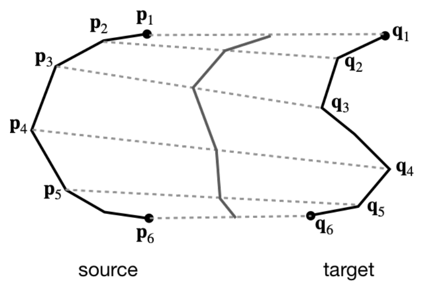

Figure 1: Tweening

1 Keyframing and Interpolation

1. Consider the in-betweening scenario shown in Fig. 1 where the source shape at $t = 0$ is morphed to the target shape at $t = 1$ via linear interpolation. On the figure, sketch the shape corresponding to $t = 0.5$.

   Solution. The shape corresponding to $t = 0.5$ is sketched on Fig. 1. It is obtained by joining the mid-points of the line segments, i.e. the points $\frac{1}{2}(\mathbf{p}_i + \mathbf{q}_i)$. ■

2. A Hermite spline segment corresponding to the control points $\mathbf{p}_0$, $\mathbf{p}_1$ and tangents $\mathbf{t}_0$ and $\mathbf{t}_1$ is defined by the parametric equation:

   $$\mathbf{C}(t) = \mathbf{p}_0(2t^3 - 3t^2 + 1) + \mathbf{p}_1(-2t^3 + 3t^2) + \mathbf{t}_0(t^3 - 2t^2 + t) + \mathbf{t}_1(t^3 - t^2) \quad (0 \le t \le 1)$$

   First verify that the segment interpolates $\mathbf{p}_0$ at $t = 0$ and $\mathbf{p}_1$ at $t = 1$. Next, verify that the tangents of the segment are $\mathbf{t}_0$ at $t = 0$ and $\mathbf{t}_1$ at $t = 1$.

   Solution.

   To verify we simply plug in $t = 0$ and $t = 1$ into $\mathbf{C}(t)$ and $\mathbf{C}'(t)$.

   For $t = 0$, we have:

   $$\mathbf{C}(0) = p_0(0 - 0 + 1) + p_1(-0 + 0) + t_0(0 - 0 + 0) + t_1(0 - 0).$$

   Making $\mathbf{C}(0) = p_0$.

   For $t = 1$, we have:

   $$\mathbf{C}(1) = p_0(2 - 3 + 1) + p_1(-2 + 3) + t_0(0 - 0 + 0) + t_1(0 - 0).$$

   Making $\mathbf{C}(1) = p_1$.

   To verify the tangents, first differentiate with respect to t to obtain the parametric equation for the tangent.

   $$\mathbf{C}'(t) = \mathbf{p}_0(6t^2 - 6t) + \mathbf{p}_1(-6t^2 + 6t) + \mathbf{t}_0(3t^2 - 4t + 1) + \mathbf{t}_1(3t^2 - 2t)$$

   At $t = 0$, the tangent is

   $$\mathbf{C}'(0) = \mathbf{p}_0(0 - 0) + \mathbf{p}_1(0 + 0) + \mathbf{t}_0(0 - 0 + 1) + \mathbf{t}_1(0 - 0)$$

   Making $\mathbf{C}'(0) = t_0$.

   At $t = 1$, the tangent is

   $$\mathbf{C}'(1) = \mathbf{p}_0(6 - 6) + \mathbf{p}_1(-6 + 6) + \mathbf{t}_0(3 - 4 + 1) + \mathbf{t}_1(3 - 2)$$

   Making $\mathbf{C}'(1) = t_1$. ■

3. A cubic Bézier segment corresponding to the control points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ is defined by the parametric equation:

   $$\mathbf{C}(t) = \mathbf{p}_0(1 - t)^3 + \mathbf{p}_1 3t(1 - t)^2 + \mathbf{p}_2 3t^2(1 - t) + \mathbf{p}_3 t^3 \quad (0 \le t \le 1)$$

   Show that:

   (a) The segment interpolates $\mathbf{p}_0$ at $t = 0$ and $\mathbf{p}_3$ at $t = 1$.
   (b) The tangents at $\mathbf{p}_0$ and $\mathbf{p}_3$ are $3(\mathbf{p}_1 - \mathbf{p}_0)$ and $3(\mathbf{p}_3 - \mathbf{p}_2)$ respectively.

   Solution. (a) As in question 2, this is verified by simply plugging in $t = 0$ and $t = 1$ into the equation. For $t = 0$, we have $\mathbf{C}(0) = \mathbf{p}_0$ and for $t = 1$, $\mathbf{C}(1) = \mathbf{p}_3$.

   (b) First, we differentiate with respect to t to obtain a parametric equation for the tangent.

   $$\mathbf{C}'(t) = -3\mathbf{p}_0(1 - t)^2 + 3\mathbf{p}_1((1 - t)^2 - 2t(1 - t)) + 3\mathbf{p}_2(2t(1 - t) - t^2) + 3\mathbf{p}_3 t^2$$

   At $t = 0$, the curve interpolates $\mathbf{p}_0$. So that tangent at $\mathbf{p}_0$ is:

   $$\mathbf{C}'(0) = -3\mathbf{p}_0 + 3\mathbf{p}_1 = 3(\mathbf{p}_1 - \mathbf{p}_0).$$

   At $t = 1$, the curve interpolates $\mathbf{p}_3$. So the tangent at $\mathbf{p}_3$ is:

   $$\mathbf{C}'(1) = -3\mathbf{p}_2 + 3\mathbf{p}_3 = 3(\mathbf{p}_3 - \mathbf{p}_2).$$

   ■

4. Consider an animation of cars racing on a narrow track. The midline of the track is described parametrically as:

   $$\mathbf{C}(u) = \big(x(u), y(u), z(u)\big)$$

(a) What mathematical condition should the three functions $x(u)$, $y(u)$ and $z(u)$ satisfy for the parametrization to be an arc-length parametrization?

(b) From an animation perspective, why is it convenient for this parametrization to be an arc-length parametrization?

Solution. (a) For this parametrization to be an arc-length parametrization, the speed has to be 1 everywhere, i.e. $\|\mathbf{C}'(u)\| = \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2} = 1$.

(b) Arc-length parametrization provides control over the speed at which the curve is traversed. An animator can design a speed-profile $s(t)$ that specifies the distance s (arc-length) along the curve in terms of animation time t.

■

5. Consider arc-length parametrization of the polyline in Fig. 2. Assume that point A has arc-length parametrization $s = 0$.

(a) What are the (x, y) coordinates of point P with the arc-length parameter value $s = 13$? Plot the point on the figure.

(b) What is the arc-length of the entire curve?

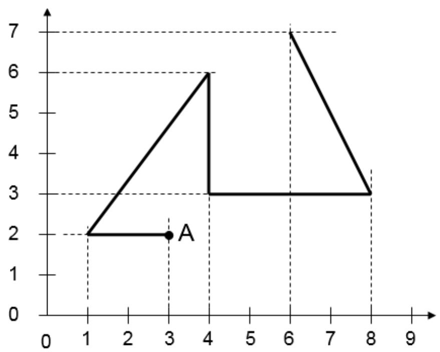

Figure 2: A polygonal line

Solution. (a) Calculating the length of each line segment we obtain $\{2, 5, 3, 4, 2\sqrt{5}\}$. Adding the arc-lengths starting at A, we see that $(4, 3)$ has an arc-length parameter value of 10 and $(8, 3)$ has an arc-length parameter value of 14. Thus, the point P with arc-length parameter value $s = 13$ has coordinates $(7, 3)$.

(b) Summing the lengths of each line segment gives the total arc-length of the curve. We calculated the arc-length for the first four segments in part (a). Thus the arc-length for the entire curve is:

$$s = 14 + 2\sqrt{5} \approx 18.47.$$

■

6. A circle is given by the parametric equations $x(t) = r\cos\omega t$, $y(t) = r\sin\omega t$. What is the relationship between ω and r for this to be an arc-length parametrization?

Solution. For the parametrization to be an arc-length parametrization, the speed has to be 1 everywhere. In other words,

$$\begin{aligned}\sqrt{x'(t)^2 + y'(t)^2} &= 1\\ \sqrt{(-r\omega\sin\omega t)^2 + (r\omega\cos\omega t)^2} &= 1\\ \sqrt{r^2\omega^2} &= 1\\ r\omega &= 1.\end{aligned}$$

Thus $\omega = \frac{1}{r}$ for an arc-length parametrization. ■

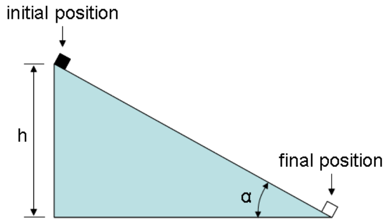

Figure 3: A cube sliding down an inclined plane

2 Single Particle Kinematics

1. In an animation, a small cube slides down an incline of height h with an angle at the base of α, as shown in Fig. 3. At what time t_0 should this cube be released, so that it reaches the final position at the bottom of the incline at a given time t_1? Assume that the acceleration due to gravity g is known, and there is no friction.

   Solution. The acceleration *down* the inclined plane is $g \sin\alpha$. The length of the plane is $\frac{h}{\sin\alpha}$. Assuming that the cube is initially at rest, its position at time t along the plane is given by
   $$x(t) = \tfrac{1}{2} g t^2 \sin\alpha,$$
   where $x(0)$ corresponds to the top of the plane. When the cube reaches the bottom, we have:
   $$\tfrac{h}{\sin\alpha} = \tfrac{1}{2} g t^2 \sin\alpha.$$
   From here, we see that the cube will be at the bottom when $t = \frac{1}{\sin\alpha}\sqrt{\frac{2h}{g}}$. So, if the cube is to be at the bottom at t_1, it should be released at $t_0 = t_1 - \frac{1}{\sin\alpha}\sqrt{\frac{2h}{g}}$. ■

2. In the 2001 animated movie 'Pipe Dream', a musical composition is played on different percussive instruments hit by steel balls shot out of PVC pipes (see Fig. 4: (left)). Consider the configuration of the tube and the instrument shown schematically in Fig. 4 (right). At what time t_0 and with what speed v should the ball be projected from the pipe, so that the instrument plays at a given time t_1? Assume that the ball is projected at an angle of $\alpha = 45°$.

   Solution. For this setup the position (x, y) is described by the equation:
   $$(x(t), y(t)) = (0, h) + (\tfrac{v}{\sqrt{2}}t, \tfrac{v}{\sqrt{2}}t - \tfrac{1}{2}gt^2).$$
   When the steel ball hits the target instrument, we have:
   $$s = \tfrac{v}{\sqrt{2}}t$$
   $$0 = h + \tfrac{v}{\sqrt{2}}t - \tfrac{1}{2}gt^2$$

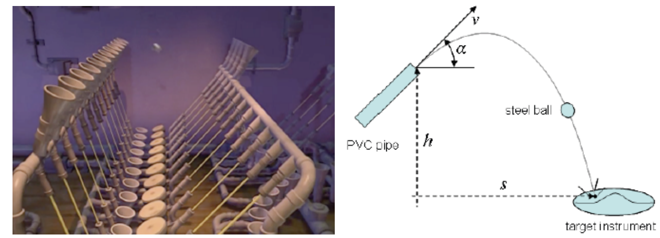

Figure 4: Left: A frame from the movie 'Pipe Dream'. Right: Schematic representation.

   This can be solved to yield $v = s\sqrt{\frac{g}{h+s}}$ and $t = \sqrt{2}\sqrt{\frac{h+s}{g}}$. Thus, the steel ball must be releasted at $t_0 = t_1 - \sqrt{2}\sqrt{\frac{h+s}{g}}$ with speed $s\sqrt{\frac{g}{h+s}}$. ■

3. Given the following quadratic curve segment defined by the control points $\mathbf{p}_0$, $\mathbf{p}_1$ and $\mathbf{p}_2$:
   $$\mathbf{x}(t) = (1-t)^2\mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2\mathbf{p}_2, \quad (0 \le t \le 1).$$
   Determine the following:
   (a) the velocity $\mathbf{v}(t)$ for $\mathbf{x}(t)$.
   (b) the direction of acceleration $\hat{\mathbf{n}} := \mathbf{a}/\|\mathbf{a}\|$ for $\mathbf{x}(t)$.

   Solution. (a) The velocity is obtained by differentiating the position equation once with respect to t. It is given by:
   $$\mathbf{v}(t) = -2(1-t)\mathbf{p}_0 + (2-4t)\mathbf{p}_1 + 2t\mathbf{p}_2,$$
   (b) The acceleration is obtained by differentiating the position equation twice, or velocity once. It is given by:
   $$\mathbf{a} = 2(\mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2).$$
   Thus, $\hat{\mathbf{n}} = \frac{\mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2}{\|\mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2\|}$.
   ■

4. In an animation of a roller coaster, the track includes a circular loop of diameter $2R$ (see Fig. 5). When a car reaches the top of the loop what is its centripetal acceleration?

   Solution. The magnitude of centripetal acceleration when the car is on the loop is v^2/R. We can determine the speed v at the top of the loop using conservation of energy.
   $$gH = 2gR + \tfrac{1}{2}v^2.$$

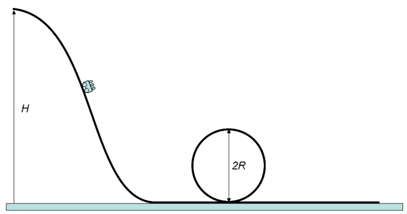

Figure 5: A roller coaster with a loop.

Thus, the speed squared is $v^2 = 2g(H - 2R)$. This makes the magnitude of centripetal acceleration at the top of the loop:

$$\frac{2g(H-2R)}{R}.$$

The direction of centripetal acceleration is towards the centre of the circular loop, and at the top of the loop will point towards the ground. Assuming that height varies along the y-axis, we obtain the following centripetal acceleration at the top of the loop:

$$\left(0, -\frac{2g(H-2R)}{R}, 0\right).$$

■

5. In a computer game, a cyclist rides a bicycle on a circular path of radius $r = 10\,\mathsf{m}$ with speed $v = 36\,\mathsf{km/h}$. What is the angle α at which the cyclist (and the bicycle) should be inclined with respect to the ground? Assume for simplicity that the acceleration due to gravity is $10\,\mathsf{ms}^{-2}$.

   Solution. First we need to ensure that all units are consistent. The speed is $v = 36\,\mathsf{km/h} = 36 \times \frac{1000}{3600}\,\mathsf{m/s} = 10\,\mathsf{m/s}$. The horizontal centripetal acceleration is $v^2/r = 100/10 = 10\,\mathsf{m/s^2}$. The vertical acceleration (due to gravity) is also $10\,\mathsf{m/s^2}$. Thus, the cyclist should be inclined at a 45 degree angle. ■

3 Rotations

1. An object A was rotated via Euler angles (i.e. with respect to its local coordinate frame). The first rotation was around the y axis by angle θ_{y_1}. The second rotation was around the x axis by angle θ_x. The third rotation was (again) around the y axis by angle θ_{y_2}. What is the sequence of rotations in the global (fixed) reference frame that would yield the same result?

   Solution. We simply need to reverse the order. In the fixed reference frame, the first rotation would be around the y axis through angle θ_{y_2}, followed by a rotation around the x axis through angle θ_x, followed by a rotation around the y axis through angle θ_{y_1}. ■

2. Quaternion $\mathbf{q}$ represents rotation by an angle α about the axis $\hat{\mathbf{n}}$ (note that $||\hat{\mathbf{n}}|| = 1$). What rotations are given by the quaternions given below:

   (a) $\bar{\mathbf{q}}$

   (b) $\mathbf{q}\bar{\mathbf{q}}$

   Solution. In axis-angle representation, the quaternion is $\mathbf{q} = [\cos\frac{\alpha}{2}, \hat{\mathbf{n}}\sin\frac{\alpha}{2}]$.

   (a) $\bar{\mathbf{q}} = [\cos\frac{\alpha}{2}, -\hat{\mathbf{n}}\sin\frac{\alpha}{2}] = [\cos\frac{-\alpha}{2}, \hat{\mathbf{n}}\sin\frac{-\alpha}{2}]$.
   This represents a rotation through angle $-\alpha$ about axis $\hat{\mathbf{n}}$.

   (b) Note that $\mathbf{q}\bar{\mathbf{q}} = [1, \mathbf{0}]$ which is the identity quaternion. Thus, this represents the identity rotation.

   ■

3. Rotate the point $P := (2, 0, 0)$ about the vector $\mathbf{v} = (0, 1, 0)$ through an angle θ. Perform your calculation thrice using:

   (a) quaternion multiplication,

   (b) Rodrigues’ axis-angle rotation formula,

   (c) a fixed-angle rotation matrix.

   Verify that you get the same answer in all cases.

   Solution. (a) The rotation quaternion is $\mathbf{q} = [\cos\frac{\theta}{2}, \sin\frac{\theta}{2}(0, 1, 0)]$ and the quaternion corresponding to the point is $\mathbf{p} = [0, (2, 0, 0)]$.
   Thus, we need to compute $\mathbf{qpq}^{-1}$. We can proceed either by either (1) using the formula for quaternion multiplication in scalar-vector form, or (2) directly multiplying the quaternions.
   (1) Using our scalar-vector multiplication equation we first compute $\mathbf{qp}$:

$$
\begin{aligned}
\mathbf{qp} &= [\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}(0,1,0)] \cdot [0,(2,0,0)] \\
&= [0-0, \cos\tfrac{\theta}{2}(2,0,0) + 0 + \sin\tfrac{\theta}{2}(0,1,0)\times(2,0,0)] \\
&= [0, 2(\cos\tfrac{\theta}{2}, 0, -\sin\tfrac{\theta}{2})]
\end{aligned}
$$

We can now compute $\mathbf{qpq}^{-1}$:

$$
\begin{aligned}
\mathbf{qpq}^{-1} &= [0, 2(\cos\tfrac{\theta}{2}, 0, -\sin\tfrac{\theta}{2})] \cdot [\cos\tfrac{\theta}{2}, -\sin\tfrac{\theta}{2}(0,1,0)] \\
&= [0, 2\cos\tfrac{\theta}{2}(\cos\tfrac{\theta}{2}, 0, -\sin\tfrac{\theta}{2}) - 2\sin\tfrac{\theta}{2}(\cos\tfrac{\theta}{2}, 0, -\sin\tfrac{\theta}{2}) \times (0,1,0)] \\
&= [0, 2\cos\tfrac{\theta}{2}(\cos\tfrac{\theta}{2}, 0, -\sin\tfrac{\theta}{2}) - 2\sin\tfrac{\theta}{2}(\sin\tfrac{\theta}{2}, 0, \cos\tfrac{\theta}{2})] \\
&= [0, 2(\cos\tfrac{\theta}{2}\cos\tfrac{\theta}{2} - \sin\tfrac{\theta}{2}\sin\tfrac{\theta}{2}, 0, -\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2} - \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}] \\
&= [0, 2(\cos\theta, 0, -\sin\theta)]
\end{aligned}
$$

Thus, the rotated point is $2(\cos\theta, 0, -\sin\theta)$
(2) Using the algebraic identities for i, j, k cross multiplication (derived from $i^2 = j^2 = k^2 = ijk = -1$), we can compute the rotation directly using quaternions:

$$
\begin{aligned}
\mathbf{qpq}^{-1} &= [\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}(0,1,0)] \cdot [0,(2,0,0)] \cdot [\cos\tfrac{\theta}{2}, -\sin\tfrac{\theta}{2}(0,1,0)] \\
&= (\cos\tfrac{\theta}{2} + j\sin\tfrac{\theta}{2}) \cdot 2i \cdot (\cos\tfrac{\theta}{2} - j\sin\tfrac{\theta}{2}) \\
&= 2(i\cos\tfrac{\theta}{2} - k\sin\tfrac{\theta}{2}) \cdot (\cos\tfrac{\theta}{2} - j\sin\tfrac{\theta}{2}) \\
&= 2(i\cos^2\tfrac{\theta}{2} - k\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} - k\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2} - i\sin^2\tfrac{\theta}{2}) \\
&= 2(i\cos\theta - k\sin\theta) \\
&= 2[0, 2(\cos\theta, 0, \sin\theta)]
\end{aligned}
$$

The rotated point is again $2(\cos\theta, 0, -\sin\theta)$

(b) Using Rodrigues' axis-angle formula $P' = cos\theta P + (1-cos\theta)(P\cdot\mathbf{v})\mathbf{v} + sin\theta(\mathbf{v}\times P)$, we get:

$$\cos\theta(2,0,0) + (1-\cos\theta)(0)(0,1,0) + \sin\theta(0,0,-2) = 2(\cos\theta, 0, -\sin\theta).$$

(c) Using a fixed-angle rotation matrix, we can simply rotate about the Y-axis, yielding the result:

$$
\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}
\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} =
\begin{pmatrix} 2\cos\theta \\ 0 \\ -2\sin\theta \end{pmatrix}
$$

■

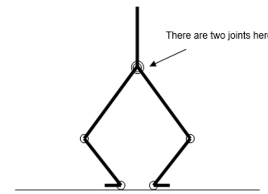

Figure 6: A bipedal stick figure.

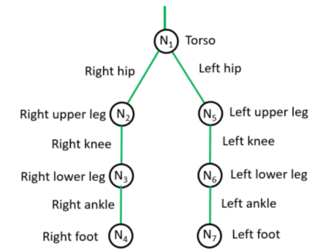

Figure 7: Tree data structure for Fig.6
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4 Articulated Structures

1. Consider the animation of the bipedal stick figure shown in Fig. 6.

   (a) How many degrees of freedom will this articulated structure have when:

   i. standing on one leg?

   Solution. It will have 6 degrees of freedom, as all joints are still free to rotate. ■

   ii. standing on two legs?

   Solution. The degrees of freedom are reduced to 3. With both feet on the ground, the linkages have additional constraints on them. The stick figure is allowed to move horizontally and vertically, and the torso can rotate left or right. ■

   (b) Suppose the torso is the root of the articulated structure. First, draw the corresponding tree data structure. Next, provide the transformation matrix that will convert the local coordinates of the right foot to global coordinates in the *rest-pose*.

   Solution. The corresponding tree data structure is shown in Fig. 7. Let T_1 be the transformation positioning the root in global coordinates. Now, let T_2, T_3 and T_4 be the respective transformations from the local reference frames of the child to the parent stored in the edges corresponding to the right hip, knee and ankle joints. To convert the local coordinates of the foot into global coordinates we multiply the matrices left-to-right to obtain:

$$T_1 \cdot T_2 \cdot T_3 \cdot T_4.$$

■

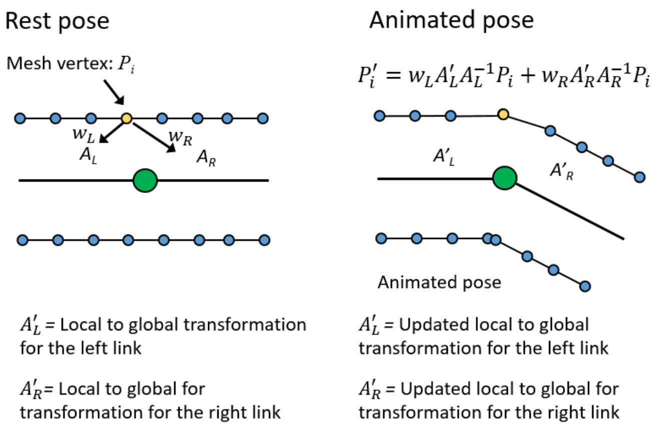

Figure 8: Linear blend skinning applied to an articulated structure with two links.

2. The vertex i shown in Figure 8 is animated using linear blend skinning. In this case, the animated position of the vertex P_i is given by:

$$P'_i = w_L A'_L A_L^{-1} P_i + w_R A'_R A_R^{-1} P_i$$

where w_L and w_R are the weights used to associate P_i with the left and right bones and A_L, A_R, A'_L, A'_R and P_i are as indicated in the diagram. In this case, why is it important that the weights w_L and w_R sum to 1?

Solution. When the articulated structure is in the rest pose, we want $P'_i = P_i$. In this case, we have $A_L = A'_L$ and $A_R = A'_R$. Thus:

$$P'_i = (w_L + w_R)P_i$$

If $P'_i = P_i$ then:

$$P_i = (w_L + w_R)P_i.$$

This equality only holds when $w_L + w_R = 1$. Thus, to reproduce the rest pose the weights must sum to 1. ■