Delaunay Triangulation (DT)

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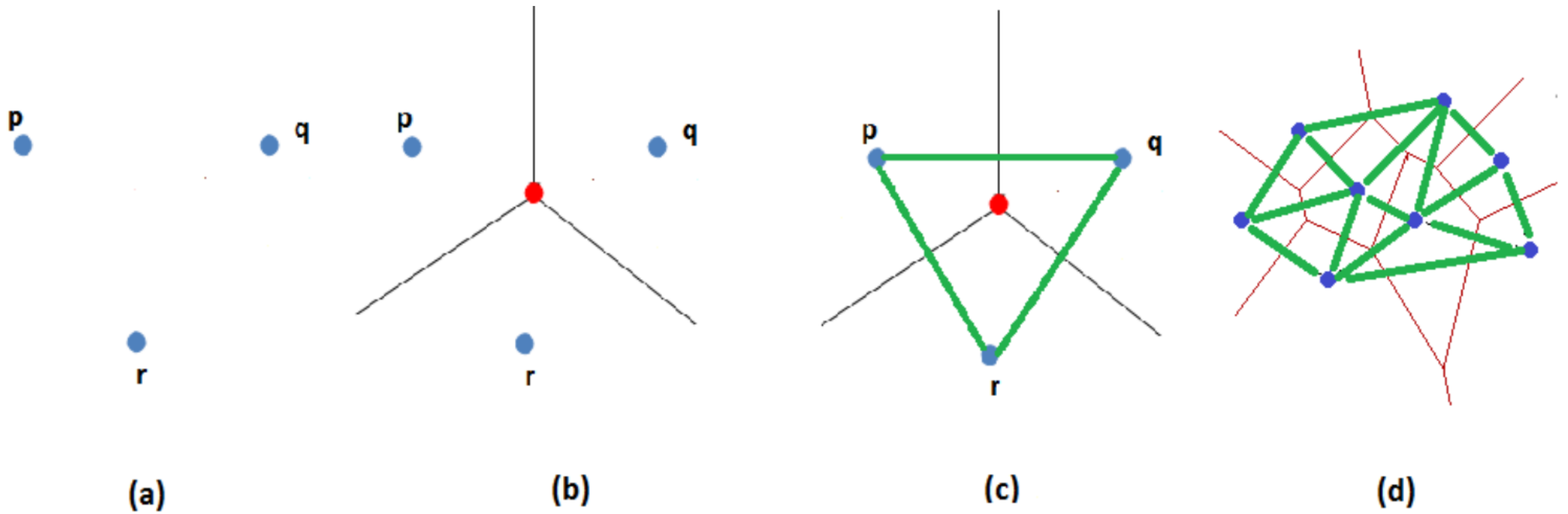
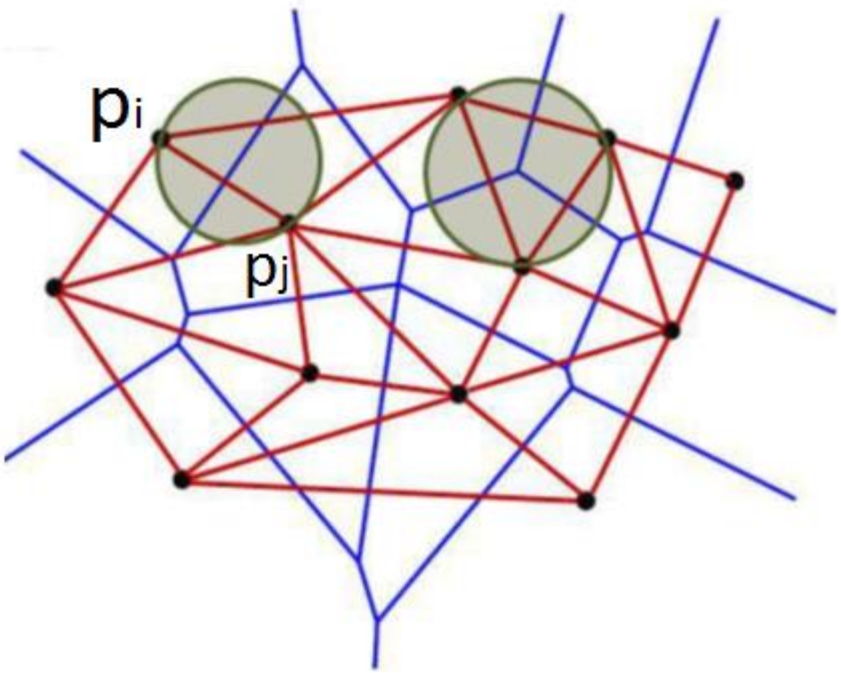
- Straight line dual of a Voronoi diagram
- Example : 3 points, its VD, its DT



Properties : Delaunay triangulation

- **Empty circle property**: Two points p_i and p_j are connected by an edge in the Delaunay triangulation, if and only if there is an empty circle passing through p_i and p_j .
- **Circumcircle property:** The circumcircle of any triangle in the Delaunay triangulation is empty (contains no other points of P).



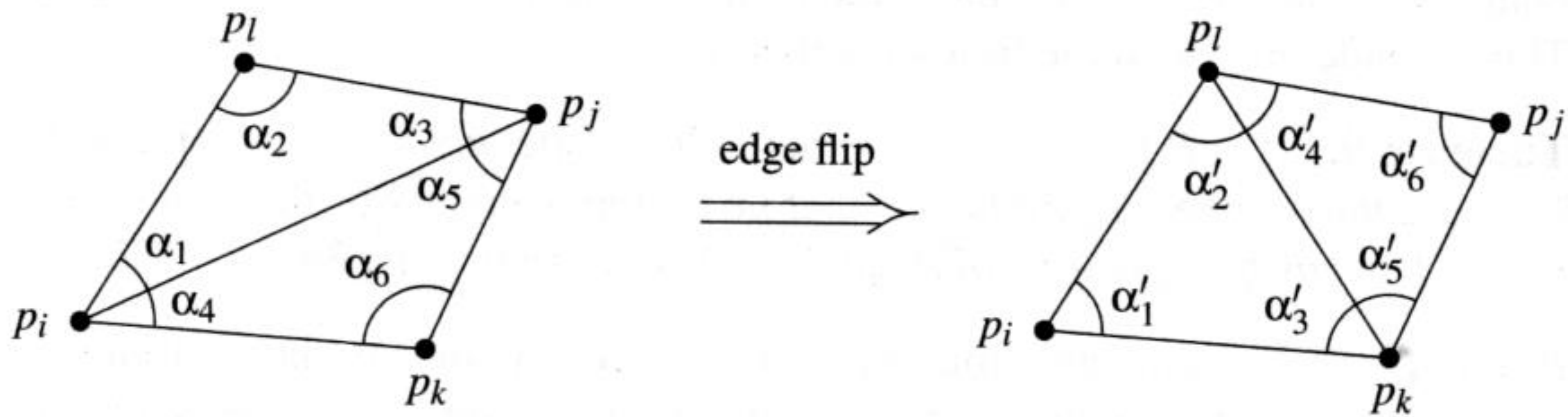
Angle Optimal Triangulations

- Create *angle vector* of the sorted angles of triangulation *T*, $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_m) = A(T)$, with α_1 being the smallest angle.
- Let $(\alpha'_1, \alpha'_2, \alpha'_3, \ldots \alpha'_m) = A(T')$
- $A(T)$ is larger than $A(T')$ iff there exists an i

  such that $\alpha_j = \alpha'_j$ for all $j < i$ and $\alpha_i > \alpha'_i$
- Best triangulation is a triangulation that is *angle optimal,* i.e. has the largest angle vector.

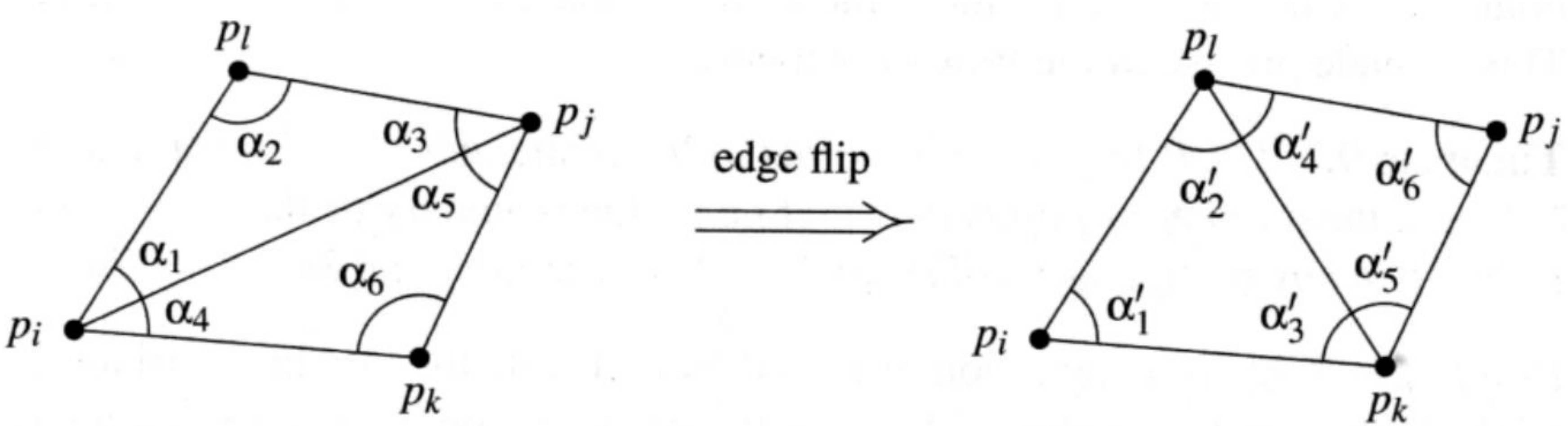
Angle Optimal Triangulations

- Consider two adjacent triangles of a Triangulation:
- If the two triangles form a convex quadrilateral, we could have an alternate triangulation by performing an *edge flip* on their shared edge



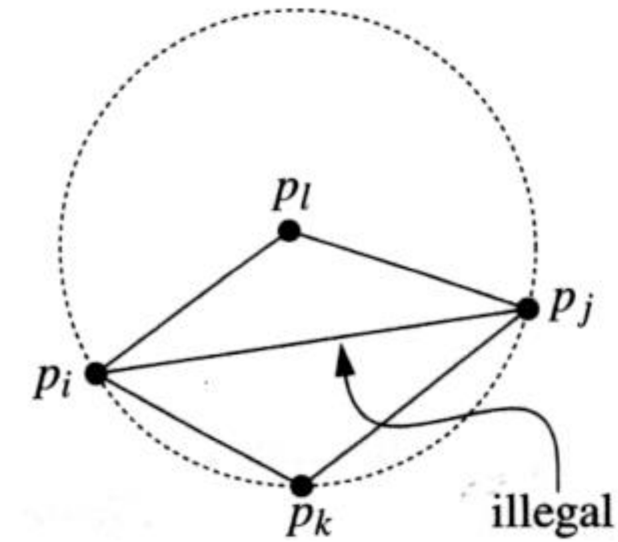
Illegal edges

- Edge e is illegal if: $$\min_{1\leqslant i\leqslant 6} \alpha_i < \min_{1\leqslant i\leqslant 6} \alpha'_i.$$
- Only difference between T containing e and T' with e flipped are the six angles of the quadrilateral.
- If triangulation T contains an illegal edge e, we can make A(T) larger by flipping e.
- In this case, T is an *illegal triangulation*.



Illegal edges

- If p_i, p_j, p_k, p_l form a convex quadrilateral and do not lie on a common circle, exactly one of $p_i p_j$ and $p_k p_l$ is an illegal edge.



- The edge $p_i p_j$ is illegal iff p_l lies inside the circle

Illegal Triangulation

- If triangulation *T* contains an illegal edge e, we can make A(*T*) larger by flipping e.
- In this case, *T* is an *illegal triangulation*.

Computing Legal Triangulations

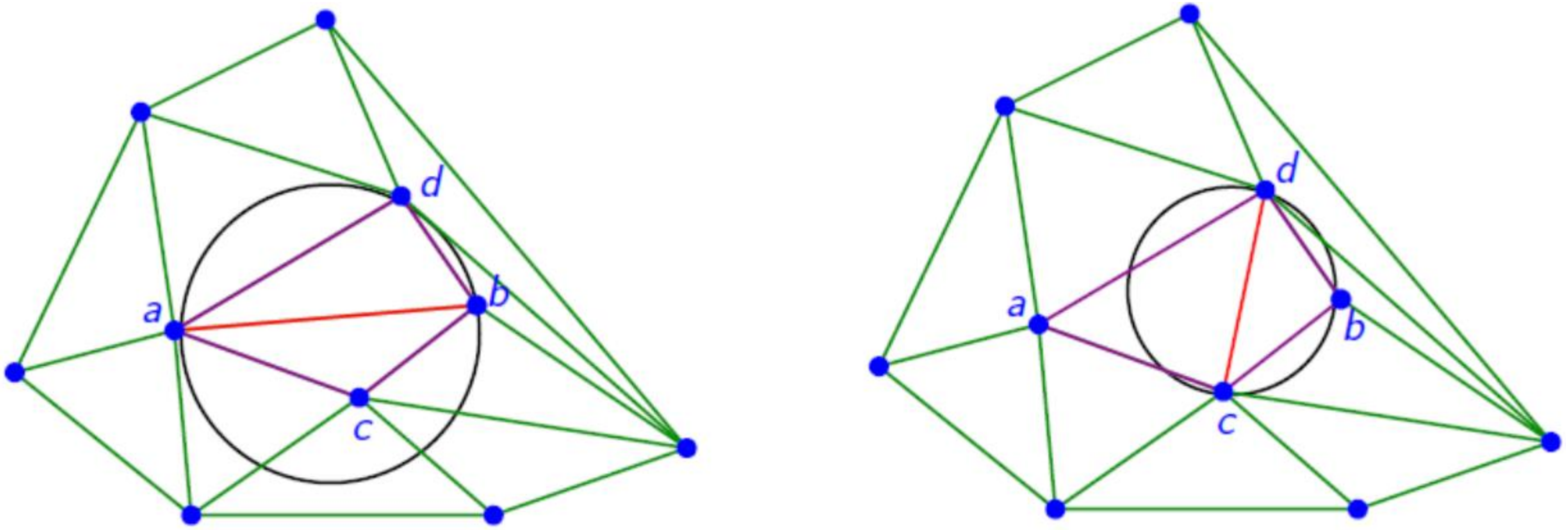
- Compute a triangulation of input points *P*.
- Flip illegal edges of this triangulation until all edges are legal.
- Algorithm terminates because there is a finite number of triangulations.

Back to Delaunay Triangulation

- A triangulation *T* of *P* is legal iff *T* is a DT(P).
- The angle optimal triangulation is a DT, where an angle optimal triangulation is the one with the largest angle vector.

Edge flip

- if ab is illegal, we can perform an edge flip: remove ab from the triangulation and insert cd



- now cd is locally Delaunay

How do we construct DT(*P*)?

- Compute VD(*P*) then construct the dual of that to obtain DT(*P*).
- DT(P) without using VD(P)

Algorithm to construct DT :DT(P) without using VD(P)

- Construct a triangulation T of P
- If all the edges of T are locally Delaunay, then it is a Delaunay triangulation
- Otherwise, pick an illegal edge and flip it
- Repeat this process, until all edges are locally Delaunay

Algorithm *SlowDelaunay*(P)
Input: a set P of n points in $\mathbb{R}^2$
Output: $\mathcal{DT}(P)$

compute a triangulation $\mathcal{T}$ of P
initialize a stack containing all the edges of $\mathcal{T}$
while stack is non–empty
    do pop ab from stack and unmark it
        if ab is illegal then
            do flip ab to cd
                for $xy \in \{ac, cb, bd, da\}$
                    do if xy is not marked
                        then mark xy and push it on stack
return $\mathcal{T}$

Analysis

- There are $O(n^2)$ edges
- One edge will be flipped only once.
- **The algorithm runs in $O(n^2)$ time :**
- Delaunay Triangulation is represented as a DCEL
- All the operations on a DCEL is done in constant time

We already know

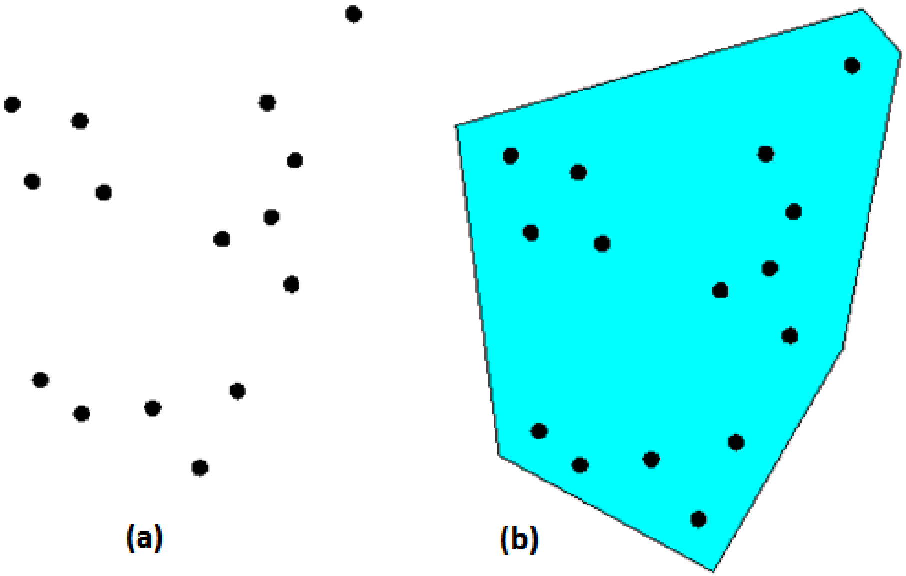
- **Applications of Voronoi diagram :**
- Fire Observation towers
- Towers on fire
- Facility location
- Path planning
- Crystallography

Application of Delaunay Triangulation

- One of the important area where Delaunay Triangulation is used:
- **RECONSTRUCTION / CONTOURING / BOUNDARY DETECTION**

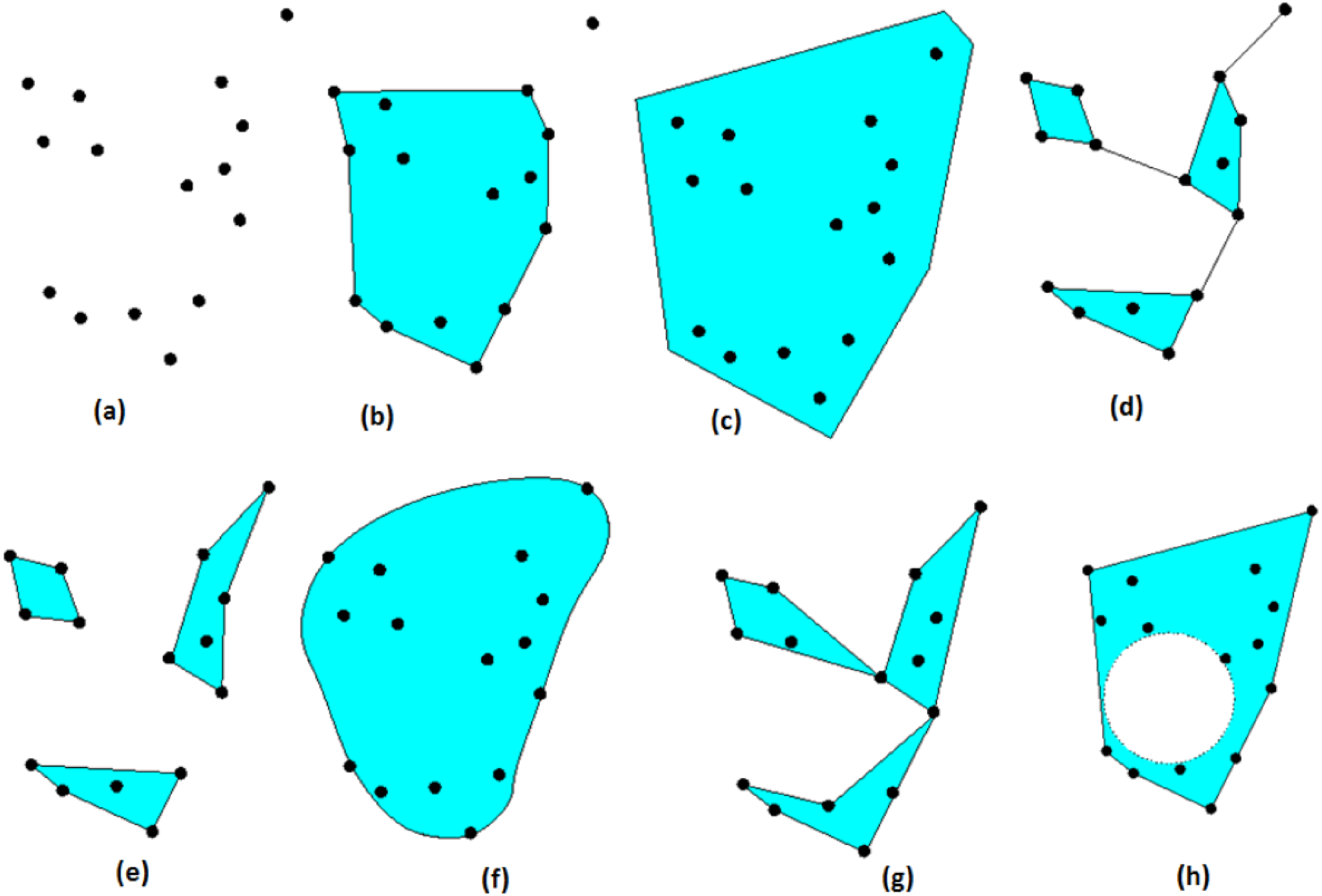
Reconstruction problem

- To construct a region that characterizes the shape of a given finite set of points (P) in the plane
- Alternate Definition: Computation of an approximation of the unknown shape induced by a given point set



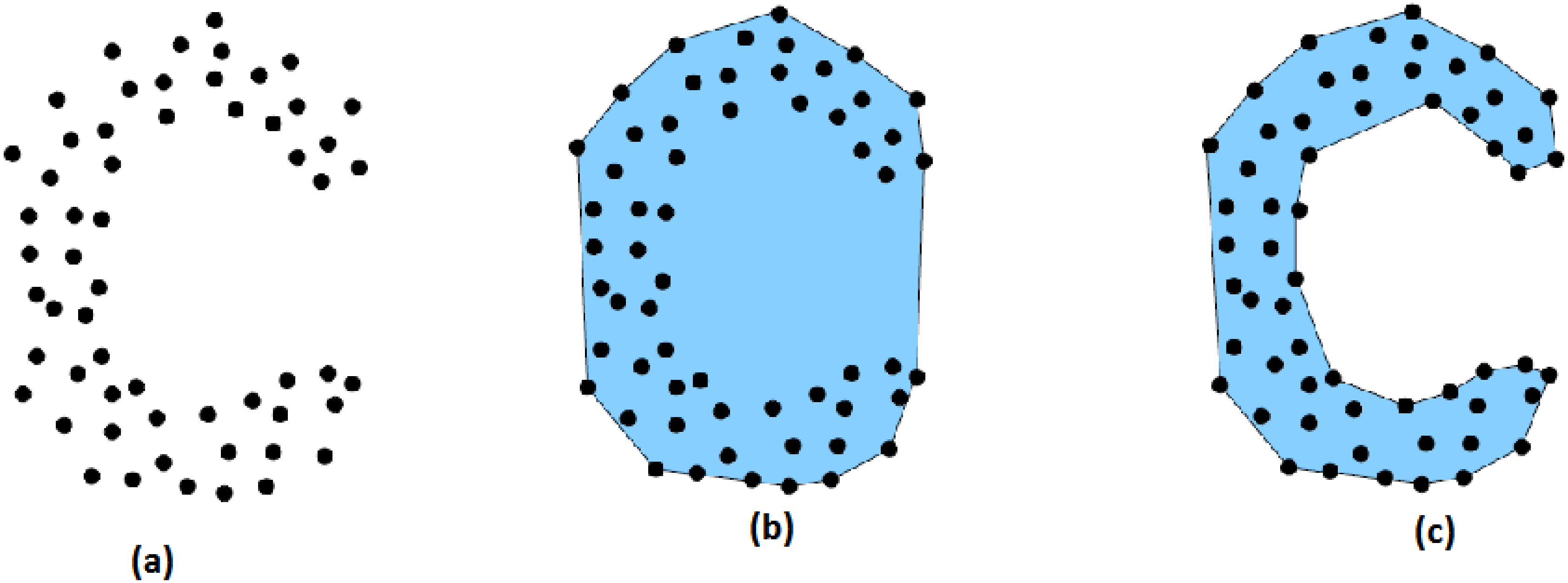
- Is this the only region occupied by P?

Different regions for the same point set



- Which region characterizes P?
- Is convex hull the solution?

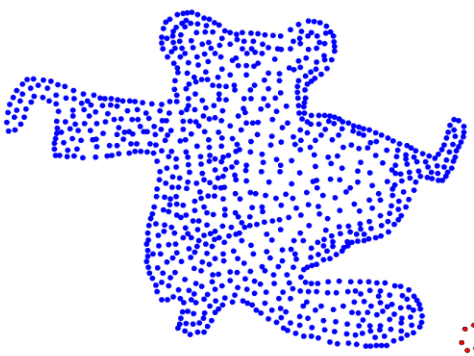
Convex hull

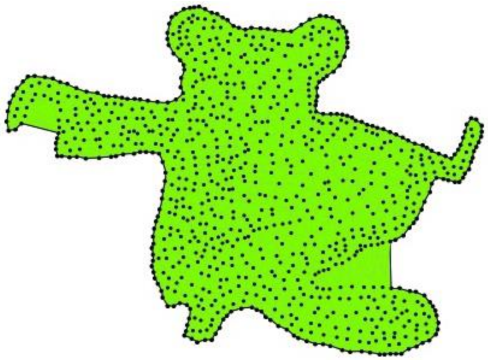


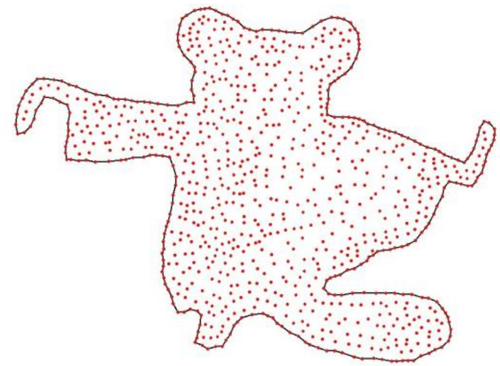
- Convex hull does not capture the shape characterized by P

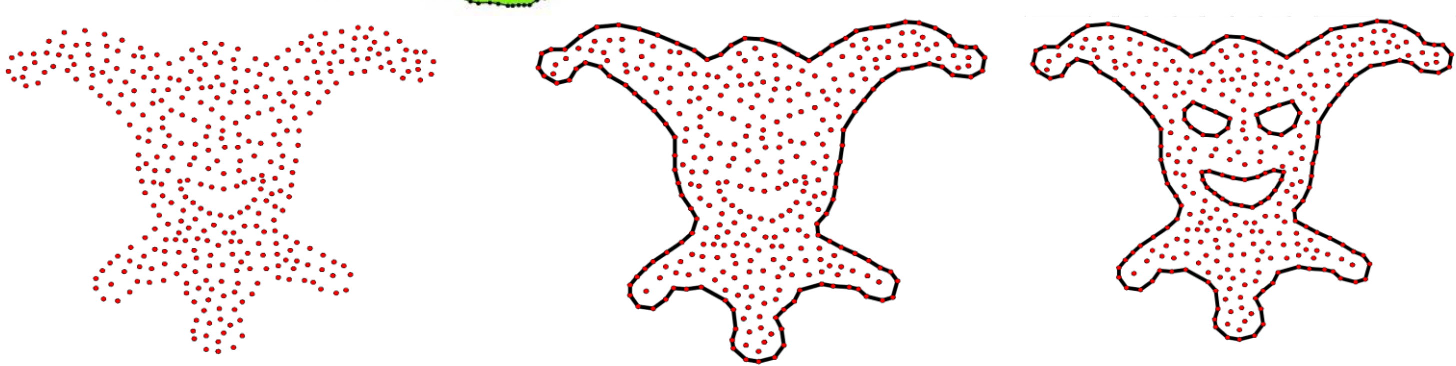
Challenges of the problem

- Reconstruction is an ill-posed problem
  - Different outputs for the same input
- Quantifying how output approximates the input is a difficult task
- Output differs with human cognition and perception
- Output is dependent on heterogeneity in input point density and distribution
- Application specific nature of the output
- Varied applications









Applications of Reconstruction

- Product design - eg: Initial design of an aircraft
- Geographical information systems -eg: Map generalization
- Computer graphics - eg: Point set matching , Geometric modeling
- Bio medical image analysis
- Routing in networks
- Island formation in power systems

Methods for reconstruction

- **Delaunay Triangulation -based methods :**
- χ-shape
- α-shape
- Automatic Surface Reconstruction
- Geometric Structures for 3D shape
- Crust and NN Crust
- Optimal transport driven approach
- RGG from planar point set
- **Non-Delaunay Triangulation methods**
- Ball Pivoting algorithm
- Simple Shape
- Methods using implicit functions

References

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THANK YOU