Homework 5

Obliviate

May 2021

1 Solution:

We will transform this vertex-flow-network to a equivalent edge-flow-network.

For every vertex $u \in V$, we create two vertices u_{in} and u_{out} in the new network. If vertex u has capacity u_c in the original network, then there is an edge (u_{in}, u_{out}) with capacity u_c. And for every edge (x, y) in the original network, we create an edge (x_{out}, y_{in}) in the new network with infinite capacity. Obviously the new network is equivalent to the original network.

A flow in the original network from a to b is equivalent to the flow in the new network from a_{in} to b_{out}. Therefore we successfully transformed the network to a edge-flow-network. At last we apply the conclusion in Exercise 1 from Homework 4 on it.

3 Solution:

Firstly, we prove that not both cases can occur. Suppose that there are vertices $v_1, \cdots, v_{k-1} \in V \backslash \{s, t\}$ such that $G - \{v_1, \cdots, v_{k-1}\}$ contains no s-t-path. Then every s-t-path must contain at least one of $v_1, \cdots, v_{k-1}$. Hence there will be at most $k-1$ disjoint s-t-paths.

Secondly, we prove one of them must occur. We construct a network G' as following. For each $v \in V \backslash \{s, t\}$, there are two corresponding vertices v^i and v^j and an edge (v^i, v^j) with 1 unit capacity in G'. And for each $(v_1, v_2) \in E$, there is a corresponding edge (v_1^j, v_2^i) with infinite capacity in G'. It's easy to see that the maximum number of disjoint s-t-paths in G is equal to the maximum flow in G' and the minimum number of vertices cut in G is equal to the minimum cut in G'. By the max-flow min-cut theorem, the maximum number of disjoint s-t-paths in G is equal to the minimum cut in G'. If it's greater than $k-1$, then we can find at least k disjoint s-t-paths in G. Otherwise $k-1$ vertices are sufficient to cut all s-t-paths in G'.

4 Solution:

For any $X \subseteq L_i$, let $\Gamma(X)$ denote the neighbors of vertex set X in L_{i+1}. By Hall Theorem, there is a matching of size $\binom{n}{i}$ if and only if $\forall X \subseteq L_i \ |X| \leq |\Gamma(X)|$.

Since $i < \frac{n}{2}$, we have $i + 1 \leq n - i$.

For any $X \subseteq L_i$, we generate a subgraph $H_n[X \cup \Gamma(X)]$ by X. There are exactly $|X| \cdot (n-i)$ edges because each vertex in L_i has $n - i$ edges. And there are at most $|\Gamma(X)| \cdot (i+1)$ edges.

Then we have $|X| \cdot (n-i) \leq |\Gamma(X)| \cdot (i+1)$ and then $|\Gamma(X)| \geq |X| \cdot \dfrac{n-i}{i+1} \geq |X|$.

This finishes the proof.

5 Solution:

We will construct a graph H_n'. For every vertex $x \in L_j$ in H_n, we split it to two vertices x_{in} and x_{out} in H_n' and add an edge from x_{in} to x_{out} with capacity 1. $\forall x \in L_j, i \le j < n-i$, we add an edge from x_{out} to $\Gamma(x)_{in}$ with capacity $+\infty$. We add an edge from s to every vertex $x_{in} \in L_i$ and from every vertex $x_{out} \in L_{n-i}$ to t, which has $+\infty$ capacity. Obviously, the maximum flow of H_n' is the number of disjoint paths from L_i to L_{n-i}.

Now we construct the flow as follows. Every edge from s has 1 unit flow and every edge to t has 1 unit flow. Every edge from x_{in} to x_{out} with $x \in L_j$ have $\frac{\binom{n}{i}}{\binom{n}{j}}$ unit flow. Every edge from x_{out} with $x \in L_j$ have $\frac{\binom{n}{i}}{\binom{n}{j}} \cdot \frac{1}{j}$ unit flow.

Obviously, the flow is well-defined because all vertices except for s and t satisfy flow conservation. Our total flow is $\binom{n}{i}$ and the maximum flow is not bigger than $\binom{n}{i}$. So the maximum flow is $\binom{n}{i}$.

6 Solution:

G is a bipartite graph, so the size of minimum vertex cover is equal to the maximum matching $\nu(G)$.

First we find a minimum vertex cover set X. Let $Y = G \backslash X$. $|X| = \nu(G)$. There are only 2 kinds of edges in G. One is from X to X while the other is from X to Y. If there is an edge from Y to Y then X is not a vertex cover of G.

For any subset $D \subseteq X$, we try to build a set $E \subseteq Y$ such that $D + E$ is a minimum vertex cover of G. Obviously, E is unique. $y \in E$ if and only if $y \in Y$ and some edge to y hasn't been covered by D. Then for any subset $D \subseteq X$, there is at most one minimum vertex cover R such that $R \cap X = D$. There are $2^{\nu(G)}$ subsets of X which implies that there are at most $2^{\nu(G)}$ minimum vertex covers.

7 Solution:

We could find a bound of $f(k) = 3^k$.

Consider any maximum matches M. We first construct a set of vertices S which covers all edges in M. For every edge (u, v) in M, there are three possible cases. The first case is that only u is in S. The second case is that only v is in S. The third case is that both u and v are in S. So there are 3^k different sets S.

Then we consider every unmatched vertices and construct another set S'. For each unmatched vertex x, if at least one of its neighbor is not in S, we should add x to S' to cover that edge. Notice that S' is uniquely determined by S.

Let $C = S \cup S'$. We now have 3^k different C, but not every C is a cover. We will prove that every minimum cover is exactly some C.

For each minimum cover C', let S be the set of all matched vertices in C'. Then S' is uniquely determined by S and $C = S \cup S' = C'$. Since $|\{C'\}| \le |\{C\}| = 3^k$, we have finished the proof.