CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Spring 2021

Handed out on Thursday, 2021-05-06
First submission and questions due on Thursday, 2021-05-13
You will receive feedback from the TA.
Final submission due on Thursday, 2021-05-20

6 Matching LP and Vertex Cover LP

Let $G = (V, E)$ be a graph and consider the Vertex Cover Linear Program VCLP(G):

$$\text{VCLP}(G): \qquad \begin{array}{lrll} \text{minimize} & \sum_{u \in V} y_u & & \\ \text{subject to} & y_u + y_v & \geq 1 & \forall \text{ edges } \{u, v\} \in E \\ & \mathbf{y} & \geq \mathbf{0} & \end{array}$$

Every vertex cover of G corresponds to a feasible solution $\mathbf{y} \in \text{sol}(\text{VCLP}(G))$, but not vice versa. However, every $\mathbf{y} \in \text{sol}(\text{VCLP}(G)) \cap \{0, 1\}^V$ does. Let $\tau(G)$ denote the size of a minimum vertex cover of G. In class, we showed that $\tau(G) = \text{val}(\text{VCLP}(G))$ for all *bipartite* graphs G. We achieved this by taking an arbitrary feasible solution $\mathbf{y}$ and "shaking" it until it becomes integral, while making sure its value does not go up.

Next, recall the Matching Linear Program MLP(G):

$$\text{MLP}(G): \qquad \begin{array}{lrll} \text{maximize} & \sum_{e \in E} x_e & & \\ \text{subject to} & \sum_{e \in E: u \in e} x_e & \leq 1 & \forall\, u \in V \\ & \mathbf{x} & \geq \mathbf{0} & \end{array}$$

Every matching of G corresponds to a feasible solution $\mathbf{x} \in \text{sol}(\text{MLP}(G))$, but not vice versa. However, every $\mathbf{x} \in \text{sol}(\text{MLP}(G)) \cap \{0, 1\}^E$ does.

Exercise 1. [Matching LP in bipartite graphs] Let $\nu(G)$ denote the size of a maximum matching of G. Obviously, $\mathrm{val}(\mathrm{MLP}(G)) \geq \nu(G)$ for all graphs. Show that $\nu(G) = \mathrm{val}(\mathrm{MLP}(G))$ for all *bipartite* graphs G. Do this without referring to Kőnig's Theorem, i.e., solely by working on a given solution to the linear program. **Hint.** This is similar to what we did to the VCLP in class: take some solution; if this contains fractional numbers (i.e., from $\mathbb{R} \setminus \mathbb{Z}$), find a way to "wiggle" the solution, improving it and making it "more integral".

Exercise 2. [Half-integrality of VCLP in general graphs] Let G be a graph, not necessarily bipartite. Show that its vertex cover linear program $\mathrm{VCLP}(G)$ has an optimal solution $\mathbf{y}$ for which $y_u \in \left\{0, \frac{1}{2}, 1\right\}$ for every vertex u.

Exercise 3. For a graph $G = (V, E)$, let $\tau(G)$ denote the size of a minimum vertex cover, and $\nu(G)$ the size of a maximum matching. Recall the two linear programs VCLP and MLP (see above). Let $\tau_f(G) := \mathrm{opt}(\mathrm{VCLP}(G))$ and $\nu_f(G) := \mathrm{opt}(\mathrm{MLP}(G))$. Note that

$$\nu(G) \leq \nu_f(G) = \tau_f(G) \leq \tau(G) \ , \tag{1}$$

where the equality in the middle follows from Strong LP Duality. Also, if G is bipartite, then equality holds throughout in (1). Let us say a graph G is *VCLP-exact* if $\tau(G) = \tau_f(G)$, and *MLP-exact* if $\nu(G) = \nu_f(G)$. As we already know, a bipartite graph G is both VCLP exact and MLP exact.

From now on, suppose that G is *not* bipartite but $\tau(G) = \tau_f(G)$.

1. Give an example of such a graph G that is not bipartite but still VCLP-exact.

2. Give an example of a graph G that is MLP-exact but not VCLP-exact.

3. Suppose G is VCLP-exact. Let $Y \subseteq V(G)$ be a minimum vertex cover. Let $\mathbf{x}$ be an optimal solution of $\mathrm{MLP}(G)$. Show that $x_e = 0$ if $e \subseteq Y$ (i.e., if both endpoints of e are in the cover).

4. Show that such a graph G has a matching of size $|Y|$, and thus is MLP-exact, too.

Exercise 4. Show that $\nu(G) \geq \frac{2}{3}\nu_f(G)$ for every graph G (note that this is tight for the triangle, where $\nu(G) = 1$ and $\nu_f(G) = \frac{3}{2}$).

I will introduce to you the main tool to solve this exercise: Tutte's theorem. For a set $U \subseteq V$ of vertices, let $G - U$ denote the graph arising from G by deleting the vertices of U and its incident edges. Note that $G - U$ might have several connected components. Let $\mathcal{C}_i$ denote the set of components of $G - U$ that have size i. Let $\mathcal{C}_{\text{odd}} := \mathcal{C}_1 \cup \mathcal{C}_3 \cup \mathcal{C}_5 \cup \ldots$, i.e., the set of all odd-sized connected components of $G - U$.

If M is a matching of G, it is quite obvious that every odd-sized component $C \in \mathcal{C}_{\text{odd}}$ contains a vertex that is either (1) unmatched by M or (2) matched to a vertex in U. Note that case (2) can only for at most $|U|$ vertices, and therefore M leaves at least $|\mathbf{C}_{\text{odd}}| - |U|$ vertices unmatched. Tutte's theorem states that this is optimal.

Theorem 5 (Tutte's theorem)**.** *There is a set $U \subseteq V$ and a matching M such that M leaves exactly $|\mathbf{C}_{\text{odd}}| - |U|$ vertices of G unmatched.*

Obviously, such a matching M must be a maximum matching. To prove $\nu(G) \geq \frac{2}{3}\nu_f(G)$, take an optimal fractional solution $\mathbf{x}$ of the MLP, take a maximum matching M, and a set U as guaranteed by Tutte's theorem. Then start playing around with $\mathbf{x}$ and M.