CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Spring 2021

Handed out on Monday, 2021-04-26
First submission and questions due on Monday, 2021-05-03
You will receive feedback from the TA.
Final submission due on Monday, 2021-05-10

5 Flows with Vertex Capacities

Exercise 1. Let $G = (V, c)$ be a flow network. Prove that flow is "transitive" in the following sense: if r, s, t are vertices, and there is an r–s-flow of value k and an s–t-flow of value k, then there is an r–t-flow of value k.

5.1 Vertex Disjoint Paths

Let G be a directed graph. Two paths p_1, p_2 from s to t are called *vertex disjoint* if they don't share any vertices except s and t.

Theorem 2 (Menger's Theorem)**.** *Let G be a graph and $s \neq t$ two vertices therein. Let $k \in \mathbf{N}_0$. Then exactly one of the following is true:*

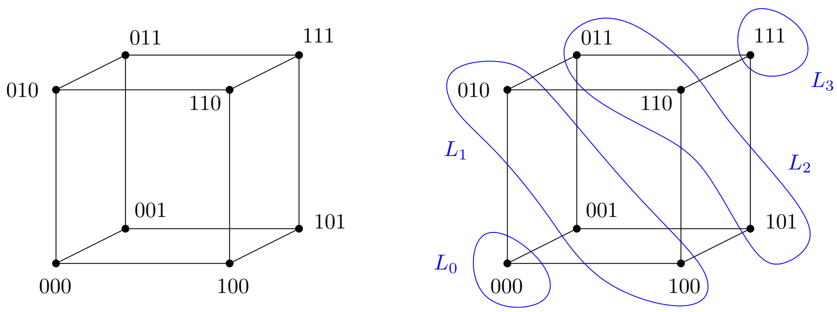
1. *There are k vertex disjoint paths $p_1, \ldots, p_k$ from s to t; that is, no two p_i, p_j share any vertex besides s and t.*
2. *There are vertices $v_1, \ldots, v_{k-1} \in V \setminus \{s, t\}$ such that $G - \{v_1, \ldots, v_{k-1}\}$ contains no s–t-path.*

Exercise 3. Prove Menger's Theorem. You have to prove two things: first, not both cases above can occur (this is rather easy); second, one of them must occur (this requires a tool from the lecture).

Let $V = \{0,1\}^n$. The n-dimensional Hamming cube H_n is the graph (V, E) where $\{u, v\} \in E$ if u, v differ in exactly one coordinate. Define the i^{th} level of H_n as

$$L_i := \{u \in V \mid \|u\|_1 = i\} \ ,$$

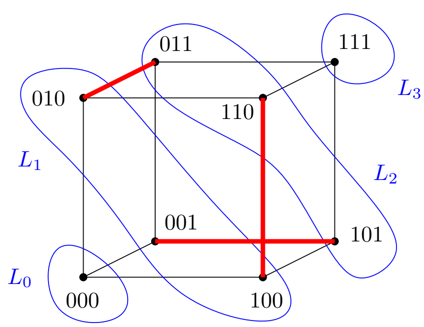
i.e., those vertices u having exactly i coordinates which are 1.



The 3-dimensional Hamming cube and the four sets L_0, L_1, L_2, L_3.

Exercise 4. [Matchings in H_n] Consider the induced bipartite subgraph $H_n[L_i \cup L_{i+1}]$. This is the graph on vertex set $L_i \cup L_{i+1}$ where two edges are connected by an edge if and only if they are connected in H_n.

Show that for $i < n/2$ the graph $H_n[L_i \cup L_{i+1}]$ has a matching of size $|L_i| = \binom{n}{i}$.



A matching of size 3 between L_1 and L_2.

Exercise 5. Let H_n be the n-dimensional Hamming cube. For $i < n/2$ consider L_i and L_{n-i}. Note that $|L_i| = \binom{n}{i} = \binom{n}{n-i} = L_{n-i}$, so the L_i and L_{n-i} have the same size. Show that there are $\binom{n}{i}$ paths $p_1, p_2, \ldots, p_{\binom{n}{i}}$ in H_n

such that (i) each p_i starts in L_i and ends in L_{n-i}; (ii) two different paths p_i, p_j do not share any vertices. **Hint 1.** Model this problem as a network flow with vertex capacities. What would the maximum flow be in this network? **Hint 2.** It's not *that* easy. If you try to work from both sides towards the middle by combining matchings between levels, you will certainly run into problems as how to glue things together in the middle. I have never seen any "meet in the middle" proof that works. **Hint 3.** There is a "direct" proof by induction that does not require anything about network flows.

5.2 Matchings and Vertex Covers

The following exercise was on the final exam of CS 499 (mathematical foundations of computer science) in spring 2019.

Exercise 6. Let $\nu(G)$ denote the size of a maximum matching of G. Show that a bipartite graph G has at most $2^{\nu(G)}$ minimum vertex covers.

Obviously, this is not true for general (non-bipartite) graphs: the triangle K_3 has $\nu(K_3) = 1$ but it has three minimum vertex covers. The five-cycle C_5 has $\nu(C_5) = 2$ but has five minimum vertex covers.

Exercise 7. Is there a function $f : \mathbf{N}_0 \to \mathbf{N}_0$ such that every graph with $\nu(G) = k$ has at most $f(k)$ minimum vertex covers? How small a function f can you obtain?