Ecole Centrale de Nantes

MASTER CORO-IMARO

Humanoid Robotics

Dynamic modeling of a compass

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1 Introduction

We consider a compass robot made of 2 links walking along a slope. The legs of our robot are outlined in Fig.1. For each leg we have the following parameters:

- l, length of the leg segments
- s distance between G and the hip
- m mass
- I inertia parameter

The black frame shown in Fig.1 indicates the reference frame R_0, which it is attached to the ground.

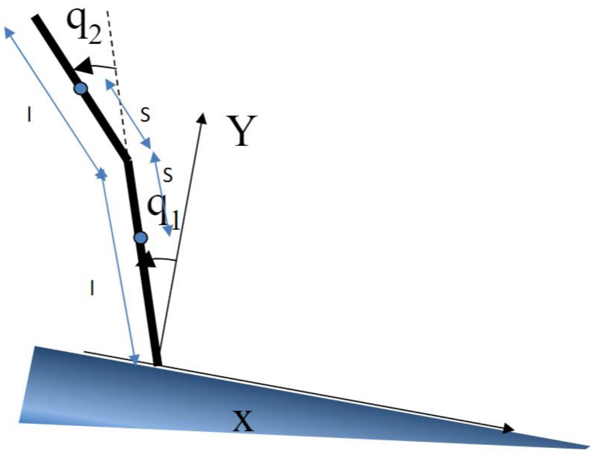

Figure 1: Compass robot

The objective is to define three Matlab functions, that will be used in the next lab:

- the dynamic model of the biped in single support
- the Reaction force
- the impact model

2 Dynamic model of the biped in single support

We define the Matlab function $[A, H] = function\_dyn(q_1, q_2, \dot{q}_1, \dot{q}_2, \theta)$. The matrices defined allow to write the dynamic model under the form:

$$A \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_1 \end{bmatrix} + H = B\Gamma \tag{1}$$

The matrix B will depend on the actuation and is not defined in this function.
Our model will be defined by the Lagrange approach.

2.1 Position mass centres G_1 and G_2

The position of G1 is given by the vector $\overrightarrow{OG_1}$

$$\overrightarrow{OG_1} = (l - s)\mathbf{u}_{\text{G1}} \tag{2}$$

where:

$$\mathbf{u}_{\text{G1}} = \begin{pmatrix} \cos(q_1 + \frac{\pi}{2}) \\ \sin(q_1 + \frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} -\sin q_1 \\ \cos q_1 \end{pmatrix} \tag{3}$$

The position of G2 is given by the vector $\overrightarrow{OG_1}$

$$\overrightarrow{OG_2} = l\mathbf{u}_{\text{G1}} + s\mathbf{u}_{\text{G2}} \tag{4}$$

where $\mathbf{u_{G1}}$ is given by (3) and

$$\mathbf{u}_{\text{G2}} = \begin{pmatrix} -\sin(q_1 + q_2) \\ \cos(q_1 + q_2) \end{pmatrix} \tag{5}$$

2.2 Velocities of the mass centres G_1 and G_2

The velocity of G1 is given by:

$$\mathbf{v_{G1}} = (l - s)\dot{\mathbf{u}}_{\mathbf{G1}} = (l - s)\dot{q}_1\mathbf{E}\mathbf{u_{G1}} \tag{6}$$

where $\mathbf{E}$ is the rotation matrix given by:

$$\mathbf{E} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

The velocity of G2 is given by:

$$\mathbf{v_{G2}} = l\dot{\mathbf{u}}_{\mathbf{G1}} + s\dot{\mathbf{u}}_{\mathbf{G2}} = l\dot{q}_1\mathbf{E}\mathbf{u_{G1}} + s(\dot{q}_1 + \dot{q}_2)\mathbf{E}\mathbf{u_{G2}} \tag{8}$$

2.3 Angular velocities

The angular velocity of each leg are defined and don't need to be computed.

2.4 Kinetic energy of the system

The kinetic energy of each link is given by

$$E_{Ci} = \frac{1}{2}(m\mathbf{v_{Gi}}^T\mathbf{v_{Gi}} + \mathbf{!}_{\text{Gi}}{}^T I\mathbf{!}_{\text{Gi}}) \tag{9}$$

The kinetic energy of the first leg is then given by:

$$E_{C1} = \frac{1}{2}(m(l - s)^2 + I)\dot{q}_1{}^2 \tag{10}$$

The kinetic energy of the second leg is then given by:

$$E_{C2} == \frac{1}{2}(m(l^2\dot{q}_1{}^2 + s^2(\dot{q}_1 + \dot{q}_2)^2 + 2ls\dot{q}_1(\dot{q}_1 + \dot{q}_2)cosq_2) + I(\dot{q}_1 + \dot{q}_2)^2) \tag{11}$$

The total kinetic energy is given by $E_C = E_{C1} + E_{C2}$

$$E_C = \frac{1}{2}((m(l-s)^2+I)\dot{q}_1{}^2+m(l^2\dot{q}_1{}^2+s^2(\dot{q}_1+\dot{q}_2)^2+2ls\dot{q}_1(\dot{q}_1+\dot{q}_2)cosq_2)+I(\dot{q}_1+\dot{q}_2)^2) \tag{12}$$

2.5 Inertia matrix A

The inertia matrix **A** is deduces by the expression

$$E_C = E_{C1} + E_{C2} = \frac{1}{2}\begin{bmatrix}\dot{q_1} & \dot{q_2}\end{bmatrix}\mathbf{A}\begin{bmatrix}\dot{q_1}\\ \dot{q_2}\end{bmatrix} \tag{13}$$

We can rewrite linear velocities as follows:

$$\mathbf{v_{G1}} = (l-s)\dot{q_1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-sin(q_1)\\ cos(q_1)\end{pmatrix} = \begin{pmatrix}(s-l)cos(q_1) & 0\\ (s-l)sin(q_1) & 0\end{pmatrix}\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix}$$

obtaining:

$$\mathbf{v_{G1}} = B_1\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} \tag{14}$$

where

$$B_1 = \begin{pmatrix}(s-l)cos(q_1) & 0\\ (s-l)sin(q_1) & 0\end{pmatrix} \tag{15}$$

We repeat the same for $\mathbf{v_{G2}}$

$$\mathbf{v_{G2}} = l\begin{pmatrix}-cos(q_1) & 0\\ -sin(q_1) & 0\end{pmatrix}\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} + s\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-sin(q_1+q_2)\\ cos(q_1+q_2)\end{pmatrix}(\dot{q_1}+\dot{q_2}) =$$

$$= l\begin{pmatrix}-cos(q_1) & 0\\ -sin(q_1) & 0\end{pmatrix}\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} + s\begin{pmatrix}-cos(q_1+q_2) & -cos(q_1+q_2)\\ -sin(q_1+q_2) & -sin(q_1+q_2)\end{pmatrix}\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} =$$

$$= \begin{pmatrix}-lcos(q_1) - scos(q_1+q_2) & -scos(q_1+q_2)\\ -lcos(q_1) - scos(q_1+q_2) & -ssin(q_1+q_2)\end{pmatrix}\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix}$$

obtaining:

$$\mathbf{v_{G2}} = B_2\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} \tag{16}$$

where

$$B_2 = \begin{pmatrix}-lcos(q_1) - scos(q_1+q_2) & -scos(q_1+q_2)\\ -lcos(q_1) - scos(q_1+q_2) & -ssin(q_1+q_2)\end{pmatrix} \tag{17}$$

We can rewrite angular velocities as follows:

$$\mathbf{w_{G1}} = \dot{q_1}\mathbf{z_0} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} = C_1\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} \tag{18}$$

$$\mathbf{w_{G2}} = (\dot{q_1}+\dot{q_2})\mathbf{z_0} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} = C_2\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} \tag{19}$$

Rewriting the expression of the kinetic energies:

$$E_{C1} = \frac{1}{2}(m\begin{pmatrix}\dot{q_1} & \dot{q_2}\end{pmatrix})B_1^T B_1\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} + \begin{pmatrix}\dot{q_1} & \dot{q_2}\end{pmatrix} IC_1^T C_1\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} \tag{20}$$

$$E_{C2} = \frac{1}{2}(m\begin{pmatrix}\dot{q_1} & \dot{q_2}\end{pmatrix})B_2^T B_2\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} + \begin{pmatrix}\dot{q_1} & \dot{q_2}\end{pmatrix} IC_2^T C_2\begin{pmatrix}\dot{q_1}\\ \dot{q_2}\end{pmatrix} \tag{21}$$

We finally obtain the expression of A as:

$$\mathbf{A} = mB_1^T B_1 + IC_1^T C_1 + mB_2^T B_2 + IC_2^T C_2 \tag{22}$$

The matrix **A** can be expressed as:

$$\mathbf{A} = \begin{pmatrix} m(s-l)^2 + 2I + ml^2 + 2mlscos(q_2) + ms^2 + I & ms(lcos(q_2) + s) + I \\ ms(lcos(q_2) + s) + I & ms^2 + 1 \end{pmatrix} \tag{23}$$

2.6 Potential energies

The potential energy of the link i is given by

$$U_i = -m({}^0\mathbf{g}^T)^0\overrightarrow{OG_1} \tag{24}$$

The vector g in the reference frame depends on the angle θ. We know that g expressed in the absolute reference frame F_abs is shown in Fig.2 and it is given by

$${}^{\text{abs}}\mathbf{g} = -g\mathbf{y} \tag{25}$$

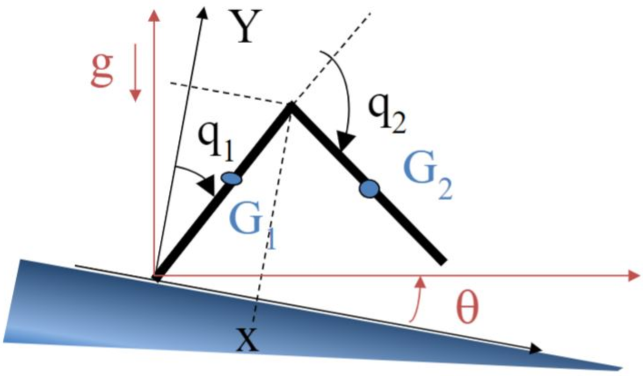

Figure 2: Compass robot

In order to express it in frame F_0 we take the rotation matrix

$${}^0R_{\text{abs}} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{26}$$

We then obtain

$${}^0\mathbf{g} = -g(-sin\theta\mathbf{x} + cos\theta\mathbf{y}) \tag{27}$$

The potential energy of link 1 is then given by

$$U_1 = -mg\begin{pmatrix} sin\theta & -cos\theta \end{pmatrix}\begin{pmatrix} (s-l)sinq_1 \\ (l-s)cosq_1 \end{pmatrix} = mg(l-s)(-sin\theta sinq_1 - cos\theta cosq_1)$$
$$= mg(l-s)cos(q_1 - \theta)$$

We then obtain:

$$U_1 = mg(l-s)cos(q_1 - \theta) \tag{28}$$

The potential energy of link 2 is then given by

$U_2 = -mg\begin{pmatrix} sin\theta & -cos\theta \end{pmatrix}\begin{pmatrix} -lsinq_1 - ssin(q_1+q_2) \\ lcosq_1 + scos(q_1+q_2) \end{pmatrix} = mg(lcos(q_1-\theta)+scos(\theta-q_1-q_2) = mg(l-s)cos(q_1-\theta)$

We then obtain:

$$U_2 = mg(lcos(q_1-\theta)+scos(\theta-q_1-q_2)) \tag{29}$$

2.7 Gravity effects Q_i

We now deduce the gravity effects which are given by

$$Q_1 = \frac{\partial U}{\partial q_1} = -mg((2l-s)sin(q_1-\theta)+ssin(\theta-q_1-q_2)) \tag{30}$$

$$Q_2 = \frac{\partial U}{\partial q_2} = (mgs)sin(\theta-q_1-q_2) \tag{31}$$

2.8 Vector H

We compute the vector H as

$$\mathbf{H} = B\dot{\mathbf{q}}\dot{\mathbf{q}} + C\dot{\mathbf{q}}^{\mathbf{2}} \tag{32}$$

where $\dot{\mathbf{q}}\dot{\mathbf{q}} = [\dot{q}_1\dot{q}_2\ldots\dot{q}_1\dot{q}_n\dot{q}_2\dot{q}_3q_{\text{n-1}}\dot{q}_n]^T = [\dot{q}_1\dot{q}_2]$ and $\dot{\mathbf{q}}^{\mathbf{2}} = [\dot{q}_1{}^2\ldots\dot{q}_n]^T = [\dot{q}_1{}^2\dot{q}_2{}^2]^T$

Fist of all we compute B_{ij}

$$B_{\text{ijk}} = \frac{\partial A_{\text{ij}}}{\partial q_k} + \frac{\partial A_{\text{ik}}}{\partial q_j} - \frac{\partial A_{\text{ik}}}{\partial q_i} \tag{33}$$

where i=1,...,n, j=1,...,n-1, k=j+1,...,n

The coefficients of matrix B are then given by:

$$B_{112} = \frac{\partial A_{11}}{\partial q_2} + \frac{\partial A_{12}}{\partial q_1} - \frac{\partial A_{12}}{\partial q_1} = -2(mls)sinq_2$$
$$B_{212} = \frac{\partial A_{21}}{\partial q_2} + \frac{\partial A_{22}}{\partial q_1} - \frac{\partial A_{12}}{\partial q_2} = -(mls)sinq_2 + (mls)sinq_2 = 0$$

We then obtain

$$B = \begin{pmatrix} -(mls)sinq_2 & 0 \end{pmatrix} \tag{34}$$

Secondly, we compute

$$C_{\text{ij}} = \frac{\partial A_{\text{ii}}}{\partial q_i} - \frac{1}{2}\frac{\partial A_{\text{ij}}}{\partial q_i} \tag{35}$$

where i=1,...,n, j=1,...,n

The coefficients of matrix C are then given by:

$$C_{11} = \frac{\partial A_{11}}{\partial q_1} - \frac{1}{2}\frac{\partial A_{11}}{\partial q_1} = 0$$
$$C_{21} = \frac{\partial A_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial A_{11}}{\partial q_2} = (mls)sinq_2$$
$$C_{12} = \frac{\partial A_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial A_{22}}{\partial q_1} = -(mls)sinq_2$$
$$C_{22} = \frac{\partial A_{22}}{\partial q_2} - \frac{1}{2}\frac{\partial A_{22}}{\partial q_2} = 0$$

We then obtain

$$C = \begin{pmatrix} 0 & -(mls)sinq_2 \\ (mls)sinq_2 & 0 \end{pmatrix} \tag{36}$$

The vector $\mathbf{H}$ is then given by:

$$\mathbf{H} = B\dot{\mathbf{q}}\dot{\mathbf{q}} + C\dot{\mathbf{q}}^{\mathbf{2}} = \begin{pmatrix} -(mls)sinq_2\dot{q}_1\dot{q}_2 - (mls)sinq_2\dot{q_2}^2 \\ (mls)sinq_2\dot{q_1}^2 \end{pmatrix} \tag{37}$$

3 Reaction force

We define the Matlab function $[F] = function\_dyn(q_1, q_2, \dot{q}_1, \dot{q}_2, \ddot{q}_1, \ddot{q}_2)$.

3.1 Mass center position

We compute the position of the mass center x_G of the robot as function of q_1 and q_2:

$$\overrightarrow{OG} = \frac{m\overrightarrow{OG}_1 + m\overrightarrow{OG}_2}{2m} \tag{38}$$

The expression of $\overrightarrow{OG}_1$ and $\overrightarrow{OG}_2$ are given by respectively 2 and 4.

3.2 Mass center velocity and acceleration

We derive the expression of $\overrightarrow{OG}_1$ and $\overrightarrow{OG}_2$:

$$\mathbf{v_{G_1}} = (l-s)\dot{q}_1\mathbf{Eu_{G1}} \tag{39}$$

$$\mathbf{v_{G_2}} = l\dot{q}_1\mathbf{Eu_{G1}} + s(\dot{q}_1 + \dot{q}_2)\mathbf{Eu_{G2}} \tag{40}$$

$$\mathbf{v_G} = \frac{\mathbf{v_{G_1}} + \mathbf{v_{G_2}}}{2} \tag{41}$$

Finally we derive these expressions to obtain the acceleration of the center of mass:

$$\dot{\mathbf{v}}_{\mathbf{G_1}} = -(l-s)\dot{q}_1^2\mathbf{u_{G1}} + (l-s)\ddot{q}_1\mathbf{Eu_{G1}} \tag{42}$$

$$\dot{\mathbf{v}}_{\mathbf{G_2}} = -l\dot{q}_1^2\mathbf{u_{G1}} - s*(\dot{q}_1 + \dot{q}_2)^2\mathbf{u_{G2}} + l\ddot{q}_1\mathbf{Eu_{G1}} + s(\ddot{q}_1 + \ddot{q}_2)\mathbf{Eu_{G2}} \tag{43}$$

$$\dot{\mathbf{v}}_{\mathbf{G}} = \frac{\dot{\mathbf{v}}_{\mathbf{G_1}} + \dot{\mathbf{v}}_{\mathbf{G_2}}}{2} \tag{44}$$

3.3 Reaction force value

Finally we compute :

$$\mathbf{F} = M\dot{\mathbf{v}}_{\mathbf{G}} - M\mathbf{g}; \tag{45}$$

4 Impact model

We define the Matlab function $[A_1, J_{\mathrm{R2}}] = function_impact(q_1, q_2)$. The matrices A_1 and J_{R2} can be used to define the state of the robot after impact and the impulsive forces, according to the following equation:

$$\begin{bmatrix} A_1 - {J_{\mathrm{R2}}}^T \\ {J_{\mathrm{R2}}}^T 0_{\mathrm{2x2}} \end{bmatrix} \begin{bmatrix} \dot{x}_+ \\ \dot{y}_+ \\ \dot{q}_{1+} \\ \dot{q}_{2+} \\ I_{\mathrm{R_{2x}}} \\ I_{\mathrm{R_{2y}}} \end{bmatrix} = \begin{bmatrix} A_1 \\ 0_{\mathrm{2x4}} \end{bmatrix} \begin{bmatrix} \dot{x}_{\text{-}} \\ \dot{y}_{\text{-}} \\ \dot{q}_{1\text{-}} \\ \dot{q}_{2\text{-}} \end{bmatrix} \tag{46}$$

In order to define matrix A_1 we use the same procedure we used for matrix A

4.1 Position of mass centres G_1 and G_2

This time we have to take into account the position of the hip which is given by

$$\dot{\mathbf{X}_\mathbf{G}} = \begin{bmatrix} \dot{x_G} \\ \dot{y_G} \end{bmatrix} \tag{47}$$

The center mass G_1 is given by

$$\overrightarrow{OG_1} = \overrightarrow{OG} - s\mathbf{u}_{\mathrm{G1}} = \begin{bmatrix} x \\ y \end{bmatrix} - s \begin{bmatrix} -sinq_1 \\ cosq_1 \end{bmatrix} \tag{48}$$

The center mass G_2 is given by

$$\overrightarrow{OG_2} = \overrightarrow{OG} + s\mathbf{u}_{\mathrm{G2}} = \begin{bmatrix} x \\ y \end{bmatrix} + s \begin{bmatrix} -sin(q_1 + q_2) \\ cos(q_1 + q_2) \end{bmatrix} \tag{49}$$

4.2 Velocities of the mass centres G_1 and G_2

The velocity of G1 is given by:

$$\mathbf{v_{G1}} = \dot{\mathbf{X}_\mathbf{G}} - s\dot{q_1}\mathbf{E}\mathbf{u_{G1}} \tag{50}$$

where $\mathbf{E}$ is the rotation matrix given by:

$$\mathbf{E} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{51}$$

The velocity of G2 is given by:

$$\mathbf{v_{G2}} = \dot{\mathbf{X}_\mathbf{G}} + s(\dot{q_1} + \dot{q_2})\mathbf{E}\mathbf{u_{G2}} \tag{52}$$

4.3 Kinetic energy of the system

The kinetic energy of each leg is given by formula (9)
The kinetic energy of the first leg is then given by:

$$E_{C1} = \frac{1}{2}(m((\dot{x} + s\dot{q_1}cosq_1)^2 + (\dot{y} + s\dot{q_1}sinq_1)) + I\dot{q_1}^2) \tag{53}$$

The kinetic energy of the second leg is then given by:

$$E_{C2} = \frac{1}{2}(m(\dot{x} - s(\dot{q_1} + \dot{q_2})cos(q_1 + q_2))^2 + (\dot{y} - s(\dot{q_1} + \dot{q_2})^2 sin(q_1 + q_2))^2 + I(\dot{q_1} + \dot{q_2})^2) \tag{54}$$

The total kinetic energy can then be rewritten as $E_C = E_{\mathrm{C1}} + E_{\mathrm{C2}}$

$$\begin{aligned} E_C = & \frac{1}{2}(m((\dot{x} + s\dot{q_1}cosq_1)^2 + (\dot{y} + s\dot{q_1}sinq_1)) + I\dot{q_1}^2) \\ & + \frac{1}{2}(m(\dot{x} - s(\dot{q_1} + \dot{q_2})cos(q_1 + q_2))^2 + (\dot{y} - s(\dot{q_1} + \dot{q_2})^2 sin(q_1 + q_2))^2 + I(\dot{q_1} + \dot{q_2})^2) \end{aligned} \tag{55}$$

4.4 Inertia matrix A_1

The inertia matrix $\mathbf{A_1}$ is deduces by the expression

$$E_C = E_{C1} + E_{C2} = \frac{1}{2}\begin{bmatrix}\dot{x} & \dot{y} & \dot{q_1} & \dot{q_2}\end{bmatrix} \mathbf{A_1} \begin{bmatrix}\dot{x} \\ \dot{y} \\ \dot{q_1} \\ \dot{q_2}\end{bmatrix} \tag{56}$$

Matrix A_1 is a matrix $(4x4)$ given by the following coefficients:

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \tag{57}$$

Developing the expression (48) we obtain:

$$\begin{aligned} E_C = \frac{1}{2}(& a_{11}\dot{x}^2 + a_{12}\dot{x}\dot{y} + a_{13}\dot{x}\dot{q_1} + a_{14}\dot{x}\dot{q_2} + a_{21}\dot{y}\dot{x} + a_{22}\dot{y}^2 + a_{23}\dot{y}\dot{q_1} + a_{24}\dot{y}\dot{q_2} + a_{31}\dot{q_1}\dot{x} \\ & + a_{32}\dot{q_1}\dot{y} + a_{33}\dot{q_1}^2 + a_{34}\dot{q_1}\dot{q_2} + a_{41}\dot{q_2}\dot{x} + a_{42}\dot{q_2}\dot{y} + a_{43}\dot{q_2}\dot{q_1} + a_{44}\dot{q_2}^2) \end{aligned} \tag{58}$$

Confronting equations (47) and (50) we obtain:

- $a_{11} = 2m$
- $a_{12} = 0$
- $a_{13} = mscosq_1 - mscos(q_1 + q_2)$
- $a_{14} = -mscos(q_1 + q_2)$
- $a_{21} = 0$

- $a_{22} = 2m$
- $a_{23} = msinq_1 - mssin(q_1 + q_2)$
- $a_{24} = -mssin(q_1 + q_2)$
- $a_{31} = mscosq_1 - mscos(q_1 + q_2)$
- $a_{32} = msinq_1 - mssin(q_1 + q_2)$
- $a_{33} = 2ms^2 + 2I$
- $a_{34} = I + 2ms^2$
- $a_{41} = -mscos(q_1 + q_2)$
- $a_{42} = -mssin(q_1 + q_2)$
- $a_{43} = I + 2ms^2$
- $a_{44} = ms^2 + I$

4.5 Matrix $\mathbf{J_{R2}}$

We now consider an impact such that the leg 1 is in support, while the leg 2 arrives; when the leg 1 takes off, leg 2 stay on the ground without sliding. We write the contact conditions after impact knowing that the the extremity of the leg 2 is fixed and we express it as function of x,y,q1,q2.
The coordinates of the extremity of leg 2 are given by

$$F_2 = \mathbf{X} + l\mathbf{u}_{\mathrm{G2}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + l \begin{pmatrix} -sin(q_1 + q_2) \\ cos(q_1 + q_2) \end{pmatrix} \tag{59}$$

The velocity of the extremity of leg 2 is given by :

$$\mathbf{v}_{\mathrm{F2}} = \dot{\mathbf{X}} + l\dot{\mathbf{u}_{\mathrm{G2}}} = \dot{\mathbf{X}} + l(\dot{q_1} + \dot{q_2})\mathbf{E}\mathbf{u}_{\mathrm{G2}} \tag{60}$$

We follow the same procedure used to compute $\mathbf{A_1}$ knowing that:

$$\mathbf{v}_{\mathrm{F2}} = \mathbf{J}_{\mathrm{R2}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{q_1} \\ \dot{q_2} \end{pmatrix} \tag{61}$$

and we obtain:

$$\mathbf{J}_{\mathrm{R2}} = \begin{pmatrix} 1 & 0 & -lcos(q_1 + q_2) & -lcos(q_1 + q_2) \\ 0 & 1 & -lsin(q_1 + q_2) & -lcos(q_1 + q_2) \end{pmatrix} \tag{62}$$