hurricane_posterior

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Model

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \mathbf{X}_i\gamma + \epsilon_i(t)$$

$$\begin{bmatrix} \beta_{0,i} \\ \beta_{1,i} \\ \beta_{2,i} \\ \beta_{3,i} \\ \beta_{4,i} \end{bmatrix} \sim MVN(\begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}, \Sigma)$$

$$\epsilon_i \sim N(0, \sigma^2)$$

Priors

1.

$$\begin{bmatrix} \mu_{0,i} \\ \mu_{1,i} \\ \mu_{2,i} \\ \mu_{3,i} \\ \mu_{4,i} \end{bmatrix} \sim MVN(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, V)$$

$$f_{\mu_i}(\mu_i) \propto det(V)^{\frac{-1}{2}} e^{\frac{-1}{2}\mu_i^\mathsf{T}V^{-1}\mu_i} \propto e^{\frac{-1}{2}\mu_i^\mathsf{T}V^{-1}\mu_i}$$

2.

$$\Sigma \sim W^{-1}(S, \nu = 5)$$

$$f_{\Sigma^{-1}}(\Sigma) \propto |\Sigma^{-1}|^{n+1} e^{\frac{-1}{2}tr(\Sigma^{-1})}$$

where n is the dimension of dataset... this formula might be wrong meh

determinant n dim of of sigma

Wishart

Due to property of Wishart distribution,

$$\Sigma^{-1} \sim W(S^{-1}, \nu = 5)$$

$$f_{\Sigma^{-1}}(\Sigma^{-1}) = |\Sigma^{-1}|^{\frac{\nu-d-1}{2}} exp(-\frac{tr(S\Sigma^{-1})}{2}) \propto |\Sigma^{-1}|^{\frac{\nu-5-1}{2}} exp(-\frac{tr(S\Sigma^{-1})}{2})$$

3.

$$\gamma \sim MVN(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, 0.005^2 I_3)$$

$$f_\gamma(\gamma) = (3 * 0.005^2)^{-1/2} exp(-\frac{\gamma^\intercal 0.005^2 I_3 \gamma}{2}) \propto exp(-\frac{400\gamma^\intercal \gamma}{2})$$

4.

$$\sigma \sim Half - Cauchy(0, 10)$$

$$f_\sigma(\sigma) = \frac{2 * 10}{\pi(\sigma^2 + 10^2)}$$

By transformation theorem

$$f_{\sigma^2}(\sigma^2) = \frac{2 * 10}{\pi(\sigma^2 + 10^2)} \frac{1}{2\sigma} \propto \frac{1}{\pi(\sigma^2 + 10^2)\sigma}$$

Likelihood

Because random effects coefficients β_i is normal, $Y_i|\beta_i$ also follows a normal distribution by property of normal distribution.

For each hurricane Y_i

$$\begin{aligned} Y_i|\beta_i, \mu_i, \sigma^2, \Sigma, \gamma \sim &MVN(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \mathbf{X}_i\gamma, \sigma^2 I_i) \\ = &MVN(D_i\beta_i + X_i\gamma, \sigma^2 I_{n_i}) \end{aligned}$$

where

$$Y_i = \begin{bmatrix} Y_i(t_0 + 6) \\ Y_i(t_1 + 6) \\ \vdots \\ Y_i(t = t_j + 6) \\ \vdots \\ Y_i(t = t_{n_{i-1}} + 6) \end{bmatrix}_{n_i \times 1}$$

$$D_i(t) = \begin{bmatrix} 1 & Y_i(t) & \Delta_{i,1}(t) & \Delta_{i,2}(t) & \Delta_{i,3}(t) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & Y_i(t = t_0) & \Delta_{i,1}(t = t_0) & \Delta_{i,2}(t = t_0) & \Delta_{i,3}(t = t_0) \\ 1 & Y_i(t = t_1) & \Delta_{i,1}(t = t_1) & \Delta_{i,2}(t = t_1) & \Delta_{i,3}(t = t_1) \\ \ldots & & & & \\ 1 & Y_i(t = t_j) & \Delta_{i,1}(t = t_j) & \Delta_{i,2}(t = t_j) & \Delta_{i,3}(t = t_j) \\ \ldots & & & & \\ 1 & Y_i(t = t_{n_{i-1}}) & \Delta_{n,1}(t = t_{n_{i-1}}) & \Delta_{n,2}(t = t_{n_{i-1}}) & \Delta_{n,3}(t = t_{n_{i-1}}) \end{bmatrix}_{n_i \times 5}$$

$$\beta_i = \begin{bmatrix} \beta_{0,i} \\ \beta_{1,i} \\ \beta_{2,i} \\ \beta_{3,i} \\ \beta_{4,i} \end{bmatrix}$$

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$$X_i = \begin{bmatrix} x_{i,1} & x_{i,2} & x_{i,3} \end{bmatrix}_{1 \times 3}$$

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}_{3 \times 1}$$

Likelihood for the ith hurricane is

$$f(Y_i|\beta_i, \mu_i, \sigma^2, \Sigma, \gamma) = det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^{\intercal}(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))$$

To calculate the joint likelihood for $Y = \begin{bmatrix} Y_1 & Y_2 \dots Y_i \dots Y_H \end{bmatrix}^{\intercal}$, we denote total number of observations for all hurricanes as $N = \sum_{i=0}^{H} n_i$ where n_i is the total number of observation for the ith hurricane and H is the total number of hurricanes.

All random effects coefficients β_i in

$$B = \begin{bmatrix} \beta_1 & \beta_2 \dots & \beta_i & \dots \beta_H \end{bmatrix}$$

$$= \begin{bmatrix} \beta_{0,1} & \beta_{0,2} & \dots & \beta_{0,i} & \dots & \beta_{0,H} \\ \beta_{1,1} & \beta_{1,2} & \dots & \beta_{1,i} & \dots & \beta_{1,H} \\ \beta_{2,1} & \beta_{2,2} & \dots & \beta_{2,i} & \dots & \beta_{2,H} \\ \beta_{3,1} & \beta_{3,2} & \dots & \beta_{3,i} & \dots & \beta_{3,H} \\ \beta_{4,1} & \beta_{4,2} & \dots & \beta_{4,i} & \dots & \beta_{4,H} \end{bmatrix}_{5 \times H}$$

Design matrix for random effects for all hurricanes are in D. $D = \begin{bmatrix} D_1(t) \\ D_2(t) \\ \vdots \\ D_i(t) \\ \vdots \\ D_H(t) \end{bmatrix}_{N \times 5}$

Due to independence of each hurricane, the joint likelihood is

$$L_Y(B, \mu, \sigma^2, \Sigma, \gamma) = \prod_{i=1}^{H} L_{Y_i}(\beta_i, \mu, \sigma^2, \Sigma, \gamma)$$

$$= \prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^{\intercal}(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))$$

$$= \frac{1}{\sigma^N} \prod_{i=1}^{H} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^{\intercal}(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))$$

Posterior

By Baye's Rule

$$f(B, \mu, \sigma^2, \Sigma, \gamma|Y) \propto f(Y|B, \mu, \sigma^2, \Sigma, \gamma) \times f(B|\mu, \Sigma) \times f(\mu) \times f(\Sigma) \times f(\sigma^2) \times f(\gamma)$$

where

$$\mu = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}_{5 \times 1}$$

$$f(B|\mu, \Sigma) = \prod_{i=1}^{H} f(\beta_i|\mu, \Sigma)$$

$$= \prod_{i=1}^{H} det(\Sigma)^{-1/2} exp(-\frac{(\beta_i - \mu)^{\mathsf{T}}\Sigma^{-1}(\beta_i - \mu)}{2})$$

$$= det(A)^{H/2} \prod_{i=1}^{H} exp(-\frac{(\beta_i - \mu)^{\mathsf{T}}A(\beta_i - \mu)}{2})$$

where $A = \Sigma^{-1}$

$$f(\mu) = det(V)^{\frac{-1}{2}} exp(-\frac{\mu^{\mathsf{T}}V^{-1}\mu}{2})$$

We'll only use $f_{\Sigma^{-1}}$ because only Σ^{-1} shows up in the likelihood equation. We denote $A = \Sigma^{-1}$ in the posterior.

$$f_{\Sigma^{-1}}(\Sigma^{-1}) \propto |\Sigma^{-1}|^{\frac{\nu-5-1}{2}} exp(-\frac{tr(S\Sigma^{-1})}{2})$$

$$f_{\sigma^2}(\sigma^2) = \frac{2*10}{\pi(\sigma^2 + 10^2)} \frac{1}{2\sigma} \propto \frac{1}{\pi(\sigma^2 + 10^2)\sigma}$$

$$f_\gamma(\gamma) = exp(-\frac{400\gamma^{\mathsf{T}}\gamma}{2})$$

Final posterior

$$f(B, \mu, \sigma^2, \Sigma, \gamma|Y) \propto f(Y|B, \mu, \sigma^2, \Sigma, \gamma) \times f(B|\mu, \Sigma) \times f(\mu) \times f(\Sigma^{-1}) \times f(\sigma^2) \times f(\gamma)$$

$$= f(Y|B, \mu, \sigma^2, \Sigma, \gamma) \times f(B|\mu, A) \times f(\mu) \times f(A) \times f(\sigma^2) \times f(\gamma)$$

$$= \prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^{\mathsf{T}}(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma)) \times$$

$$det(A)^{H/2} \prod_{i=1}^{H} exp(-\frac{(\beta_i - \mu)^{\mathsf{T}}A(\beta_i - \mu)}{2}) \times$$

$$det(V)^{\frac{-1}{2}} exp(-\frac{\mu^{\mathsf{T}}V^{-1}\mu}{2}) \times$$

$$|A|^{\frac{\nu-5-1}{2}} exp(-\frac{tr(SA)}{2}) \times$$

$$\frac{1}{\pi(\sigma^2 + 10^2)\sigma} \times$$

$$exp(-\frac{400\gamma^{\mathsf{T}}\gamma}{2})$$

Conditional Posterior

For B,

$$f(B|\mu,\sigma^2,\Sigma,\gamma,Y) \propto \prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma)) \times$$

$$det(A)^{H/2} \prod_{i=1}^{H} exp(-\frac{(\beta_i - \mu)^\intercal A(\beta_i - \mu)}{2})$$

$$\propto \prod_{i=1}^{H} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma))exp(-\frac{(\beta_i - \mu)^\intercal A(\beta_i - \mu)}{2})$$

$$= exp(\frac{-1}{2}\sum_{i=1}^{H}\beta_i^\intercal(D_i^\intercal\sigma^{-2}I_{n_i}D_i + A)\beta_i - 2\beta_i^\intercal(D_i^\intercal\sigma^{-2}I_{n_i}Y_i - D_i^\intercal\sigma^{-2}I_{n_i}X_i\gamma + A\mu)$$

$$+ Y_i^\intercal\sigma^{-2}I_{n_i}Y_i - 2Y_i^\intercal\sigma^{-2}I_{n_i}X_i\gamma + \gamma^\intercal X_i^\intercal\sigma^{-2}I_{n_i}X_i\gamma + \mu^\intercal A\mu)$$

Let $M = D_i^\intercal\sigma^{-2}I_{n_i}D_i + A$ and $N = D_i^\intercal\sigma^{-2}I_{n_i}Y_i - D_i^\intercal\sigma^{-2}I_{n_i}X_i\gamma + A\mu$,

Finally, we have $f(B|\mu,\sigma^2,\Sigma,\gamma,Y) \sim MVN(M^{-1}N, M^{-1})$

For μ,

$$f(\mu|B,\sigma^2,\Sigma,\gamma,Y) \propto \prod_{i=1}^{H} exp(-\frac{(\beta_i - \mu)^\intercal A(\beta_i - \mu)}{2})exp(-\frac{\mu^\intercal V^{-1}\mu}{2})$$

$$= exp(\sum_{i=1}^{H} -\frac{1}{2}(\mu^\intercal(A - \frac{1}{H}V^{-1})\mu - 2\mu^\intercal A\beta_i + \beta_i^\intercal A\beta_i))$$

$$= exp(-\frac{1}{2}(\mu^\intercal(HA - V^{-1})\mu - 2\mu^\intercal\sum_{i=1}^{H}(A\beta_i) + \beta_i^\intercal A\beta_i))$$

Let $M = HA - V^{-1}$ and $N = \sum_{i=1}^{H}(A\beta_i)$,

Finally, we have $f(\mu|B,\sigma^2,\Sigma,\gamma,Y) \sim MVN(M^{-1}N, M^{-1})$

For σ^2,

$$f(\sigma^2|B,\mu,\Sigma,\gamma,Y) \propto \prod_{i=1}^{H} det(\sigma^2 I_{n_i})^{-1/2} exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma)) \times$$

$$\frac{1}{\pi(\sigma^2 + 10^2)\sigma}$$

$$\approx (\sigma^2)^{-1/2\sum_{i=1}^{H}n_i} exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{H}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(Y_i - D_i\beta_i - X_i\gamma)) \times \frac{1}{\sigma}$$

$$= (\sigma^2)^{-\frac{N+1}{2}} exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{H}(Y_i - D_i\beta_i - X_i\gamma)^\intercal(Y_i - D_i\beta_i - X_i\gamma))$$

Note: since σ^2 is much smaller than 10^2, we consider use $\frac{1}{\pi k'\sigma}$, where k' is a constant, to approximate $\frac{1}{\pi(\sigma^2+10^2)\sigma}$.

Let $shape = \frac{N-1}{2}$ and $rate = \frac{1}{2}\sum_{i=1}^{H}(Y_i - D_i\beta_i - X_i\gamma)^{\intercal}(Y_i - D_i\beta_i - X_i\gamma)$,

Finally, we have $f(\sigma^2|B, \mu, \Sigma, \gamma, Y) \sim InverseGamma(shape, rate)$

For $A = \Sigma^{-1}$,

$$f(\Sigma^{-1}|B, \mu, \sigma^2, \gamma, Y) \propto det(A)^{H/2}\prod_{i=1}^{H}exp(-\frac{(\beta_i - \mu)^{\intercal}A(\beta_i - \mu)}{2}) \times$$

$$|A|^{\frac{\nu-5-1}{2}}exp(-\frac{tr(SA)}{2})$$

$$= det(A)^{\frac{H+\nu-5-1}{2}}exp(-\frac{1}{2}tr(SA + \sum_{i=1}^{H}(\beta_i - \mu)^{\intercal}A(\beta_i - \mu)))$$

$$= det(A)^{\frac{H+\nu-5-1}{2}}exp(-\frac{1}{2}tr[(S + \sum_{i=1}^{H}(\beta_i - \mu)(\beta_i - \mu)^{\intercal})A])$$

Note: $Tr((k)_{1\times 1}) = k$.

Let degree of freedom $= H + \nu$ and scale matrix $= (S + \sum_{i=1}^{H}(\beta_i - \mu)(\beta_i - \mu)^{\intercal})^{-1}$,

Finally, we have $f(\Sigma^{-1}|B, \mu, \sigma^2, \gamma, Y) \sim Wishart(df, \text{scale matrix})$

For γ,

$$f(\gamma|B, \mu, \sigma^2, \Sigma, Y) \propto \prod_{i=1}^{H}exp(-\frac{1}{2}(Y_i - D_i\beta_i - X_i\gamma)^{\intercal}(\sigma^2 I_{n_i})^{-1}(Y_i - D_i\beta_i - X_i\gamma)) \times$$

$$exp(-\frac{400\gamma^{\intercal}\gamma}{2})$$

$$= exp(-\frac{1}{2}\sum_{i=1}^{H}\gamma^{\intercal}(X_i^{\intercal}\sigma^{-2}I_{n_i}X_i + 400\frac{1}{H}I_3)\gamma - 2\gamma^{\intercal}(X_i\sigma^{-2}I_{n_i}Y_i - X_i^{\intercal}\sigma^{-2}I_{n_i}D_i\beta_i) +$$

$$Y_i^{\intercal}\sigma^{-2}I_{n_i}Y_i - 2Y_i^{\intercal}\sigma^{-2}I_{n_i}D_i\beta_i + \beta_i^{\intercal}D_i^{\intercal}\sigma^{-2}I_{n_i}D_i\beta_i)$$

Let $M = \sum_{i=1}^{H}X_i^{\intercal}\sigma^{-2}I_{n_i}X_i + 400I_3$ and $N = \sum_{i=1}^{H}(X_i\sigma^{-2}I_{n_i}Y_i - X_i^{\intercal}\sigma^{-2}I_{n_i}D_i\beta_i)$,

Finally, we have $f(\gamma|B, \mu, \sigma^2, \Sigma, Y) \sim MVN(M^{-1}N, M^{-1})$