hurricane_posterior

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Model

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \mathbf{X}_i\gamma + \epsilon_i(t)$$

$$\begin{bmatrix}\beta_{0,i}\\ \beta_{1,i}\\ \beta_{2,i}\\ \beta_{3,i}\\ \beta_{4,i}\end{bmatrix} \sim MVN(\begin{bmatrix}\mu_0\\ \mu_1\\ \mu_2\\ \mu_3\\ \mu_4\end{bmatrix}, \Sigma)$$

$$\epsilon_i \sim N(0, \sigma^2)$$

Priors

1.

$$\begin{bmatrix}\mu_{0,i}\\ \mu_{1,i}\\ \mu_{2,i}\\ \mu_{3,i}\\ \mu_{4,i}\end{bmatrix} \sim MVN(\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}, V)$$

$$f_{\mu_i}(\mu_i) \propto det(V)^{\frac{-1}{2}} e^{\frac{-1}{2}\mu_i^{\intercal}V^{-1}\mu_i} \propto e^{\frac{-1}{2}\mu_i^{\intercal}V^{-1}\mu_i}$$

2.

$$\Sigma \sim W^{-1}(S, \nu = 5)$$

$$f_{\Sigma^{-1}}(\Sigma) \propto |\Sigma^{-1}|^{n+1} e^{\frac{-1}{2}tr(\Sigma^{-1})}$$

where n is the dimension of dataset… this formula might be wrong meh

determinant n dim of of sigma

Wishart

Due to property of Wishart distribution,

$$\Sigma^{-1} \sim W(S^{-1}, \nu = 5)$$

$$f_{\Sigma^{-1}}(\Sigma^{-1}) = |\Sigma^{-1}|^{\frac{\nu-d-1}{2}} exp(-\frac{tr(S\Sigma^{-1})}{2}) \propto |\Sigma^{-1}|^{\frac{\nu-5-1}{2}} exp(-\frac{tr(S\Sigma^{-1})}{2})$$

3.

$$\gamma \sim MVN(\begin{bmatrix}0\\0\\0\end{bmatrix}, 0.005^2 I_3)$$

$$f_\gamma(\gamma) = (3 * 0.005^2)^{-1/2} exp(-\frac{\gamma^\intercal 0.005^2 I_3 \gamma}{2}) \propto exp(-\frac{400\gamma^\intercal \gamma}{2})$$

4.

$$\sigma \sim Half - Cauchy(0, 10)$$
$$f_\sigma(\sigma) = \frac{2 * 10}{\pi(\sigma^2 + 10^2)}$$

By transformation theorem

$$f_{\sigma^2}(\sigma^2) = \frac{2 * 10}{\pi(\sigma^2 + 10^2)} \frac{1}{2\sigma} \propto \frac{1}{\pi(\sigma^2 + 10^2)\sigma}$$

Likelihood

Because random effects coefficients β_i is normal, $Y_i|\beta_i$ also follows a normal distribution by property of normal distribution.

For each hurricane Y_i

$$\begin{aligned} Y_i|\beta_i, \mu_i, \sigma^2, \Sigma, \gamma \sim & MVN(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \mathbf{X}_i\gamma, \sigma^2 I_i) \\ = & MVN(D_i\beta_i + X_i\gamma, \sigma^2 I_{n_i}) \end{aligned}$$

where

$$Y_i = \begin{bmatrix} Y_i(t_0 + 6) \\ Y_i(t_1 + 6) \\ \vdots \\ Y_i(t = t_j + 6) \\ \vdots \\ Y_i(t = t_{n_i - 1} + 6) \end{bmatrix}_{n_i \times 1}$$

$$\begin{aligned} D_i(t) &= \begin{bmatrix} 1 & Y_i(t) & \Delta_{i,1}(t) & \Delta_{i,2}(t) & \Delta_{i,3}(t) \end{bmatrix} \\ &= \begin{bmatrix} 1 & Y_i(t = t_0) & \Delta_{i,1}(t = t_0) & \Delta_{i,2}(t = t_0) & \Delta_{i,3}(t = t_0) \\ 1 & Y_i(t = t_1) & \Delta_{i,1}(t = t_1) & \Delta_{i,2}(t = t_1) & \Delta_{i,3}(t = t_1) \\ \ldots \\ 1 & Y_i(t = t_j) & \Delta_{i,1}(t = t_j) & \Delta_{i,2}(t = t_j) & \Delta_{i,3}(t = t_j) \\ \ldots \\ 1 & Y_i(t = t_{n_i - 1}) & \Delta_{n,1}(t = t_{n_i - 1}) & \Delta_{n,2}(t = t_{n_i - 1}) & \Delta_{n,3}(t = t_{n_i - 1}) \end{bmatrix}_{n_i \times 5} \end{aligned}$$

$$\beta_i = \begin{bmatrix} \beta_{0,i} \\ \beta_{1,i} \\ \beta_{2,i} \\ \beta_{3,i} \\ \beta_{4,i} \end{bmatrix}$$

$$X_i = \begin{bmatrix} x_{i,1} & x_{i,2} & x_{i,3} \end{bmatrix}_{1\times 3}$$
$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}_{3\times 1}$$

Likelihood for the ith hurricane is

$$f(Y_i|\beta_i, \mu_i, \sigma^2, \Sigma, \gamma) = det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal (\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))$$

To calculate the joint likelihood for $Y = \begin{bmatrix} Y_1 & Y_2 \dots Y_i \dots Y_H \end{bmatrix}^\intercal$, we denote total number of observations for all hurricanes as $N = \sum_{i=0}^{H} n_i$ where n_i is the total number of observation for the ith hurricane and H is the total number of hurricanes.

All random effects coefficients β_i in

$$\begin{aligned} B &= \begin{bmatrix} \beta_1 & \beta_2 \dots & \beta_i & \dots \beta_H \end{bmatrix} \\ &= \begin{bmatrix} \beta_{0,1} & \beta_{0,2} & \dots & \beta_{0,i} & \dots & \beta_{0,H} \\ \beta_{1,1} & \beta_{1,2} & \dots & \beta_{1,i} & \dots & \beta_{1,H} \\ \beta_{2,1} & \beta_{2,2} & \dots & \beta_{2,i} & \dots & \beta_{2,H} \\ \beta_{3,1} & \beta_{3,2} & \dots & \beta_{3,i} & \dots & \beta_{3,H} \\ \beta_{4,1} & \beta_{4,2} & \dots & \beta_{4,i} & \dots & \beta_{4,H} \end{bmatrix}_{5\times H} \end{aligned}$$

Design matrix for random effects for all hurricanes are in D. $D = \begin{bmatrix} D_1(t) \\ D_2(t) \\ \vdots \\ D_i(t) \\ \vdots \\ D_H(t) \end{bmatrix}_{N\times 5}$

Due to independence of each hurricane, the joint likelihood is

$$\begin{aligned} L_Y(B, \mu, \sigma^2, \Sigma, \gamma) &= \prod_{i=1}^{H} L_{Y_i}(\beta_i, \mu, \sigma^2, \Sigma, \gamma) \\ &= \prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal (\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma)) \\ &= \frac{1}{\sigma^N} \prod_{i=1}^{H} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal (\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma)) \end{aligned}$$

Posterior

By Baye's Rule

$$f(B, \mu, \sigma^2, \Sigma, \gamma|Y) \propto f(Y|B, \mu, \sigma^2, \Sigma, \gamma) \times f(B|\mu, \Sigma) \times f(\mu) \times f(\Sigma) \times f(\sigma^2) \times f(\gamma)$$

where

$$\mu = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}_{5\times 1}$$

$$
\begin{aligned}
f(B|\mu,\Sigma) =& \prod_{i=1}^{H} f(\beta_i|\mu,\Sigma) \\
=& \prod_{i=1}^{H} det(\Sigma)^{-1/2} exp(-\frac{(\beta_i-\mu)^\intercal\Sigma^{-1}(\beta_i-\mu)}{2}) \\
=& det(A)^{H/2}\prod_{i=1}^{H} exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2})
\end{aligned}
$$

where $A = \Sigma^{-1}$

$$
f(\mu) = det(V)^{\frac{-1}{2}} exp(-\frac{\mu^\intercal V^{-1}\mu}{2})
$$

We'll only use $f_{\Sigma^{-1}}$ because only Σ^{-1} shows up in the likelihood equation. We denote $A = \Sigma^{-1}$ in the posterior.

$$
\begin{aligned}
f_{\Sigma^{-1}}(\Sigma^{-1}) \propto& |\Sigma^{-1}|^{\frac{\nu-5-1}{2}} exp(-\frac{tr(S\Sigma^{-1})}{2}) \\
f_{\sigma^2}(\sigma^2) =& \frac{2*10}{\pi(\sigma^2+10^2)}\frac{1}{2\sigma} \propto \frac{1}{\pi(\sigma^2+10^2)\sigma} \\
f_\gamma(\gamma) =& exp(-\frac{400\gamma^\intercal\gamma}{2})
\end{aligned}
$$

Final posterior

$$
\begin{aligned}
f(B,\mu,\sigma^2,\Sigma,\gamma|Y) \propto& f(Y|B,\mu,\sigma^2,\Sigma,\gamma) \times f(B|\mu,\Sigma) \times f(\mu) \times f(\Sigma^{-1}) \times f(\sigma^2) \times f(\gamma) \\
=& f(Y|B,\mu,\sigma^2,\Sigma,\gamma) \times f(B|\mu,A) \times f(\mu) \times f(A) \times f(\sigma^2) \times f(\gamma) \\
=& \prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))\times \\
& det(A)^{H/2}\prod_{i=1}^{H} exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2})\times \\
& det(V)^{\frac{-1}{2}} exp(-\frac{\mu^\intercal V^{-1}\mu}{2})\times \\
& |A|^{\frac{\nu-5-1}{2}} exp(-\frac{tr(SA)}{2})\times \\
& \frac{1}{\pi(\sigma^2+10^2)\sigma}\times \\
& exp(-\frac{400\gamma^\intercal\gamma}{2})
\end{aligned}
$$

Conditional Posterior

For B,

$$
\begin{aligned}
f(B|\mu,\sigma^2,\Sigma,\gamma,Y) \propto & \prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal (\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))\times \\
& det(A)^{H/2}\prod_{i=1}^{H} exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2}) \\
\propto & \prod_{i=1}^{H} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal (\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma)) exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2}) \\
= & exp(\frac{-1}{2}\sum_{i=1}^{H}\beta_i^\intercal (D_i{}^\intercal \sigma^{-2} I_{n_i} D_i + A)\beta_i - 2\beta_i^\intercal (D_i^\intercal \sigma^{-2} I_{n_i} Y_i - D_i^\intercal \sigma^{-2} I_{n_i} X_i\gamma + A\mu) \\
& + Y_i^\intercal \sigma^{-2} I_{n_i} Y_i - 2Y_i^\intercal \sigma^{-2} I_{n_i} X_i\gamma + \gamma^\intercal X_i^\intercal \sigma^{-2} I_{n_i} X_i \gamma + \mu^\intercal A\mu)
\end{aligned}
$$

Let $M = D_i{}^\intercal \sigma^{-2} I_{n_i} D_i + A$ and $N = D_i^\intercal \sigma^{-2} I_{n_i} Y_i - D_i^\intercal \sigma^{-2} I_{n_i} X_i\gamma + A\mu$,

Finally, we have $f(B|\mu,\sigma^2,\Sigma,\gamma,Y) \sim MVN(M^{-1}N, M^{-1})$

For μ,

$$
\begin{aligned}
f(\mu|B,\sigma^2,\Sigma,\gamma,Y) \propto & \prod_{i=1}^{H} exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2}) exp(-\frac{\mu^\intercal V^{-1}\mu}{2}) \\
= & exp(\sum_{i=1}^{H} -\frac{1}{2}(\mu^\intercal (A - V^{-1})\mu - 2\mu^\intercal A\beta_i + \beta_i^\intercal A\beta_i)) \\
= & exp(-\frac{1}{2}(\mu^\intercal H(A - V^{-1})\mu - 2\mu^\intercal \sum_{i=1}^{H}(A\beta_i) + \beta_i^\intercal A\beta_i))
\end{aligned}
$$

Let $M = H(A - V^{-1})$ and $N = \sum_{i=1}^{H}(A\beta_i)$,

Finally, we have $f(\mu|B,\sigma^2,\Sigma,\gamma,Y) \sim MVN(M^{-1}N, M^{-1})$

For σ^2,

$$
\begin{aligned}
f(\sigma^2|B,\mu,\Sigma,\gamma,Y) \propto & \prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal (\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))\times \\
& \frac{1}{\pi(\sigma^2+10^2)\sigma} \\
\approx & (\sigma^2)^{-1/2\sum_{i=1}^{H} n_i} exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{H}(Y_i - D_i\beta_i - X_i\gamma)^\intercal (Y_i - D_i\beta_i - X_i\gamma)) \times \frac{1}{\sigma} \\
= & (\sigma^2)^{-\frac{N+1}{2}} exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{H}(Y_i - D_i\beta_i - X_i\gamma)^\intercal (Y_i - D_i\beta_i - X_i\gamma))
\end{aligned}
$$

Note: since σ^2 is much smaller than 10^2, we consider use $\frac{1}{\pi k'\sigma}$, where k' is a constant, to approximate $\frac{1}{\pi(\sigma^2+10^2)\sigma}$.

Let $shape = \frac{N-1}{2}$ and $rate = \frac{1}{2}\sum_{i=1}^{H}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(Y_i - D_i\beta_i - X_i\gamma)$,

Finally, we have $f(\sigma^2|B,\mu,\Sigma,\gamma,Y) \sim InverseGamma(shape, rate)$

For $A = \Sigma^{-1}$,

$$\begin{aligned}
f(\Sigma^{-1}|B,\mu,\sigma^2,\gamma,Y) \propto & det(A)^{H/2}\prod_{i=1}^{H} exp(-\frac{(\beta_i-\mu)^\intercal A(\beta_i-\mu)}{2})\times \\
& |A|^{\frac{\nu-5-1}{2}} exp(-\frac{tr(SA)}{2}) \\
= & det(A)^{\frac{H+\nu-5-1}{2}} exp(-\frac{1}{2}tr(SA + \sum_{i=1}^{H}(\beta_i-\mu)^\intercal A(\beta_i-\mu))) \\
= & det(A)^{\frac{H+\nu-5-1}{2}} exp(-\frac{1}{2}tr[(S + \sum_{i=1}^{H}(\beta_i-\mu)(\beta_i-\mu)^\intercal)A])
\end{aligned}$$

Note: $Tr((k)_{1\times1}) = k$.

Let degree of freedom $= H + \nu$ and scale matrix $= (S + \sum_{i=1}^{H}(\beta_i-\mu)(\beta_i-\mu)^\intercal)^{-1}$,

Finally, we have $f(\Sigma^{-1}|B,\mu,\sigma^2,\gamma,Y) \sim Wishart(df, \text{scale matrix})$

For γ,

$$\begin{aligned}
f(\gamma|B,\mu,\sigma^2,\Sigma,Y) \propto & \prod_{i=1}^{H} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))\times \\
& exp(-\frac{400\gamma^\intercal\gamma}{2}) \\
= & exp(-\frac{1}{2}\sum_{i=1}^{H}\gamma^\intercal(X_i^\intercal\sigma^{-2}I_{n_i}X_i - 400\frac{1}{H}I_3)\gamma - 2\gamma^\intercal(X_i\sigma^{-2}I_{n_i}Y_i - X_i^\intercal\sigma^{-2}I_{n_i}D_i\beta_i) + \\
& Y_i^\intercal\sigma^{-2}I_{n_i}Y_i - 2Y_i^\intercal\sigma^{-2}I_{n_i}D_i\beta_i + \beta_i^\intercal D_i^\intercal\sigma^{-2}I_{n_i}D_i\beta_i)
\end{aligned}$$

Let $M = \sum_{i=1}^{H} X_i^\intercal\sigma^{-2}I_{n_i}X_i - 400I_3$ and $N = \sum_{i=1}^{H}(X_i\sigma^{-2}I_{n_i}Y_i - X_i^\intercal\sigma^{-2}I_{n_i}D_i\beta_i)$,

Finally, we have $f(\gamma|B,\mu,\sigma^2,\Sigma,Y) \sim MVN(M^{-1}N, M^{-1})$