

hurricane_posterior

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Model

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \mathbf{X}_i\gamma + \epsilon_i(t)$$

$$\begin{bmatrix} \beta_{0,i} \\ \beta_{1,i} \\ \beta_{2,i} \\ \beta_{3,i} \\ \beta_{4,i} \end{bmatrix} \sim MVN\left(\begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}, \Sigma\right)$$

$$\epsilon_i \sim N(0, \sigma^2)$$

Priors

1.

$$\begin{bmatrix} \mu_{0,i} \\ \mu_{1,i} \\ \mu_{2,i} \\ \mu_{3,i} \\ \mu_{4,i} \end{bmatrix} \sim MVN\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, V\right)$$

$$f_{\mu_i}(\mu_i) \propto \det(V)^{\frac{-1}{2}} e^{\frac{-1}{2}\mu_i^T V^{-1} \mu_i} \propto e^{\frac{-1}{2}\mu_i^T V^{-1} \mu_i}$$

2.

$$\Sigma \sim W^{-1}(S, \nu = 5)$$

$$f_{\Sigma^{-1}}(\Sigma) \propto |\Sigma^{-1}|^{n+1} e^{\frac{-1}{2}tr(\Sigma^{-1})}$$

where n is the dimension of dataset... this formula might be wrong meh

determinant n dim of of sigma

Wishart

Due to property of Wishart distribution,

$$\Sigma^{-1} \sim W(S^{-1}, \nu = 5)$$

$$f_{\Sigma^{-1}}(\Sigma^{-1}) = |\Sigma^{-1}|^{\frac{\nu-d-1}{2}} \exp\left(-\frac{tr(S\Sigma^{-1})}{2}\right) \propto |\Sigma^{-1}|^{\frac{\nu-5-1}{2}} \exp\left(-\frac{tr(S\Sigma^{-1})}{2}\right)$$

3.

$$\gamma \sim MVN\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, 0.005^2 I_3\right)$$

$$f_\gamma(\gamma) = (3 * 0.005^2)^{-1/2} \exp\left(-\frac{\gamma^\top 0.005^2 I_3 \gamma}{2}\right) \propto \exp\left(-\frac{400 \gamma^\top \gamma}{2}\right)$$

4.

$$\sigma \sim Half - Cauchy(0, 10)$$

$$f_\sigma(\sigma) = \frac{2 * 10}{\pi(\sigma^2 + 10^2)}$$

By transformation theorem

$$f_{\sigma^2}(\sigma^2) = \frac{2 * 10}{\pi(\sigma^2 + 10^2)} \frac{1}{2\sigma} \propto \frac{1}{\pi(\sigma^2 + 10^2)\sigma}$$

Likelihood

Because random effects coefficients β_i is normal, $Y_i|\beta_i$ also follows a normal distribution by property of normal distribution.

For each hurricane Y_i

$$Y_i|\beta_i, \mu_i, \sigma^2, \Sigma, \gamma \sim MVN(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \mathbf{X}_i\gamma, \sigma^2 I_i)$$

$$= MVN(D_i\beta_i + X_i\gamma, \sigma^2 I_{n_i})$$

where

$$Y_i = \begin{bmatrix} Y_i(t_0 + 6) \\ Y_i(t_1 + 6) \\ \vdots \\ Y_i(t = t_j + 6) \\ \vdots \\ Y_i(t = t_{n_i-1} + 6) \end{bmatrix}_{n_i \times 1}$$

$$D_i(t) = \begin{bmatrix} 1 & Y_i(t) & \Delta_{i,1}(t) & \Delta_{i,2}(t) & \Delta_{i,3}(t) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & Y_i(t = t_0) & \Delta_{i,1}(t = t_0) & \Delta_{i,2}(t = t_0) & \Delta_{i,3}(t = t_0) \\ 1 & Y_i(t = t_1) & \Delta_{i,1}(t = t_1) & \Delta_{i,2}(t = t_1) & \Delta_{i,3}(t = t_1) \\ \cdots & & & & \\ 1 & Y_i(t = t_j) & \Delta_{i,1}(t = t_j) & \Delta_{i,2}(t = t_j) & \Delta_{i,3}(t = t_j) \\ \cdots & & & & \\ 1 & Y_i(t = t_{n_i-1}) & \Delta_{i,1}(t = t_{n_i-1}) & \Delta_{i,2}(t = t_{n_i-1}) & \Delta_{i,3}(t = t_{n_i-1}) \end{bmatrix}_{n_i \times 5}$$

$$\beta_i = \begin{bmatrix} \beta_{0,i} \\ \beta_{1,i} \\ \beta_{2,i} \\ \beta_{3,i} \\ \beta_{4,i} \end{bmatrix}$$

$$X_i = \begin{bmatrix} x_{i,1} & x_{i,2} & x_{i,3} \end{bmatrix}_{1 \times 3}$$

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}_{3 \times 1}$$

Likelihood for the i th hurricane is

$$f(Y_i|\beta_i, \mu_i, \sigma^2, \Sigma, \gamma) = \det(\sigma^2 I_{n_i})^{-1/2} \exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\top (\sigma^2 I_{n_i})^{-1} (Y_i - D_i\beta_i - X_i\gamma))$$

To calculate the joint likelihood for $Y = [Y_1 \ Y_2 \dots Y_i \dots Y_H]^\top$, we denote total number of observations for all hurricanes as $N = \sum_{i=0}^H n_i$ where n_i is the total number of observation for the i th hurricane and H is the total number of hurricanes.

All random effects coefficients β_i in

$$B = \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_i & \dots & \beta_H \end{bmatrix}$$

$$= \begin{bmatrix} \beta_{0,1} & \beta_{0,2} & \dots & \beta_{0,i} & \dots & \beta_{0,H} \\ \beta_{1,1} & \beta_{1,2} & \dots & \beta_{1,i} & \dots & \beta_{1,H} \\ \beta_{2,1} & \beta_{2,2} & \dots & \beta_{2,i} & \dots & \beta_{2,H} \\ \beta_{3,1} & \beta_{3,2} & \dots & \beta_{3,i} & \dots & \beta_{3,H} \\ \beta_{4,1} & \beta_{4,2} & \dots & \beta_{4,i} & \dots & \beta_{4,H} \end{bmatrix}_{5 \times H}$$

Design matrix for random effects for all hurricanes are in D . $D = \begin{bmatrix} D_1(t) \\ D_2(t) \\ \vdots \\ D_i(t) \\ \vdots \\ D_H(t) \end{bmatrix}_{N \times 5}$

Due to independence of each hurricane, the joint likelihood is

$$\begin{aligned} L_Y(B, \mu, \sigma^2, \Sigma, \gamma) &= \prod_{i=1}^H L_{Y_i}(\beta_i, \mu, \sigma^2, \Sigma, \gamma) \\ &= \prod_{i=1}^H \det(\sigma^2 I_{n_i})^{-1/2} \exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\top (\sigma^2 I_{n_i})^{-1} (Y_i - D_i\beta_i - X_i\gamma)) \\ &= \frac{1}{\sigma^N} \prod_{i=1}^H \exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\top (\sigma^2 I_{n_i})^{-1} (Y_i - D_i\beta_i - X_i\gamma)) \end{aligned}$$

Posterior

By Baye's Rule

$$f(B, \mu, \sigma^2, \Sigma, \gamma|Y) \propto f(Y|B, \mu, \sigma^2, \Sigma, \gamma) \times f(B|\mu, \Sigma) \times f(\mu) \times f(\Sigma) \times f(\sigma^2) \times f(\gamma)$$

where

$$\mu = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}_{5 \times 1}$$

$$\begin{aligned}
f(B|\mu, \Sigma) &= \prod_{i=1}^H f(\beta_i|\mu, \Sigma) \\
&= \prod_{i=1}^H \det(\Sigma)^{-1/2} \exp\left(-\frac{(\beta_i - \mu)^\top \Sigma^{-1} (\beta_i - \mu)}{2}\right) \\
&= \det(A)^{H/2} \prod_{i=1}^H \exp\left(-\frac{(\beta_i - \mu)^\top A (\beta_i - \mu)}{2}\right)
\end{aligned}$$

where $A = \Sigma^{-1}$

$$f(\mu) = \det(V)^{-\frac{1}{2}} \exp\left(-\frac{\mu^\top V^{-1} \mu}{2}\right)$$

We'll only use $f_{\Sigma^{-1}}$ because only Σ^{-1} shows up in the likelihood equation. We denote $A = \Sigma^{-1}$ in the posterior.

$$\begin{aligned}
f_{\Sigma^{-1}}(\Sigma^{-1}) &\propto |\Sigma^{-1}|^{\frac{\nu-5-1}{2}} \exp\left(-\frac{\text{tr}(S\Sigma^{-1})}{2}\right) \\
f_{\sigma^2}(\sigma^2) &= \frac{2 * 10}{\pi(\sigma^2 + 10^2)} \frac{1}{2\sigma} \propto \frac{1}{\pi(\sigma^2 + 10^2)\sigma} \\
f_{\gamma}(\gamma) &= \exp\left(-\frac{400\gamma^\top \gamma}{2}\right)
\end{aligned}$$

Final posterior

$$\begin{aligned}
f(B, \mu, \sigma^2, \Sigma, \gamma|Y) &\propto f(Y|B, \mu, \sigma^2, \Sigma, \gamma) \times f(B|\mu, \Sigma) \times f(\mu) \times f(\Sigma^{-1}) \times f(\sigma^2) \times f(\gamma) \\
&= f(Y|B, \mu, \sigma^2, \Sigma, \gamma) \times f(B|\mu, A) \times f(\mu) \times f(A) \times f(\sigma^2) \times f(\gamma) \\
&= \prod_{i=1}^H \det(\sigma^2 I_{n_i})^{-1/2} \exp\left(-\frac{1}{2}(Y_i - D_i \beta_i - X_i \gamma)^\top (\sigma^2 I_{n_i})^{-1} (Y_i - D_i \beta_i - X_i \gamma)\right) \times \\
&\quad \det(A)^{H/2} \prod_{i=1}^H \exp\left(-\frac{(\beta_i - \mu)^\top A (\beta_i - \mu)}{2}\right) \times \\
&\quad \det(V)^{-\frac{1}{2}} \exp\left(-\frac{\mu^\top V^{-1} \mu}{2}\right) \times \\
&\quad |A|^{\frac{\nu-5-1}{2}} \exp\left(-\frac{\text{tr}(SA)}{2}\right) \times \\
&\quad \frac{1}{\pi(\sigma^2 + 10^2)\sigma} \times \\
&\quad \exp\left(-\frac{400\gamma^\top \gamma}{2}\right)
\end{aligned}$$

Conditional Posterior

For B,

$$\begin{aligned}
f(B|\mu, \sigma^2, \Sigma, \gamma, Y) &\propto \prod_{i=1}^H \det(\sigma^2 I_{n_i})^{-1/2} \exp(-\frac{1}{2}(Y_i - D_i \beta_i - X_i \gamma)^\top (\sigma^2 I_{n_i})^{-1} (Y_i - D_i \beta_i - X_i \gamma)) \times \\
&\quad \det(A)^{H/2} \prod_{i=1}^H \exp(-\frac{(\beta_i - \mu)^\top A(\beta_i - \mu)}{2}) \\
&\propto \prod_{i=1}^H \exp(-\frac{1}{2}(Y_i - D_i \beta_i - X_i \gamma)^\top (\sigma^2 I_{n_i})^{-1} (Y_i - D_i \beta_i - X_i \gamma)) \exp(-\frac{(\beta_i - \mu)^\top A(\beta_i - \mu)}{2}) \\
&= \exp(-\frac{1}{2} \sum_{i=1}^H \beta_i^\top (D_i^\top \sigma^{-2} I_{n_i} D_i + A) \beta_i - 2\beta_i^\top (D_i^\top \sigma^{-2} I_{n_i} Y_i - D_i^\top \sigma^{-2} I_{n_i} X_i \gamma + A\mu) \\
&\quad + Y_i^\top \sigma^{-2} I_{n_i} Y_i - 2Y_i^\top \sigma^{-2} I_{n_i} X_i \gamma + \gamma^\top X_i^\top \sigma^{-2} I_{n_i} X_i \gamma + \mu^\top A\mu)
\end{aligned}$$

Let $M = D_i^\top \sigma^{-2} I_{n_i} D_i + A$ and $N = D_i^\top \sigma^{-2} I_{n_i} Y_i - D_i^\top \sigma^{-2} I_{n_i} X_i \gamma + A\mu$,

Finally, we have $f(B|\mu, \sigma^2, \Sigma, \gamma, Y) \sim MVN(M^{-1}N, M^{-1})$

For μ ,

$$\begin{aligned}
f(\mu|B, \sigma^2, \Sigma, \gamma, Y) &\propto \prod_{i=1}^H \exp(-\frac{(\beta_i - \mu)^\top A(\beta_i - \mu)}{2}) \exp(-\frac{\mu^\top V^{-1} \mu}{2}) \\
&= \exp(\sum_{i=1}^H -\frac{1}{2}(\mu^\top (A - V^{-1})\mu - 2\mu^\top A\beta_i + \beta_i^\top A\beta_i)) \\
&= \exp(-\frac{1}{2}(\mu^\top H(A - V^{-1})\mu - 2\mu^\top \sum_{i=1}^H (A\beta_i) + \beta_i^\top A\beta_i))
\end{aligned}$$

Let $M = H(A - V^{-1})$ and $N = \sum_{i=1}^H (A\beta_i)$,

Finally, we have $f(\mu|B, \sigma^2, \Sigma, \gamma, Y) \sim MVN(M^{-1}N, M^{-1})$

For σ^2 ,

$$\begin{aligned}
f(\sigma^2|B, \mu, \Sigma, \gamma, Y) &\propto \prod_{i=1}^H \det(\sigma^2 I_{n_i})^{-1/2} \exp(-\frac{1}{2}(Y_i - D_i \beta_i - X_i \gamma)^\top (\sigma^2 I_{n_i})^{-1} (Y_i - D_i \beta_i - X_i \gamma)) \times \\
&\quad \frac{1}{\pi(\sigma^2 + 10^2)\sigma} \\
&\approx (\sigma^2)^{-1/2} \sum_{i=1}^H n_i \exp(-\frac{1}{2\sigma^2} \sum_{i=1}^H (Y_i - D_i \beta_i - X_i \gamma)^\top (Y_i - D_i \beta_i - X_i \gamma)) \times \frac{1}{\sigma} \\
&= (\sigma^2)^{-\frac{N+1}{2}} \exp(-\frac{1}{2\sigma^2} \sum_{i=1}^H (Y_i - D_i \beta_i - X_i \gamma)^\top (Y_i - D_i \beta_i - X_i \gamma))
\end{aligned}$$

Note: since σ^2 is much smaller than 10^2 , we consider use $\frac{1}{\pi k' \sigma}$, where k' is a constant, to approximate $\frac{1}{\pi(\sigma^2 + 10^2)\sigma}$.

Let $shape = \frac{N-1}{2}$ and $rate = \frac{1}{2} \sum_{i=1}^H (Y_i - D_i \beta_i - X_i \gamma)^\top (Y_i - D_i \beta_i - X_i \gamma)$,

Finally, we have $f(\sigma^2 | B, \mu, \Sigma, \gamma, Y) \sim \text{InverseGamma}(shape, rate)$

For $A = \Sigma^{-1}$,

$$\begin{aligned} f(\Sigma^{-1} | B, \mu, \sigma^2, \gamma, Y) &\propto \det(A)^{H/2} \prod_{i=1}^H \exp\left(-\frac{(\beta_i - \mu)^\top A (\beta_i - \mu)}{2}\right) \times \\ &\quad |A|^{\frac{\nu-5-1}{2}} \exp\left(-\frac{\text{tr}(SA)}{2}\right) \\ &= \det(A)^{\frac{H+\nu-5-1}{2}} \exp\left(-\frac{1}{2} \text{tr}(SA + \sum_{i=1}^H (\beta_i - \mu)^\top A (\beta_i - \mu))\right) \end{aligned}$$

Note: $\text{Tr}((k)_{1\ddot{O}1}) = k$.

Let degree of freedom = $H + \nu$ and scale matrix = ,

Finally, we have $f(\Sigma^{-1} | B, \mu, \sigma^2, \gamma, Y) \sim \text{Wishart}(df, \text{scale matrix})$

For γ ,

$$\begin{aligned} f(\gamma | B, \mu, \sigma^2, \Sigma, Y) &\propto \prod_{i=1}^H \exp\left(-\frac{(Y_i - D_i \beta_i - X_i \gamma)^\top (Y_i - D_i \beta_i - X_i \gamma)}{2\sigma^2}\right) \times \\ &\quad \exp\left(-\frac{400\gamma^\top \gamma}{2}\right) \\ &= \exp\left(\sum_{i=1}^H \right) \end{aligned}$$