hurricane_posterior

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Model

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \mathbf{X}_i\gamma + \epsilon_i(t)$$

$$\begin{bmatrix}\beta_{0,i}\\ \beta_{1,i}\\ \beta_{2,i}\\ \beta_{3,i}\\ \beta_{4,i}\end{bmatrix} \sim MVN(\begin{bmatrix}\mu_0\\ \mu_1\\ \mu_2\\ \mu_3\\ \mu_4\end{bmatrix}, \Sigma)$$

$$\epsilon_i \sim N(0, \sigma^2)$$

Priors

1.

$$\begin{bmatrix}\mu_0\\ \mu_1\\ \mu_2\\ \mu_3\\ \mu_4\end{bmatrix} \sim MVN(\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}, V)$$

$$f_\mu(\mu) \propto det(V)^{\frac{-1}{2}} e^{\frac{-1}{2}\mu^\intercal V^{-1}\mu} \propto e^{\frac{-1}{2}\mu^\intercal V^{-1}\mu}$$

2.

$$\Sigma \sim W^{-1}(S, \nu = 5)$$

Due to property of Wishart distribution,

$$\Sigma^{-1} \sim W(S^{-1}, \nu = 5)$$

$$f_{\Sigma^{-1}}(\Sigma^{-1}) = |\Sigma^{-1}|^{\frac{\nu-d-1}{2}} exp(-\frac{tr(S\Sigma^{-1})}{2}) \propto |\Sigma^{-1}|^{\frac{\nu-5-1}{2}} exp(-\frac{tr(S\Sigma^{-1})}{2})$$

3.

$$\gamma \sim MVN(\begin{bmatrix} 0 \\ 0 \\ ... \\ 0 \end{bmatrix}, 0.005^2 I_{14})$$

Since Month and Nature are categorical variables, the dimension of the γ is 14×1.

$$f_\gamma(\gamma) = (14 \times 0.005^2)^{-1/2} exp(-\frac{\gamma^\intercal 0.005^2 I_{14} \gamma}{2}) \propto exp(-\frac{400\gamma^\intercal \gamma}{2})$$

4.

$$\sigma \sim Half - Cauchy(0, 10)$$
$$f_\sigma(\sigma) = \frac{2 \times 10}{\pi(\sigma^2 + 10^2)}$$

By transformation theorem

$$f_{\sigma^2}(\sigma^2) = \frac{2 \times 10}{\pi(\sigma^2 + 10^2)} \frac{1}{2\sigma} \propto \frac{1}{\pi(\sigma^2 + 10^2)\sigma}$$

Likelihood

Because random effects coefficients β_i is normal, $Y_i|\beta_i$ also follows a normal distribution by property of normal distribution.

For each hurricane Y_i

$$\begin{aligned} Y_i|\beta_i, \mu, \sigma^2, \Sigma, \gamma \sim & MVN(\beta_{0,i} + \beta_{1,i} Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \mathbf{X}_i\gamma, \sigma^2 I_{n_i}) \\ = & MVN(D_i\beta_i + X_i\gamma, \sigma^2 I_{n_i}) \end{aligned}$$

where

$$Y_i = \begin{bmatrix} Y_i(t_2) \\ Y_i(t_3) \\ \vdots \\ Y_i(t = t_j) \\ \vdots \\ Y_i(t = t_{n_i}) \end{bmatrix}_{n_i \times 1}$$

$$\begin{aligned} D_i(t) = & \begin{bmatrix} 1 & Y_i(t) & \Delta_{i,1}(t) & \Delta_{i,2}(t) & \Delta_{i,3}(t) \end{bmatrix} \\ = & \begin{bmatrix} 1 & Y_i(t = t_1) & \Delta_{i,1}(t_1, t_0) & \Delta_{i,2}(t_1, t_0) & \Delta_{i,3}(t_1, t_0) \\ 1 & Y_i(t = t_2) & \Delta_{i,1}(t_2, t_1) & \Delta_{i,2}(t_2, t_1) & \Delta_{i,3}(t_2, t_1) \\ \dots \\ 1 & Y_i(t = t_{j-1}) & \Delta_{i,1}(t_{j-1}, t_{j-2}) & \Delta_{i,2}(t_{j-1}, t_{j-2}) & \Delta_{i,3}(t_{j-1}, t_{j-2}) \\ \dots \\ 1 & Y_i(t = t_{n_{i-1}}) & \Delta_{n,1}(t_{n_{i-1}} - t_{n_{i-2}}) & \Delta_{n,2}(t_{n_{i-1}} - t_{n_{i-2}}) & \Delta_{n,3}(t_{n_{i-1}} - t_{n_{i-2}}) \end{bmatrix}_{n_i \times 5} \end{aligned}$$

$$\beta_i = \begin{bmatrix} \beta_{0,i} \\ \beta_{1,i} \\ \beta_{2,i} \\ \beta_{3,i} \\ \beta_{4,i} \end{bmatrix}_{5\times 1}$$

$$X_i = \begin{bmatrix} X_{i,month} & x_{i,season} & X_{i,type} \end{bmatrix}_{n_i \times 14}$$

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ ... \\ \gamma_{14} \end{bmatrix}_{14\times 1}$$

Likelihood for the ith hurricane is

$$f(Y_i|\beta_i, \mu, \sigma^2, \Sigma, \gamma) = det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))$$

To calculate the joint likelihood for $Y = \begin{bmatrix} Y_1 & Y_2 \dots Y_i \dots Y_H \end{bmatrix}^\intercal$, we denote total number of observations for all hurricanes as $N = \sum_{i=0}^{H} n_i$ where n_i is the total number of observation for the ith hurricane and H is the total number of hurricanes.

All random effects coefficients β_i in

$$\begin{aligned} B &= \begin{bmatrix} \beta_1 & \beta_2 \dots & \beta_i & \dots \beta_H \end{bmatrix} \\ &= \begin{bmatrix} \beta_{0,1} & \beta_{0,2} & \cdots & \beta_{0,i} & \cdots & \beta_{0,H} \\ \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,i} & \cdots & \beta_{1,H} \\ \beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2,i} & \cdots & \beta_{2,H} \\ \beta_{3,1} & \beta_{3,2} & \cdots & \beta_{3,i} & \cdots & \beta_{3,H} \\ \beta_{4,1} & \beta_{4,2} & \cdots & \beta_{4,i} & \cdots & \beta_{4,H} \end{bmatrix}_{5\times H} \end{aligned}$$

Design matrix for random effects for all hurricanes are in D. $D = \begin{bmatrix} D_1(t) \\ D_2(t) \\ \vdots \\ D_i(t) \\ \vdots \\ D_H(t) \end{bmatrix}_{N\times 5}$

Due to independence of each hurricane, the joint likelihood is

$$\begin{aligned} L_Y(B, \mu, \sigma^2, \Sigma, \gamma) &= \prod_{i=1}^{H} L_{Y_i}(\beta_i, \mu, \sigma^2, \Sigma, \gamma) \\ &= \prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma)) \\ &= \frac{1}{\sigma^N}\prod_{i=1}^{H} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma)) \end{aligned}$$

Posterior

By Baye's Rule

$$f(B, \mu, \sigma^2, \Sigma, \gamma|Y) \propto f(Y|B, \mu, \sigma^2, \Sigma, \gamma) \times f(B|\mu, \Sigma) \times f(\mu) \times f(\Sigma) \times f(\sigma^2) \times f(\gamma)$$

where

$$\mu = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}_{5\times 1}$$

$$\begin{aligned} f(B|\mu,\Sigma) &= \prod_{i=1}^{H} f(\beta_i|\mu,\Sigma) \\ &= \prod_{i=1}^{H} det(\Sigma)^{-1/2} exp(-\frac{(\beta_i-\mu)^\intercal \Sigma^{-1}(\beta_i-\mu)}{2}) \\ &= det(A)^{H/2} \prod_{i=1}^{H} exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2}) \end{aligned}$$

where $A = \Sigma^{-1}$

$$f(\mu) = det(V)^{\frac{-1}{2}} exp(-\frac{\mu^\intercal V^{-1}\mu}{2})$$

We'll only use $f_{\Sigma^{-1}}$ because only Σ^{-1} shows up in the likelihood equation. We denote $A = \Sigma^{-1}$ in the posterior.

$$\begin{aligned} f_{\Sigma^{-1}}(\Sigma^{-1}) &\propto |\Sigma^{-1}|^{\frac{\nu-5-1}{2}} exp(-\frac{tr(S\Sigma^{-1})}{2}) \\ f_{\sigma^2}(\sigma^2) &= \frac{2\times 10}{\pi(\sigma^2+10^2)}\frac{1}{2\sigma} \propto \frac{1}{\pi(\sigma^2+10^2)\sigma} \\ f_\gamma(\gamma) &= exp(-\frac{400\gamma^\intercal\gamma}{2}) \end{aligned}$$

Final posterior

$$\begin{aligned} f(B,\mu,\sigma^2,\Sigma,\gamma|Y) \propto & f(Y|B,\mu,\sigma^2,\Sigma,\gamma)\times f(B|\mu,\Sigma)\times f(\mu)\times f(\Sigma^{-1})\times f(\sigma^2)\times f(\gamma) \\ =& f(Y|B,\mu,\sigma^2,\Sigma,\gamma)\times f(B|\mu,A)\times f(\mu)\times f(A)\times f(\sigma^2)\times f(\gamma) \\ =& \prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))\times \\ & det(A)^{H/2}\prod_{i=1}^{H} exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2})\times \\ & det(V)^{\frac{-1}{2}} exp(-\frac{\mu^\intercal V^{-1}\mu}{2})\times \\ & |A|^{\frac{\nu-5-1}{2}} exp(-\frac{tr(SA)}{2})\times \\ & \frac{1}{\pi(\sigma^2+10^2)\sigma}\times \\ & exp(-\frac{400\gamma^\intercal\gamma}{2}) \end{aligned}$$

Conditional Posterior

For B,

$$\begin{aligned}
f(B|\mu,\sigma^2,\Sigma,\gamma,Y) \propto &\prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))\times \\
&det(A)^{H/2}\prod_{i=1}^{H} exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2}) \\
\propto &\prod_{i=1}^{H} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2}) \\
= &exp(\frac{-1}{2}\sum_{i=1}^{H}\beta_i^\intercal(D_i{}^\intercal\sigma^{-2}I_{n_i}D_i + A)\beta_i - 2\beta_i^\intercal(D_i^\intercal\sigma^{-2}I_{n_i}Y_i - D_i^\intercal\sigma^{-2}I_{n_i}X_i\gamma + A\mu) \\
&+Y_i^\intercal\sigma^{-2}I_{n_i}Y_i - 2Y_i^\intercal\sigma^{-2}I_{n_i}X_i\gamma + \gamma^\intercal X_i^\intercal\sigma^{-2}I_{n_i}X_i\gamma + \mu^\intercal A\mu)
\end{aligned}$$

Let $M = D_i{}^\intercal\sigma^{-2}I_{n_i}D_i + A$ and $N = D_i^\intercal\sigma^{-2}I_{n_i}Y_i - D_i^\intercal\sigma^{-2}I_{n_i}X_i\gamma + A\mu$,

Finally, we have $f(B|\mu,\sigma^2,\Sigma,\gamma,Y) \sim MVN(M^{-1}N, M^{-1})$

For μ,

$$\begin{aligned}
f(\mu|B,\sigma^2,\Sigma,\gamma,Y) \propto &exp(-\frac{\mu^\intercal V^{-1}\mu}{2})\prod_{i=1}^{H} exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2}) \\
= &exp(\sum_{i=1}^{H} -\frac{1}{2}(\mu^\intercal(A - \frac{1}{H}V^{-1})\mu - 2\mu^\intercal A\beta_i + \beta_i^\intercal A\beta_i)) \\
= &exp(-\frac{1}{2}(\mu^\intercal(HA - V^{-1})\mu - 2\mu^\intercal\sum_{i=1}^{H}(A\beta_i) + \beta_i^\intercal A\beta_i))
\end{aligned}$$

Let $M = HA - V^{-1}$ and $N = \sum_{i=1}^{H}(A\beta_i)$,

Finally, we have $f(\mu|B,\sigma^2,\Sigma,\gamma,Y) \sim MVN(M^{-1}N, M^{-1})$

For σ^2,

$$\begin{aligned}
f(\sigma^2|B,\mu,\Sigma,\gamma,Y) \propto &\prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))\times \\
&\frac{1}{\pi(\sigma^2 + 10^2)\sigma}
\end{aligned}$$

Note: the conditional probability density function of σ^2 is not a known distribution, so we need to introduce metropolis hasting here.

For $A = \Sigma^{-1}$,

$$
\begin{aligned}
f(\Sigma^{-1}|B,\mu,\sigma^2,\gamma,Y) \propto & det(A)^{H/2}\prod_{i=1}^{H} exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2})\times \\
& |A|^{\frac{\nu-5-1}{2}} exp(-\frac{tr(SA)}{2}) \\
= & det(A)^{\frac{H+\nu-5-1}{2}} exp(-\frac{1}{2}tr(SA+\sum_{i=1}^{H}(\beta_i-\mu)^\intercal A(\beta_i-\mu))) \\
= & det(A)^{\frac{H+\nu-5-1}{2}} exp(-\frac{1}{2}tr[(S+\sum_{i=1}^{H}(\beta_i-\mu)(\beta_i-\mu)^\intercal)A])
\end{aligned}
$$

Note: $Tr((k)_{1\times 1}) = k$.

Let degree of freedom $= H+\nu$ and scale matrix $= (S+\sum_{i=1}^{H}(\beta_i-\mu)(\beta_i-\mu)^\intercal)^{-1}$,

Finally, we have $f(\Sigma^{-1}|B,\mu,\sigma^2,\gamma,Y) \sim Wishart(df, \text{scale matrix})$

For γ,

$$
\begin{aligned}
f(\gamma|B,\mu,\sigma^2,\Sigma,Y) \propto & \prod_{i=1}^{H} exp(-\frac{1}{2}(Y_i-D_i\beta_i-X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i-D_i\beta_i-X_i\gamma))\times \\
& exp(-\frac{400\gamma^\intercal\gamma}{2}) \\
= & exp(-\frac{1}{2}\sum_{i=1}^{H}\gamma^\intercal(X_i^\intercal\sigma^{-2}I_{n_i}X_i+400\frac{1}{H}I_3)\gamma-2\gamma^\intercal(X_i^\intercal\sigma^{-2}I_{n_i}Y_i-X_i^\intercal\sigma^{-2}I_{n_i}D_i\beta_i)+ \\
& Y_i^\intercal\sigma^{-2}I_{n_i}Y_i-2Y_i^\intercal\sigma^{-2}I_{n_i}D_i\beta_i+\beta_i^\intercal D_i^\intercal\sigma^{-2}I_{n_i}D_i\beta_i)
\end{aligned}
$$

Let $M=\sum_{i=1}^{H}X_i^\intercal\sigma^{-2}I_{n_i}X_i+400I_3$ and $N=\sum_{i=1}^{H}(X_i^\intercal\sigma^{-2}I_{n_i}Y_i-X_i^\intercal\sigma^{-2}I_{n_i}D_i\beta_i)$,

Finally, we have $f(\gamma|B,\mu,\sigma^2,\Sigma,Y) \sim MVN(M^{-1}N, M^{-1})$