed and undirected edges is called a mixed graph

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Graph Terminologies

- Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G
- The set of all neighbors of a vertex v of G(V , E), denoted by N(v), is called the neighborhood of v
- The degree of a vertex in an undirected graph is the number of edges incident with it, and is denoted by deg(v)
- A vertex of degree zero is called isolated
- A vertex is pendant if and only if it has degree one



G

The Hanshaking Theorem

Let G (V , E) be an undirected graph with m edges. Then
$2m = \sum_{v \in V} deg(v)$

1 How many edges are there in a graph with 10 vertices each of degree six?
2 What is the sum of the degree of vertices in G?

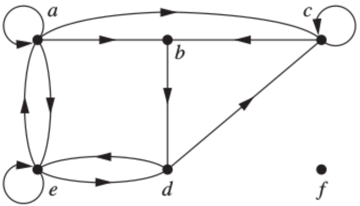
An undirected graph has an **even** number of vertices of **odd degree**

Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph G (V , E) with m edges. Then

$$2m = \sum_{v \in V} deg(v) = \sum_{v \in V_1} deg(v) + \sum_{v \in V_2} deg(v)$$

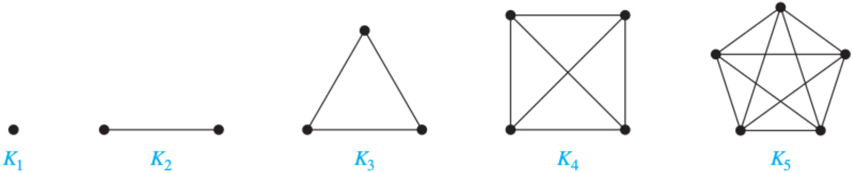
Directed Graph Terminologies

- Consider a directed edge from u to v. The vertex u is called the initial vertex of (u, v), and v is called the terminal or end vertex of (u, v)
- In a graph with directed edges the in-degree of a vertex v, denoted by $\deg^{-}(v)$, is the number of edges with v as their **terminal vertex**
- The out-degree of v, denoted by $\deg^{+}(v)$, is the number of edges with v as their **initial vertex**
- In a directed graph G(V,E), $\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$

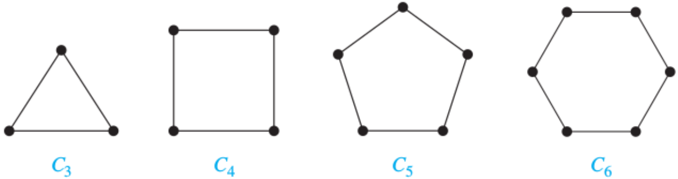


G

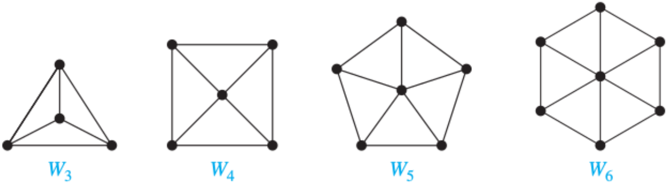
- A complete graph on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices



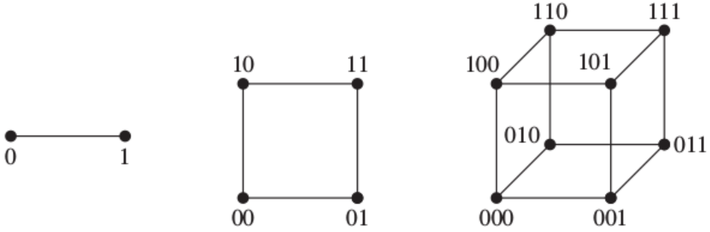
- A cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$



- ⋆ We obtain a wheel W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n , by new edges



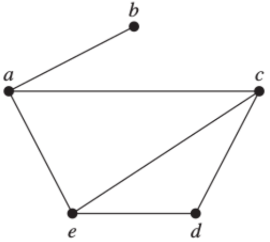
- An n-dimensional hypercube, or n-cube, denoted by Q_n, is a graph that has vertices representing the 2^n bit strings of length n.



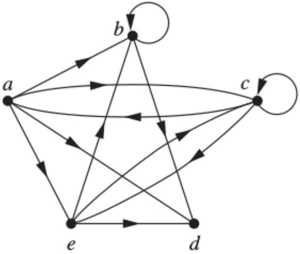
Q_1 Q_2 Q_3

- Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position

Adjacency Lists



Vertex	*Adjacent Vertices*
a	*b*, *c*, *e*
b	*a*
c	*a*, *d*, *e*
d	*c*, *e*
e	*a*, *c*, *d*

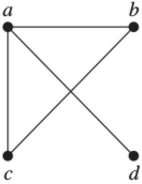


Initial Vertex	*Terminal Vertices*
a	*b*, *c*, *d*, *e*
b	*b*, *d*
c	*a*, *c*, *e*
d	
e	*b*, *c*, *d*

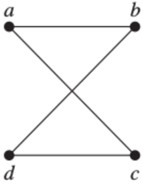
Adjacency Matrices

- Suppose that G (V , E) is a simple graph where $|V| = n$
- Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$
- Adjacency matrix is A = [a$_{ij}$] where

$$a_{ij} = \begin{cases} 1 \text{ if } \{v_i, v_j\} \text{ is an edge of G} \\ 0 \text{ otherwise} \end{cases}$$



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

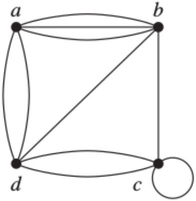


$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

- The adjacency matrix for an undirected graph is symmetric

Adjacency Matrices

- ⋆ Adjacency matrices can also be used to represent undirected multigraphs
- ⋆ Such matrices are not zero–one matrices when there are multiple edges connecting two vertices



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

- ⋆ Zero-one matrices could be used to represent directed graphs
- ⋆ For the directed graph, Adjacency matrix is A = [a_{ij}] where

$$a_{ij} = \begin{cases} 1 \text{ if } (v_i, v_j) \text{ is an edge of G} \\ 0 \text{ otherwise} \end{cases}$$

- ⋆ The adjacency matrix for a directed graph does not have to be symmetric

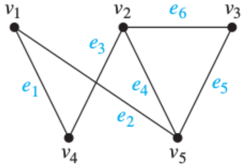
- **Trade-Offs**: When a simple graph contains relatively few edges, that is, when it is sparse, it is usually preferable to use adjacency lists rather than an adjacency matrix to represent the graph
  Practice Questions

1. Represent each of these graphs with an adjacency matrix
   K_5 C_6 W_5 Q_3
2. Draw graphs with the following adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

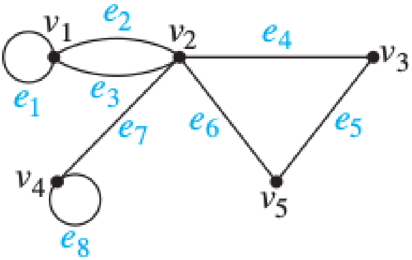
- Let G (V , E) be an undirected graph
- Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G
  Then the incidence matrix with respect to this ordering of V and E is the n × m matrix M = $[m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ when edge } e_j \text{ is incident with } v_i \\ 0 \text{ otherwise} \end{cases}$$



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Incidence Matrices



$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

Practice Questions

1. True or False: If every vertex of a simple graph G has degree 2, then G is a cycle.
2. True or False: Every n-vertex graph with at least n edges contains a cycle.
3. How can you obtain the degree of vertices from adjacency / incidence matrices?
4. Does there exists a graph G for which the incidence matrix of G is symmetric?
5. An arbitrary $n \times n$ matrices (binary matrix) may not represent a valid simple undirected graph. Given n, how many different $n \times n$ adjacency matrices can be constructed which corresponds to a valid simple undirected graph.