Homework 6

Obliviate

May 2021

1 Solution:

For every $\mathbf{x} \in \text{sol}(\text{MLP}(G))$, let's find a solution $\mathbf{y} \in \text{sol}(\text{MLP}(G)) \cap \{0,1\}^E$ so that $\text{val}(\mathbf{y}) \geq \text{val}(\mathbf{x})$. Then $\nu(G) = \text{val}(\text{MLP}(G))$ because every $\mathbf{y} \in \text{sol}(\text{MLP}(G)) \cap \{0,1\}^E$ corresponds a matching of G.

For edges with non-integer value, we call them bad edges. We call a cycle which only contains bad edges a bad cycle. We use similar way to define "bad path" and "longest bad path". Let's denote the sum of values of edges from one vertex t to be $sum(t)$.

For any $\mathbf{x} \in \text{sol}(\text{MLP}(G))$, we just consider bad edges because our goal is to change the values of bad edges into integers 0 or 1. If there is a bad cycle, we can change at least one of bad edges in the cycle to good edge by using the following method.

Since G is a bipartite graph, the length of the cycle is even. We can color the edges white and black alternately in the cycle. For white edges, we increase their values by d. And for black edges, we decrease their values by d. At least one edge will become 0 or 1 by choosing d properly.

After eliminating all bad cycles, there may be many bad paths. Each time we choose the longest bad path and change the values of edges in it. We again color the edges white and black with the first edge on the path being white and use the same way to change the values of white edges and black edges. Because the path is the longest bad path, values of other edges of the endpoints must be integers and less than 1, i.e. 0. That is to say, $sum(endpoint)$ is equal to the value of its neighboring bad edge in this path. So the $sum(endpoint) \leq 1$. At the same time, sum of other vertices wouldn't change. We use this method to change the longest bad path repeatedly until there are no bad edges. Each time we eliminate at least one bad edge. So the number of bad edges is always decreasing and finally will be zero.

Both methods above can ensure any vertex's sum value is never greater than 1 and sum of edge values don't decrease. And the number of bad edges will decrease by at least one in each step, which implies that the algorithm will end in finite steps. This finishes the proof.

2 Solution:

To prove this we firstly construct a bipartite graph B based on G.

For each vertex $v \in V_G$, there are two corresponding vertices v_X and v_Y in V_B. For each edge $(u,v) \in E_G$, there are two corresponding edges (u_X, v_Y) and (v_X, u_Y) in E_B.

Rewrite the linear program in B as follows,

$$\text{VCLP}(B): \begin{array}{lll} \text{minimize} & \sum_{u \in V_B} z_v & \\ \text{subject to} & z_u + z_v & \geq 1 \ \forall \text{ edges } \{u,v\} \in E_B \\ & \mathbf{z} & \geq \mathbf{0} \end{array}.$$

As we have proved in the class, there is an integral optimal solution $\mathbf{z}$ to VCLP(B).

Let $y_u = \dfrac{z_{u_X} + z_{u_Y}}{2}$.

It's easy to see that $\mathbf{y} \in \left\{0, \frac{1}{2}, 1\right\}^{|V|}$. Hence $\forall (u,v) \in E_G, y_u + y_v = \dfrac{z_{u_X} + z_{v_Y} + z_{v_X} + z_{u_Y}}{2} \geq \dfrac{2}{2} = 1$.

This provides us with a mapping

$$\begin{aligned} \phi_1 : \quad & \text{sol}\left(\text{VCLP}(B)\right) \to \text{sol}\left(\text{VCLP}(G)\right) \\ & \mathbf{z} \mapsto \mathbf{y} \end{aligned}$$

so that the value of $\mathbf{z}$ is twice the value of $\mathbf{y}$ and $\mathbf{y} \in \left\{0, \frac{1}{2}, 1\right\}^{|V|}$.

To finish the proof, we need to show that there is a mapping

$$\begin{aligned} \phi_2 : \quad & \text{sol}\left(\text{VCLP}(G)\right) \to \text{sol}\left(\text{VCLP}(B)\right) \\ & \mathbf{y} \mapsto \mathbf{z} \end{aligned}$$

so that the value of $\mathbf{y}$ is half the value of $\mathbf{z}$. By ϕ_1 and ϕ_2, $\text{val}\left(\text{VCLP}(G)\right) = 1/2 \, \text{val}\left(\text{VCLP}(B)\right)$.

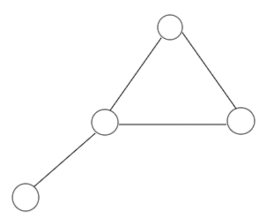
ϕ_2 can be simply generated by letting $z_{u_X} = z_{u_Y} = y_u$ for each $u \in V_G$.

Now let's put everything together. We construct a bipartite graph B based on G, find an optimal integral solution of VCLP(B) and compute its corresponding solution in VCLP(G), which is both optimal and half-integral.

3 Solution:

1.

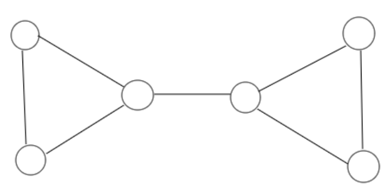
Here is the example.



Since the graph above has an odd cycle, it is not bipartite. And $\tau(G) = 2$, $\tau_f(G) = 2$, so it is VCLP-exact.

2.

Here is the example.



According to the graph above , we can see that $\nu(G) = 3, \nu_f(G) = 3, \tau_f(G) = 3$ (with every vertex's value being $\frac{1}{2}$) , $\tau(G) = 4$, so it is MLP-exact and not VCLP-exact.

3.

Let $\boldsymbol{b} = \boldsymbol{1}, \boldsymbol{c} = \boldsymbol{1}$, then MLP : maximize $\boldsymbol{b}^T\boldsymbol{x}$, subject to $\boldsymbol{A}^T\boldsymbol{x} \leq \boldsymbol{c}$, $x \geq 0$; VCLP : minimize $\boldsymbol{c}^T\boldsymbol{y}$, subject to $\boldsymbol{Ay} \geq \boldsymbol{b}$, $y \geq 0$.

Let $\boldsymbol{tx}$ be an optimal solution of MLP(G), $\boldsymbol{ty}$ be a solution satisfying that $y_u = 1$ if $u \in Y$, $y_u = 0$ if $u \in V/Y$. Because G is VCLP-exact, $\boldsymbol{ty}$ is an optimal solution of VCLP(G). By strong duality, $\boldsymbol{b}^T\boldsymbol{tx} = \boldsymbol{tx}^T\boldsymbol{Aty} = \boldsymbol{c}^T\boldsymbol{ty}$.

Since $\boldsymbol{tx}^T\boldsymbol{b} = \boldsymbol{tx}^T\boldsymbol{Aty} \Longrightarrow \boldsymbol{b}^T\boldsymbol{tx} = \boldsymbol{ty}^T\boldsymbol{A}^T\boldsymbol{tx} \Longrightarrow (\boldsymbol{b}^T - \boldsymbol{ty}^T\boldsymbol{A}^T)\boldsymbol{tx} = 0$, and for $e \subseteq Y$, $\boldsymbol{b}_e - \boldsymbol{A}\cdot\boldsymbol{ty}_e = 1 - 2 = -1 \neq 0$, as a result, $\boldsymbol{tx}_e$ must be zero.

4.

Because $Y \subseteq V(G)$ is a minimum vertex cover of G, there's no edge between nodes in V/Y. So after removing edges between nodes in Y, the new graph becomes bipartite with Y and V/Y on two sides.

According to Hall's Theorem, if we can prove that $\forall S \subseteq Y, |S| \leq |N(S)|$, then G has a matching of size $|Y|$, and thus is MLP-exact. ($N(S)$ means the set of neighbours of S in V/Y)

Assume that we can find $S_0 \subseteq Y : |S_0| > |N(S_0)|$. Then there's a solution for VCLP:

$$y_u = \begin{cases} \frac{1}{2}, u \in S_0 \bigcup N(S_0) \\ 1, u \in Y/S_0 \\ 0, u \in V/(Y \bigcup N(S_0)) \end{cases}$$

Let's check the solution. For $\forall$ edges $\{u, v\} \in E$, if $u, v \in Y$, $y_u + y_v \geq \frac{1}{2} + \frac{1}{2} = 1$; if $u \in Y/S_0, v \in V/Y$, $y_u + y_v \geq 1 + 0 = 1$; if $u \in S_0, v \in N(S_0)$, then $y_u + y_v = \frac{1}{2} + \frac{1}{2} = 1$. So this is a solution of VCLP(G), and $\sum_{u \in V} y_u = |Y| - \frac{|S_0| - |N(S_0)|}{2} < \tau(G) = |Y|$. This is contradictory to the given fact that G is VCLP-exact. Therefore, we can't find such $S_0 \subseteq Y : |S_0| > |N(S_0)|$. That is to say, $\forall S \subseteq Y, |S| \leq |N(S)|$. By Hall's Theorem, we have proved that G has a matching of size $|Y|$, and thus is MLP-exact.

4 Solution:

Firstly, we introduce another tool we will use to prove the main result.

For a set $U \subseteq V$ of vertices, let $\mathcal{I} := \mathcal{C}_1$, the set of all isolated vertices.

Lemma 0.1.

$$\nu_f(G) \leq \frac{1}{2}\big(n - \max\{|\mathcal{I}| - |U|\}\big),\ \forall U \subseteq V$$

Proof.

Firstly, we prove that there is an optimal fractional solution $\mathbf{x}$, $\sum_{e \in E: u \in e} x_e \in \{0, 1\},\ \forall u \in \mathcal{I}$. If not, take an edge $\{u, v\}$ with $e_{\{u,v\}} \in (0, 1), u \in \mathcal{I}, v \in U$, set all $e_{\{w,v\}}$ with $w \neq u$ to 0 and set $e_{\{u,v\}}$ to 1. The modified solution is obviously not smaller than the original one.

There are $|\mathcal{I}|$ isolated vertices. Clearly no more than $|U|$ of them satisfy that $\sum_{e\in E:u\in e} x_e = 1$. Now we bound $\nu_f(G)$ by simply assuming that each x_e can reach its maximum possible value, i.e.

$$\begin{aligned}\nu_f(G) &= \max \sum_{e\in E} x_e \\ &= \frac{1}{2} \max \sum_{u\in V} \sum_{e\in E:u\in e} x_e \\ &\le \frac{1}{2}\big(n\cdot 1 - (|\mathcal{I}| - |U|)\big)\end{aligned}$$

.

By the arbitrariness of U, the lemma is proved. □

Remark. *In fact, the upper bound here is tight by the fractional analogue of the Berge-Tutte formula. Although this lemma is weak, it's still enough for our following proof.*

Now take a maximum matching M, and a **maximum** set S under maximum $|\mathcal{C}_{odd}| - |S|$.

Lemma 0.2. *All connected components of $G - S$ are odd-sized.*

Proof. Suppose that there is an even-sized connected component C in $G - S$. Pick an arbitrary vertex $c \in C$. It's easy to see that $\mathcal{C}_{odd}$ will increase by at least 1 while S will increase by exactly 1 after adding c to S, which implies that S is not maximized under maximum $|\mathcal{C}_{odd}| - |S|$. □

Denote $x = |\mathcal{I}(G-S)|, y = |\mathcal{C}_{odd}(G-S)| - x$. Since $G-S$ has no even components, $n \ge |S| + x + 3y$.

$$\begin{aligned}\frac{\nu_f(G)}{\nu(G)} &= \frac{\nu_f(G)}{\frac{1}{2}\big(\max_U\{|\mathcal{C}_{odd}| - |U|\}\big)} && \text{(by Tutte's theorem)} \\ &\le \frac{\frac{1}{2}\big(\max_U\{|\mathcal{I}| - |U|\}\big)}{\frac{1}{2}\big(\max_U\{|\mathcal{C}_{odd}| - |U|\}\big)} && \text{(by Lemma 0.1.)} \\ &\le \frac{n - x + |S|}{n - (x+y) + |S|} && \text{(by choosing } S \text{ as } U \text{ in both max function)} \\ &\le 1 + \frac{y}{2\big(y + |S|\big)} && \text{(by } n \ge |S| + x + 3y\text{)} \\ &\le \frac{3}{2}\end{aligned}$$

Hence $\nu(G) \ge \frac{2}{3}\nu_f(G)$.