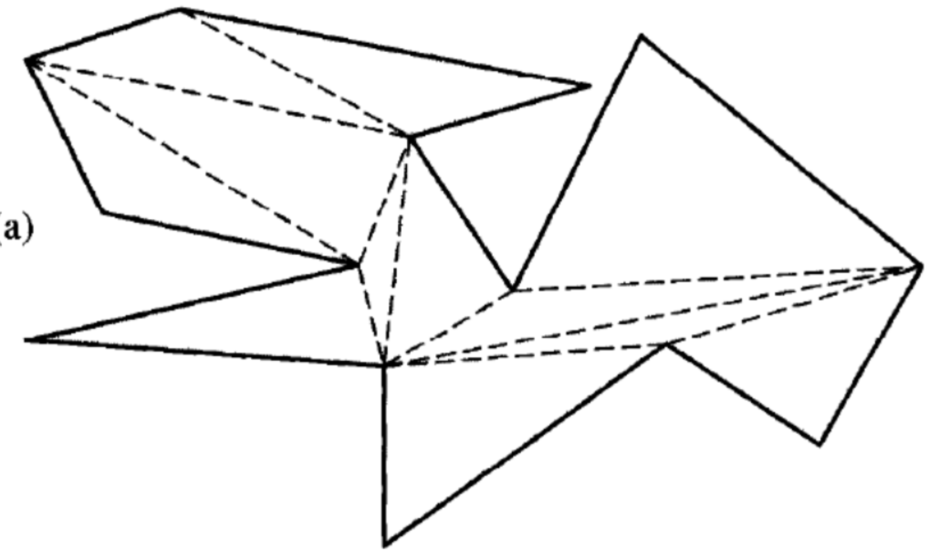
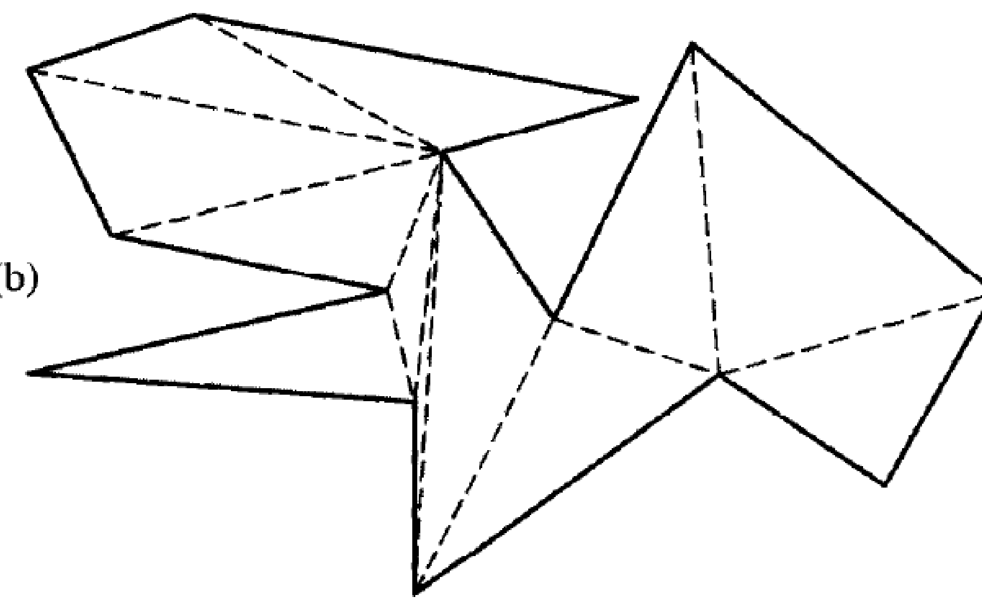
Art Gallery Problem [Victor Klee, 1973]

- Input : Art Gallery (Represented by a polygon)
- Output : Minimum number of guards that can safe-guard or cover the interior walls of the gallery
- Minimum # of guards G(n) = Floor (n/3)
- Empirical exploration
- General structure : Chavtal’s comb
- Chavtal’s proof : complex one
- Fisk’s proof

(a)



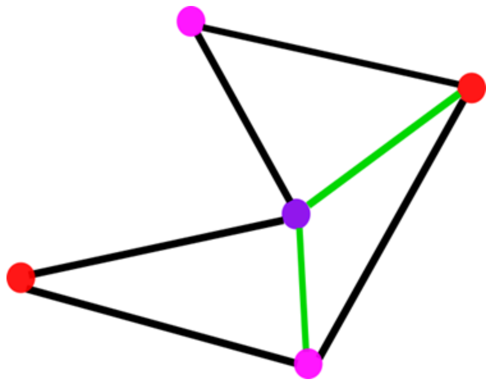
(b)



Our graph : Triangulation of P

- In this case, what is G = (V, E)?
- V = vertices of P
- E = edges of P + diagonals of P
- What is a k-coloring of G?
- k-coloring of G :
  – Assign k colors to the vertices of G
  – No two vertices connected by an edge are assigned the same color

Fisk's proof for sufficiency of floor (n/3) guards for any P



- Steps in Fisk's Proof :
- 1. Triangulate P
- 2. Three-color the triangulation
- What do we do after 3-coloring of T ?
- What do we observe after 3-coloring?
- Placing guards on any of the colors covers the whole polygon

At least one color should not be more than *n*/3

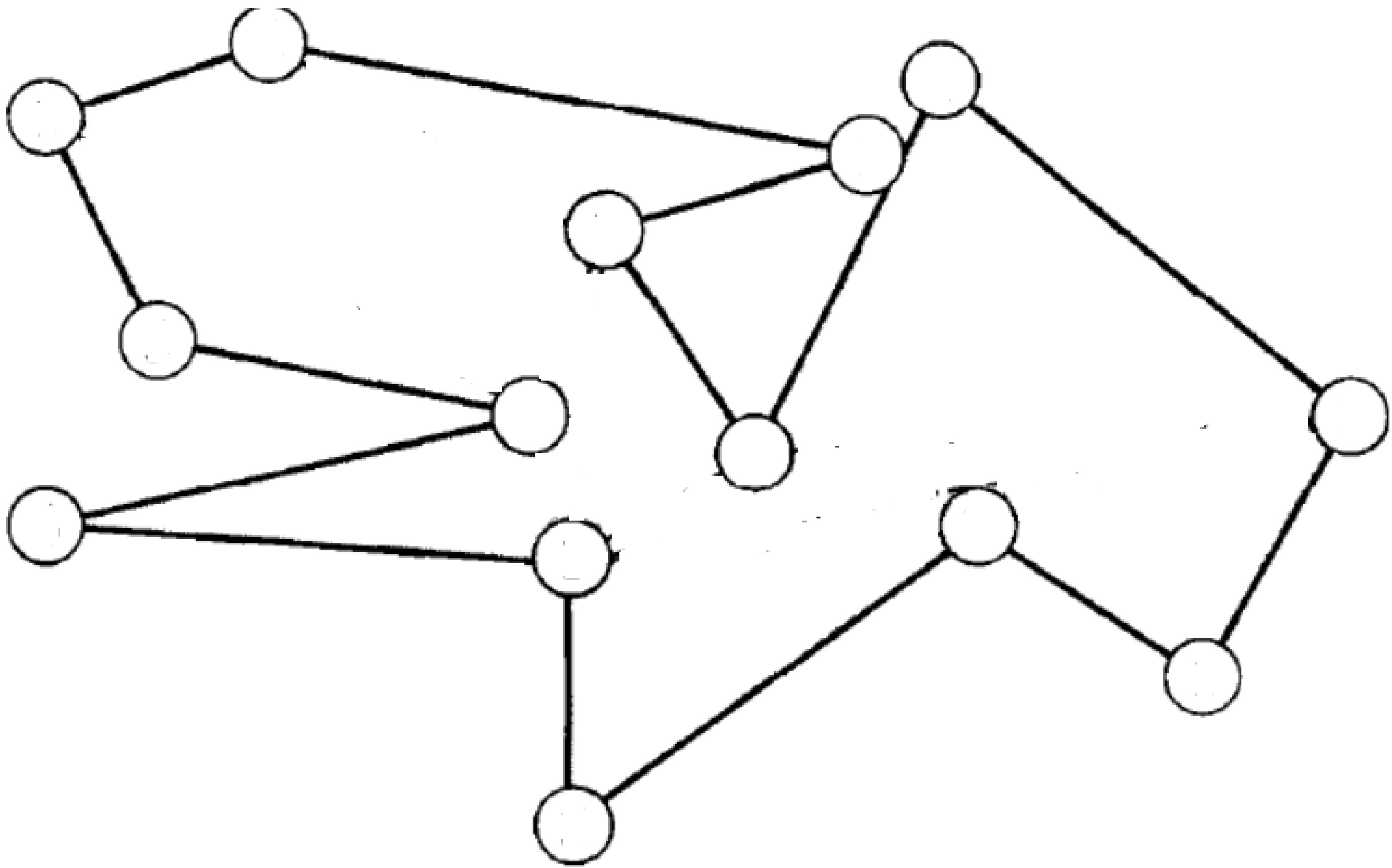
- There are *n* vertices for the T of P
- There are 3 colors to color the vertices
- Our requirement: At least one color must not be used more than *n*/3 times
- Since n is an integer, at least one color must not be used more than floor(*n*/3) times
- **Pigeon-hole principle**

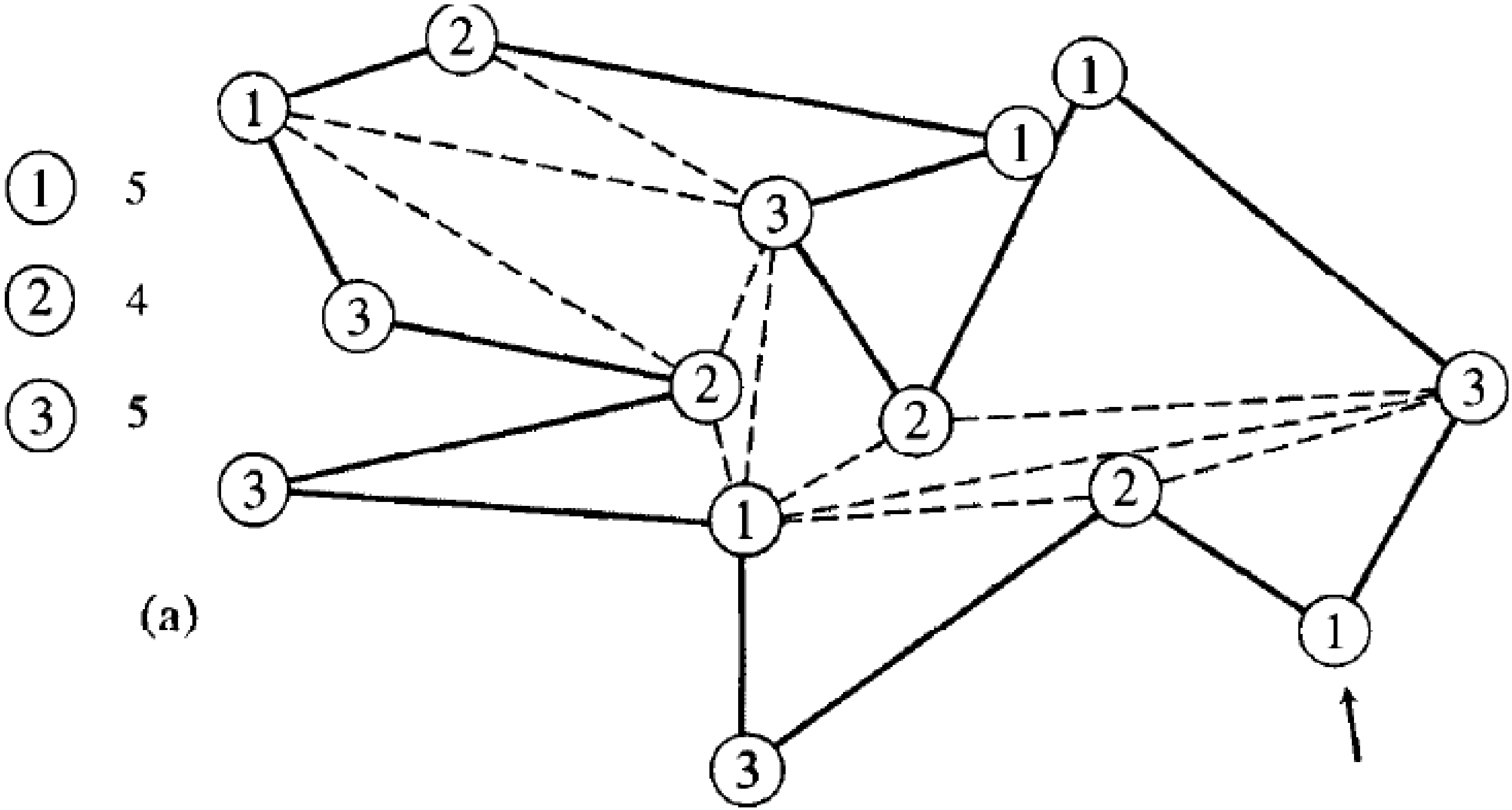
Pigeon-hole principle

- If n objects to be placed in to k pigeon holes, then at least one hole must contain no more than n/k objects
- If each one of the k holes contain more than n/k objects, the total number of objects would exceed n
- In our case n is the # of vertices of T of P and k is the # of colors used (k =3)
- **At least one color must not be used more than floor(n/3) times**

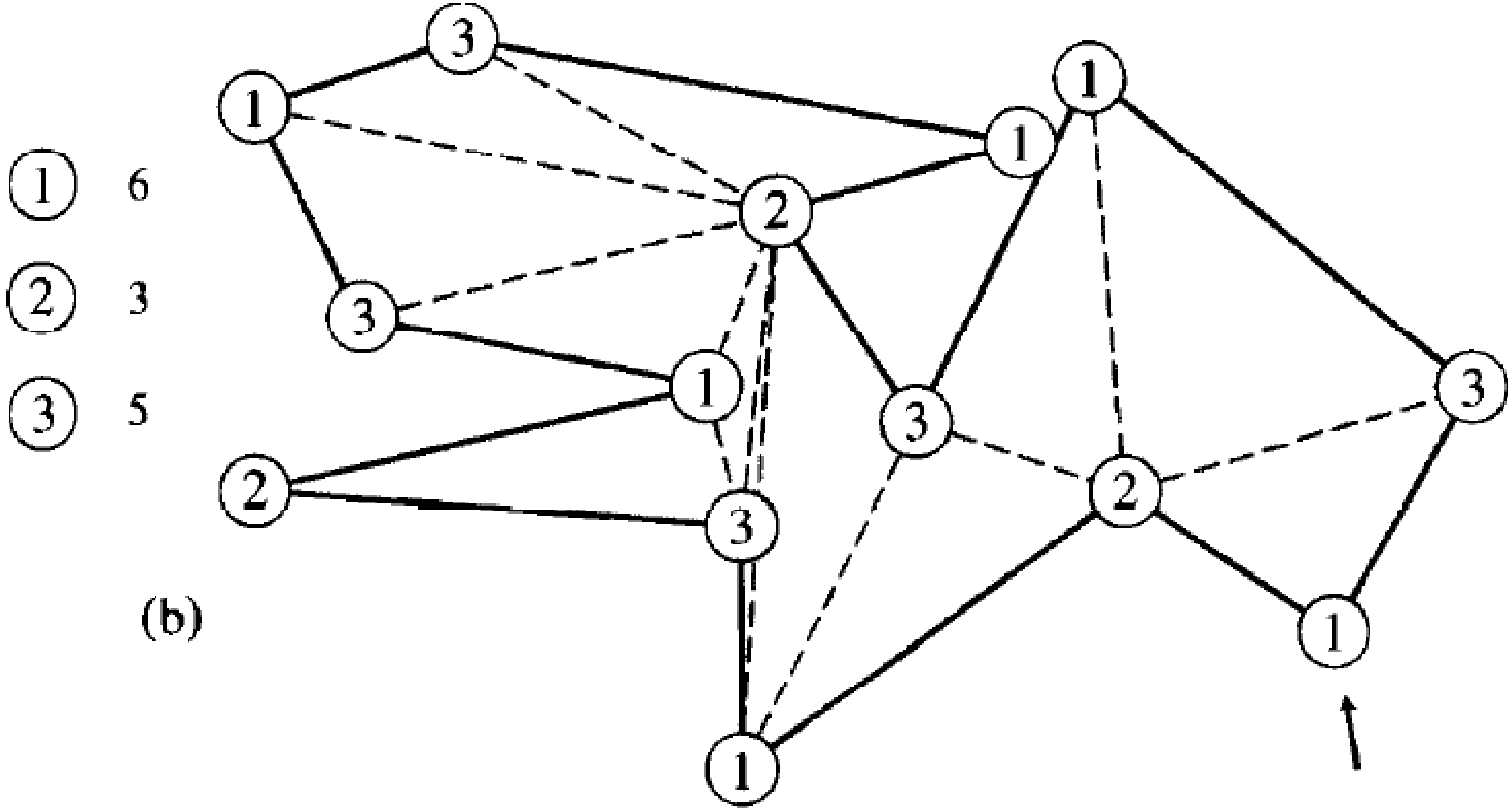
Exercise

- Try Fisk's proof on a polygon of 12 vertices
- With different triangulations of same P, do we get the same # of guards ?





(a)

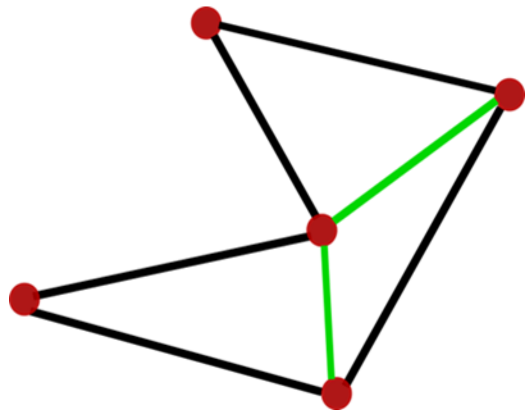


(b)

Fisk’s proof for sufficiency of floor (n/3) guards for any P

- Place guards at vertices with the least frequently used color in the 3-coloring
- This covers the polygon with no more than
  G(n)= floor (n/3) guards
- However, Fisk’s proof does not always guarantee minimum number of guards

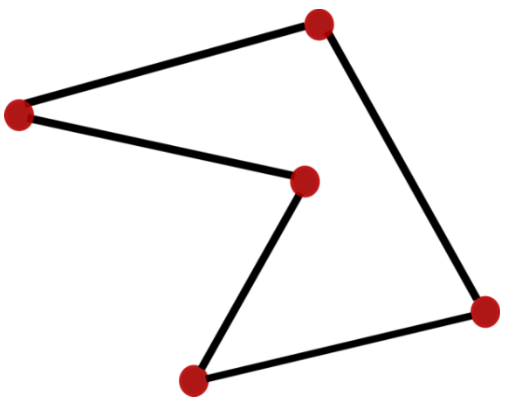
Triangulation: Theory



- Must every polygon has a triangulation?
- How could a polygon not have a triangulation?
- What about the equivalent question in 3D?
- Theorem: **Every polygon *P* of *n* vertices may be partitioned into Δs by the addition of (zero or more) diagonals**
  - We have to prove the existence of a diagonal
- Lemma 1: **Every polygon of *n* ≥ 4 vertices has a diagonal**
  - For a diagonal to exist, all the points should not be collinear
- Lemma 2: **Every polygon *P* must have at least one strictly convex vertex**

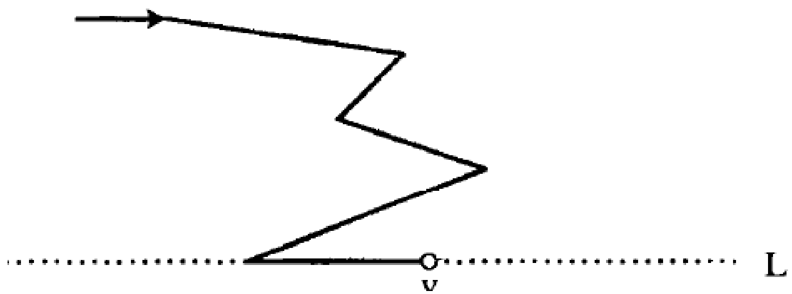
Proof of Lemma2

- Lemma 2: **Every polygon *P* must have at least one strictly convex vertex**
- Suppose a hypothetical walker walks along the boundary of P in a counter clock wise direction
- The interior of P is always to the left of the walker
- A strictly convex vertex is made on ∂P by a left turn
- A reflex vertex is made on ∂P by a right turn



Proof of Lemma 2

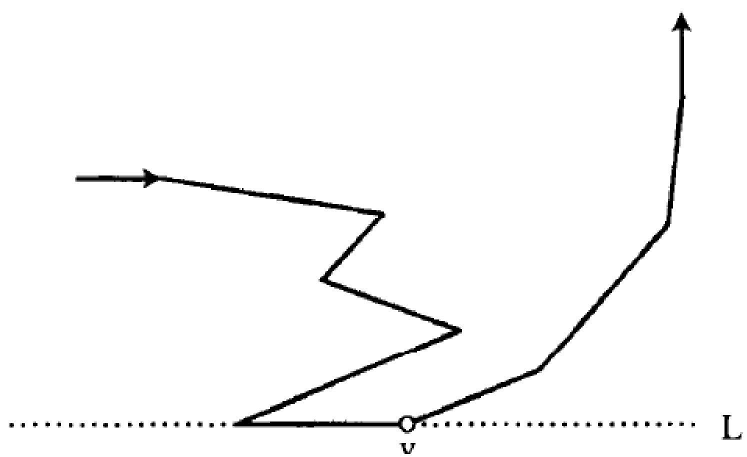
- Suppose a hypothetical walker walks along the boundary as shown below



- Let L be a line through the lowest vertex (vertex with minimum y coordinate with respect to the coordinate system)
- If there are more than one lowest vertex, let v be the right most
- The interior of P must be above L
- The edge following v must be above L. If not ?

Proof of Lemma2

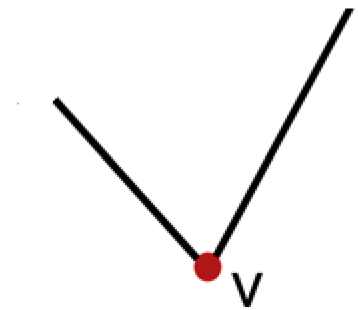
- Hence, the walker should make a left turn after v



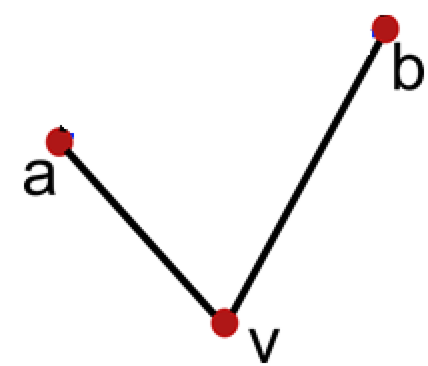
- **∃ a rightmost lowest vertex (v) which is strictly convex**
- Hence, Lemma 2 (**Every polygon *P* must have at least one strictly convex vertex**) is proved.

Proof of Lemma 1

- Lemma 1: **Every polygon of *n* ≥ 4 vertices has a diagonal**
- Let v be a strictly convex vertex

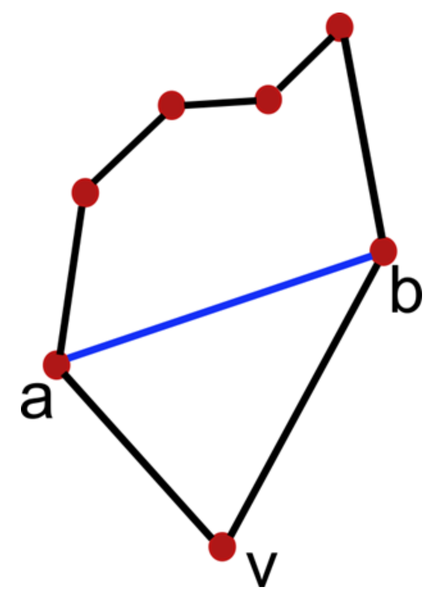


- Let a & b are the vertices adjacent to v
- Case-1: If ab is a diagonal, then the proof is complete
- Draw an example of P in Case 1



Proof of Lemma 1

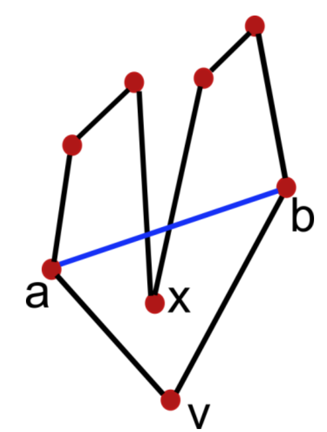
- Case-1: If ab is a diagonal
- An example polygon where ab is a diagonal



- Case-1 : **ab is a diagonal**, **hence trivially proved**

Proof of Lemma 1

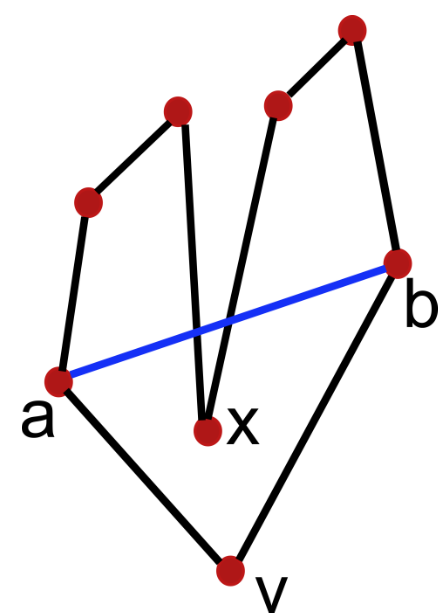
- Will there be a case where ab is not a diagonal?
- Case- 2: ab is not a diagonal
- Draw a polygon of Case 2
- Case 2: (a)Either ab is exterior to P or (b) ab intersects ∂P
- Draw a polygon of Case 2 (a)
- Draw a polygon of Case 2 (b)



Proof of Lemma 1

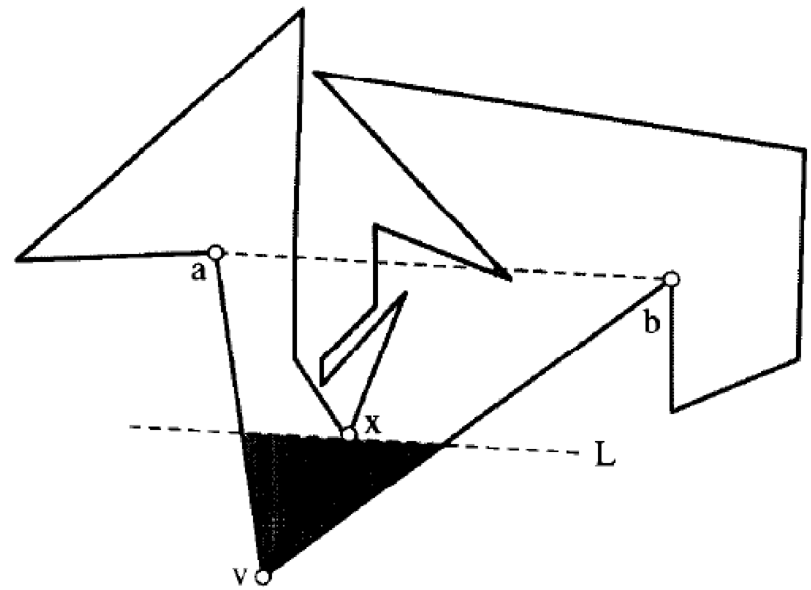
- Case-2: **ab is not a diagonal**
- Since $n > 3$, The closed Δavb contains at least one vertex of P other than a,v,b
- Let x be the vertex of P in Δavb that is closest to v
- Draw a P illustrating a,v,b & x
- x is the first vertex in Δavb hit

  by a line L moving from v to ab



Proof of Lemma 1

- vx is a diagonal. Why



- The shaded part (interior of Δavb bounded by L that includes v) is empty of points of ∂P
- Hence, vx cannot intersect ∂P except v & x, hence it is a diagonal.
- Recall: Case 2: (a)Either ab is exterior to P or (b) ab intersects ∂P
- **vx is a diagonal** is true in both Case 2(a) and Case 2(b)
- **Both case 1 & 2, ∃ a diagonal, hence Lemma 1 is proved**

Reference

- J. O Rourke, *Computational Geometry in C*, 2/e, Cambridge University Press, 1998

Thank you