Context-Free Grammars

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Context-Free Grammars

- The topic of context-free languages is perhaps the most important aspect of formal language theory as it applies to programming languages.
- Actual programming languages have many features that can be described elegantly by means of context-free languages.
- What formal language theory tells us about context-free languages has important applications in the design of programming languages as well as in the construction of efficient compilers.

Context-Free Grammars

- The productions in regular grammar are restricted in two ways:
  - The left side must be a single non-terminal.
  - The right side has a special form
    - ★ left linear or right linear
    - ★ at most one non-terminal appears on the right side of any production
- To create grammars that are more powerful, we must relax some of these restrictions.
- By retaining the restriction on the left side, by permitting anything on the right side, we get context-free grammars.

Context-Free Grammars

- **Definition : Context-Free Grammar:**

  A grammar $G = (N, T, S, P)$ is said to be context-free if all productions in P have the form

  $$A \to x$$

  where $A \in N$ and $x \in (N \cup T)^*$

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- Every regular grammar is context-free, so a regular language is also a context-free one.

Context-Free Grammars

- **Example 1:** The grammar G = ($\{S\}$, {a, b}, S, P), with P given by

$$1.\ S \to aSb,$$
$$2.\ S \to ab$$

is context-free.

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- **Example 1:** The grammar G = ({S}, {a, b}, S, P), with P given by

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  - Some of the derivations of this grammar is:
  - $S \overset{2}{\Rightarrow} ab$ $\quad ab \in L(G)$
  - $S \overset{1}{\Rightarrow} aSb \overset{2}{\Rightarrow} aabb$ or $S \overset{*}{\Rightarrow} a^2b^2$ $\quad a^2b^2 \in L(G)$
  - $S \overset{1}{\Rightarrow} aSb \overset{1}{\Rightarrow} aaSbb \overset{2}{\Rightarrow} aaabbb$ or $S \overset{*}{\Rightarrow} a^3b^3$ $\quad a^3b^3 \in L(G)$
  - $S \overset{1}{\Rightarrow} aSb \overset{1}{\Rightarrow} aaSbb \overset{1}{\Rightarrow} aaaSbbb \overset{2}{\Rightarrow} aaaabbbb$ or $S \overset{*}{\Rightarrow} a^4b^4$ $\quad a^4b^4 \in L(G)$

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Thus, G can derive only strings of the form a^nb^n.

So, $L(G) = \{a^nb^n : n \geq 1\}$ and the language is context free.

Context-Free Grammars

- **Example 2:** The grammar G = ($\{S\}$, {a, b}, S, P), with P given by

$$1.\ S \to aSa,$$
$$2.\ S \to bSb,$$
$$3.\ S \to \epsilon$$

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  From these derivation it is clear that $L(G) = \{ww^R : w \in a, b^*\}$ and the language is context free.

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- From example 1 and example 2, we can say that the family of regular language is a **proper subset** of the family of the context-free language.

Context-Free Grammars

- **Example 3:** The grammar G with productions

$$S \to abB, \qquad B \to bbAa,$$
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is a context-free grammar. The corresponding language is:

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- **Example 4:** The grammar G with productions
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- Both the grammar generate context-free language, but what is the difference between these two grammar ?
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    - ★ A **linear grammar** is a grammar in which at most one non-terminal can occur on the right side of any production.

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  - First one is linear, but second one is not linear. Why?
    - ⋆ A **linear grammar** is a grammar in which at most one non-terminal can occur on the right side of any production.
- Regular and linear grammars are context-free, but a context-free grammar is not necessarily linear.

Context-Free Grammars

- **Example 5:** Consider the grammar G with production

$$S \to aSb|SS|\epsilon$$

  - This is another grammar that is context-free, but not linear. Some strings in $L(G)$ are $abaabb$, $aababb$, and $ababab$. So, $L(G) = \{w \in \{a, b\}^* : n_a(w) = n_b(w)$ and $n_a(v) \geq n_b(v)$,where v is any prefix of w$\}$.

- We can see the connection with programming languages clearly if we replace a and b with left and right parenthesis, respectively.

- The language $L(G)$ includes such strings as $(())$ and $()()()$ and is in fact the set of all properly nested parenthesis structures for the common programming languages.

- So, the language generated is $L(G)$ consists of well formed strings of parenthesis.
  - The language of well formed strings of parenthesis is called the **Dyck set**.

Leftmost and Rightmost Derivations

- In a grammar that is not linear, a derivation may involve sentential forms with more than one variable (or non-terminal).
- In such cases, we have a choice in the order in which variables are replaced.

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- In a grammar that is not linear, a derivation may involve sentential forms with more than one variable (or non-terminal).
- In such cases, we have a choice in the order in which variables are replaced.
- **Definition :** Leftmost and Rightmost derivations :
  - A derivation is said to be **leftmost** if in each step the leftmost variable in the sentential form is replaced.
  - A derivation is said to be **rightmost** if in each step rightmost variable in the sentential form is replaced.

Leftmost and Rightmost Derivations

- **Example 6:** Consider the grammar G with productions

$$1.\ S \rightarrow aAB,$$
$$2.\ A \rightarrow bBb,$$
$$3.\ B \rightarrow A|\epsilon$$

  - Then,

$$S \overset{1}{\Rightarrow} aAB \overset{2}{\Rightarrow} abBbB \overset{3}{\Rightarrow} abAbB \overset{2}{\Rightarrow} abbBbbB \overset{3}{\Rightarrow} abbbbB \overset{3}{\Rightarrow} abbbb$$

is a **leftmost derivation** of the string $abbbb$.

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    is a **leftmost derivation** of the string $abbbb$.

  - A **rightmost derivation** of the same string is

    $S \overset{1}{\Rightarrow} aAB \overset{3}{\Rightarrow} aA \overset{2}{\Rightarrow} abBb \overset{3}{\Rightarrow} abAb \overset{2}{\Rightarrow} abbBbb \overset{3}{\Rightarrow} abbbb$

Derivation Tree or Parse Tree

- A second way of showing derivation, independent of the order in which productions are used, is by a **derivation** or **parse tree**.
- A derivation tree is an ordered tree in which nodes are labeled with the left side of productions and in which the children of a node represent its corresponding right sides.

Derivation Tree or Parse Tree

- **Definition :** Derivation Tree

  Let $G = (N, T, P, S)$ be a context-free grammar. An ordered tree is a derivation tree for G if and only if it has the following properties.

  1. The root is labeled S.
  2. Every leaf has a label from $T \cup \{\epsilon\}$
  3. Every interior vertex (a vertex that is not a leaf) has a label from N.
  4. If a vertex has label $A \in N$, and its children are labeled (from left to right) $a_1, a_2, \cdots, a_n$, then P must contain a production of the form
  $$A \to a_1 a_2 \cdots a_n.$$
  5. A leaf labeled ϵ has no sibling, that is, a vertex with a child labeled ϵ can have no other children.

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- A tree that has properties 3, 4, and 5, but in which 1 does not necessarily hold and in which property 2 is replaced by

  2a. Every leaf has a label from $N \cup T \cup \{\epsilon\}$,

  is said to be a **partial derivation tree**.

Derivation Tree or Parse Tree

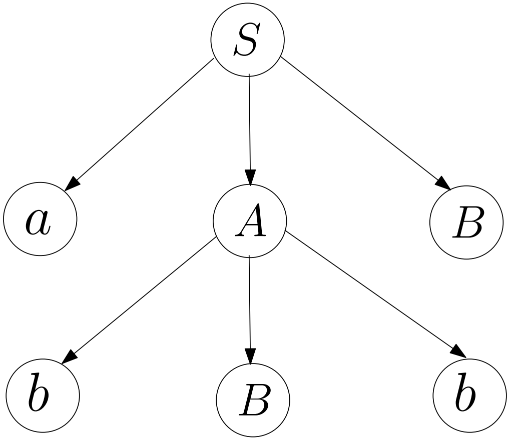
- **Example 7:** Consider the grammar G with productions

$$S \to aAB,$$

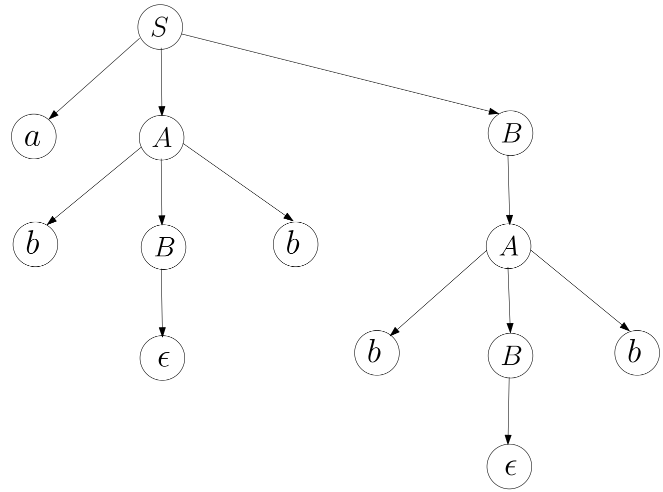
$$A \to bBb,$$

$$B \to A|\epsilon$$

The partial derivation tree for G



The derivation tree for G



- The string $abBbB$, which is yield of the first tree, is a sentential form of G. The yield of the second tree, $abbbb$, is a sentence of $L(G)$.
  - The string of symbols obtained by reading the leaves of the tree from left to right, omitting any ϵ's encountered, is said to be the **yield** of the tree.