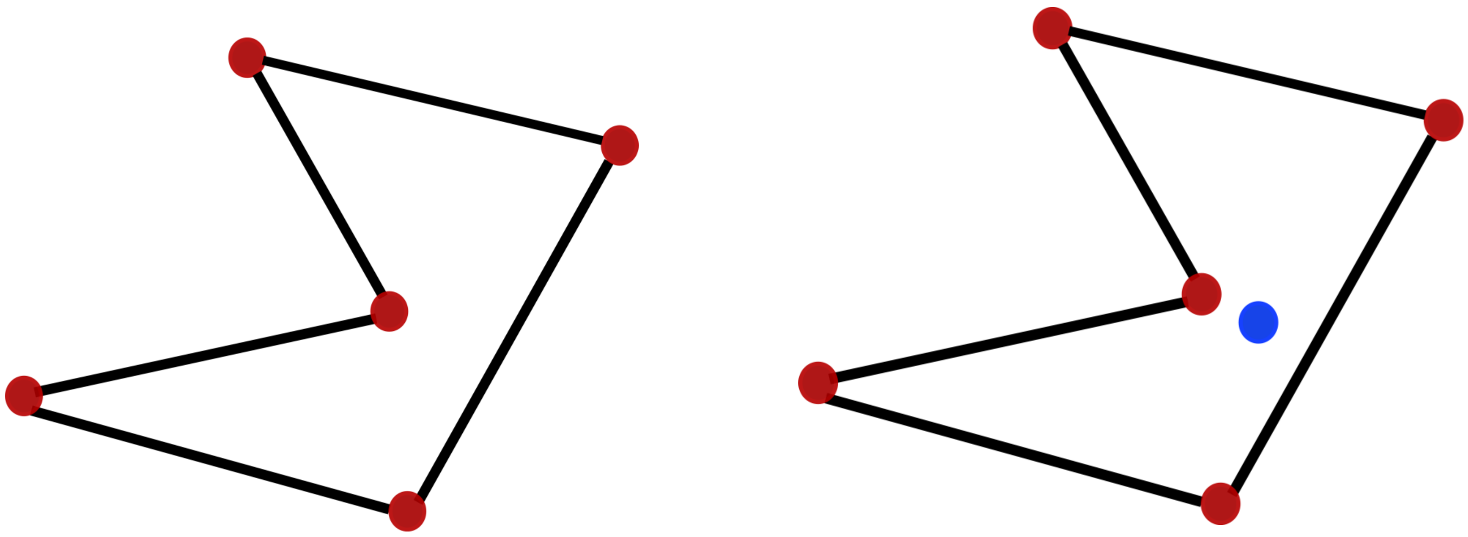
Art Gallery Problem [Victor Klee, 1973]

- Input : Art Gallery
- Output : Minimum number of guards that can safe-guard or cover the interior walls of the gallery

- The Guggenheim Museum in Bilbao: hard to supervise (Image courtesy: BBC.com)

Placing Guards



Visibility : Covering a polygon

- A set of guards cover a polygon if every point in the polygon is visible to some guard
- What we have to do is: Given a simple Polygon *P* with *n* vertices, compute the minimum number of guards which cover *P*

Example

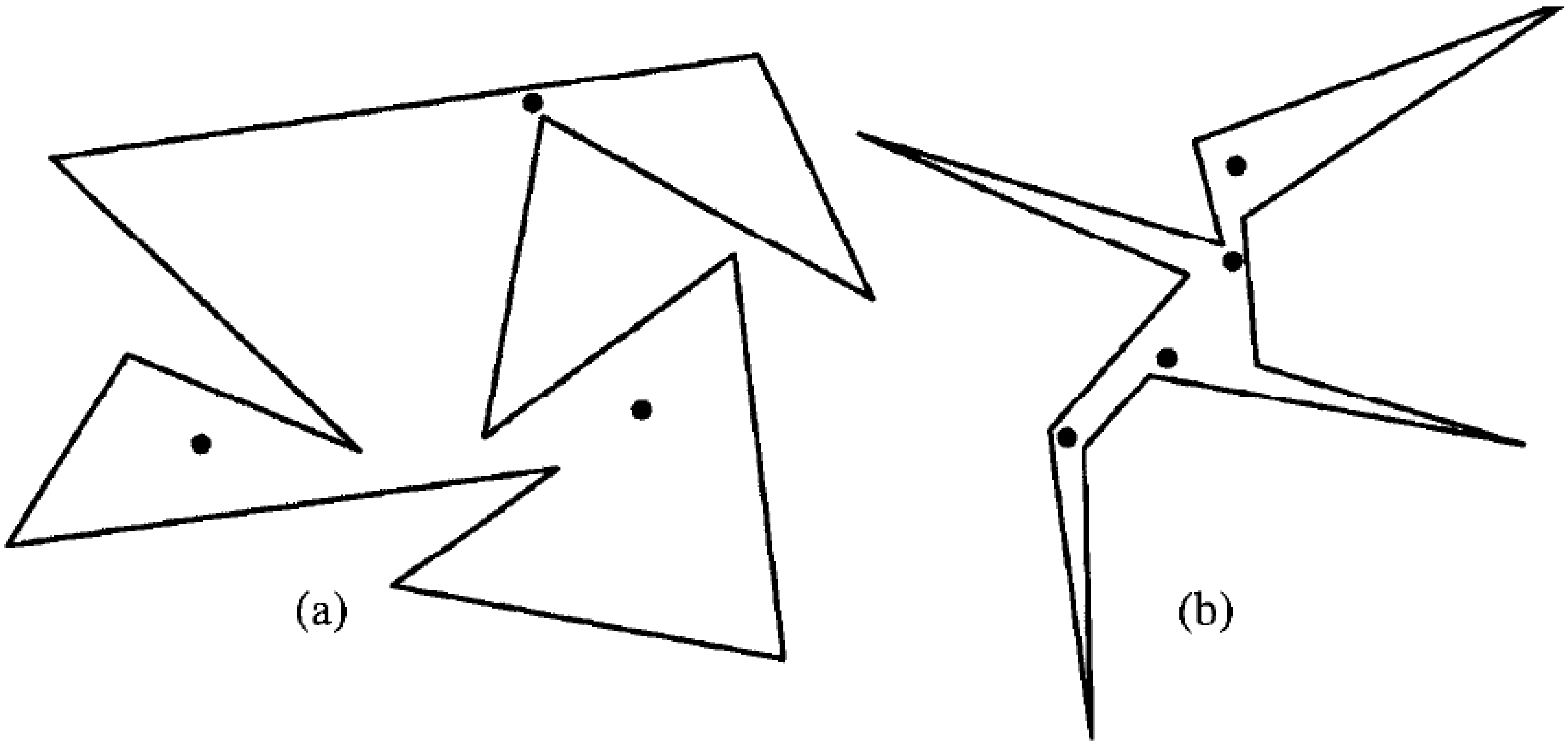


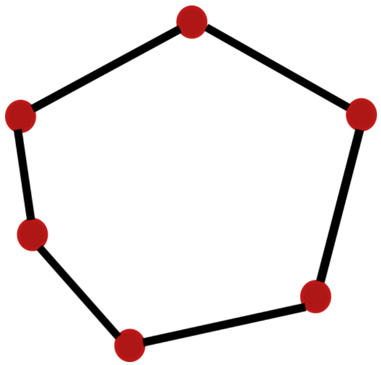
FIGURE 1.3 Two polygons of $n = 12$ vertices: (a) requires 3 guards; (b) requires 4 guards.

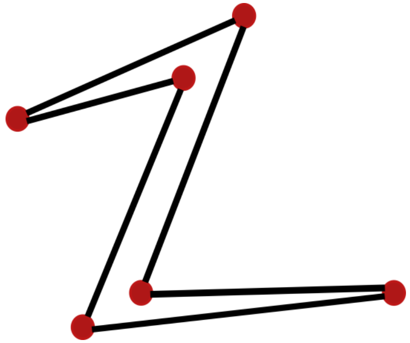
Our objective

- Find out the largest number of guards that any polygon of *n* vertices needs, where *n* is an integer known to us
- This number of guards is said to be sufficient and necessary to cover P
  - Necessary because at least that many guards are needed for some polygons
  - Sufficient because that many always suffice for any polygon

Our objective in simple words

- There can be different polygons with *n* vertices





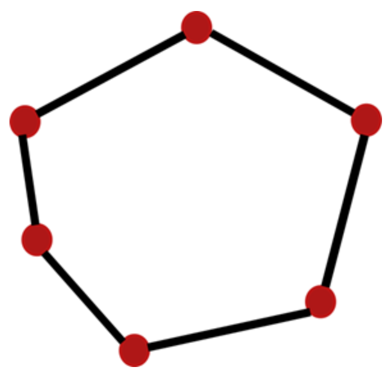
- Find the minimum # of guards for each *P*
- $g(P) = \min_S \{S \text{ covers } P\}$
- Get the maximum # among those
- $G(n) = \max_{P_n} g(P_n)$
- **Max over Min Formulation**

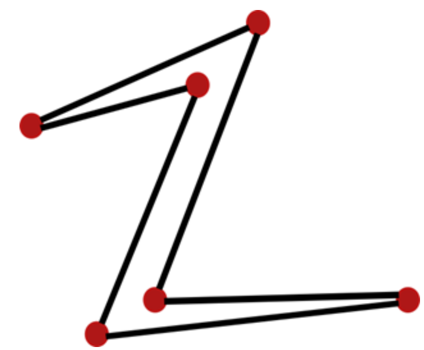
Empirical exploration to compute G(n)

- For a triangle, $G(3) = 1$
- For a quadrilateral, $G(4) = 1$
- For a pentagon, $G(5) = 1$
- For a hexagon, G(6) = 2
- For a heptagon, G(7) = 2
- For a octagon, G(8) = 2
- For a nonagon (enneagon), G(9)=3
- For a decagon, G(10) = 3
- For a hendecagon (undecagon or endecagon), G(11) = 3
- For a dodecagon, G(12) = 4

We already know that

- For a strictly convex polygon, $G(n)$ =1
- A strictly convex polygon is one with all the interior angles less than 180°





- A non-convex polygon is one with at least one reflex vertex (a reflex vertex is the one which makes an interior angle greater than 180°)

Bounds on $G(n)$

- Certainly at least one guard is always necessary
  - $1 <= G(n)$, hence lower bound of $G(n)$ is 1
- n guards suffice for any P with n vertices
  - stationing point guards near/on every vertex of P
  - $G(n) <= n$, hence upper bound of $G(n)$ is *n or more*

Bounds of G(n)

- Lower bound of $G(n)$ is 1
- Upper bound of $G(n)$ is n ?
- Is there a tight bound of G(n) ?
- From the empirical exploration:
- G(n) = Floor (n/3)

Bounds on $G(n)$ in general

- n guards suffice for any P with n vertices
  - stationing point guards near/on every vertex of P
  - $G(n)$ <= n, hence upper bound of $G(n)$ is *n or more*
- Is the above intuition in general true?

Seidel Polyhedron

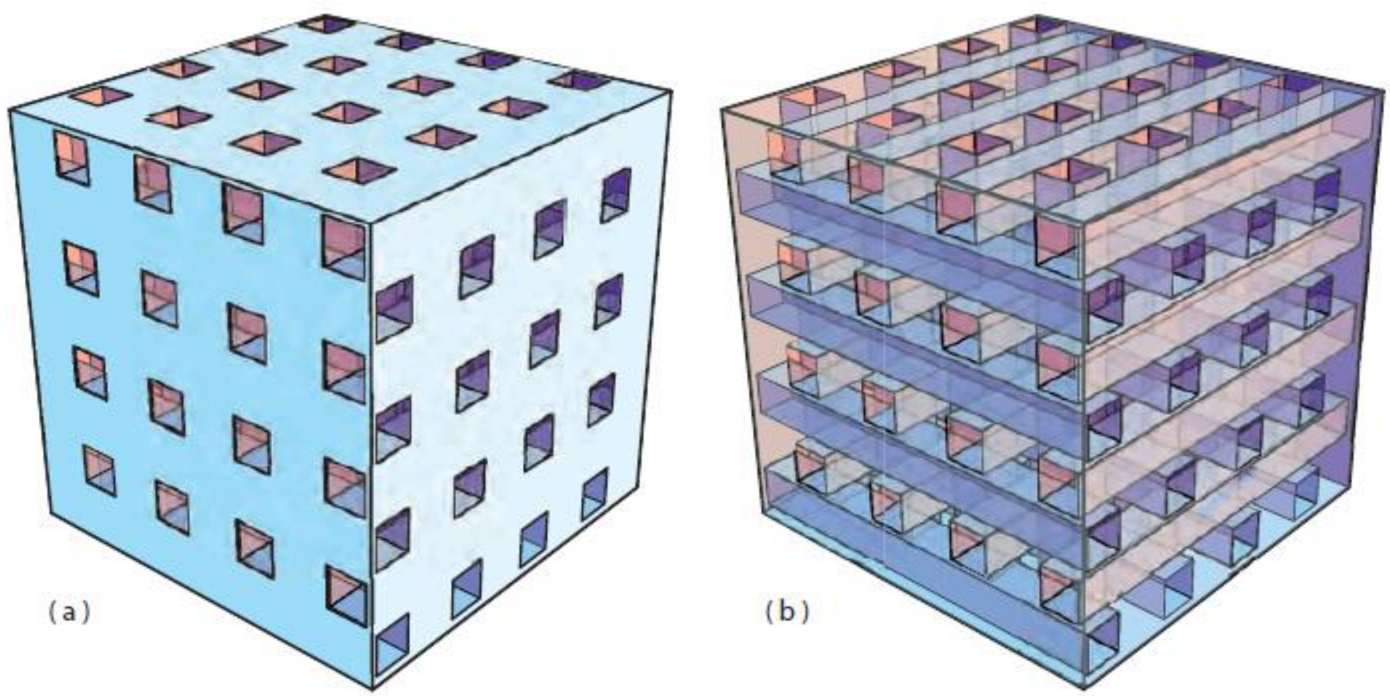


Figure 1.17. (a) The Seidel polyhedron with (b) three faces removed to reveal the interior.

Seidel Polyhedron

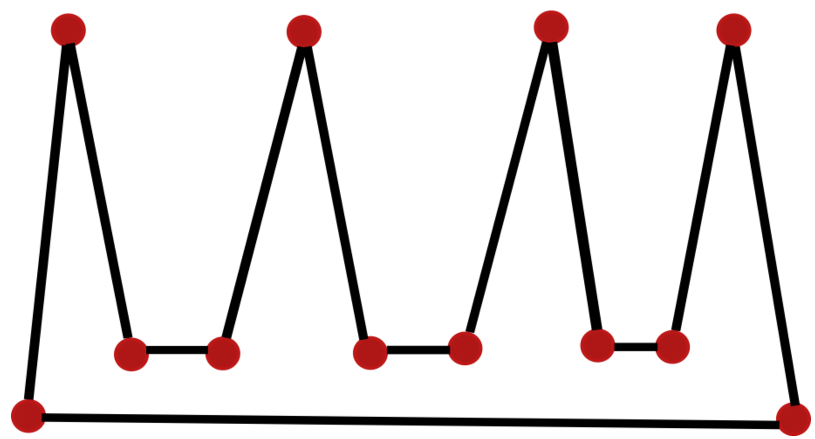
can be constructed as follows. Start with a large cube and let $\varepsilon \ll 1$ be a very small positive number. On the front side of the cube, create an $n \times n$ array of 1×1 squares, with a separation of $1 + \varepsilon$ between their rows and columns. Create a tunnel into the cube at each square that does not quite go all the way through to the back face of the cube, but instead stops ε short of that back face. The result is a deep dent at each square of the front face. Repeat this procedure for the top face and the right face, staggering the squares so their respective dents do not intersect. Now imagine standing deep in the interior, surrounded by dent faces above and below, left and right, fore and aft. From a sufficiently central point, no vertex can be seen!

Recall : Bounds of G(n)

- Lower bound of $G(n)$ is 1
- Upper bound of $G(n)$ is n ?
- Is there a tight bound of G(n) ?
- From the empirical exploration:
- G(n) = Floor (n/3)
- We will generalize the result

Generalize : $G(n)$ = Floor (n/3)

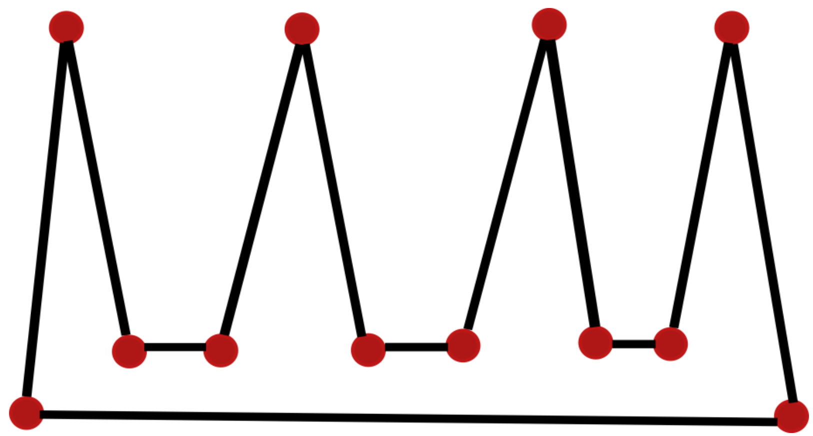
- Consider the comb structure:



- What is the minimum number of guards?
- One guard for each prong?

Empirically $G(n)$ = Floor (n/3)

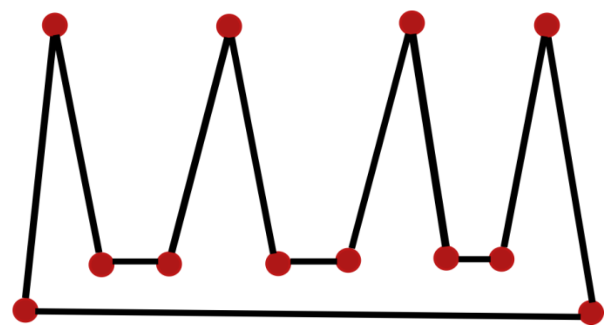
- Chavtal's comb



- The comb shape consists of *k* prongs
  - Each prong composed of 2 edges
  - Adjacent prongs are separated by an edge

Number of edges in a comb of *k* prongs

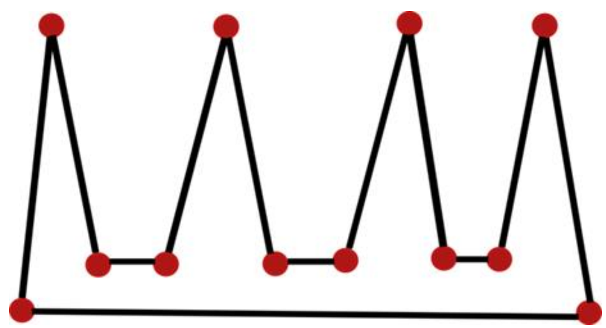
- Associate each prong with the separating edge to its right
- Associate the bottom edge to the rightmost prong



- No. of edges in a *k* prong comb = 3*k*
- No. of vertices in a *k* prong comb = *3k, (n=3k)*

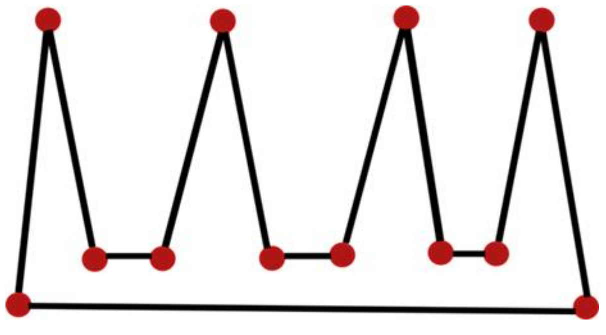
Generalize : $G(n)$ = Floor ($n/3$)

- $n = 3k$, where n is the no. of vertices and k is the no. of guards
- Each prong requires its own guard
- $k = n / 3$



$G(n)$ = Floor ($n/3$)

- Conjecture: Floor ($n/3$) $= G(n)$
- Proof for **$G(n)$ = Floor ($n/3$)**
- First proof was by Chavtal in 1975

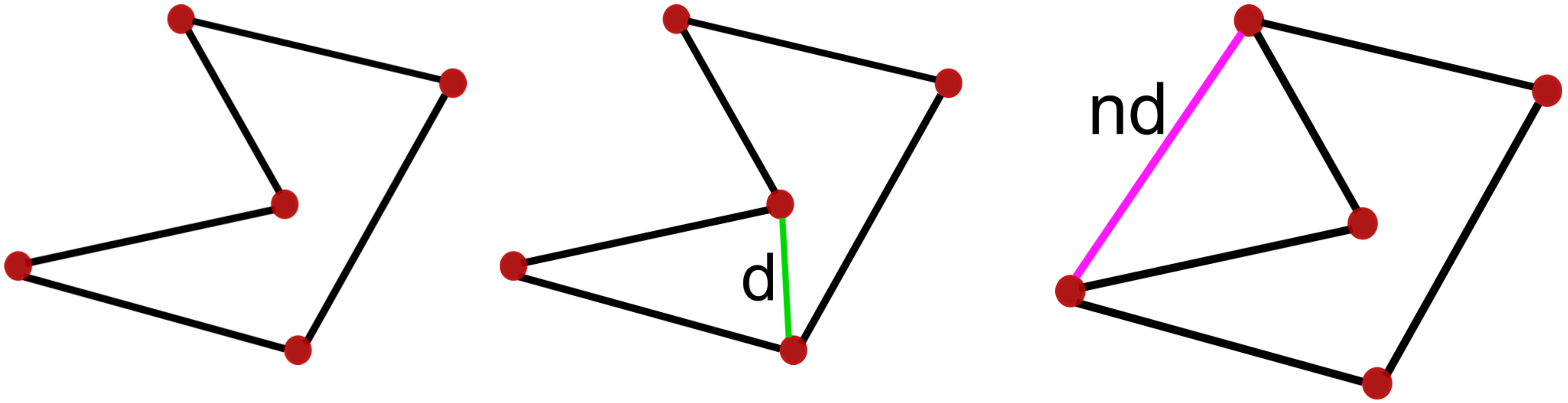


Outline on Chavtal’s proof

- Mathematical induction
  - By removing a part of polygon & reducing the no. of vertices
  - Applying induction hypotheses
  - Reattach the removed part of the polygon
- Complex proof with many cases

A simpler proof by Fisk in 1978

- Frisk's proof (1978): Partitioning a polygon to triangles with a diagonal
- What is a diagonal?
- Diagonal of a *P* is a line segment between its vertices which are clearly visible to each other

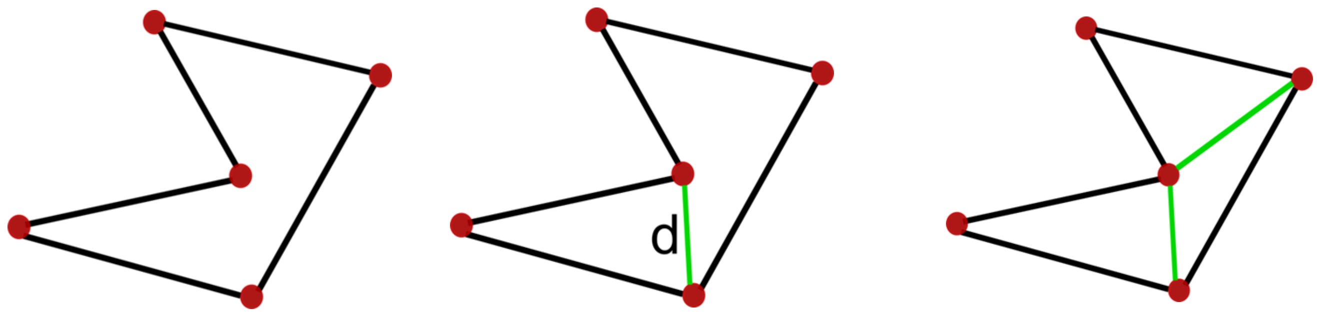


Diagonal of P

- Diagonal of a *P* is a line segment between its vertices a, b which are clearly visible to one another
- Intersection of the closed segment *ab* with ∂P ?
- $ab \cap \partial P = \{a,b\}$
- Open segment ab does not intersect with ∂P
- A diagonal can not make grazing contact with ∂P

Partitioning P into triangles

- Add / Insert non-crossing diagonals
- Non-crossing diagonals are those whose intersection is a subset of their endpoints
  - They share no interior points
- If we add as many non-crossing diagonals in to P as possible, the interior is partitioned in to Δs



- Such partition is called a **Triangulation of P**

Reference

- J. O Rourke, *Computational Geometry in C*, 2/e, Cambridge University Press, 1998

Thank you