

Random Variables

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1 Discrete Random Variables

1.1 Bernoulli

An experiment that results in "success" or "failure."

$$\begin{aligned}X &\sim \text{Ber}(p) \\P(X = 0) &= 1 - p \\P(X = 1) &= p \\E[X] &= p \\\text{Var}(X) &= p(1 - p) \\M(t) &= e^t p + 1 - p\end{aligned}$$

1.2 Binomial

The number of successes in an experiment with n trials and p probability of success on each trial.

$$\begin{aligned}X &\sim \text{Bin}(n, p) \\P(X = i) &= p(i) = \binom{n}{i} p^i (1 - p)^{n-i} \quad \text{where } i = 0, 1, \dots, n \\E[X] &= np \\\text{Var}(X) &= np(1 - p) \\M(t) &= (pe^t + 1 - p)^n\end{aligned}$$

If $X_i \sim \text{Bin}(n_i, p)$ for $1 \leq i \leq N$, then

$$\left(\sum_{i=1}^N X_i \right) \sim \text{Bin} \left(\sum_{i=1}^N n_i, p \right)$$

1.3 Poisson

Approximates the binomial random variable when n is large and p is small enough to make np "moderate"—generally when $n > 20$ and $p < 0.05$.

$$\begin{aligned}X &\sim \text{Poi}(\lambda) \quad \text{where } \lambda = np \\P(X = i) &= e^{-\lambda} \frac{\lambda^i}{i!} \quad \text{where } i = 0, 1, 2, \dots \\E[X] &= \lambda \\ \text{Var}(X) &= \lambda \\M(t) &= e^{\lambda(e^t - 1)}\end{aligned}$$

The approximations also works to a certain extent when the successes in the trials are not entirely independent, and when the probability of success in each trial varies slightly.

If $X_i \sim \text{Poi}(\lambda_i)$ for $1 \leq i \leq N$, then

$$\left(\sum_{i=1}^N X_i \right) \sim \text{Poi} \left(\sum_{i=1}^N \lambda_i \right)$$

1.4 Geometric

The number of independent trials until a success, where the probability of success is p .

$$\begin{aligned}X &\sim \text{Geo}(p) \\P(X = n) &= (1 - p)^{n-1} p \quad \text{where } n = 1, 2, \dots \\E[X] &= 1/p \\ \text{Var}(X) &= (1 - p)/p^2\end{aligned}$$

1.5 Negative Binomial

The number of independent trials until r successes, with probability p of success.

$$\begin{aligned}X &\sim \text{NegBin}(r, p) \\P(X = n) &= \binom{n-1}{r-1} p^r (1-p)^{n-r} \quad \text{where } n = r, r+1, \dots \\E[X] &= r/p \\ \text{Var}(x) &= r(1-p)/p^2 \\ \text{Geo}(p) &\sim \text{NegBin}(1, p)\end{aligned}$$

1.6 Hypergeometric

The number of white balls drawn after drawing n balls (without replacement) from an urn containing N balls, with m white balls and $N - m$ other ("black") balls.

$$\begin{aligned} X &\sim \text{HypG}(n, N, m) \\ P(X = i) &= \frac{\binom{m}{i} \binom{N-m}{n-i}}{\binom{N}{n}} \quad \text{where } i = 0, 1, \dots, n \\ E[X] &= n(m/N) \\ \text{Var}(X) &= \frac{nm(N-n)(N-m)}{N^2(N-1)} \end{aligned}$$

$\text{HypG}(n, N, m) \rightarrow \text{Bin}(n, m/N)$, as $N \rightarrow \infty$ and m/N stays constant

1.7 Multinomial

Given an experiment with n independent trials, where each trial results in one of m outcomes, with respective probabilities p_1, p_2, \dots, p_m such that $\sum_{i=1}^m p_i = 1$, then if X_i denotes the number of trials with outcome i we have

$$P(X_1 = c_1, X_2 = c_2, \dots, X_m = c_m) = \binom{n}{c_1, c_2, \dots, c_m} p_1^{c_1} p_2^{c_2} \dots p_m^{c_m}$$

where $\sum_{i=1}^m c_i = n$ and $\binom{n}{c_1, c_2, \dots, c_m} = \frac{n!}{c_1! c_2! \dots c_m!}$.

2 Continuous Random Variables

If Y is a non-negative continuous random variable

$$E[Y] = \int_0^{\infty} P(Y > y) dy$$

2.1 Uniform

$$\begin{aligned} X &\sim \text{Uni}(\alpha, \beta) \\ f(x) &= \begin{cases} \frac{1}{\beta - \alpha} & \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases} \\ E[X] &= \frac{\alpha + \beta}{2} \\ \text{Var}(X) &= \frac{(\beta - \alpha)^2}{12} \end{aligned}$$

2.2 Normal

For values in common natural phenomena, especially when resulting from the sum of multiple variables.

$$\begin{aligned}
 X &\sim N(\mu, \sigma^2) \\
 f(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{where } -\infty < x < \infty \\
 E[X] &= \mu \\
 \text{Var}(X) &= \sigma^2 \\
 M(t) &= e^{\left(\frac{\sigma^2 t^2}{2} + \mu t\right)}
 \end{aligned}$$

Letting $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, we have

$$\begin{aligned}
 Y &\sim N(a\mu + b, a^2\sigma^2) \\
 F_Y(x) &= F_X\left(\frac{x-b}{a}\right)
 \end{aligned}$$

The Standard (Unit) Normal Random Variable $Z \sim N(0, 1)$ has a cumulative distribution function (CDF) commonly labeled $\Phi(x) = P(Z \leq x)$ that has some useful properties.

$$\begin{aligned}
 \Phi(z) &= \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
 \Phi(-z) &= 1 - \Phi(z) \\
 P(Z \geq -z) &= P(Z > z)
 \end{aligned}$$

Given $X \sim N(\mu, \sigma^2)$ where $\sigma > 0$, we can then compute the CDF of X using the CDF of the standard normal variable.

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

By the de Moivre-Laplace Limit Theorem, the normal variable can approximate the binomial when $\text{Var}(X) = np(1-p) \geq 10$. If we let S_n denote the number of successes (with probability p) in n independent trials, then

$$P\left(a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right) \xrightarrow{n \rightarrow \infty} \Phi(b) - \Phi(a)$$

If $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, \dots, n$, then

$$\left(\sum_{i=1}^n X_i\right) \sim N\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$$

2.3 Exponential

Represents time until some event, with rate $\lambda > 0$.

$$\begin{aligned}X &\sim \text{Exp}(\lambda) \\f(x) &= \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \\E[X] &= \frac{1}{\lambda} \\\text{Var}(X) &= \frac{1}{\lambda^2} \\F(x) &= 1 - e^{-\lambda x} \quad \text{where } x \geq 0\end{aligned}$$

Exponentially distributed random variables are memoryless.

$$P(X > s + t | X > s) = P(X > t)$$

2.4 Beta

$$\begin{aligned}X &\sim \text{Beta}(a, b) \\f(x) &= \begin{cases} \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \\B(a, b) &= \int_0^1 x^{a-1} (1-x)^{b-1} dx \\E[X] &= \frac{a}{a+b} \\\text{Var}(X) &= \frac{ab}{(a+b)^2(a+b+1)}\end{aligned}$$

If $X \sim \text{Uni}(0, 1)$ and N denotes the number of heads resulting from a number of coin flips with some unknown probability of getting heads, then

$$X | (N = n, m + n \text{ trials}) \sim \text{Beta}(n + 1, m + 1)$$