Generating the Transformation Matrix

Scaling: $\mathbf{S} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Shear: $\mathbf{H} = \mathbf{H}_{xy}\mathbf{H}_{xz}\mathbf{H}_{zy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ h_{xy} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ h_{xz} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ h_{xy} & 1 & 0 & 0 \\ h_{xz} & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rotation: $\mathbf{R} = \mathbf{R}_x\mathbf{R}_y\mathbf{R}_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x & 0 \\ 0 & -\sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 & 0 \\ -\sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\mathbf{R} = \begin{bmatrix} \cos\theta_y\cos\theta_z & \cos\theta_y\sin\theta_z & -\sin\theta_y & 0 \\ \sin\theta_x\sin\theta_y\cos\theta_z - \cos\theta_x\sin\theta_z & \sin\theta_x\sin\theta_y\sin\theta_z + \cos\theta_x\cos\theta_z & \sin\theta_x\cos\theta_y & 0 \\ \cos\theta_x\sin\theta_y\cos\theta_z + \sin\theta_x\sin\theta_z & \cos\theta_x\sin\theta_y\sin\theta_z - \sin\theta_x\cos\theta_z & \cos\theta_x\cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation: $\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & z & 1 \end{bmatrix}$

Full Transformation: $\mathbf{M}(x, y, z) = \begin{bmatrix} x & y & z & 1 \end{bmatrix}\mathbf{SHRT}$

$$\mathrm{M}(x,y,z) = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \mathrm{SHRT}$$

Condensing the representation of the rotation matrix, we have:

$$\mathrm{SHRT} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ h_{xy} & 1 & 0 & 0 \\ h_{xz} & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & z & 1 \end{bmatrix}$$

$$\mathrm{M}(x,y,z) = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 & 0 \\ S_y h_{xy} & S_y & 0 & 0 \\ S_z h_{xz} & S_z h_{zy} & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & z & 1 \end{bmatrix}$$

$$\mathrm{M}(x,y,z) = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} S_x(R_{11}) & S_x(R_{12}) & S_x(R_{13}) & 0 \\ S_y h_{xy}(R_{11}) + S_y(R_{21}) & S_y h_{xy}(R_{12}) + S_y(R_{22}) & S_y h_{xy}(R_{13}) + S_y(R_{23}) & 0 \\ S_z h_{xz}(R_{11}) + S_z h_{yz}(R_{21}) + S_z(R_{31}) & S_z h_{xz}(R_{12}) + S_z h_{yz}(R_{22}) + S_z(R_{32}) & S_z h_{xz}(R_{13}) + S_z h_{yz}(R_{23}) + S_z(R_{33}) & 0 \\ x & y & z & 1 \end{bmatrix}$$

$$\mathrm{M}(x,y,z) = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} S_x(R_{11}) & S_x(R_{12}) & S_x(R_{13}) & 0 \\ S_y\left[h_{xy}(R_{11}) + (R_{21})\right] & S_y\left[h_{xy}(R_{12}) + (R_{22})\right] & S_y\left[h_{xy}(R_{13}) + (R_{23})\right] & 0 \\ S_z\left[h_{xz}(R_{11}) + h_{yz}(R_{21}) + (R_{31})\right] & S_z\left[h_{xz}(R_{12}) + h_{yz}(R_{22}) + (R_{32})\right] & S_z\left[h_{xz}(R_{13}) + h_{yz}(R_{23}) + (R_{33})\right] & 0 \\ x & y & z & 1 \end{bmatrix}$$

Extracting Scale, Shear and Rotation from the Transformation Matrix

We can rewrite $\mathbf{M}$ as $\begin{bmatrix} S_x\mathbf{R}_1 \\ S_y\left(h_{xy}\mathbf{R}_1+\mathbf{R}_2\right) \\ S_z\left(h_{xz}\mathbf{R}_1+h_{zy}\mathbf{R}_2+\mathbf{R}_3\right) \end{bmatrix}$

Note that because $\mathbf{R}$ is linearly independent, $\mathbf{R}_i \bullet \mathbf{R}_j = 0$. Also note that $\left\|\mathbf{R}_i\right\| = \mathbf{R}_i \bullet \mathbf{R}_i = 1$.

Isolate S_x: $\left\|\mathbf{M}_1\right\| = S_x\left\|\mathbf{R}_1\right\| = S_x$

$$\begin{bmatrix} \mathbf{R}_1 \\ S_y\left(h_{xy}\mathbf{R}_1+\mathbf{R}_2\right) \\ S_z\left(h_{xz}\mathbf{R}_1+h_{zy}\mathbf{R}_2+\mathbf{R}_3\right) \end{bmatrix} \qquad \frac{1}{S_x}r1 \rightarrow r1$$

Isolate S_yh_{xy}:

$$\begin{aligned} \mathbf{R}_1 \bullet \mathbf{M}_2 &= \mathbf{R}_1 \bullet \left(S_yh_{xy}\mathbf{R}_1 + S_y\mathbf{R}_2\right) \\ &= S_yh_{xy}\left(\mathbf{R}_1 \bullet \mathbf{R}_1\right) + S_y\left(\mathbf{R}_1 \bullet \mathbf{R}_2\right) \\ &= S_yh_{xy}(1) + S_y(0) \\ &= S_yh_{xy} \end{aligned}$$

$$\begin{bmatrix} \mathbf{R}_1 \\ S_y\mathbf{R}_2 \\ S_z\left(h_{xz}\mathbf{R}_1+h_{zy}\mathbf{R}_2+\mathbf{R}_3\right) \end{bmatrix} \qquad -S_yh_{xy}r1 + r2 \rightarrow r2 \qquad \begin{array}{r} -S_yh_{xy}\mathbf{R}_1 \\ +S_yh_{xy}\mathbf{R}_1 + S_y\mathbf{R}_2 \\ \hline S_y\mathbf{R}_2 \end{array}$$

Isolate S_y: $\left\|r2\right\| = \left\|S_y\mathbf{R}_2\right\| = S_y\left\|\mathbf{R}_2\right\| = S_y$

$$\begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ S_z\left(h_{xz}\mathbf{R}_1+h_{zy}\mathbf{R}_2+\mathbf{R}_3\right) \end{bmatrix} \qquad \frac{1}{S_y}r2 \rightarrow r2$$

Isolate $S_z h_{xz}$: $$\begin{aligned}\mathbf{R}_1 \bullet \mathbf{M}_3 &= \mathbf{R}_1 \bullet \left(S_z h_{xz} \mathbf{R}_1 + S_z h_{zy} \mathbf{R}_2 + S_z \mathbf{R}_3\right) \\ &= S_z h_{xz}\left(\mathbf{R}_1 \bullet \mathbf{R}_1\right) + S_z h_{zy}\left(\mathbf{R}_1 \bullet \mathbf{R}_2\right) + S_z\left(\mathbf{R}_1 \bullet \mathbf{R}_3\right) \\ &= S_z h_{xz}(1) + S_z h_{zy}(0) + S_z(0) \\ &= S_z h_{xz}\end{aligned}$$

$$\begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ S_z h_{zy} \mathbf{R}_2 + S_z \mathbf{R}_3 \end{bmatrix} \qquad -S_z h_{xz} r1 + r3 \rightarrow r3 \qquad \begin{array}{r} -S_z h_{xz} \mathbf{R}_1 \\ +S_z h_{xz} \mathbf{R}_1 + S_z h_{zy} \mathbf{R}_2 + S_z \mathbf{R}_3 \\ \hline S_z h_{zy} \mathbf{R}_2 + S_z \mathbf{R}_3 \end{array}$$

Isolate $S_z h_{zy}$: $$\begin{aligned}\mathbf{R}_2 \bullet r3 &= \mathbf{R}_2 \bullet \left(S_z h_{zy} \mathbf{R}_2 + S_z \mathbf{R}_3\right) \\ &= S_z h_{zy}\left(\mathbf{R}_2 \bullet \mathbf{R}_2\right) + S_z\left(\mathbf{R}_2 \bullet \mathbf{R}_3\right) \\ &= S_z h_{zy}(1) + S_z(0) \\ &= S_z h_{zy}\end{aligned}$$

$$\begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ S_z \mathbf{R}_3 \end{bmatrix} \qquad -S_z h_{zy} r2 + r3 \rightarrow r3 \qquad \begin{array}{r} -S_z h_{zy} \mathbf{R}_2 \\ +S_z h_{zy} \mathbf{R}_2 + S_z \mathbf{R}_3 \\ \hline S_z \mathbf{R}_3 \end{array}$$

Isolate S_z: $$\|r3\| = \|S_z \mathbf{R}_3\| = S_z \|\mathbf{R}_3\| = S_z$$

$$\begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \frac{1}{S_z} r3 \rightarrow r3$$

We have now extracted the scale terms, from which we can easily compute the shear terms. We have also isolated the rotation matrix **R**. We now turn our attention to extracting the individual rotation angles $\left(\theta_x, \theta_y, \theta_z\right)$.

Decomposing the Rotation Matrix (X-Y-Z order)

$$R_{13} = -\sin\theta_y$$
$$\theta_y = \sin^{-1}(-R_{13})$$

This is always defined.

$$\underbrace{R_{11} = \cos\theta_y \cos\theta_z \quad R_{12} = \cos\theta_y \sin\theta_z \quad \cos\theta_y = \frac{R_{11}}{\cos\theta_z} \quad \cos\theta_y = \frac{R_{12}}{\sin\theta_z}}$$
$$\cos\theta_y = \frac{R_{11}}{\cos\theta_z} = \frac{R_{12}}{\sin\theta_z}$$
$$\frac{\sin\theta_z}{\cos\theta_z} = \frac{R_{12}}{R_{11}}$$
$$\tan\theta_z = \frac{R_{12}}{R_{11}}$$
$$\theta_z = \tan^{-1}\left(\frac{R_{12}}{R_{11}}\right)$$

$$\underbrace{R_{23} = \sin\theta_x \cos\theta_y \quad R_{33} = \cos\theta_x \cos\theta_y \quad \cos\theta_y = \frac{R_{23}}{\sin\theta_x} \quad \cos\theta_y = \frac{R_{33}}{\cos\theta_x}}$$
$$\cos\theta_y = \frac{R_{23}}{\sin\theta_x} = \frac{R_{33}}{\cos\theta_x}$$
$$\frac{R_{23}}{R_{33}} = \frac{\sin\theta_x}{\cos\theta_x}$$
$$\tan\theta_x = \frac{R_{23}}{R_{33}}$$
$$\theta_x = \tan^{-1}\left(\frac{R_{23}}{R_{33}}\right)$$

As long as $\cos\theta_y \neq 0$, we can easily compute θ_x and θ_z. However, a singularity occurs when $\cos\theta_y = 0$, leaving us unable to determine the other two angles as described above. In this case, we must be a little more resourceful:

When $\theta_y = \frac{\pi}{2}$, $\cos\theta_y = 0$ and $\sin\theta_y = 1$. So,

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \sin\theta_x \cos\theta_z - \cos\theta_x \sin\theta_z & \sin\theta_x \sin\theta_z + \cos\theta_x \cos\theta_z & 0 & 0 \\ \cos\theta_x \cos\theta_z + \sin\theta_x \sin\theta_z & \cos\theta_x \sin\theta_z - \sin\theta_x \cos\theta_z & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & -1 & 0 \\ \sin(\theta_x - \theta_z) & \cos(\theta_x - \theta_z) & 0 & 0 \\ \cos(\theta_x - \theta_z) & \sin(\theta_x - \theta_z) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $(\theta_x - \theta_z) = \tan^{-1}\left(\frac{R_{32}}{R_{22}}\right)$

When $\theta_y = -\frac{\pi}{2}$, $\cos\theta_y = 0$ and $\sin\theta_y = -1$. So,

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -\sin\theta_x \cos\theta_z - \cos\theta_x \sin\theta_z & -\sin\theta_x \sin\theta_z + \cos\theta_x \cos\theta_z & 0 & 0 \\ -\cos\theta_x \cos\theta_z + \sin\theta_x \sin\theta_z & -\cos\theta_x \sin\theta_z - \sin\theta_x \cos\theta_z & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & -1 & 0 \\ -\sin(\theta_x + \theta_z) & \cos(\theta_x + \theta_z) & 0 & 0 \\ -\cos(\theta_x + \theta_z) & -\sin(\theta_x + \theta_z) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $(\theta_x + \theta_z) = \tan^{-1}\left(\frac{-R_{32}}{R_{22}}\right)$

In this case, the two angles act as one, so we can simply let one of the angles be zero.

Composing the Rotation Matrix (Z-Y-X order)

Rotations are not commutative. An x-y-z rotation is not the same as a z-y-x rotation with the same angles.

Rotation (z-y-x):

$$\mathbf{R} = \mathbf{R}_z\mathbf{R}_y\mathbf{R}_x = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 & 0 \\ -\sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x & 0 \\ 0 & -\sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \cos\theta_y\cos\theta_z & \sin\theta_x\sin\theta_y\cos\theta_z + \cos\theta_x\sin\theta_z & -\cos\theta_x\sin\theta_y\cos\theta_z + \sin\theta_x\sin\theta_z & 0 \\ -\cos\theta_y\sin\theta_z & -\sin\theta_x\sin\theta_y\sin\theta_z + \cos\theta_x\cos\theta_z & \cos\theta_x\sin\theta_y\sin\theta_z + \sin\theta_x\cos\theta_z & 0 \\ \sin\theta_y & -\sin\theta_x\cos\theta_y & \cos\theta_x\cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A single rotation matrix can be decomposed into a number of different angle components depending on their multiplication order. However, the rotation itself is unique even though there are multiple ways to represent it. Consequently, once we have a rotation matrix —no matter how it was arrived at—we can choose to decompose it into any angle order we wish.

Decomposing the Rotation Matrix (Z-Y-X order)

$$R_{31} = \sin\theta_y$$
$$\theta_y = \sin^{-1}\left(R_{31}\right)$$

This is always defined.

$$R_{32} = -\sin\theta_x \cos\theta_y \qquad R_{33} = \cos\theta_x \cos\theta_y$$
$$\underbrace{\cos\theta_y = \frac{-R_{32}}{\sin\theta_x} \qquad \cos\theta_y = \frac{R_{33}}{\cos\theta_x}}$$
$$\cos\theta_y = \frac{-R_{32}}{\sin\theta_x} = \frac{R_{33}}{\cos\theta_x}$$
$$\frac{\sin\theta_x}{\cos\theta_x} = -\frac{R_{32}}{R_{33}}$$
$$\tan\theta_x = -\frac{R_{32}}{R_{33}}$$
$$\theta_x = \tan^{-1}\left(-\frac{R_{32}}{R_{33}}\right)$$

$$R_{11} = \cos\theta_y \cos\theta_z \qquad R_{21} = -\cos\theta_y \sin\theta_z$$
$$\underbrace{\cos\theta_y = \frac{R_{11}}{\cos\theta_z} \qquad \cos\theta_y = \frac{-R_{21}}{\sin\theta_z}}$$
$$\cos\theta_y = \frac{R_{11}}{\cos\theta_z} = \frac{-R_{21}}{\sin\theta_z}$$
$$\frac{\sin\theta_z}{\cos\theta_z} = -\frac{R_{21}}{R_{11}}$$
$$\tan\theta_z = -\frac{R_{21}}{R_{11}}$$
$$\theta_z = \tan^{-1}\left(-\frac{R_{21}}{R_{11}}\right)$$

As long as $\cos\theta_y \neq 0$, we can easily compute θ_x and θ_z. However, a singularity occurs when $\cos\theta_y = 0$, leaving us unable to determine the other two angles as described above. In this case, we must be a little more resourceful:

When $\theta_y = \frac{\pi}{2}$, $\cos\theta_y = 0$ and $\sin\theta_y = 1$. So,

$$\mathbf{R} = \begin{bmatrix} 0 & \sin\theta_x \cos\theta_z + \cos\theta_x \sin\theta_z & -\cos\theta_x \cos\theta_z + \sin\theta_x \sin\theta_z & 0 \\ 0 & -\sin\theta_x \sin\theta_z + \cos\theta_x \cos\theta_z & \cos\theta_x s\sin\theta_z + \sin\theta_x \cos\theta_z & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \sin\left(\theta_x + \theta_z\right) & -\cos\left(\theta_x + \theta_z\right) & 0 \\ 0 & \cos\left(\theta_x + \theta_z\right) & \sin\left(\theta_x + \theta_z\right) & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $\left(\theta_x + \theta_z\right) = \tan^{-1}\left(\frac{R_{12}}{R_{22}}\right)$

When $\theta_y = -\frac{\pi}{2}$, $\cos\theta_y = 0$ and $\sin\theta_y = -1$. So,

$$\mathbf{R} = \begin{bmatrix} 0 & -\sin\theta_x \cos\theta_z + \cos\theta_x \sin\theta_z & \cos\theta_x \cos\theta_z + \sin\theta_x \sin\theta_z & 0 \\ 0 & \sin\theta_x \sin\theta_z + \cos\theta_x \cos\theta_z & -\cos\theta_x \sin\theta_z + \sin\theta_x \cos\theta_z & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -\sin\left(\theta_x - \theta_z\right) & \cos\left(\theta_x - \theta_z\right) & 0 \\ 0 & \cos\left(\theta_x - \theta_z\right) & \sin\left(\theta_x - \theta_z\right) & 0 \\ -1 & -0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $\left(\theta_x - \theta_z\right) = \tan^{-1}\left(\frac{-R_{23}}{R_{13}}\right)$

In this case, the two angles act as one, so we can simply let one of the angles be zero.

Deriving the Normal Vector Transform

Normal vectors are not transformed by the same matrix as vertices.

Consider a vector $\bar{v}$ on a surface and its normal $\bar{n}$. They are transformed to a new coordinate system by $\bar{v}' = \bar{v}M$ and $\bar{n}' = \bar{n}G$.
Since $\bar{v}$ and $\bar{n}$ are perpendical, $\bar{v} \bullet \bar{n} = 0$.

We also want the transformed $\bar{v}' \bullet \bar{n}' = 0$. (Transformed normals remain perpendicular to their transformed surface.)

So $\bar{v}' \bullet \bar{n}' = (\bar{v}M) \bullet (\bar{n}G) = 0$. The dot product can be re-written as a matrix multiplication:

$$
\begin{aligned}
(\bar{v}M) \bullet (\bar{n}G) &= 0 \\
(\bar{v}M)(\bar{n}G)^T &= 0 \\
\bar{v}MG^T\bar{n}^T &= 0
\end{aligned}
$$

Note that since $\bar{v} \bullet \bar{n} = \bar{v}\bar{n}^T = 0$, it must hold true that $MG^T = I_4$
Now we isolate G, the matrix which transforms normals.

$$
\begin{aligned}
MG^T &= I_4 \\
M^{-1}MG^T &= M^{-1}I_4 \\
G^T &= M^{-1} \\
G &= \left(M^{-1}\right)^T
\end{aligned}
$$