Solución a los Ejercicios de Haberman

Cap 1

1.2.1 a) $\boxed{\dfrac{\partial e}{\partial t} = -\dfrac{\partial \Phi}{\partial t} + \cancel{Q}}$ → 0 without sources

time rate of change of thermal energy per unit volume.

positional rate of change of heat flow per unit area per unit time.

if the heat energy flux is increasing then more heat flows to the right than in from the left and thus the thermal energy must decrease.

if the heat energy flux is decreasing then more heat flows in the left than out through the right and thus the thermal energy must increase

b) $\boxed{\Phi = -K_0 \dfrac{\partial u}{\partial x}}$ → positional rate of change of temperature

↳ Heat flow per unit area per unit time.

if the temperature is increasing with position, then since heat flows from hotter regions to cooler regions Φ must negative.

c) $\boxed{c\rho \dfrac{\partial u}{\partial t} = -\dfrac{\partial \Phi}{\partial t} + \cancel{Q}}$ → 0 without sources

time rate of change of temperature

positional rate of change of heat flow per unit time per unit area.

if the heat energy flux is increasing then mor heat flow out the right than in from the left and thus the temperature must decrease with time.

d) $\boxed{\Phi = -K_0 \dfrac{\partial u}{\partial x}}$ The explanation is similar than for part b)

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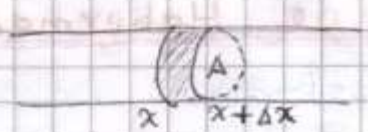
(1.2.2) a)



Heat energy for the slice $= e(x,t) A \Delta x$ (density of energy)

$\Rightarrow$ (Rate of change of energy in slice) = (flow of energy throughout the slice without sources)

$$\frac{\partial}{\partial t} e(x,t) \cancel{A} \Delta x \approx [\phi(x,t) - \phi(x+\Delta x, t)] \cancel{A}$$

$$\frac{\partial}{\partial t} e(x,t) \approx \frac{\phi(x,t) - \phi(x+\Delta x, t)}{\Delta x}$$

When take $\lim \Delta x \to 0 \Rightarrow \boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}}$ (1)

Energy per unit mass $= c\, u(x,t)$
↳ specific heat

Total mass $= \rho A \Delta x$
↳ mass density

$\Rightarrow$ total thermal energy $= c\, u(x,t) \rho A \Delta x$

$e(x,t) \cancel{A \Delta x} = c\, u(x,t) \rho \cancel{A \Delta x} \rightarrow e = \rho c u$

$$\Rightarrow \boxed{\rho(x) c(x) \frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}} \quad (2)$$

But $\boxed{\phi = -K_0 \frac{\partial u}{\partial x}}$ → Fourier's law of heat conductivity

$$\Rightarrow \rho(x) c \frac{\partial u}{\partial t} = +K_0 \frac{\partial^2 u}{\partial x^2} \rightarrow \boxed{\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}} \quad (3) \qquad K = \frac{K_0}{\rho c}$$

b) Total Energy $= \int_a^b e(x,t) A\, dx$ → Sum of infinitesimal slices.

$$\frac{\partial}{\partial t} \int_a^b e(x,t) \cancel{A}\, dx \simeq [\phi(a,t) - \phi(b,t)] \cancel{A}$$

$$\int_a^b \frac{\partial e}{\partial t} dx = -\int_a^b \frac{\partial \phi}{\partial x} dx$$

$$\int_a^b \left(\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x}\right) dx = 0 \longrightarrow \boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}} \quad (1)$$

by similar way than a) and applying Fourier's Law we can obtain (3).

(1.2.3) With no sources, but with variable cross section $A(x)$, eq (1.2.4) is:

$$\frac{d}{dt}\int_a^b e(x,t)A(x)\,dx = [\phi(a,t)A(a) - \phi(b,t)A(b)] = -\int_a^b \frac{\partial}{\partial x}\left(\phi(x,t)A(x)\right)dx$$

$$\Rightarrow \int_a^b \left\{\frac{\partial[e(x,t)A(x)]}{\partial t} + \frac{\partial}{\partial x}\left(\phi(x,t)A(x)\right)\right\}dx = 0$$

$$\Rightarrow \frac{\partial[e(x,t)A(x)]}{\partial t} = -\frac{\partial}{\partial x}[\phi(x,t)A(x)]$$

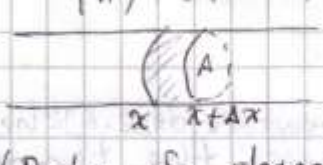
Fourier's Law still holds: $\phi(x,t) = -K_0\frac{\partial u}{\partial x}$

$$\Rightarrow \frac{\partial[e(x,t)A(x)]}{\partial t} = +K_0\frac{\partial}{\partial x}[A(x)u(x)]$$ and using $e = c\rho u$

$K = K_0/c\rho$

$$c\rho A(x)\frac{\partial u(x,t)}{\partial t} = K_0\frac{\partial}{\partial x}[A(x)u(x)] \longrightarrow \frac{\partial u}{\partial t} = \frac{K}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial u}{\partial x}\right)$$

(1.2.4) This problem is similar to (1.2.2), only change the physical interpretation:

a) 

$u(x)$ give directly density = Concentration of the chemical per unit volume.

$\Rightarrow$ (Rate of change of concentration) = (flow of concentration through the slice with no sources)

$$\frac{\partial}{\partial t}u(x,t)\cancel{A}\Delta x \approx [\phi(x,t) - \phi(x+\Delta x,t)]\cancel{A}$$

$$\frac{\partial u}{\partial t} = \lim_{\Delta x\to 0}\left[\frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x}\right] = -\frac{\partial \phi}{\partial x}$$

diffusivity

using Fick's law of diffusion: $\phi = -K\frac{\partial u}{\partial x} \Rightarrow \boxed{\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}}$

b) Total concentration $= \int_a^b u(x,t)A\,dx$

$$\frac{\partial}{\partial u}\int_a^b u(x,t)\cancel{A}\,dx \approx [\phi(a,t) - \phi(b,t)]\cancel{A}$$

$$\int_a^b \frac{\partial u}{\partial t}dx = -\int_a^b \frac{\partial \phi}{\partial x}dx \longrightarrow \frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}$$

applying Fick's law $\Rightarrow$ $\boxed{\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}}$

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source of concentration by chemical reaction.

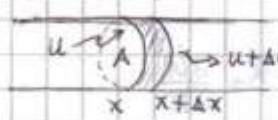
1.2.5 $$\frac{d}{dt}\int_a^b e\,\cancel{A}\,dx = [\phi(a,t) - \phi(b,t)]\cancel{A} + \int_a^b \alpha u(\beta - u)\cancel{A}\,dx$$

$$\Rightarrow \int_a^b \left[\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - \alpha u(\beta - u)\right] dx = 0 \rightarrow \frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + \alpha u(\beta - u)$$

but $e = c\rho u$ and since $\phi = -K_0 \frac{\partial u}{\partial x}$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + \alpha' u(\beta - u)} \qquad K = \frac{K_0}{c\rho} \qquad \alpha' = \frac{\alpha}{c\rho}$$

1.2.6



a) For small interval of temperatures from $u(x,t)$ to $u(x,t)+\Delta u$:

$$\Delta\begin{pmatrix}\text{Heat energy}\\ \text{per unit mass}\end{pmatrix} = \underbrace{\begin{pmatrix}\text{Heat energy per unit}\\ \text{mas per unit temperature}\end{pmatrix}}_{c(x,u)} \times \Delta u$$

$$\boxed{\begin{pmatrix}\text{Heat energy}\\ \text{per unit mass}\end{pmatrix} = \int_0^u c(x,u)\,du}$$

b) from (1.2.6): $e(x,t) = \rho(x)\,c(x)\,u(x,t)$, however now the heat energy per unit mass necessary to rise the temperature is not $c(x)u(x,t)$, but instead $\int_0^u c(x,\bar{u})\,d\bar{u}$; then the expression (1.2.6) changes to

$$e(x,t) = \rho(x)\int_0^u c(x,\bar{u})\,d\bar{u}$$

$$\Rightarrow \frac{\partial e}{\partial t} = \rho\frac{\partial}{\partial t}\int_0^u c(x,\bar{u})\,d\bar{u} = \rho\int_0^u \frac{\partial c}{\partial t}\,du = \rho\int_0^u \frac{\partial c}{\partial u}\frac{\partial u}{\partial t}\,du$$

$$= \rho\frac{\partial u}{\partial t}\int_0^u \frac{\partial c}{\partial u}\,du = \rho\frac{\partial u}{\partial t}\int_0^u dc = c\rho\frac{\partial u}{\partial t}$$

$\boxed{\frac{\partial e}{\partial t} = c\rho\frac{\partial u}{\partial t}}$ the insert this in (1.2.5) for obtain:

$c\rho\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$ and Fourier's law: $\phi = -K_0\frac{\partial u}{\partial x}$

$$\Rightarrow \boxed{c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q}$$

(1.2.7) $\frac{\partial}{\partial b}\int_a^b f(x)\,dx = f(b)$ → Fundamental theorem of calculus.

$\underbrace{\frac{d}{dt}\int_a^b e\,dx}_{} = \phi(a,t) - \phi(b,t) + \int_a^b Q\,dx$ → Ec of conservation of heat energy.

$\frac{\partial}{\partial b}\int_a^b \frac{\partial e(x,t)}{\partial t}dx = \cancel{\frac{\partial \phi(a,t)}{\partial b}}^{\,0} - \frac{\partial \phi(b,t)}{\partial b} + \frac{\partial}{\partial b}\int_a^b Q\,dx$ by Fundamental theorem of calculus:

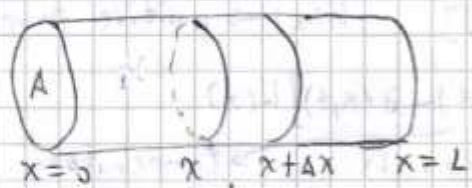
$\therefore \frac{\partial e(b,t)}{\partial t} = -\frac{\partial}{\partial b}\phi(b,t) + Q(b,t)$

change $b \to x \Rightarrow \frac{\partial}{\partial t} e(x,t) = -\frac{\partial}{\partial x}\phi(x,t) + Q(x,t)$

$e = c\rho u$ and $\phi(x,t) = -K_0 \frac{\partial u}{\partial x}$ (Fourier's law).

$\Rightarrow \boxed{c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q(x,t)}$

*(1.2.8)



$e(x,t) \equiv$ thermal energy density

↳ $e(x,t)$ may be aproximated as a constant throughout the volume between x and $x+\Delta x$ since Δx is exceedingly small.

$\Rightarrow$ heat energy for slice $= e(x,t)\,A\,\Delta x$

by otherwise, the total thermal energy in any thin slice is

$c(x)\,u(x,t)\cdot\rho A\Delta x$

$\Rightarrow \quad e(x,t)\,\cancel{A\Delta x} = c(x)\,u(x,t)\cdot\rho\,\cancel{A\Delta x}$

$\rightarrow \begin{pmatrix}\text{heat energy}\\ \text{by slice}\end{pmatrix} = c(x)\,u(x)\,\rho(x)\,A\,\Delta x$

$\boxed{\begin{pmatrix}\text{Total thermal energy}\\ \text{contained in a rod}\end{pmatrix} = A\int_0^L c(x)\,u(x)\,\rho(x)\,dx}$

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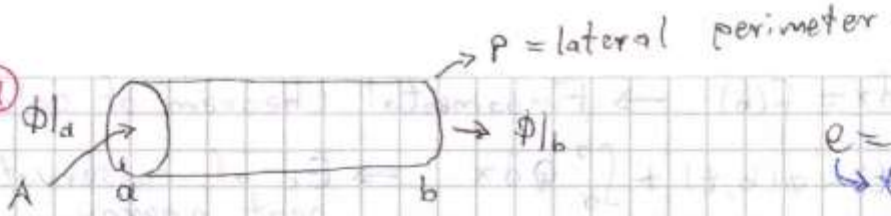
1.2.9

a)



$e = c\rho u$ ↳ thermal energy density

$$\frac{\partial}{\partial t}\int_a^b eA\,dx = [\phi(a,t) - \phi(b,t)]A - \underbrace{P\int_a^b w\,dx}_{\text{Flux through sides}} + A\int_a^b \cancel{Q}\,dx$$

0 no sources.

$$\Rightarrow A\int_a^b \frac{\partial e}{\partial t}dx = -A\int_a^b \frac{\partial \phi}{\partial x}dx - P\int_a^b w\,dx$$

$$c\rho\int_a^b \frac{\partial u}{\partial t}dx = -\int_a^b\left(\frac{\partial \phi}{\partial x} - \frac{P}{A}w\right)dx \rightarrow \int_a^b\left(c\rho\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} + \frac{P}{A}w\right)dx = 0$$

$$c\rho\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} - \frac{P}{A}w(x,t) \qquad \text{but } \phi = -K_0\frac{\partial u}{\partial x}$$

$$\Rightarrow \boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}w(x,t)}$$

b) If $w(x) = h(x)[u(x,t) - \gamma(x,t)]$ $\quad h(x) \rightarrow$ positive proporcionality

$$\Rightarrow c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}[u(x,t) - \gamma(x,t)]h(x)$$

$\rightarrow$ temperature dependent

c) with heat sources the equation is:

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \frac{1}{c\rho}Q(x,t) \rightarrow \text{Contribution of sources}$$

d) $A = \pi R^2 \quad P = 2\pi R \quad \gamma = 0 \rightarrow$ outside temperature.

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{2\pi R}{\pi R^2}[u - 0]h$$

$$\boxed{\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} - \beta u} \qquad \text{where } K = K_0/c\rho \qquad \beta = \frac{2h}{c\rho R}$$

e) If $u(x,t) = u(t)$ and $u(0) = u_0$

$$\Rightarrow \frac{\partial u}{\partial t} = K\cancel{\frac{\partial^2 u}{\partial x^2}} - \beta u \longrightarrow u(t) = Ce^{-\beta t}$$

0 $u(x,t) = u(t)$ no dependent of x

$$\boxed{u(t) = u_0 e^{-\beta t}} \qquad \text{with } \beta = 2h/c\rho R$$

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1.3.1 By Newton's law of cooling $\phi(L,t) = H[u(L,t) - u_B(t)]$

if $u(L,t) > u_B(t)$, then heat flows to the right, and hence

$\phi(L,t) > 0$, thus $H > 0$

Since $\phi = -K_0 \frac{\partial u}{\partial x}$ → Fourier's law

$$\Rightarrow \boxed{-K_0(L)\frac{\partial u(L,t)}{\partial x} = H[u(L,t) - u_B]} \quad \text{where } H > 0.$$

* **1.3.2** if the temperature is continuous at $x = x_0$:

$u(x_{0-}, t) = u(x_{0+}, t)$ and no heat energy is lost at $x = x_0$ (i.e. the heat energy flowing out of one flows into the other)

$\Rightarrow \phi(x_{0-}, t) = \phi(x_{0+}, t)$

Since $\phi = -K_0 \frac{\partial u}{\partial x} \Rightarrow -K_0(x_{0-})\frac{\partial u}{\partial x}(x_{0-}, t) = -K_0(x_{0+})\frac{\partial u}{\partial x}(x_{0+}, t)$

thus, the latter condition at $x = x_0$ is this.

Under the condition that $K_0(x_{0-}) = K_0(x_{0+})$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x}(x_{0-}, t) = \frac{\partial u}{\partial x}(x_{0+}, t)}$$ that is $\frac{\partial u}{\partial x}$ is continuous at $x = x_0$.

* **1.3.3** $\phi(L_-, t) = \phi(L_+, t)$, since $\phi = -K_0 \frac{\partial u}{\partial x}$ Fourier's law

$$\begin{cases} \underbrace{-K_0(L_-)\frac{\partial u(L_-,t)}{\partial x}} = -K_0(L_+)\frac{\partial u(L_+,t)}{\partial x} \\ C(L_-)\rho(L_-)\frac{\partial u(L_-,t)}{\partial t} \end{cases}$$

→ assuming no energy lost (i.e, the heat energy flowing out rod flows into fluid)

→ By heat equation

~~$k) \frac{\partial u}{\partial x}(L_-,t) = H[u(L_-,t) - u_f(t)]$~~

~~$-K_0(L_+)\frac{\partial u}{\partial x}(L_+,t) = -H[u_f(L_+,t) - u(L_+,t)]$~~ ⇒ ~~this is similar to (1.3.4) assuming in L_+ start fluid~~

~~by the continuity condition~~

Since $u(L_-,t) = u(L_+,t)$ and $C(L_-) = C_f$, $\rho(L_-) = C_f$ f: fluid

$\partial u/\partial x|_{L_-} = \partial u/\partial x|_{L_+}$ $K_0(L_+) = K_0(L)$

$$\Rightarrow \boxed{C_f \rho_f \frac{\partial u}{\partial x}(L,0) = -K_0(L)\frac{\partial u}{\partial x}(L,t)}$$

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→ steady-state

1.4.1 For equilibrium solution to be a temperature distribution that does not depend on time, that is,

$$u(x,t) = u(x) \rightarrow \partial u/\partial t = 0$$

$$\Rightarrow c\rho \cancel{\frac{\partial u}{\partial t}}^{\,0} = K_0 \frac{\partial^2 u}{\partial x^2} + Q \rightarrow \boxed{\frac{d^2u}{dx^2} = -\frac{Q}{K_0}}$$

* a) $Q=0 \rightarrow \frac{d^2u}{dx^2} = 0 \rightarrow u(x) = C_1 x + C_2$

$u(0) = 0 \rightarrow C_2 = 0$

$u(L) = T = C_1 L \rightarrow C_1 = T/L$

$$\Rightarrow \boxed{u(x) = \frac{T}{L}x}$$

b) $Q=0 \rightarrow \frac{d^2u}{dx^2} = 0 \rightarrow u(x) = C_1 x + C_2$

$u(0) = T \rightarrow C_2 = T$

$u(L) = 0 = C_1 L + T \rightarrow C_1 = -T/L$

$$\Rightarrow \boxed{u(x) = -\frac{T}{L}x + T}$$

c) $Q=0 \rightarrow \frac{d^2u}{dx^2} = 0 \rightarrow u(x) = C_1 x + C_2$

$\frac{\partial u(0)}{\partial x} = C_1 = 0$ and $u(L) = T = C_2$

$$\Rightarrow \boxed{u(x) = T}$$

* d) $Q=0 \rightarrow \frac{d^2u}{dx^2} = 0 \rightarrow u(x) = C_1 x + C_2$

$u(0) = T \rightarrow C_2 = T$

$\frac{\partial u(L)}{\partial x} = \alpha = C_1 \rightarrow \boxed{u(x) = \alpha x + T}$

e) $\frac{Q}{K_0} = 1 \rightarrow \frac{d^2u}{dx^2} = -1 \rightarrow \frac{du}{dx} = -x + C_1 \rightarrow u = -\frac{1}{2}x^2 + C_1 x + C_2$.

$u(0) = T_1 = C_2$, $u(L) = T_2 = -\frac{1}{2}L^2 + C_1 L + T_1$

$0 = -\frac{1}{2}L^2 + C_1 L + T_1 - T_2$

$$\rightarrow C_1 = \frac{T_2 - T_1}{L} + \frac{1}{2}L \Rightarrow \boxed{u(x) = -\frac{1}{2}x^2 + \left[\frac{T_2 - T_1}{L} + \frac{L}{2}\right]x + T_1}$$

* f) $\frac{Q}{K_0} = x^2 \rightarrow \frac{d^2u}{dx^2} = -x^2 \rightarrow \frac{du}{dx} = -\frac{1}{3}x^3 + C_1 \rightarrow u = -\frac{1}{12}x^4 + C_1 x + C_2$

$u(0) = T = C_2$ and $\frac{\partial u(L)}{\partial x} = -\frac{1}{3}L^3 + C_1 = 0 \rightarrow C_1 = \frac{1}{3}L^3$

$$\Rightarrow \boxed{u(x) = -\frac{1}{12}x^4 + \frac{1}{3}L^3 x + T}$$

g) $Q=0 \rightarrow \frac{d^2u}{dx^2}=0 \rightarrow u(x)=c_1x+c_2$

$u(0)=T=c_2 \rightarrow u(x)=c_1x+T$

$\frac{\partial u(L)}{\partial x}+u(L)=0=c_1+c_1L+T \rightarrow c_1=\frac{-T}{1+L}$

$\Rightarrow \boxed{u(x)=-\frac{T}{1+L}x+T}$

* h) $Q=0 \rightarrow \frac{d^2u}{dx^2}=0 \rightarrow u(x)=c_1x+c_2$

$\frac{\partial u(L)}{\partial x}=\alpha=c_1 \rightarrow u(x)=\alpha x+c_2$

$\frac{\partial u}{\partial x}(0)-[u(0)-T]=0=\alpha-[c_2-T] \rightarrow c_2=+(T+\alpha)$

$\Rightarrow \boxed{u(x)=\alpha x+T+\alpha}$

⊙ 1.4.2 $c\rho\frac{\partial u}{\partial t}=K_0\frac{\partial^2 u}{\partial x^2}+Q$ → 0 at equilibrium → because $u(x,t)=u(x)$ in steady-state

$\Rightarrow \frac{d^2u}{dx^2}=-\frac{Q}{K_0}=-x$ where $Q=K_0x$

* a) $Q \rightarrow$ heat energy per unit volume generated per unit time. $\Rightarrow \underbrace{\int_0^L Q\,dx=\int_0^L K_0x\,dx=\frac{K_0L^2}{2}}_{\text{Energy generated per unit time inside the entire rod.}}$

b) $\frac{du}{dx}=-\frac{1}{2}x^2+c_1 \rightarrow u(x)=-\frac{1}{6}x^3+c_1x+c_2$

$u(0)=0=c_2$

$u(L)=0=-\frac{1}{6}L^3+c_1L \rightarrow c_1=L^2/6 \Rightarrow \frac{du}{dx}=-\frac{1}{2}x^2+\frac{L^2}{6}$

and the heat energy flowing of the rod per unit time is

$\phi=-K_0\frac{du}{dx}=\frac{K_0}{2}x^2-\frac{K_0}{6}L^2 \Rightarrow \boxed{\begin{array}{l}\phi(0)=-K_0L^2/6\\ \phi(L)=K_0L^2/3\end{array}}$

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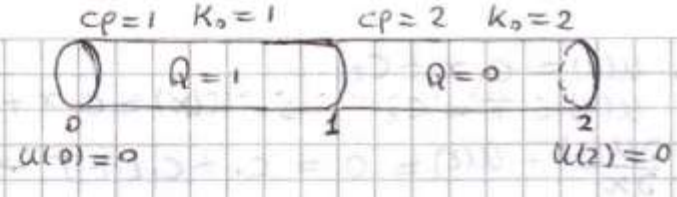
1.4.3



$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q$$

For $0 < x < 1$: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1$ but $u(x,t) = u(x)$ in equilibrium

$\Rightarrow \frac{\partial u}{\partial t} = 0 \rightarrow \frac{d^2 u}{dx^2} = -1 \rightarrow \frac{du}{dx} = -x + c_1 \rightarrow u = -\frac{1}{2}x^2 + c_1 x + c_2$

$u(0) = 0 = c_2 \rightarrow u(x) = -\frac{1}{2}x^2 + c_1 x$

For $1 < x < 2$: $\cancel{2}\frac{\partial u}{\partial t} = \cancel{2}\frac{\partial^2 u}{\partial x^2} + 0$ but $u(x,t) = u(x)$ in equilibrium

$\Rightarrow \frac{\partial u}{\partial t} = 0 \rightarrow \frac{d^2 u}{dx^2} = 0 \rightarrow \frac{du}{dx} = d_1 \rightarrow u(x) = d_1 x + d_2$

$u(2) = 0 = 2d_1 + d_2 \rightarrow d_2 = -2d_1 \Rightarrow u(x) = d_1(x-2)$

at $x = 1$ by the perfect thermal contact there are conditions of continuity:

$u(1^-) = u(1^+)$ and $K_0(1^-)\frac{\partial u}{\partial x}\Big|_{x=1^-} = K_0(1^+)\frac{\partial u}{\partial x}\Big|_{x=1^+}$

$-\frac{1}{2} + c_1 = -d_1$

$1 \cdot (-1 + c_1) = 2 \cdot (d_1)$

$-1 + c_1 = 2 \cdot (\frac{1}{2} - c_1)$

$\boxed{c_1 = 2/3}$

$\rightarrow d_1 = +\frac{1}{2} - \frac{2}{3} \rightarrow \boxed{d_1 = -\frac{1}{6}}$

$$\Rightarrow \boxed{u(x) = \begin{cases} -\frac{1}{2}x^2 + \frac{2}{3}x & 0 < x \leq 1 \\ -\frac{1}{6}(x-2) & 1 \leq x < 2 \end{cases}}$$

1.4.4 $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \rightarrow$ Heat Eq with no sources

Since the rod is insulated $\rightarrow \frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$

Total energy $\rightarrow E(t) = \int_0^L e \, dx = c\rho \int_0^L u(x,t)\, dx$

rod insulated

$$\frac{dE}{dt} = c\rho \int_0^L \frac{\partial u}{\partial t} dx = c\rho K \int_0^L \frac{\partial^2 u}{\partial x^2} dx = c\rho K \left[\cancel{\frac{\partial u}{\partial x}(L,t)}^{\,0} - \cancel{\frac{\partial u}{\partial x}(0,t)}^{\,0}\right]$$

$\Rightarrow \boxed{\frac{dE}{dt} = 0} \rightarrow E = cte \rightarrow$ Conservation of energy.

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1.4.5 $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ and $u(x,t) = u(x)$ in steady-state

$$\Rightarrow \frac{\partial u}{\partial t} = 0 = K\frac{\partial^2 u}{\partial x^2} \rightarrow \frac{d^2u}{dx^2} = 0 \rightarrow u(x) = C_1 x + C_2$$

$$\boxed{u(0) = C_2}$$

By Fourier's law: $\varphi(0) = -K_0 \frac{\partial u}{\partial x}(0) = -K_0 \cdot C_1 \rightarrow \boxed{C_1 = -\frac{\varphi(0)}{K_0}}$

$$\Rightarrow u(x) = -\frac{\varphi(0)}{K_0}x + u(0) \rightarrow \boxed{u(L) = -\frac{\varphi(0)}{K_0}\cdot L + u(0)}$$

⊙ **1.4.6** $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + \frac{Q_0}{c\rho}$ $\quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$

↳ ends insulated.

a)

$$E(t) = \int_0^L e\,dx = c\rho\int_0^L u(x,t)\,dx$$

$$\frac{dE}{dt} = c\rho\int_0^L \frac{\partial u}{\partial t} = c\rho\int_0^L\left[K\frac{\partial^2 u}{\partial x^2} + \frac{Q_0}{c\rho}\right]dx = c\rho\left[K\left(\cancel{\frac{\partial u}{\partial x}(L)}^{\,0} - \cancel{\frac{\partial u}{\partial x}(0)}^{\,0}\right) + \frac{Q_0}{c\rho}L\right]$$

$\boxed{\frac{dE}{dt} = Q_0 L \neq 0}$ → No conservation of energy ⇒ does not exist any equilibrium temperature.

b) $$\frac{d}{dt}\underbrace{\int_0^L e\,dx} = \cancel{\phi(0,t)}^{\,0} - \cancel{\phi(L,t)}^{\,0} + \int_0^L Q_0\,dx$$

→ ends insulated, then there are not flux through the ends.

$$\frac{d}{dt}E(t) = Q_0 x\Big|_0^L = Q_0 L$$

↳ flux or variation of energía is only caused by the source.

$\Rightarrow E(t) = Q_0 L t + C_1$ when $t = 0$ $\quad E(0) = C_1$

by otherhand $E = c\rho\int_0^L u(x,t)\,dx$, but in $t=0$ $u(x,t) = f(x)$

$$\rightarrow E(0) = c\rho\int_0^L f(x)\,dx = C_1$$

$$\Rightarrow \boxed{E(t) = Q_0 L t + c\rho\int_0^L f(x)\,dx}$$

1.4.7 a) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1$, $u(x,0) = f(x)$ $\frac{\partial u}{\partial x}(0,t) = 1$ $\frac{\partial u}{\partial x}(L,t) = \beta$

in equilibrium $u(x,t) = u(x) \rightarrow \frac{\partial u}{\partial t} = 0 = \frac{\partial^2 u}{\partial x^2} + 1$

$\Rightarrow \frac{d^2 u}{dx^2} = -1 \rightarrow \frac{du}{dx} = -x + c_1 \rightarrow \boxed{u_e(x) = -\frac{x^2}{2} + c_1 x + c_2} \rightarrow$ Solution for the equilibrium

$\frac{\partial u}{\partial x}(0) = 1 = c_1$

$\frac{\partial u}{\partial x}(L) = -L + 1 = \beta \rightarrow \boxed{\beta = 1 - L} \rightarrow$ Condition for the equilibrium.

In order to determine the constant C_2, we integrate the full equation:

$$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + 1\right) dx = \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) + L$$

$$= \cancel{\beta} - \cancel{1} + L = 0$$

$\Rightarrow \frac{d}{dt}\int_0^L u(x,t)\,dx = 0 \rightarrow$ This says that the quantity $\int_0^L u(x,t)\,dx = cte$ for all t.

in particular $\int_0^L u(x,0)\,dx = \int_0^L u_e(x)\,dx$

$$\int_0^L f(x)\,dx = \int_0^L \left(-\frac{x^2}{2} + x + C_2\right) dx = -\frac{L^3}{6} + \frac{L^2}{2} + C_2 L$$

$$\Rightarrow C_2 = \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx$$

$$\rightarrow \boxed{u_e(x) = -\frac{x^2}{2} + x + \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx}$$

The condition $\beta = 1 - L$ ensures that the rate of heat loss through the boundaries exactly equals the rate of heat production in the interval $0 < x < L$

The PDE is the heat eq (with $K_0 = c\rho$) with a constant source $Q = c\rho$, so heat is being generated at each x in the interval $0 < x < L$. The total rate at which heat is generated in the interval is $A\int_0^L Q(x,t)\,dx = Ac\rho L$.
At the left end, we have $\frac{\partial u}{\partial x}(0,t) = 1$. From Fourier's law, we know that $\phi = -K_0 \frac{\partial u}{\partial x}$, so the left boundary condition says heat flow rate through the left end is $A\phi = -AK_0 = -Ac\rho$.
Heat is also being forced to move through the right end at a constant rate, with the total heat flow through the right being $-Ac\rho\beta$. (The direction depends on the sign of β. If $\beta > 0$, heat is flowing into the region, while if $\beta < 0$, heat is flowing out of the region.)

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The net rate of heat flow out of the region is:

$$Ac\rho - Ac\rho\beta$$

In order to have an equilibrium solution, the rate at which heat is generated within the interval $0<x<L$ must equal the rate at which heat is removed through the boundaries. Thus we must have

$$Ac\rho - Ac\rho\beta = (QL)A = Ac\rho L$$

$$\Rightarrow \boxed{1-\beta = L}$$ condition for the equilibrium.

b) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\quad u(x,0)=f(x)$ $\quad \frac{\partial u}{\partial x}(0,t)=1$ $\quad \frac{\partial u}{\partial x}(L,t)=\beta$

in equilibrium $u(x,t)=u(x) \rightarrow \frac{\partial u}{\partial t} = 0 = \frac{\partial^2 u}{\partial x^2} \rightarrow \frac{d^2u}{dx^2}=0$

$$\rightarrow u_e(x) = c_1 x + c_2 \qquad \left.\begin{array}{l} \frac{\partial u}{\partial x}(x=0) = c_1 = 1 \\ \frac{\partial u}{\partial x}(x=L) = c_1 = \beta \end{array}\right\} \Rightarrow \boxed{\beta = 1}$$ Condition for the equilibrium

$$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) = \beta - 1 = 0$$

$\frac{d}{dt}\int_0^L u(x,t)\,dx = 0 \rightarrow$ This says that the quantity $\int_0^L u(x,t)\,dx = cte$ for all t.

in particular: $\int_0^L u(x,0)\,dx = \int_0^L u_e(x)\,dx$

$$\int_0^L f(x)\,dx = \int_0^L (x+c_2)\,dx = \frac{L^2}{2} + c_2 L$$

$$\Rightarrow c_2 = -\frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx \rightarrow \boxed{u_e(x) = x - \frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx}$$

c) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x - \beta$ $\quad u(x,0)=f(x)$ $\quad \frac{\partial u}{\partial x}(0,t)=0$ $\quad \frac{\partial u}{\partial x}(L,t)=0$

in equilibrium $u(x,t)=u(x) \rightarrow \frac{\partial u}{\partial t} = 0 = \frac{\partial^2 u}{\partial x^2} + x - \beta$

$$\rightarrow \frac{d^2u}{dx^2} = \beta - x \rightarrow \frac{du}{dx} = \beta x - \frac{1}{2}x^2 + c_1 \rightarrow u = \frac{1}{2}\beta x^2 - \frac{1}{6}x^3 + c_1 x + c_2$$

$$\frac{\partial u}{\partial x}(x=0) = 0 = c_1$$

$$\frac{\partial u}{\partial x}(x=L) = 0 = \beta L - \frac{1}{2}L^2 \rightarrow \boxed{\beta = \frac{1}{2}L}$$

Condition for the equilibrium.

$$\Rightarrow \quad u_e(x) = \frac{1}{4} L x^2 - \frac{1}{6} x^3 + C_2$$

Integrating the full equation:

$$\int_0^L \frac{\partial u}{\partial t}\, dx = \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + x - \beta \right) dx = \cancel{\frac{\partial}{\partial x}(L,t)}^{\,0} - \cancel{\frac{\partial u}{\partial x}(0,t)}^{\,0} + \cancel{\frac{1}{2}L^2} - \cancel{\beta} = 0$$

$\Rightarrow \frac{d}{dt}\int_0^L u(x,t)\,dx = 0 \longrightarrow$ This says that the quantity $\int_0^L u(x,t)\,dx = cte$ for all t.

in particular $\int_0^L u(x,0)\,dx = \int_0^L u_e(x)\,dx$

$$\int_0^L f(x)\,dx = \int_0^L \left(\frac{1}{4} L x^2 - \frac{1}{6} x^3 + C_2 \right) dx$$

$$= \frac{1}{12} L^4 - \frac{1}{24} L^4 + C_2 L$$

$$\Rightarrow C_2 = -\frac{1}{24} L^3 + \frac{1}{L}\int_0^L f(x)\,dx \longrightarrow \boxed{u(x) = \frac{1}{4} L x^2 - \frac{1}{6} x^3 - \frac{1}{24} L^3 + \frac{1}{L}\int_0^L f(x)\,dx}$$

$$\cancel{\frac{d}{dt}\int_0^L c\rho u(x,t)\,dx = \phi(0,t) - \phi(L,t) + \int_0^L Q\,dx} \longrightarrow \text{Conservation of energy}$$

$$c\rho \frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2} + Q \longrightarrow \text{Heat equation.}$$

a) $Q = 0 \longrightarrow \frac{d}{dt}\int_0^L c\rho u\,dx = \phi_0 - \phi_L$ and $c\rho \frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2}$

in equilibrium $\frac{\partial u}{\partial t} = 0 \longrightarrow \frac{\partial^2 u}{\partial x^2} = 0 \longrightarrow \frac{\partial u}{\partial x} = C_1$

$$\left.\begin{aligned} k_0 \frac{\partial u}{\partial x}(x=0) &= \phi_0 = k_0 C_1 \longrightarrow C_1 = \phi_0/k_0 \\ k_0 \frac{\partial u}{\partial x}(x=L) &= \phi_L = k_0 C_1 \longrightarrow C_1 = \phi_L/k_0 \end{aligned}\right\} \boxed{\phi_L = \phi_0} \quad \text{condition for equilibrium}$$

$$\Rightarrow \frac{d}{dt}\int_0^L c\rho u\,dx = 0 \longrightarrow E = \int_0^L c\rho u\,dx = cte$$

in particular $E = cte = \int_0^L c\rho \underbrace{u(x,0)}_{f(x) \to \text{C.I}}\,dx \longrightarrow \boxed{E = c\rho \int_0^L f(x)\,dx}$

b) $Q = Q_0 \longrightarrow \frac{d}{dt}\int_0^L c\rho u(x,t)\,dx = \phi_0 - \phi_L + \int_0^L Q_0\,dx$

and $c\rho \frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2} + Q_0$ in equilibrium $\frac{\partial u}{\partial t} = 0$

$\Rightarrow$ in ~~equilibrium~~ $\cancel{\frac{\partial^2 u}{\partial x^2} = -\frac{Q_0}{k_0} \Rightarrow \frac{\partial u}{\partial x} = -\frac{Q_0}{k_0} x + C_1 \Rightarrow}$

(1.4.8) $\frac{d}{dt}\int_0^L c\rho u(x,t)\,dx = \phi(0,t) - \phi(L,t) + \int_0^L Q\,dx$

a) $Q=0 \rightarrow \frac{d}{dt}\int_0^L c\rho u(x,t)\,dx = \phi_0 - \phi_L$ → flux through the ends is cte.

$$\int_0^L c\rho u(x,t)\,dx = (\phi_0-\phi_L)t + C_1$$

$\rightarrow E(t) = \int_0^L c\rho u(x,t) A\,dx = (\phi_0-\phi_L)At + AC_1$

$E(0) = \int_0^L c\rho \underbrace{u(x,0)}_{f(x)\rightarrow C.I} \cancel{A}\,dx = \cancel{A}C_1 \rightarrow C_1 = c\rho\int_0^L f(x)\,dx$

$\Rightarrow \boxed{E(t) = (\phi_0-\phi_L)At + Ac\rho\int_0^L f(x)\,dx}$

b) if $Q=Q_0$ it is similar, only change the cte $\phi_0-\phi_L$ by $\phi_0-\phi_L+Q_0L$

$\frac{d}{dt}\int_0^L c\rho u\,dx = \phi_0-\phi_L + \int_0^L Q_0\,dx = \phi_0-\phi_L+Q_0L$

$\Rightarrow \boxed{E(t) = (\phi_0-\phi_L+Q_0L)At + Ac\rho\int_0^L f(x)\,dx}$

(1.4.9) $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \rightarrow \int_a^b \frac{\partial u}{\partial t}\,dx = K\int_a^b \frac{\partial^2 u}{\partial x^2}\,dx$

$\Rightarrow \frac{d}{dt}\int_a^b u\,dx = K\left(\frac{\partial u}{\partial x}(b,t) - \frac{\partial u}{\partial x}(a,t)\right) = K\left(-\frac{1}{K_0}\phi(b,t) - \frac{-1}{K_0}\phi(a,t)\right)$

by Fourier's law $\phi = -K_0\frac{\partial u}{\partial x}$. Using $K = K_0/c\rho$

$\rightarrow \frac{d}{dt}\int_a^b u\,dx = \frac{1}{c\rho}\left(\phi(a,t)-\phi(b,t)\right)$ $\quad e = c\rho u$

$\boxed{\frac{d}{dt}\int_a^b e\,dx = \phi(a,t) - \phi(b,t)}$

(1.4.10) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4 \qquad u(x,0) = f(x) \quad \frac{\partial u}{\partial x}(0,t) = 5 \quad \frac{\partial u}{\partial x}(L,t) = 6$

$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L\left(\frac{\partial^2 u}{\partial x^2}+4\right)dx = \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) + 4L = 1+4L$

$\frac{d}{dt}\int_0^L u\,dx = 1+4L \rightarrow \int_0^L u(x,t)\,dx = (1+4L)t + C$

if $t=0 \rightarrow \int_0^L \underbrace{u(x,0)}_{f(x)}\,dx = C \rightarrow \int_0^L u(x,t) = (1+4L)t + \int_0^L f(x)\,dx \rightarrow$

$$E(t) = \int_0^L eA\,dx = c\rho A \int_0^L u\,dx \quad \text{because} \quad e = c\rho u$$

$$\Rightarrow \boxed{E(t) = c\rho A\left[(1+4L)t + \int_0^L f(x)\,dx\right]}$$

⊙ (1.4.11) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x \qquad u(x,0) = f(x) \qquad \frac{\partial u}{\partial x}(0,t) = \beta \qquad \frac{\partial u}{\partial x}(L,x) = 7$

a) $$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + x\right)dx = \underbrace{\frac{\partial u}{\partial x}(L,t)}_{7} - \underbrace{\frac{\partial u}{\partial x}(0,t)}_{\beta} + \frac{1}{2}L^2$$

$$\Rightarrow \frac{d}{dt}\int_0^L u\,dx = 7 - \beta + \frac{1}{2}L^2$$

$$\int_0^L u(x,t)\,dx = \left(7 - \beta + \frac{1}{2}L^2\right)t + c$$

if $t = 0 \rightarrow \int_0^L \underbrace{u(x,0)}_{f(x)}\,dx = c \rightarrow \int_0^L u(x,t)\,dx = \left(7 - \beta + \frac{1}{2}L^2\right)t + \int_0^L f(x)\,dx$

$$E(t) = \int_0^L eA\,dx = \int_0^L c\rho u A\,dx = c\rho\, A \int_0^L u(x,t)\,dx$$

$$\boxed{E(t) = c\rho A\left[\left(7 - \beta + \frac{1}{2}L^2\right)t + \int_0^L f(x)\,dx\right]}$$

b) For an equilibrium, the energy don't change with t, then: $\left(7 - \beta + \frac{1}{2}L\right) = 0 \rightarrow \boxed{\beta = 7 + \frac{1}{2}L^2}$ Equilibrium condition

for equilibrium $\frac{\partial u}{\partial t} = 0 = \frac{\partial^2 u}{\partial x^2} + x \rightarrow \frac{d^2 u}{dx^2} = -x$

$$\rightarrow \frac{du}{dx} = -\frac{1}{2}x^2 + c_1 \rightarrow u(x) = -\frac{1}{6}x^3 + c_1 x + c_2$$

$$\frac{\partial u}{\partial x}(0) = \beta = c_1$$

$$\frac{\partial u}{\partial x}(L) = -\frac{1}{2}L^2 + c_1 = 7 \rightarrow c_1 = \boxed{\frac{1}{2}L^2 + 7 = \beta}$$ Equilibrium condition again.

$$\Rightarrow E(t) = c\rho A\left[\cancelto{0}{\left(7 - \beta + \frac{1}{2}L\right)}t + \int_0^L f(x)\,dx\right] = 0 \rightarrow E(t) = cte$$

in particular for $t = 0 \rightarrow E(0) = c\rho A \int_0^L f(x)\,dx = cte$

$$\cancel{c\rho A}\int_0^L u(x,0)\,dx = \cancel{c\rho A}\int_0^L f(x)\,dx \rightarrow$$

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$$\int_0^L f(x)\,dx = \int_0^L u(x)\,dx = \int_0^L \left(-\frac{1}{6}x^3 + \overset{\beta = \frac{1}{2}L^2+7}{c_1}x + c_2\right)dx$$

$$= -\frac{1}{24}L^4 + \left(\frac{1}{2}L^2+7\right)\cdot\frac{1}{2}L^2 + c_2 L$$

$$\Rightarrow c_2 = \frac{1}{L}\int_0^L f(x)\,dx + \frac{1}{24}L^3 - \frac{1}{2}\left(\frac{1}{2}L^2+7\right)\cdot L$$

$$\Rightarrow \boxed{u(x) = -\frac{1}{6}x^3 + \left(\frac{1}{2}L^2+7\right)x + \frac{1}{L}\int_0^L f(x)\,dx + \frac{1}{24}L^3 - \frac{1}{2}\left(\frac{1}{2}L^2+7\right)L}$$

$\lim_{t\to\infty} u(x,t) = u(x)$ since physically the system reaches the equilibrium in $t\to\infty$.

1.4.12 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad u(x,0) = f(x) \quad -k\frac{\partial u}{\partial x}(0,t) = \alpha \quad -k\frac{\partial u}{\partial x}(L,t) = \beta$

a) $$\int_0^L \frac{\partial u}{\partial t}\,dx = k\int_0^L \frac{\partial^2 u}{\partial x^2}\,dx = k\left[\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t)\right] = \alpha - \beta$$

$$\boxed{\frac{d}{dt}\int_0^L u(t,x)\,dx = \alpha - \beta}$$ → Conservation of energy (with no sources)

b) $\int_0^L u(t,x)\,dx = (\alpha-\beta)t + c \Rightarrow$ si $t=0$ $\quad \int_0^L \underbrace{u(0,x)}_{f(x)}\,dx = c$

$$\Rightarrow \boxed{\int_0^L u(t,x)\,dx = (\alpha-\beta)t + \int_0^L f(x)\,dx}$$

c) The steady solution need $\int_0^L u(t,x)\,dx = cte \Rightarrow \boxed{\alpha = \beta}$

In this case $\frac{\partial u}{\partial t} = 0 = k\frac{\partial^2 u}{\partial x^2} \to \frac{d^2u}{dx^2} = 0 \to \frac{du}{dx} = c_1$

$$\left.\begin{aligned} -k\frac{\partial u}{\partial x}(0) &= \alpha = -kc_1 \\ -k\frac{\partial u}{\partial x}(L) &= \beta = \alpha = -kc_1 \end{aligned}\right\} \quad c_1 = -\alpha/k \quad \to u = -\frac{\alpha}{k}x + c_2$$

$\alpha = \beta$ in equilibrium

By conservation of energy $\int_0^L u\,dx = \cancel{(\alpha-\beta)}^{\,0}t + \int_0^L f(x)\,dx$

$$\int_0^L\left(-\frac{\alpha}{k}x + c_2\right)dx = -\frac{\alpha}{k}\frac{L^2}{2} + c_2 L = \int_0^L f(x)\,dx$$

$$\to c_2 = \frac{1}{2}\int_0^L f(x)\,dx + \frac{\alpha}{2k}L \to \boxed{u(x) = -\frac{\alpha}{k}x + \frac{1}{L}\int_0^L f(x)\,dx + \frac{\alpha}{2k}L}$$

(1.13) The mathematical steps are identical, then exercise (1.12) except that $\alpha = \alpha(t)$ and $\beta = \beta(t)$, then:

a) $$\boxed{\frac{d}{dt}\int_0^L u(t,x)\,dx = \alpha(t) - \beta(t)}$$

b) $$\int_0^L u(t,x)\,dx = \int [\alpha(t) - \beta(t)]\,dt$$

c) The steady solution need $\int_0^L u(t,x)\,dx = cte \rightarrow \alpha(t) = \beta(t)$

$\Rightarrow u = -\frac{\alpha(t)}{K}x + C_2$ and similarity

$$\boxed{u(x) = -\frac{\alpha}{K}x + \frac{1}{L}\int_0^L f(x)\,dx + \frac{\alpha}{2K}L}$$

Since $\frac{\partial u}{\partial t} = 0$ because $u(x)$ is solution in equilibrium, then in this case needs $\alpha(t) = cte$ ✓

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1.5.1 a) Total amount of pollutant $= \iiint_R C(x,y,z,t)\,dv$

b) $\vec{J} = -K\vec{\nabla}C$ this is an extension for three dimensions of Fick's law $J_x = -K\frac{\partial c}{\partial x}$

$$\underbrace{\frac{d}{dt}\iiint_R C(x,y,z,t)\,dv}_{\text{Rate of change of the amount of pollutant in time in R.}} = \underbrace{-\oiint \vec{J}\cdot\hat{n}\,ds}_{\text{Amount of pollant flowing across the boundaries of the R per unit time}} + \underbrace{\iiint_R Q\,dv}_{\text{Amount of pollant generated by internal sources per unit time}}$$

$Q \rightarrow$ amount of pollant per unit ~~time~~ volume per unit time generated inside R.

c) Applying divergence theorem to conservation of heat energy:

$$\frac{d}{dt}\int_R C(x,y,z,t)\,dv = -\int_R \vec{\nabla}\cdot\vec{J}\,dv + \int_R Q\,dv$$

$$\Rightarrow \int_R \frac{\partial C}{\partial t}\,dv = \int_R\left[-\vec{\nabla}\cdot\vec{J} + Q\right]dv \rightarrow \frac{\partial C}{\partial t} = -\vec{\nabla}\cdot\vec{J} + Q$$

by $\vec{J} = -K\vec{\nabla}C$ (Fick's law) $\Rightarrow$ $\boxed{\frac{\partial C}{\partial t} = K\nabla^2 C + Q}$

* **1.5.2** $\phi = -K_0\nabla u + \underbrace{c\rho u}_{\text{thermal energy density}}\vec{v} = -K_0\nabla u + e\cdot\vec{v} = -K_0\nabla u + \underbrace{e\frac{\Delta x}{\Delta t}}_{\text{thermal energy per unit time.}}$

$$\Rightarrow \frac{d}{dt}\int_R c\rho u\,dv = -\oiint \vec{\phi}\cdot\hat{n}\,ds + \int_R Q\,dv = -\int_R(\nabla\cdot\phi)\,dv + \int_R Q\,dv$$

$$= -\int_R \vec{\nabla}\cdot(-K_0\nabla u + c\rho u\vec{v})\,dv + \int_R Q\,dv$$

$$\int_R c\rho\frac{\partial u}{\partial t}\,dv = +\int_R\left[(K_0\nabla^2 u - c\rho\vec{v}\,\nabla u) + Q\right]dv$$

$$c\rho\frac{\partial u}{\partial t} = K_0\nabla^2 u - c\rho\vec{v}\,\nabla u + Q$$

$$\boxed{\frac{\partial u}{\partial t} = K\nabla^2 u - \vec{v}\cdot\vec{\nabla}u + \frac{1}{c\rho}Q}$$

1.5.3 $x = r\cos\theta \qquad y = r\sin\theta$

a) Since $r^2 = x^2 + y^2$ then:

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2) \rightarrow \cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x = \cancel{2}r\cos\theta$$

$$\boxed{\frac{\partial r}{\partial x} = \cos\theta}$$

$$\frac{\partial}{\partial y}(r^2) = \frac{\partial}{\partial y}(x^2+y^2) \rightarrow \cancel{2}r\frac{\partial r}{\partial y} = \cancel{2}y = \cancel{2}r\sin\theta$$

$$\boxed{\frac{\partial r}{\partial y} = \sin\theta}$$

Since $\frac{y}{x} = \tan\theta \rightarrow \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{\partial}{\partial x}(\tan\theta)$

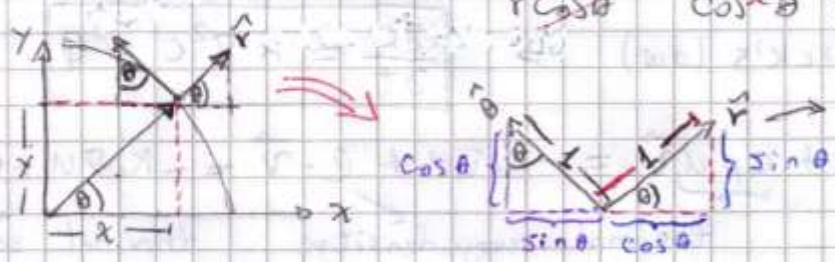
$$-\frac{y}{x^2} = \sec^2\theta\,\frac{\partial\theta}{\partial x}$$

$$\frac{-\cancel{r}\sin\theta}{r^{\cancel{2}}\cancel{\cos^2\theta}} = \frac{1}{\cancel{\cos^2\theta}}\frac{\partial\theta}{\partial x} \rightarrow \boxed{\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r}}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{\partial}{\partial y}(\tan\theta) \rightarrow \frac{1}{x} = \sec^2\theta\,\frac{\partial\theta}{\partial y}$$

$$\frac{1}{r\cancel{\cos\theta}} = \frac{1}{\cos^{\cancel{2}}\theta}\frac{\partial\theta}{\partial y} \rightarrow \boxed{\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}}$$

b)



$$\boxed{\begin{aligned}\hat{r} &= \cos\theta\,\hat{i} + \sin\theta\,\hat{j}\\ \hat{\theta} &= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\end{aligned}}$$

c) Let $u \equiv u(r,\theta) = u(r(x,y), \theta(x,y))$

$$\Rightarrow \vec{\nabla}u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} = \left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}\right)\hat{i} + \left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y}\right)\hat{j}$$

$$= \left(\frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial\theta}\left(-\frac{\sin\theta}{r}\right)\right)\hat{i} + \left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial\theta}\left(\frac{\cos\theta}{r}\right)\right)\hat{j}$$

$$= \frac{\partial u}{\partial r}\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right) + \frac{1}{r}\frac{\partial u}{\partial\theta}\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$$

$$\vec{\nabla}u = \frac{\partial u}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial u}{\partial\theta}\hat{\theta} \rightarrow \boxed{\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}}$$

d) $\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$ and $\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$$\frac{\partial \hat{r}}{\partial \theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{\theta} \qquad \frac{\partial \hat{\theta}}{\partial \theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{r}$$

$$\frac{\partial \hat{r}}{\partial r} = 0 \qquad \frac{\partial \hat{\theta}}{\partial r} = 0$$

Now let $\vec{A} = A_r\hat{r} + A_\theta\hat{\theta}$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}\right)\cdot\left(A_r\hat{r} + A_\theta\hat{\theta}\right)$$

$$= \hat{r}\frac{\partial}{\partial r}\left(A_r\hat{r} + A_\theta\hat{\theta}\right) + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\left(A_r\hat{r} + A_\theta\hat{\theta}\right)$$

$$= \hat{r}\cdot\left(\frac{\partial A_r}{\partial r}\hat{r} + A_r\cancelto{0}{\frac{\partial \hat{r}}{\partial r}} + \frac{\partial A_\theta}{\partial r}\hat{\theta} + A_\theta\cancelto{0}{\frac{\partial \hat{\theta}}{\partial r}}\right)$$

$$+ \frac{1}{r}\hat{\theta}\cdot\left(\frac{\partial A_r}{\partial \theta}\hat{r} + A_r\underbrace{\frac{\partial \hat{r}}{\partial \theta}}_{\hat{\theta}} + \frac{\partial A_\theta}{\partial \theta}\hat{\theta} + A_\theta\underbrace{\frac{\partial \hat{\theta}}{\partial \theta}}_{-\hat{r}}\right)$$

$$= \frac{\partial A_r}{\partial r}\hat{r}\cdot\hat{r} + \frac{\partial A_\theta}{\partial r}\cancelto{0}{\hat{r}\cdot\hat{\theta}} + \frac{1}{r}\frac{\partial A_r}{\partial \theta}\cancelto{0}{\hat{\theta}\cdot\hat{r}} + \frac{1}{r}A_r\,\hat{\theta}\cdot\hat{\theta}$$

$$+ \frac{1}{r}\frac{\partial A_\theta}{\partial \theta}\hat{\theta}\cdot\hat{\theta} + \frac{1}{r}A_\theta\cancelto{0}{\hat{\theta}\cdot(-\hat{r})}$$

$\hat{r}\cdot\hat{r} = 1$, $\hat{\theta}\cdot\hat{\theta} = 1$ $\rightarrow$ para vectores paralelos

$\hat{\theta}\cdot\hat{r} = 0$, $\hat{r}\cdot\hat{\theta} = 0$ $\rightarrow$ vectores ortogonales

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{\partial A_r}{\partial r} + \frac{1}{r}A_r + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta}$$

$$= \frac{1}{r}\left[r\frac{\partial A_r}{\partial r} + A_r\right] + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta}$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta)}$$

e) $\nabla^2 u = \vec{\nabla}\cdot\vec{\nabla}u = \vec{\nabla}\cdot\left(\frac{\partial u}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\theta}\right) = \vec{\nabla}\cdot\vec{A}$ if $A_r = \frac{\partial u}{\partial r}$, $A_\theta = \frac{1}{r}\frac{\partial u}{\partial \theta}$

$$\Rightarrow \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)$$

$$\boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

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1.5.4 If $u \equiv u(r) \rightarrow \frac{\partial u}{\partial \theta} = 0 \Rightarrow \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\underbrace{\frac{\partial^2 u}{\partial \theta^2}}_{0}$

1.5.5 a) Total energy $= \int e(r,t)\,dA = \int_0^{2\pi}\int_a^b c\rho u\,(r\,dr\,d\theta)$

Since $u = u(r,t)$ circularly simetric and independient of θ

$\rightarrow$ Total energy $= 2\pi \int_a^b c\rho u(r,t)\, r\,dr$

b) $\phi(r,t)$ = heat flux (thermal energy per unit time per unit surface area)

$\Rightarrow$ Heat energy flow out $= \phi(r,t) * (2\pi b) = -K_0 \frac{\partial u}{\partial r}\Big|_{r=b} * (2\pi b)$ $\rightarrow$ Fourier's Law

$= -2\pi b K_0 \frac{\partial u}{\partial r}\Big|_{r=b}$

At $r=a$, similarity the heat flowing across the circle of radius a is:

$+2\pi a K_0 \frac{\partial u}{\partial r}\Big|_{r=a}$

$\rightarrow$ flow in direction opossite to direction of flow in $r=b$.

c) $\frac{dE}{dt} = \underbrace{\phi(a,t)A_a - \phi(b,t)A_b}_{\text{Heat energy flowing out}} \rightarrow$ heat energy conservation with no sources.

$$\frac{d}{dt}\left(2\pi\int_a^b rc\rho u\,dr\right) = -2\pi a K_0 \frac{\partial u}{\partial r}\Big|_{r=a} + 2\pi b K_0 \frac{\partial u}{\partial r}\Big|_{r=b}$$

$$\cancel{2\pi}\int_a^b rc\rho \frac{\partial u}{\partial t}\,dr = \cancel{2\pi} K_0 \int_a^b \left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right] dr$$

$$\boxed{\frac{\partial u}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)} \qquad K = K_0/c\rho.$$

1.5.6 If the thermal properties depend on r:

Heat Energy flow $r=b \rightarrow \left(-K(b)\frac{\partial u}{\partial r}\Big|_{r=b}\right) * 2\pi b$

" " " $r=a \rightarrow \left(+K(a)\frac{\partial u}{\partial r}\Big|_{r=a}\right) * 2\pi a$

$$\Rightarrow \int_a^b rc\rho\frac{\partial u}{\partial t}\,dr = \int_a^b \frac{\partial}{\partial r}\left[K(r)\, r\frac{\partial u}{\partial r}\right] dr$$

$$\Rightarrow \boxed{c(r)\rho(r)\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[K(r)\, r\frac{\partial u}{\partial r}\right]}$$

1.5.7 $$\frac{d}{dt}\iint_R c\rho u\, ds = -\oint \vec{\phi}\cdot\hat{n}\, dl + \iint_R Q\, ds$$

$$\iint_R c\rho \frac{\partial u}{\partial t}\, ds = -\iint_R \vec{\nabla}\cdot\vec{\phi}\, ds + \iint_R Q\, ds = \iint_R (-\vec{\nabla}\cdot\vec{\phi} + Q)\, ds$$

$\Rightarrow c\rho \frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{\phi} + Q$ using $\vec{\phi} = -K_0 \vec{\nabla} u$ Fourier's law

$\rightarrow \boxed{c\rho \frac{\partial u}{\partial t} = K_0 \nabla^2 u + Q}$

1.5.8 $\nabla^2 u = 0 \rightarrow \iiint_R \nabla^2 u\, dv = 0$

theorem divergence

$$\iiint_R \vec{\nabla}\cdot(\vec{\nabla} u)\, dv = \oiint (\vec{\nabla} u)\cdot\hat{n}\, ds = 0$$

$\vec{\phi} = -K_0 \vec{\nabla} u \rightarrow$ using this $\Rightarrow$ $\boxed{\oiint \vec{\phi}\cdot\hat{n}\, ds = 0}$

$\Rightarrow$ if Laplace's equation is satisfied, it means that the heat energy flow out the closed surface is null.

⊙ **1.5.9** $c\rho \frac{\partial u}{\partial t} = K_0 \nabla^2 u + \cancel{Q}$ ← no sources $\quad u(r=r_1) = T_1 \quad u(r=r_2) = T_2$

* a) equilibrium temperature distribution $\rightarrow \frac{\partial u}{\partial t} = 0 \rightarrow \nabla^2 u = 0$

in polar coordinates $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

the equilibrium solution u_e will be independent of θ, because circularly symetric: $\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$.

$\rightarrow r\frac{du}{dr} = C_1 \rightarrow \frac{du}{dr} = \frac{C_1}{r} \rightarrow u(r) = C_1 \ln r + C_2$

$u(r_1) = T_1 = C_1 \ln r_1 + C_2$
$u(r_2) = T_2 = C_1 \ln r_2 + C_2$ $\rightarrow$ $T_1 - T_2 = C_1(\ln r_1 - \ln r_2) = C_1 \ln(r_1/r_2)$

$C_1 = (T_1 - T_2)/\ln(r_1/r_2)$

$C_2 = T_1 - C_1 \ln r_1 = \frac{T_1 \ln(r_1/r_2)}{\ln(r_1/r_2)} - \frac{(T_1 - T_2)}{\ln(r_1/r_2)}\ln r_1$

$\Rightarrow u(r) = \frac{T_1 - T_2}{\ln r_1 - \ln r_2}\ln r + \frac{T_1(\cancel{\ln r_1} - \ln r_2) - (\cancel{T_1} - T_2)\ln r_1}{\ln r_1 - \ln r_2}$

$$u(r) = \frac{(T_2-T_1)\ln r}{\ln(r_2/r_1)} + \frac{T_1\ln r_2 - T_2\ln r_1}{\ln(r_2/r_1)}$$

$$= \frac{(T_2-T_1)\ln r + T_1(\ln r_2 - \ln r_1) + (T_1-T_2)\ln r_1}{\ln(r_2/r_1)}$$

$$u(r) = \frac{(T_2-T_1)\ln r/r_1 + T_1\ln r_2/r_1}{(\ln r_2/r_1)} = (T_2-T_1)\frac{\ln(r/r_1)}{\ln(r_2/r_1)} + T_1$$

$$\boxed{u(r) = T_1 + (T_2-T_1)\frac{\ln(r/r_1)}{\ln(r_2/r_1)}}$$

b) Insulated boundary $\Rightarrow$ $\phi(r_2,t) = 0 = -K_0\frac{\partial u}{\partial r}(r_2,t)$ No flux through the boundary.

$\Rightarrow$ $c\rho\frac{\partial u}{\partial t} = K_0\nabla^2 u + \cancel{Q}^{\,0}$ no sources $\quad u(r=r_1)=T_1 \quad \frac{\partial u}{\partial r}(r=r_2)=0$

at the equilibrium $\frac{\partial u}{\partial t}=0 \rightarrow \nabla^2 u = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}^{\,0}$ circularly symetric

$\Rightarrow \frac{d}{dr}\left(r\frac{du}{dr}\right) = 0 \rightarrow r\frac{du}{dr} = C_1 \rightarrow \frac{du}{dr} = \frac{C_1}{r} \rightarrow u(r) = C_1\ln r + C_2$

$\frac{\partial u}{\partial r}(r=r_1) = 0 = \frac{C_1}{r_1} \rightarrow C_1 = 0$

$u(r=r_1) = C_2 = T_1 \rightarrow \boxed{u(r) = T_1}$

1.5.10 $c\rho\frac{\partial u}{\partial t} = K_0\nabla^2 u + \cancel{Q}^{\,0}$ no sources $\quad u(r=r_0)=T_0 \quad u(r\to 0)\neq\infty.$

at the equilibrium $\frac{\partial u}{\partial t}=0 \rightarrow \nabla^2 u = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}^{\,0}$ circularly symetric

$\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0 \rightarrow r\frac{du}{dr} = C_1 \rightarrow \frac{du}{dr} = \frac{C_1}{r} \rightarrow u(r) = C_1\ln r + C_2$

$u(r\to 0)\neq\infty = C_1\ln 0 + C_2 \rightarrow C_1 = 0$

$u(r=r_0) = T_0 = C_2$

$\rightarrow \boxed{u(r) = T_0}$

* 1.5.11 For equilibrium, the radial flow at $r=a$ is $2\pi a\beta$, mus equal the radial flow at $r=b$ that is $2\pi b$

$\rightarrow \boxed{\beta = b/a}$

Mathematically $\cancel{\frac{\partial u}{\partial t}}^{\,0 \text{ for equilibrium}} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \quad \frac{\partial u}{\partial r}(a,t)=\beta \quad \frac{\partial u}{\partial r}(b,t)=1$

$\frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0 \rightarrow r\frac{\partial u}{\partial r} = C_1 \rightarrow \frac{\partial u}{\partial r}(a,t) = \frac{C_1}{a} \quad \frac{\partial u}{\partial r}(b,t) = \frac{C_1}{b}$

$\Rightarrow C_1 = a\beta$ and $C_1 = b \rightarrow \boxed{\beta = b/a}$

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1.5.12 Total energy $= \int e(r,t)\, dA = \int_0^{2\pi}\int_0^{\pi}\int_a^b c\rho u\,(r^2 \sin\theta\, dr\, d\theta\, d\phi)$.

since the temperature is spherically symetric $u = u(r,t)$

$$\Rightarrow \quad \text{Total energy} = \underbrace{\int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi}\int_a^b c\rho u\, r^2 dr$$

$$\text{Total energy} = 4\pi \int_a^b c\rho\, u(r,t)\, r^2 dr$$

b) $\phi(x,t)$ = heat flux (thermal energy per unit time per unit surface area)

$$\Rightarrow \text{Heat energy flow out}\Big|_{r=b} = (4\pi b^2)\,\phi(r=b) = -4\pi b^2 K_0 \frac{\partial u}{\partial r}\Big|_{r=b}$$

Similarity at $r=a \longrightarrow \quad -4\pi a^2\, \phi(r=a) = +4\pi a^2 K_0 \frac{\partial u}{\partial r}\Big|_{r=a}$

The heat energy flowing into a spherical shell at $r=a$.

c) $$\frac{dE}{dt} = \overbrace{\phi(a,t)A_a - \phi(b,t)A_b}^{\text{Heat energy flowing out!}} \longrightarrow$$ heat energy conservation with no sources.

$$\frac{d}{dt}\left(4\pi\int_a^b c\rho u r^2 dr\right) = -4\pi a^2 K_0 \frac{\partial u}{\partial r}\Big|_{r=a} + 4\pi b^2 K_0 \frac{\partial u}{\partial r}\Big|_{r=b}$$

$$\cancel{4\pi}\int_a^b r^2 c\rho \frac{\partial u}{\partial t}\, dr = \cancel{4\pi} K_0 \int_a^b \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) dr$$

$$\Rightarrow \quad \boxed{\frac{\partial u}{\partial t} = \frac{K}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)} \qquad K = K_0/c\rho$$

1.5.13 $$\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{0 \text{ at equilibrium}} = \frac{K}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) \qquad u(1,t) = 0 \qquad u(4,t) = 80$$

$$\Rightarrow r^2 \frac{\partial u}{\partial r} = c_1 \rightarrow \frac{\partial u}{\partial r} = \frac{c_1}{r^2} \rightarrow u(r) = -\frac{c_1}{r} + c_2$$

$$u(1) = 0 = -c_1 + c_2 \rightarrow c_2 = c_1$$
$$u(4) = 80 = -\frac{c_1}{4} + c_1 = \frac{3}{4}c_1 \rightarrow c_1 = \frac{320}{3}$$

$$\Rightarrow u(r) = -\frac{320}{3r} + \frac{320}{3} \rightarrow \boxed{u(r) = \frac{320}{3}\left(1 - \frac{1}{r}\right)}$$

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1.5.14 Boundary insulated means that there is no heat flow across that portion of the boundary. Since the heat flow vector is $\vec{\phi} = -K_0 \vec{\nabla} u$, the heat flowing out will be the unit outward normal component of the heat flow vector, $\vec{\phi} \cdot \hat{n} = -K_0 \vec{\nabla} u \cdot \hat{n}$, where $\hat{n}$ is a unit outward normal to the boundary surface.

Thus, at an insulated surface: $\boxed{\vec{\nabla} u \cdot \hat{n} = 0}$

In other words, the gradient of u must be perpendicular to $\vec{n}$; this means that the gradient is parallel to the boundary.

An isobar is the same thing as a contorn line of u. Since the gradient is normal to the contour lines of u, the contorn line must be perpendicular to the boundary.

~~[Rate of change of energy] = [flow of energy through the solid 3D with no sources]~~

$$\frac{\partial e}{\partial t}(x,y,z,t) \cdot \Delta V \approx \left[\vec{\phi}(x,y,z,t) - \vec{\phi}(x+\Delta h, y+\Delta h, z+\Delta h, t)\right] \cdot \hat{n} (\Delta h)^2$$

$$\frac{\partial e}{\partial t}(\Delta h)^3 \approx \left[\phi_x(x,y,z,t) - \phi_x(x+\Delta h, y+\Delta h, z+\Delta h, t)\right] n_x (\Delta h)^2$$
$$+ \left[\phi_y(x,y,z,t) - \phi_y(x+\Delta h, y+\Delta h, z+\Delta h, t)\right] n_y (\Delta h)^2$$
$$+ \left[\phi_z(x,y,z,t) - \phi_z(x+\Delta h, y+\Delta h, z+\Delta h, t)\right] n_z (\Delta h)^2$$

for infinitesimal Δh:

$$\phi_x(x+\Delta h, y+\Delta h, z+\Delta h)\, n_x \approx \phi_x(x+\Delta h, y, z, t)$$

and similar for ϕ_y and ϕ_z

$$\Rightarrow \frac{\partial e}{\partial t} \approx \lim_{\Delta h \to 0} \frac{\left[\phi_x(x,y,z,t) - \phi_x(x+\Delta h, y, z, t)\right] n_x}{\Delta h} + \lim_{\Delta h \to 0} \left[\quad\right] + \cdots$$

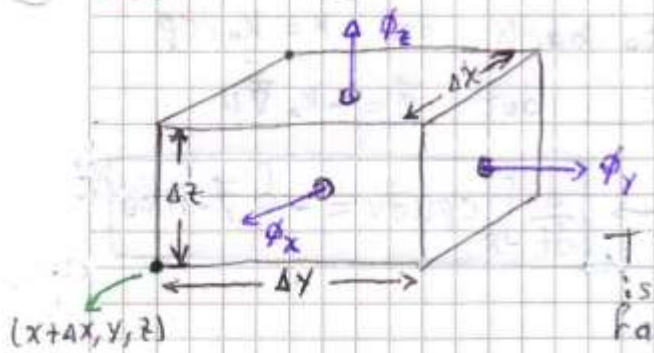
$$= -\frac{\partial \phi_x}{\partial x} n_x - \frac{\partial \phi_y}{\partial y} n_y - \frac{\partial \phi}{\partial z} n_z = -\vec{\nabla} \cdot \vec{\phi} \qquad \phi = -K_0 \nabla u$$

$$\Rightarrow \frac{\partial e}{\partial t} = -\vec{\nabla} \cdot \vec{\phi} = -\vec{\nabla} \cdot (-K_0 \vec{\nabla} u) \rightarrow \boxed{c\rho \frac{\partial u}{\partial t} = K_0 \nabla^2 u}$$

By: Herminso Villarraga Gómez

$$\vec{\phi}(x,y,z) = \left(\phi_x(x,y,z), \phi_y(x,y,z), \phi_z(x,y,z)\right)$$

1.5.15 $\left(\text{Rate of change of energy}\right) = \left(\text{flow of energy throughout the slice 3-D with no sources}\right)$



For infinitesimal slice suposse that

$$\Delta\phi_x \approx \phi(x+\Delta x, y, z) - \phi(x,y,z)$$

and similar for $\Delta\phi_y$ and $\Delta\phi_z$

Then, the total flux of energy is equal to the flux though the faces of cube.

$$\Rightarrow \frac{\partial}{\partial t} e(x,y,z,t)\,\Delta V = -\underbrace{\Delta\phi_x(\Delta y \Delta z)}_{} - \Delta\phi_y(\Delta x \Delta z) - \Delta\phi_z(\Delta x \Delta y)$$

Normal flux through faces parallel to yz plane.

$$\frac{\partial e}{\partial t}(\Delta x \Delta y \Delta z) = \left[\phi_x(x,y,z) - \phi(x+\Delta x, y, z)\right]\Delta y \Delta z$$
$$+ \left[\phi_y(x,y,z) - \phi(x, y+\Delta y, z)\right]\Delta x \Delta z$$
$$+ \left[\phi_z(x,y,z) - \phi(x, y, z+\Delta z)\right]\Delta x \Delta y$$

Dividing by $\Delta V = \Delta x \Delta y \Delta z$ and taking the limits $\Delta x, \Delta y, \Delta z \to 0$

$$\Rightarrow \frac{\partial e}{\partial t} = -\frac{\partial \phi_x}{\partial x} - \frac{\partial \phi_y}{\partial y} - \frac{\partial \phi_z}{\partial z} = -\vec{\nabla}\cdot\vec{\phi} \rightarrow c\rho\frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{\phi}$$

$$\vec{\phi} = -K_0\vec{\nabla}u \Rightarrow \boxed{c\rho\frac{\partial u(x,y,z,t)}{\partial t} = K_0\nabla^2 u(x,y,z,t)}$$

1.5.16 $$\frac{d}{dt}\int_R c\rho u\, dv = -\oint_S \vec{\phi}\cdot\vec{n}\, ds + \int_R \cancel{Q}^{\,0}\, dv$$ no sources!

$$\phi = -K_0\vec{\nabla}u \rightarrow \quad = K_0\oint_S \vec{\nabla}u\cdot\vec{n}\, ds = K_0\oint_S g(x,y,z)\, ds$$

integrating respect t: $\displaystyle\int_R c\rho u\, dv = K_0 t\oint_S g(x,y,z)\, ds + C_1$

$$\Rightarrow E(t) = \int_R e\, dv = \int_R c\rho u\, dv = K_0 t\oint_S g(x,y,z)\, ds + C_1$$

if $t=0 \rightarrow E(0) = \displaystyle\int_R c\rho \underbrace{u(x,y,z,0)}_{f(x,y,z)}\, dv = C_1$ with

$$\Rightarrow \boxed{E(t) = K_0 t\oint_S g(x,y,z)\, ds + \int_R c\rho f(x,y,z)\, dv}$$

1.5.17 $\frac{\partial u}{\partial t} = k \nabla^2 u \rightarrow \int_R \frac{\partial u}{\partial t} dv = k \int_R \vec{\nabla} \cdot \vec{\nabla} u \, dv$

using the divergence theorem to back and $k = K_0 / c\rho$

$\Rightarrow \frac{d}{dt} \int_R u \, dv = \frac{K_0}{c\rho} \oint_S \vec{\nabla} u \cdot \vec{n} \, ds$ but $\vec{\phi} = -K_0 \vec{\nabla} u$

$\Rightarrow c\rho \frac{d}{dt} \int_R u \, dv = - \oint_S \vec{\phi} \cdot \vec{n} \, ds \rightarrow \boxed{\frac{d}{dt} \int_R c\rho u \, dv = - \oint_S \vec{\phi} \cdot \vec{n} \, ds}$

By: Herminso Villarraga Gómez

Solución a los Ejercicios de Haberman

Cap 2

2.2.1 Let $\{L_1, L_2, \dots, L_n\}$ be a set of linear operator.
Let $\{c_1, c_2, \dots, c_n\}$ be a set of scalars.

$$\Rightarrow (c_1L_1 + c_2L_2 + \dots + c_nL_n)(u+v) = (c_1L_1)(u+v) + (c_2L_2)(u+v) + \dots$$
$$= c_1L_1(u+v) + c_2L_2(u+v) + \dots$$

$$= c_1L_1(u) + c_1L_1(v) + c_2L_2(u) + c_2L_2(v) + \dots + c_nL_n(u) + c_nL_n(v)$$
$$= c_1L_1(u) + c_2L_2(u) + \dots + c_nL_n(u) + c_1L_1(v) + c_2L_2(v) + \dots + c_nL_n(v)$$
$$= (c_1L_1 + c_2L_2 + \dots + c_nL_n)(u) + (c_1L_1 + c_2L_2 + \dots + c_nL_n)(v)$$

Likewise

$$(c_1L_1 + c_2L_2 + \dots + c_nL_n)(\alpha u) = (c_1L_1)(\alpha u) + (c_2L_2)(\alpha u) + \dots + (c_nL_n)(\alpha u)$$
$$= c_1L_1(\alpha u) + c_2L_2(\alpha u) + \dots + c_nL_n(\alpha u)$$
$$= \alpha c_1L_1(u) + \alpha c_2L_2(u) + \dots + \alpha c_nL_n(u)$$
$$= \alpha (c_1L_1 + c_2L_2 + \dots + c_nL_n)(u)$$

for all scalars α.

2.2.2 Let $u \equiv u(x)$ be twice differentiable

a) $L(u(x)) = K_0'(x)\frac{\partial u}{\partial x} + K_0(x)\frac{\partial^2 u}{\partial x^2}$

Thus if $v(x)$ is twice differentiable and α, β are scalars, then:

$$L(\alpha u(x) + \beta v(x)) = K_0'(x)\frac{\partial}{\partial x}(\alpha u(x) + \beta v(x)) + K_0(x)\frac{\partial^2}{\partial x^2}(\alpha u(x) + \beta v(x))$$
$$= K_0'(x)\left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial v}{\partial x}\right) + K_0(x)\left(\alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^2 v}{\partial x^2}\right)$$
$$= \underbrace{\alpha K_0'(x)\frac{\partial u}{\partial x} + \alpha K_0(x)\frac{\partial^2 u}{\partial x^2}}_{\alpha L(u)} + \underbrace{\beta K_0'(x)\frac{\partial v}{\partial x} + \beta K_0(x)\frac{\partial^2 v}{\partial x^2}}_{\beta L(v)}$$
$$= \alpha L(u) + \beta L(v)$$

$L(u) = \frac{\partial}{\partial x}\left[K_0(x,u)\frac{\partial u}{\partial x}\right]$

b) Example: Suppose $K_0(x,u) = u$. Let α, β be scalars then

$$L(\alpha u + \beta v) = \frac{\partial}{\partial x}\left((\alpha u + \beta u)\frac{\partial}{\partial x}(\alpha u + \beta v)\right)$$
$$= \left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial x}\right)^2 + (\alpha u + \beta v)\left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial v}{\partial x}\right)$$
$$\overset{?}{=} \alpha\frac{\partial}{\partial x}\left[\underbrace{K_0(x,u)}_{u}\frac{\partial u}{\partial x}\right] + \beta\frac{\partial}{\partial x}\left[\underbrace{K_0(x,u)}_{u}\frac{\partial v}{\partial x}\right] \quad ?$$
$$\neq \alpha\left[\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x^2}\right] + \beta\left[\frac{\partial v}{\partial x} + v\frac{\partial^2 v}{\partial x^2}\right]$$

No Equal. $\forall x$, $\alpha, \beta \neq 0$

2.2.3 Let $Q(u,x,t) = \alpha(x,t)u + \beta(x,t)$ and $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + Q(u,x,t)$

$\Rightarrow \frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + \alpha(x,t)u + \beta(x,t) \rightarrow \frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} - \alpha(x,t)u = \beta(x,t)$

$L(u) = \beta(x,t) \rightarrow$ The PDE is linear if $L(u)$ is linear.

Let c_1, c_2 be scalars and u_1, u_2 be differentiable functions

$$L(c_1u_1 + c_2u_2) = \frac{\partial}{\partial t}(c_1u_1 + c_2u_2) - K\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) - \alpha(x,t)(c_1u_1 + c_2u_2)$$

$$= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - c_1K\frac{\partial^2 u_1}{\partial x^2} - c_2K\frac{\partial^2 u_2}{\partial x^2} - c_1\alpha(x,t)u_1 - c_2\alpha u_2$$

$$= \underbrace{c_1\left(\frac{\partial u_1}{\partial t} - K\frac{\partial^2 u_1}{\partial x^2} - \alpha(x,t)u_1\right)}_{c_1L(u_1)} + \underbrace{c_2\left(\frac{\partial u_2}{\partial t} - K\frac{\partial^2 u_2}{\partial x^2} - \alpha(x,t)u_2\right)}_{c_2L(u_2)}$$

b) A linear equation for u it is of the form

$\boxed{L(u) = f}$ if $f = 0 \Rightarrow L(u) = 0$ called linear homogeneus equation

Since $L(u) = 0$ be a linear homogeneus PDE $\rightarrow$ $u = 0$ is always a solution of linear homogeneus equation

$\Rightarrow$ The PDE up is homogeneus if $\beta(x,t) = 0$

$\rightarrow \frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + \alpha(x,t)u$

Let $u = 0 \rightarrow 0 = K * 0 + \alpha(x,t) * 0$

$0 = 0$

2.2.4 a) Let $u = u_p + c_1u_1 + c_2u_2$, where u_i solves the homogeneus equation $L(u_i) = 0$, for $i = 1,2$ and u_p is a solution of $L(u_p) = f$. Then by the linearity of the operator L we find that

$$L(u) = L(u_p + c_1u_1 + c_2u_2) = \overset{=f}{L(u_p)} + c_1\overset{=0}{L(u_1)} + c_2\overset{=0}{L(u_2)} = f$$

b) Let u_{pi} be a particular solution of $L(u_{pi}) = f_i$ $i = 1,2$. Define $u = u_{p_1} + u_{p_2}$. By the linearity of L, the function u solves the following equation

$$L(u) = L(u_{p_1} + u_{p_2}) = L(u_{p_1}) + L(u_{p_2}) = f_1 + f_2$$

so u is a particular solution for $f_1 + f_2$.

2.2.5 To prove $L\left(\sum_{n=1}^{M} C_n u_n\right) = \sum_{n=1}^{M} C_n L(u_n)$ holds for all $M \geq 2$, we use the mathematical induction.

Clearly this formula holds for $M=2$ since L is linear by definition $L(C_p u_p + C_q u_q) = C_p L(u_p) + C_q L(u_q)$.

Assume that this formula holds for $m = K \geq 2$, then

$$L\left(\sum_{n=1}^{K+1} C_n u_n\right) = L\left(\sum_{n=1}^{K} C_n u_n + C_{K+1} u_{K+1}\right)$$

$$= L\left(\sum_{n=1}^{K} C_n u_n\right) + L(C_{K+1} u_{K+1})$$

$$= \sum_{n=1}^{K} C_n L(u_n) + C_{K+1} L(u_{K+1})$$

$$\boxed{L\left(\sum_{n=1}^{K+1} C_n u_n\right) = \sum_{n=1}^{K+1} C_n L(u_n)}$$

Thus, the formula holds for $m = K+1$ by induction the formula hold for all $m \geq 2$

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2.3-1 *a) $\frac{\partial u}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$ let $u(x,t) = h(t)f(r)$

$\Rightarrow f(r)\frac{\partial h}{\partial t} = \frac{K}{r}h\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) \rightarrow$ dividing by $Kf(r)h(t)$

$\underbrace{\frac{1}{Kh(t)}\frac{\partial h}{\partial t}}_{-\lambda} = \underbrace{\frac{1}{rf(r)}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)}_{-\lambda} \Rightarrow \boxed{\frac{dh}{dt} = -\lambda K h(t)} \rightarrow h \sim e^{-\lambda K t}$

called eq diff of Bessel radial 2-D $\leftarrow \boxed{\frac{d}{dr}\left(r\frac{df}{dr}\right) + \lambda r f(r) = 0}$

b) $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}$ Let $u(x,t) = \phi(x)G(t)$

$\Rightarrow \phi(x)\frac{\partial G}{\partial t} = KG\frac{\partial^2\phi}{\partial x^2} - v_0 G\frac{\partial\phi}{\partial x} \rightarrow$ dividing by ϕG

$\underbrace{\frac{1}{G(t)}\frac{\partial G}{\partial t}}_{-\lambda} = \underbrace{\frac{K}{\phi(x)}\frac{\partial^2\phi}{\partial x^2} - \frac{v_0}{\phi(x)}\frac{\partial\phi}{\partial x}}_{-\lambda} \Rightarrow \boxed{\frac{dG}{dt} = -\lambda G(t)} \rightarrow G \sim e^{-\lambda t}$

$\boxed{\frac{d^2\phi}{dx^2} - \frac{v_0}{K}\frac{\partial\phi}{\partial x} + \frac{\lambda}{K}\phi(x) = 0}$

*c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ let $u(x,y) = G(x)H(y)$

$H(y)\frac{\partial^2 G}{\partial x^2} + G(x)\frac{\partial^2 H}{\partial y^2} = 0 \rightarrow$ dividing by GH

$\underbrace{\frac{1}{G}\frac{\partial^2 G}{\partial x^2}}_{-\lambda} + \underbrace{\frac{1}{H}\frac{\partial^2 H}{\partial y^2}}_{\lambda} = 0 \Rightarrow \boxed{\frac{d^2G}{dx^2} = -\lambda G(x)} \rightarrow G \sim e^{\pm i\sqrt{\lambda}t}$

$\boxed{\frac{d^2H}{dy^2} = \lambda H(y)} \rightarrow H \sim e^{\pm\sqrt{\lambda}t}$

d) $\frac{\partial u}{\partial t} = \frac{K}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$ let $u(x,t) = R(r)G(t)$

$R\frac{dG}{dt} = \frac{K}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{dG}{dr}G\right) \rightarrow$ dividing by KRG

$\underbrace{\frac{1}{KG}\frac{dG}{dt}}_{-\lambda} = \underbrace{\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{-\lambda} \Rightarrow \boxed{\frac{dG}{dt} = -\lambda K G(t)} \rightarrow G \sim e^{-\lambda K t}$

called ec. diff de Bessel in 3-D radial $\leftarrow \boxed{\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \lambda r^2 R(r) = 0}$

* e) $\frac{\partial u}{\partial t} = k\frac{\partial^4 u}{\partial x^4}$ let $u(x,t) = G(t)H(x)$

$H\frac{\partial G}{\partial t} = kG\frac{\partial^4 H}{\partial x^2}$ → dividing by kHG

$\underbrace{\frac{1}{kG}\frac{\partial G}{\partial t}}_{-\lambda} = \underbrace{\frac{1}{H}\frac{\partial^4 H}{\partial x^2}}_{-\lambda}$ → $\boxed{\frac{dG}{dt} = -\lambda k G(t)}$ → $G \sim e^{-\lambda k t}$

$\boxed{\frac{d^4 H}{dx^4} = -\lambda H(x)}$

* f) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ let $u(x,t) = h(t)f(x)$

$f\frac{\partial^2 h}{\partial t^2} = c^2 h\frac{\partial^2 f}{\partial x^2}$ → dividing by $c^2 f h$

$\underbrace{\frac{1}{c^2 h}\frac{d^2 h}{dt^2}}_{-\lambda} = \underbrace{\frac{1}{f}\frac{d^2 f}{dx^2}}_{-\lambda}$ → $\boxed{\frac{d^2 h}{dt^2} = -\lambda c^2 h(t)}$ → $h \sim e^{\pm i\sqrt{\lambda}ct}$

$\boxed{\frac{d^2 f}{dx^2} = -\lambda f(x)}$ → $f \sim e^{\pm i\sqrt{\lambda}x}$

2.3.2 $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$

a) $\underline{\lambda = 0}$ → $\frac{d^2\phi}{dx^2} = 0$ → $\phi(x) = C_1 x + C_2$

$\phi(0) = 0 = C_2$

$\phi(\pi) = 0 = C_1(\pi)$ → $C_1 = 0$ → $\phi(x) = 0$ trivial solution

$\lambda = 0$ is not eigenvalue

$\underline{\lambda > 0}$ → $\frac{d^2\phi}{dx^2} = -\lambda\phi$ → $\phi(x) \sim e^{\pm i\sqrt{\lambda}x}$

$\phi(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$

$\phi(0) = 0 = C_1$

$\phi(\pi) = 0 = C_2\sin\sqrt{\lambda}\pi$ → $\begin{cases} C_2 = 0 \\ \sqrt{\lambda}\pi = n\pi \end{cases}$ → trivial solution

$\boxed{\lambda = n^2}$ $n = 1, 2, \ldots$

$\lambda < 0$ → $\frac{d^2\phi}{dx^2} = -\lambda\phi$ → $\phi(x) \sim e^{\pm\sqrt{-\lambda}x}$

$\phi(x) = C_1\cosh(\sqrt{-\lambda}x) + C_2\sinh(\sqrt{-\lambda}x)$

$\phi(0) = 0 = C_1$

$\phi(\pi) = 0 = C_2\sinh(\sqrt{-\lambda}\pi)$ → $C_2 = 0$ OR $\underbrace{\sqrt{-\lambda} = 0 \to \lambda = 0}$

Is Not autovalue because $\lambda < 0$.

⇒ $\phi(x) = 0$ → so $\lambda < 0$ is not eigenvalue.

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b) $\underline{\lambda=0} \rightarrow \frac{d^2\phi}{dx^2}=0 \rightarrow \phi(x)=C_1x+C_2$

$\phi(0)=0=C_2$

$\phi(1)=0=C_1$ $\Rightarrow \phi(x)=0 \rightarrow \lambda=0$ is not eigenvalue.

$\underline{\lambda>0} \rightarrow \frac{d^2\phi}{dx^2}=-\lambda\phi \rightarrow \phi\sim e^{\pm i\sqrt{\lambda}x}$

$\phi(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$

$\phi(0)=0=A$

$\phi(1)=0=B\sin\sqrt{\lambda} \rightarrow B=0$ OR trivial solution

$\sqrt{\lambda}=n\pi$

$\boxed{\lambda=(n\pi)^2}$ $n=1,2,3\ldots$

$\underline{\lambda<0} \rightarrow \frac{d^2\phi}{dx^2}=-\lambda\phi \rightarrow \phi\sim e^{\pm\sqrt{-\lambda}x}$

$\phi(x)=A\cosh\sqrt{-\lambda}x+B\sinh\sqrt{-\lambda}x$

$\phi(0)=0=A$

$\phi(1)=0=B\sinh\sqrt{-\lambda} \rightarrow \begin{cases} B=0 \text{ OR} \\ \sqrt{-\lambda}=0 \rightarrow \lambda=0 \text{ "}\rightarrow\leftarrow\text{"} \end{cases}$

$\Rightarrow$ only trivial sol $\phi(x)=0$ exists. No autovalues for $\lambda<0$.

c) $\underline{\lambda=0} \rightarrow \phi(x)=C_1x+C_2 \rightarrow \left.\begin{array}{l}\phi'(x)=C_1\\ \phi'(0)=0=C_1\\ \phi'(L)=0=C_1\end{array}\right\} \rightarrow \phi(x)=C_2$

$\boxed{\lambda=0}$ is eigenvalue

$\underline{\lambda>0} \rightarrow \phi(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$

$\phi'(x)=-A\sqrt{\lambda}\sin\sqrt{\lambda}x+B\sqrt{\lambda}\cos\sqrt{\lambda}x$

$\phi'(0)=0=B$

$\phi'(L)=0=-A\sqrt{\lambda}\sin\sqrt{\lambda}L \rightarrow \begin{cases} A\sqrt{\lambda}=0 \rightarrow A=0 \text{ OR } \lambda=0 \\ \sqrt{\lambda}L=n\pi \rightarrow \boxed{\lambda=\left(\frac{n\pi}{L}\right)^2} \end{cases}$ $n=1,2,3\ldots$

trivial solution

contradicts $\lambda>0$

$\lambda<0 \rightarrow \phi(x)=A\cosh\sqrt{-\lambda}x+B\sinh\sqrt{-\lambda}x$

$\phi'(x)=A\sqrt{-\lambda}\sinh\sqrt{-\lambda}x+B\sqrt{-\lambda}\cosh\sqrt{-\lambda}x$

$\phi'(0)=0=B\sqrt{-\lambda} \rightarrow B=0$ OR $\lambda=0$, but $\lambda>0 \rightarrow B=0$

$\phi'(L)=A\sqrt{-\lambda}\sinh\sqrt{-\lambda}L \rightarrow \begin{cases} A\sqrt{-\lambda}=0 \\ \sqrt{-\lambda}L=0 \end{cases}$ $\lambda=0$ is not possible $\rightarrow B=0$

$\Rightarrow$ there are not autovalues for $\lambda<0$.

d) $\underline{\lambda=0} \rightarrow \phi(x)=C_1x+C_2$ $\quad \phi'(x)=C_1$

$\phi(0)=0=C_2$ $\quad \phi'(L)=0=C_1$ $\Rightarrow \phi(x)=0$ trivial solution

$\lambda=0$ is not eigenvalue.

$\underline{\lambda>0} \rightarrow \phi(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$ $\quad \phi'(x)=-A\sqrt{\lambda}\sin\sqrt{\lambda}x+B\sqrt{\lambda}\cos\sqrt{\lambda}x$

$\phi(0)=0=A$ $\quad \phi'(L)=0=B\sqrt{\lambda}\cos(\sqrt{\lambda}L)=0$

$\rightarrow \boxed{\lambda=\left(\frac{n\pi}{L}\right)^2}$ $n=1,2,3,\ldots$

$\lambda<0 \rightarrow \phi(x)=A\cosh\sqrt{-\lambda}x+B\sinh\sqrt{-\lambda}x$ $\quad \phi'(x)=A\sqrt{-\lambda}\sinh\sqrt{-\lambda}x+B\sqrt{-\lambda}\cosh\sqrt{-\lambda}x$

$\phi(0)=0=A$ $\quad \phi'(L)=0=B\sqrt{-\lambda}\cosh\sqrt{-\lambda}L$

$\Rightarrow$ there are not eigenvalues for $\lambda<0$. $\rightarrow B=0$

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e) $\underline{\lambda=0} \rightarrow$ $\phi(x) = C_1 x + C_2$ $\quad \phi'(x) = C_1$

$\phi(L) = 0 = C_1 L + C_2$ $\quad \phi'(0) = 0 = C_1 \rightarrow C_2 = 0$ $\Big\}$ $\phi(x) = 0$, $\lambda=0$ is not eigenvalue.

$\underline{\lambda>0} \rightarrow$ $\phi(L) = 0 = A\cos\sqrt{\lambda}L + B\sin\sqrt{\lambda}L$

$\phi'(0) = 0 = -A\sqrt{\lambda}\underbrace{\sin(\sqrt{\lambda}0)}_{0} + B\sqrt{\lambda}\underbrace{\cos(\sqrt{\lambda}0)}_{1} \rightarrow B = 0$ $\Big\}$ $\sqrt{\lambda}L = (2n-1)\pi/2$ $\quad \boxed{\lambda = \left[\frac{(2n-1)\pi}{2L}\right]^2}$

$\underline{\lambda<0} \rightarrow$ $\phi(L) = A\cosh\sqrt{-\lambda}L + B\sinh\sqrt{-\lambda}L$

$\phi'(0) = 0 = A\sqrt{-\lambda}\underbrace{\sinh 0}_{0} + B\sqrt{-\lambda}\cosh 0 \rightarrow B = 0$ $\Big\}$ $A = 0 \rightarrow \phi(x) = 0$, $\lambda=0$ is not eigenvalue.

f) $\underline{\lambda=0} \rightarrow$ $\phi(a) = 0 = C_1 a + C_2 \rightarrow C_2 = -C_1 a$

$\phi(b) = 0 = C_1 b + C_2 = C_1(b-a) \rightarrow C_1 = 0,\ C_2 = 0$ if $b \neq a$

$\underline{\lambda>0} \rightarrow$ $\phi(a) = 0 = A\cos(\sqrt{\lambda}a) + B\sin(\sqrt{\lambda}a) \rightarrow A = -B\tan(\sqrt{\lambda}a)$

$\phi(b) = 0 = A\cos(\sqrt{\lambda}b) + B\sin(\sqrt{\lambda}b) \rightarrow A = -B\tan(\sqrt{\lambda}b)$

$$\tan(\sqrt{\lambda}a) = \tan(\sqrt{\lambda}b) \rightarrow \frac{\sin(\sqrt{\lambda}a)}{\cos(\sqrt{\lambda}a)} = \frac{\sin(\sqrt{\lambda}b)}{\cos(\sqrt{\lambda}b)}$$

$$\Rightarrow \underbrace{\sin\sqrt{\lambda}a\cos\sqrt{\lambda}b - \sin\sqrt{\lambda}b\cos\sqrt{\lambda}a}_{} = 0$$

$$\sin\sqrt{\lambda}(b-a) = 0 \rightarrow \sqrt{\lambda}(b-a) = n\pi \rightarrow \boxed{\lambda = \left(\frac{n\pi}{b-a}\right)^2}$$

$\lambda<0 \rightarrow$ $\phi(a) = 0 = A\cosh(\sqrt{-\lambda}a) + B\sinh(\sqrt{-\lambda}a) \rightarrow A = -B\tanh(\sqrt{-\lambda}a)$

$\phi(b) = 0 = A\cosh(\sqrt{-\lambda}b) + B\sinh(\sqrt{-\lambda}b) \rightarrow A = -B\tanh(\sqrt{-\lambda}b)$

$$\tanh\sqrt{-\lambda}a = \tanh\sqrt{-\lambda}b \rightarrow \frac{\sinh\sqrt{-\lambda}a}{\cosh\sqrt{-\lambda}a} = \frac{\sin\sqrt{-\lambda}b}{\sin\sqrt{-\lambda}b}$$

$$\Rightarrow \sinh[\sqrt{-\lambda}(b-a)] = 0 \rightarrow \sqrt{-\lambda}(b-a) = 0 \rightarrow \begin{cases} b = a \text{ OR} \\ \lambda = 0 \end{cases}$$

but $\lambda>0$ and $b=a$ it's nonsense for PDE. $\rightarrow$ No eigenvalue for $\lambda=0$.

g) $\underline{\lambda=0} \rightarrow$ $\phi(0) = 0 = C_2$ $\quad \phi'(L) + \phi(L) = 0 = C_1 + (C_1 L + \cancelto{0}{C_2}) = C_1(1+L)$

$\rightarrow C_1 = 0$ trivial solution

$\underline{\lambda>0} \rightarrow$ $\phi(0) = 0 = A$ $\quad \underbrace{\phi'(L) + \phi(L)}_{0} = B\sqrt{\lambda}\cos(\sqrt{\lambda}L) + B\sin(\sqrt{\lambda}L) = 0$

$\rightarrow \boxed{\tan\sqrt{\lambda}L = -\sqrt{\lambda}}$ $\rightarrow$ the eigenvalues are determined by this trascendental equation.

$\underline{\lambda<0} \rightarrow$ $\phi(0) = 0 = A$ $\quad \underbrace{\phi'(L) + \phi(L)}_{0} = B\sqrt{-\lambda}\cosh\sqrt{-\lambda}L + B\sinh\sqrt{-\lambda}L = 0$

$\rightarrow \boxed{\tanh(\sqrt{-\lambda}L) = -\sqrt{-\lambda}}$ $\rightarrow$ the eigenvalues are determined by this eq.

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2.3.3 $\dfrac{\partial u}{\partial t} = K\dfrac{\partial^2 u}{\partial x^2}$ $\qquad u(0,t)=0 \qquad u(L,t)=0$

Let $u(x,t)=\phi(x)G(t) \rightarrow \phi\dfrac{\partial G}{\partial t} = KG\dfrac{\partial^2\phi}{\partial x^2}$ dividing by $K\phi G$

$$\underbrace{\frac{1}{KG}\frac{\partial G}{\partial t}}_{-\lambda} = \underbrace{\frac{1}{\phi}\frac{\partial^2\phi}{\partial x^2}}_{-\lambda} \Rightarrow \frac{dG}{dt} = -\lambda K G \rightarrow \boxed{G(t)=e^{-\lambda K t}}$$

$$\boxed{\frac{d^2\phi}{dx^2}+\lambda\phi=0} \quad \text{with} \quad \phi(0)=0 \wedge \phi(L)=0$$

$\underline{\lambda=0}$: $\left.\begin{array}{l}\phi(x)=C_1x+C_2\\ \phi(0)=0=C_2\\ \phi(L)=0=C_1L\rightarrow C_1=0\end{array}\right\}$ So $\lambda=0$ is not an eigenvalue

$\underline{\lambda>0}$: $\phi(x)\sim e^{\pm i\sqrt{\lambda}x} \rightarrow \phi(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$

$\phi(0)=0=C_1$

$\phi(L)=0=C_2\sin(\sqrt{\lambda}L) \rightarrow \begin{cases}C_2=0 & \text{OR}\\ \sqrt{\lambda}L=n\pi\end{cases}$

$C_2=0$ gives trivial solution, $\lambda=\left(\dfrac{n\pi}{L}\right)^2 \quad n=1,2,3\ldots$

$\rightarrow \boxed{\phi(x)=C_2\sin\left(\dfrac{n\pi x}{L}\right)} \quad n=1,2,3,\ldots$

$\underline{\lambda<0}$: $\phi(x)\sim e^{\pm\sqrt{-\lambda}x} \rightarrow \phi(x)=A\cosh(\sqrt{-\lambda}x)+B\sinh(\sqrt{-\lambda}x)$

$\phi(0)=0=A$

$\phi(L)=0=B\sinh(\sqrt{-\lambda}L) \rightarrow \begin{cases}B=0 & \text{OR}\\ \lambda=0\end{cases}$

$B=0$ gives trivial solution and $\lambda=0$ is nonsense because $\lambda<0 \rightarrow$ Are NOT eigenvalues for $\lambda<0$.

Putting it all together and with superposition of solutions, we get:

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)e^{-K\left(\frac{n\pi}{L}\right)^2 t}}$$

At $t=0$, this must be matched to the initial condition

$$u(x,0)=f(x)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)$$

All that remains now is to determine the constants B_n. To do so, multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate from 0 to L:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx=\sum_{n=1}^{\infty}B_n\underbrace{\int_0^L\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{\frac{L}{2}\delta_{nm}}=$$

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$$\Rightarrow \int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} B_n \frac{L}{2}\delta_{nm} = B_m \frac{L}{2}$$

initial condition

$$\rightarrow \boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx} \quad \text{where } \boxed{f(x) = u(x,0)}$$

a) $u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right) \rightarrow$ Since $t=0$ in $u(x,t)$ is also a sin series, we don't even need to do the integration explicity.

We can write down the answer by inspection

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 6\sin\left(\frac{9\pi x}{L}\right) \rightarrow \begin{array}{l} B_9 = 6 \\ B_n = 0 \quad \text{if } n \neq 0 \end{array}$$

$$\Rightarrow u(x,t) = 6\sin\left(\frac{9\pi x}{L}\right) e^{-k(9\pi/L)^2 t}$$

b) $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \rightarrow \begin{array}{l} B_1 = 3 \\ B_3 = -1 \\ B_n = 0 \text{ if } n \neq 1,3 \end{array}$

$$\Rightarrow u(x,t) = 3\sin\left(\frac{\pi x}{L}\right) e^{-k\left(\frac{\pi}{L}\right)^2 t} - \sin\left(\frac{3\pi x}{L}\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$

*c) $f(x) = u(x,0) = 2\cos\left(\frac{3\pi x}{L}\right) \rightarrow B_n = \frac{2}{L}\underbrace{\int_0^L 2\cos\left(\frac{3\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx}$

integrating with Mathematica $= \frac{4}{\cancel{L}} \frac{n[1+\cos n\pi]\cancel{L}}{(n^2-9)\pi} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{8n}{(n^2-9)\pi} & n = 2,4,6,\ldots \end{cases}$

$$\Rightarrow u(x,t) = \frac{8}{\pi L}\sum_{n=2,4,6}^{\infty} \frac{n}{(n^2-9)}\sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} = \frac{8}{\pi L}\sum_{l=1}^{\infty} \frac{2l}{(4l^2-9)}\sin\left(\frac{2l\pi x}{L}\right) e^{-k\left(\frac{2l\pi}{L}\right)^2 t}$$

d) $f(x) = u(x,t) = \begin{cases} 1 & \text{if } 0 < x \leq L/2 \\ 2 & \text{if } L/2 < x < L \end{cases}$

$$\Rightarrow B_n = \frac{2}{L}\left[\int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/2}^{L} 2\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{2}{L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2} + \left\{-\frac{2L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L}\right\}\right]$$

$$= -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - \underbrace{\cos 0}_{1} + 2\left\{\underbrace{\cos(n\pi)}_{(-1)^n} - \cos\left(\frac{n\pi}{2}\right)\right\}\right]$$

$$= \frac{2}{n\pi}\left[1 + \cos\left(\frac{n\pi}{2}\right) - 2\cdot(-1)^n\right]$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[1 + \cos\left(\frac{n\pi}{2}\right) - 2\cdot(-1)^n\right]\sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

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⊙ 2.3.4 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t)=0 \quad u(L,t)=0 \quad u(x,0)=f(x)$

* a) Total heat energy $= \iiint e\,dv = A\int_0^L (c\rho u)\,dx$ for one dimension

Where we determined $u(x)$ applying the separation of variables method:

$$u(x,t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}$$

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

$$\Rightarrow \begin{pmatrix}\text{Total heat}\\ \text{Energy}\end{pmatrix} = Ac\rho\int_0^L \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}$$

$$= Ac\rho\sum_{n=0}^{\infty} B_n e^{-k(n\pi/L)^2 t}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= Ac\rho\sum_{n=0}^{\infty} B_n e^{-k(n\pi/L)^2 t}\,\frac{L}{n\pi}\left(-\cos\frac{n\pi x}{L}\right)\Big|_0^L$$

$$\boxed{\begin{pmatrix}\text{Total heat}\\ \text{Energy}\end{pmatrix} = \frac{Ac\rho}{L}\sum_{n=0}^{\infty} B_n e^{-k(n\pi/L)^2 t}\,\frac{(1-\cos n\pi)}{n\pi}}$$

b) $\phi = -K_0\frac{\partial u}{\partial x} = -K_0\frac{d}{dx}\sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-k(n\pi/L)^2 t}$

$$= -K_0\sum_{n=0}^{\infty} B_n\frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)e^{-k(n\pi/L)^2 t}$$

$$\phi(0,t) = -K_0\sum_{n=0}^{\infty}\left(\frac{n\pi}{L}\right)B_n e^{-k(n\pi/L)^2 t}$$

$$\phi(L,t) = -K_0\sum_{n=0}^{\infty}\left(\frac{n\pi}{L}\right)B_n\cos(n\pi)\,e^{-k(n\pi/L)^2 t}$$

* c) $\frac{d}{dt}\begin{pmatrix}\text{Total heat}\\ \text{Energy}\end{pmatrix} = \frac{Ac\rho}{L}\frac{d}{dt}\sum_{n=0}^{\infty} B_n e^{-k(n\pi/L)^2 t}\,\frac{(1-\cos n\pi)}{n\pi}$

$$= \frac{Ac\rho}{\cancel{L}}\sum_{n=0}^{\infty} B_n\left[-k\left(\frac{n\pi}{L}\right)^{\cancel{2}}\right]e^{-k(n\pi/L)^2 t}\,\frac{(1-\cos n\pi)}{\cancel{n\pi}}$$

$$= -c\rho k\sum_{n=0}^{\infty}\left(\frac{n\pi}{L}\right)B_n\left[\cos(n\pi)-1\right]e^{-k(n\pi/L)^2 t}\,A$$

$$\boxed{\frac{d}{dx}\int_0^L eA\,dx = -K_0\left[\Phi(L,t)-\Phi(0,t)\right]A}$$

with $\underline{k = K_0/c\rho}$

Conservation of energy !!

2.3.5 Since $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

$$\Rightarrow \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \frac{1}{2}\left[\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right]$$

$\underline{n=m\neq 0}$ $\rightarrow \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L \frac{1}{2}\left[1 - \cos\frac{2n\pi x}{L}\right]dx$

$$= \frac{1}{2}\left[\int_0^L dx - \frac{L}{2n\pi}\underbrace{\sin\frac{2n\pi x}{L}\Big|_0^L}_{0}\right]$$

$$= \frac{1}{2}L$$

$\underline{n\neq m}$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\left[\int_0^L \cos\frac{(n-m)\pi x}{L}dx - \int_0^L \cos\frac{(n+m)\pi x}{L}dx\right]$$

$$= \frac{1}{2}\left\{\frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi}{L}x\Big|_0^L - \frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi}{L}x\Big|_0^L\right\}$$

$$= \frac{L}{2(n-m)\pi}\left(\underbrace{\sin(n-m)\pi}_{0} - \underbrace{\sin 0}_{0}\right) - \frac{L}{2(n+m)\pi}\left(\underbrace{\sin(n+m)\pi}_{0} - \underbrace{\sin 0}_{0}\right)$$

$$= 0 \Rightarrow \boxed{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\delta_{nm}} \quad \begin{matrix} n>0 \\ m>0 \end{matrix}$$

* **2.3.6** Since $\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$

$$\Rightarrow \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \frac{1}{2}\left[\cos\frac{(n+m)\pi x}{L} + \cos\frac{(n-m)\pi x}{L}\right]$$

$\underline{n=m\neq 0}$ $\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\left[\int_0^L \cos\left(\frac{2\pi x}{L}\right)dx + \int_0^L dx\right]$

$$= \frac{1}{2}\left[\underbrace{\frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\Big|_0^L}_{0} + x\Big|_0^L\right] = \frac{L}{2}$$

$\underline{n\neq m}$ $\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\left\{\frac{L}{(n+m)\pi}\underbrace{\sin\frac{(n+m)\pi x}{L}\Big|_0^L}_{0} + \frac{L}{(n-m)\pi}\underbrace{\sin\frac{(n-m)\pi x}{L}\Big|_0^L}_{0}\right\}$

$$= 0$$

$$\rightarrow \boxed{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{2}\right)dx = \frac{L}{2}\delta_{nm}} \quad \begin{matrix} n>0 \\ m>0 \end{matrix}$$

$\underline{n=m=0}$ $\int_0^L 1\,dx = L \rightarrow \boxed{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = L}$

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2.3.7 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad \frac{\partial u}{\partial x}(0,t)=0 \quad \frac{\partial u}{\partial x}(L,t)=0 \quad u(x,0)=f(x)$

a) This is the heat equation that describes heat flow along a uniform, thin rod of lenght L with insulated ends.

b) let $u(x,t)=G(t)\phi(x) \rightarrow \phi\frac{\partial G}{\partial t} = kG\frac{\partial^2\phi}{\partial x^2}$ dividing by $kG\phi$

$$\Rightarrow \underbrace{\frac{1}{kG}\frac{\partial G}{\partial t}}_{-\lambda} = \underbrace{\frac{1}{\phi}\frac{\partial^2\phi}{\partial x^2}}_{-\lambda} \rightarrow \frac{dG}{dt} = -\lambda k G(t) \rightarrow \boxed{G(t)=ce^{-\lambda k t}}$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi(x) \qquad \frac{d\phi}{dx}(0)=0 \qquad \frac{d\phi}{dx}(L)=0$$

$\underline{\lambda=0} \rightarrow \frac{d^2\phi}{dx^2}=0 \rightarrow \frac{d\phi}{dx}=c_1 \rightarrow \phi(x)=c_1x+c_0$

$\frac{d\phi}{dx}(0)=0=c_1 \quad \frac{d\phi}{dx}(L)=0=c_1$ $\quad \boxed{\phi(x)=c_0}$

$\underline{\lambda>0} \rightarrow \frac{d^2\phi}{dx^2}=-\lambda\phi(x) \rightarrow \phi(x)\sim e^{\pm i\sqrt{\lambda}x}$

$\phi(x) = A\sin(\sqrt{\lambda}x)+B\cos(\sqrt{\lambda}x)$

$\phi'(x) = A\sqrt{\lambda}\cos(\sqrt{\lambda}x) - B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$

$\phi'(0)=0=A\sqrt{\lambda} \rightarrow A=0$

$\phi'(L) = -B\sqrt{\lambda}\sin(\sqrt{\lambda}L)=0 \rightarrow \sqrt{\lambda}L=n\pi$

$$\boxed{\lambda_n=\left(\frac{n\pi}{L}\right)^2} \Rightarrow \boxed{\phi(x)=B\cos\left(\frac{n\pi x}{L}\right)}$$

$\underline{\lambda<0} \rightarrow \frac{d^2\phi}{dx^2}=-\lambda\phi(x) \rightarrow \phi(x)\sim e^{\pm\sqrt{-\lambda}x}$

$\phi(x) = A\sinh(\sqrt{-\lambda}x)+B\cosh(\sqrt{-\lambda}x)$

$\phi'(x) = A\sqrt{-\lambda}\cosh(\sqrt{-\lambda}x)+B\sqrt{-\lambda}\sinh(\sqrt{-\lambda}x)$

$\phi'(0)=0=A\sqrt{-\lambda} \rightarrow A=0$

$\phi'(L) = B\sqrt{-\lambda}\sinh(\sqrt{-\lambda}L)=0 \rightarrow B=0$

$\boxed{\phi(x)=0}$ trivial solution

$$\Rightarrow \boxed{u(x,t)=A_0+\sum_{n=1}^{\infty}A_n e^{-k(n\pi/L)^2 t}\cos\left(\frac{n\pi x}{L}\right)}$$

c) $f(x)=u(x,0)=A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)$ Note that $u(x,t)$ was evaluate in $t=0$

d) $f(x)=\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)+A_0$ multiplying by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}A_n\underbrace{\int_0^L\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx}_{\frac{L}{2}\delta_{nm}} + \int_0^L A_0\cos\left(\frac{m\pi x}{L}\right)dx$$

$$\Rightarrow \int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} A_n \frac{L}{2}\delta_{nm} + A_0\int_0^L \cos\left(\frac{m\pi x}{L}\right)dx$$

if $m=0 \rightarrow \int_0^L f(x)\,dx = \sum_{n=1}^{\infty} \frac{L}{2}\cancel{\delta_{n0}}^{\,0} + A_0\int_0^L dx = A_0 L$

$$\Rightarrow \boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

if $m\neq 0 \rightarrow \int_0^L f\cos\left(\frac{m\pi x}{L}\right)dx = \underbrace{\sum_{n=1}^{\infty} A_n \frac{L}{2}\delta_{nm}} + \cancel{\frac{A_0}{(m\pi/L)}\sin\left(\frac{m\pi x}{L}\right)\Big|_0^L}^{\,0}$

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = A_m\frac{L}{2} \rightarrow \boxed{A_m = \frac{L}{2}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx}$$

e) $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\left[A_0 + \sum_{n=1}^{\infty} A_n e^{-K(n\pi/L)^2 t}\cos\left(\frac{n\pi x}{L}\right)\right]$

$$\boxed{\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

2.3.8 $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} - \alpha u \qquad u(0,t)=0 \qquad u(L,t)=0 \qquad \boxed{\alpha>0}$ ✓

a) for equilibrium $\frac{\partial u}{\partial t}=0 \rightarrow K\frac{\partial^2 u}{\partial x^2} = \alpha u \xrightarrow{\alpha>0} u \sim e^{\pm\sqrt{\alpha/K}\,x}$

$\rightarrow u(x) = A\cosh\left(\sqrt{\alpha/K}\,x\right) + B\sinh\left(\sqrt{\alpha/K}\,x\right)$

$u(0) = 0 = A$

$u(L) = 0 = B\sinh\left(\sqrt{\alpha/K}\,L\right) \rightarrow B=0 \qquad \Rightarrow \underline{u(x)=0}$ for equilibrium

b) To solve the time-dependent problem, let $u = \phi(x)h(t)$

$\Rightarrow \phi\frac{dh}{dt} = Kh\frac{d^2\phi}{dx^2} - \alpha h\phi$ dividying by ϕh

$$\underbrace{\frac{1}{h}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{\phi}\left(K\frac{d^2\phi}{dx^2} - \alpha\phi\right)}_{-\lambda} \rightarrow \frac{dh}{dt} = -\lambda h(t) \rightarrow \boxed{h(t) = ce^{-\lambda t}}$$

$$\boxed{K\frac{d^2\phi}{dx^2} = -(\lambda-\alpha)\phi(x)} \qquad \phi(0)=0,\ \phi(L)=0$$

$\alpha>0$

If $\lambda=\alpha$: $(\lambda-\alpha)=0$ $\quad K\frac{d^2\phi}{dx^2}=0 \rightarrow \phi(x) = c_1x + c_2$

$\phi(0) = 0 = c_2$

$\phi(L) = 0 = c_1L \rightarrow c_1 = 0$

$\phi(x) = 0$ so $\lambda=\alpha$ is not an eigenvalue.

If $\lambda>\alpha$: $(\lambda-\alpha)>0$ $\quad \phi(x) \sim e^{\pm i\sqrt{\frac{\lambda-\alpha}{K}}\,x} \Rightarrow \phi(x) = A\cos\sqrt{\frac{\lambda-\alpha}{K}}\,x + B\sin\sqrt{\frac{\lambda-\alpha}{K}}$

$\phi(0) = 0 = A$

$\phi(L) = 0 = B\sin\left(\sqrt{\frac{\lambda-\alpha}{K}}\,L\right) \Rightarrow$

$B\neq 0$ for non trivial solution $\Rightarrow$

eigenvalues for $\lambda > 0$.

$$\sqrt{\frac{\lambda-\alpha}{k}}\,L = n\pi \rightarrow \boxed{\lambda_n = \alpha + k\left(\frac{n\pi}{L}\right)^2} \qquad n = 1, 2, 3, \ldots$$

$$\Rightarrow \phi_n(x) = B_n \sin\left(\sqrt{\frac{\lambda_n-\alpha}{k}}\,x\right) = \boxed{B_n \sin\left(\frac{n\pi x}{L}\right)}$$

If $\lambda < \alpha$: $(\lambda-\alpha) < 0$ $\quad \phi(x) \sim e^{\pm\sqrt{\frac{\alpha-\lambda}{k}}} \rightarrow \phi(x) = A\cosh\left(\sqrt{\frac{\alpha-\lambda}{k}}\,x\right) + B\sinh\left(\sqrt{\frac{\alpha-\lambda}{k}}\,x\right)$

$$\phi(0) = 0 = A$$

$$\phi(L) = 0 = B\sinh\left(\sqrt{\frac{\alpha-\lambda}{k}}\,L\right) \rightarrow B = 0$$

$A, B = 0 \rightarrow$ trivial solution $\rightarrow$ Not eigenvalues for $\lambda < \alpha$.

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n t} \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n e^{-\left[\alpha + k\left(\frac{n\pi}{L}\right)^2\right]t} \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{u(x,t) = e^{-\alpha t} \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)}$$

Compare this to eq 2.3.30 in the text, wich is the solution to the case where $\alpha = 0$. This mean that the term $-\alpha u$ in the heat eq represent heat loss like an damped effect.

To satisfy the initial condition:

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$ multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating.

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)}_{\frac{1}{2}\delta_{nm}} = B_m \frac{L}{2}$$

$$\rightarrow \boxed{B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

The solution converges to the equilibrium solution if $t \to \infty$

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} e^{-\alpha t} \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) = 0 = \phi_{eq}(x)$$

2.3.9 $\quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u \qquad u(0,t) = 0 \qquad u(L,t) = 0 \qquad \boxed{\alpha < 0}$

a) for equilibrium $\frac{\partial u}{\partial t} = 0 \rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\alpha}{k} u \xrightarrow{\alpha<0} u \sim e^{\pm i\sqrt{\frac{-\alpha}{k}}\,x}$

$$\rightarrow u_{eq}(x) = c_1 \cos\left(\sqrt{\frac{-\alpha}{k}}\,x\right) + c_2 \sin\left(\sqrt{\frac{-\alpha}{k}}\,x\right)$$

$u(0) = 0 = c_1$

$u(L) = 0 = c_2 \sin\left(\sqrt{\frac{-\alpha}{k}}\,L\right)$ $\quad$ Non trivial solution exists only if for some integer n there is $\sqrt{\frac{-\alpha}{k}}\,L = n\pi$.

Since $\alpha = cte \rightarrow$ n is fixed and only in case that some integer n that satisfy $\sqrt{\frac{-\alpha}{K}}L = n\pi$ exists, there is a non trivial solution:

$$u_e(x) = C_2 \sin\left(\sqrt{\frac{-\alpha}{K}}\,x\right) \qquad \text{since } \sqrt{\frac{-\alpha}{K}}\,x = n\pi \text{ for some integer } n.$$

Keep in mind that K and α are physical parameters; in a real heat equation problem, we can't choose these. They are determined by the physical material in the problem. What the above calculation says is that, for a given K, α and L, we would normally expect that the only equilibrium solution is $u_{eq}(x) = 0$. If it turns out that $\sqrt{-\alpha/K}\,L$ happens to be an integer multiple of π, then there are nontrivial solution of the form give above. This does not say that, with K, α and L fixed, there is an equilibrium solution for any n.

b) To proceed like problem (2.3.8b): $h(t) = Ce^{-\lambda t}$

and $\boxed{K\frac{d^2\phi}{dx^2} = -(\lambda-\alpha)\phi(x)}$ $\quad \phi(0) = 0 \;\wedge\; \phi(L) = 0$

$\alpha < 0$

If $\lambda = \alpha$: $\quad \phi(x) = C_1 x + C_2 \quad C_1 = 0,\ C_2 = 0 \rightarrow \phi(x) = 0$ trivial solution

If $\lambda > \alpha$: $(\lambda - \alpha) > 0$

$$\phi(x) = A\cos\left(\sqrt{\frac{\lambda-\alpha}{K}}\,x\right) + B\sin\left(\sqrt{\frac{\lambda-\alpha}{K}}\,x\right)$$

$$\phi(0) = 0 = A$$

$$\phi(L) = B\sin\left(\sqrt{\frac{\lambda-\alpha}{K}}\,L\right) = 0 \rightarrow \sqrt{\frac{\lambda-\alpha}{K}}\,L = n\pi \qquad n = 1, 2, 3$$

$$\boxed{\lambda_n = \alpha + \left(\frac{n\pi}{L}\right)^2}$$

$$\Rightarrow \phi_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$$

If $\lambda < \alpha$: $(\lambda - \alpha) < 0$

$$\phi(x) = A\cosh\left(\sqrt{\frac{\alpha-\lambda}{K}}\,x\right) + B\sinh\left(\sqrt{\frac{\alpha-\lambda}{K}}\,x\right)$$

$$\phi(0) = 0 = A$$

$$\phi(L) = 0 = B\sinh\left(\sqrt{\frac{\alpha-\lambda}{K}}\,L\right) \rightarrow B = 0 \text{ trivial solution}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n t}\sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{u(x,t) = e^{-\alpha t}\sum_{n=1}^{\infty} B_n e^{-K\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)}$$

The solution when $\alpha < 0$ is the same as when $\alpha > 0$.

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Now consider the behavior of this solution as $t \to \infty$. The only dependence on t comes from the factors

$$e^{-[\alpha + K(n\pi/L)^2]t} \qquad \alpha < 0.$$

If $-\alpha < K(\pi/L)^2$: then all the exponential factors will decay, and the solution will approach 0 as $t \to \infty$

If $-\alpha > K(\pi/L)^2$: then the coefficient of t in $e^{-[\ \ ]t}$ is positive, and the first term in the series solution will grow exponentially. (Well, it will if $B_1 = 0$. For an arbitrary $f(x)$, we expect that all the coefficients B_n will be nonzero).

If $-\alpha = K(\pi/L)^2$: then the coefficient of t in the first exponential factor is zero and the first term in the series solution is:

$$B_1 \sin\left(\frac{\pi x}{L}\right)$$

The coefficients of t in the rest of the (infinitely many) exponential factors will be negative, so as $t \to \infty$.

$$\to \quad u(x,t) \to B_1 \sin\left(\frac{\pi x}{L}\right)$$

If $-\alpha = K\left(\frac{n\pi}{L}\right)^2 \to$ We know from part a) that there is a nontrivial equilibrium in this special case, and if you look again at the series solution for $u(x,t)$ given above, you will see that in this case, the coefficient of t in the exponential factor of n-thesim term will be zero. So one term in the series solution will be

$$B_n \sin\left(\frac{n\pi x}{L}\right)$$

this particular term will not grow or decay. (This is precisely the equilibrium solution that we found in part a)). However, the exponentials preceding this term in the series will have positive coefficients of t, so they will cause the solution to grow exponentially; the solution will not approach the equilibrium as $t \to \infty$. (We say the equilibrium is unstable.) In the very special case that $B_i = 0$ for $1 \le i < n$, then the solution would approach the equilibrium

$$B_n \sin\left(\frac{n\pi x}{L}\right)$$

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2.3.10 We begin by noting that for any vectors $\vec{A}$ and $\vec{B}$ and any scalars $\gamma \in \mathbb{R}$ that

a) $\|\vec{A} - \gamma\vec{B}\|^2 \geq 0$ with equality holding only if $\vec{A} = \gamma\vec{B}$

$$\rightarrow \quad 0 \leq \|\vec{A} - \gamma\vec{B}\|^2 = (\vec{A} - \gamma\vec{B})\cdot(\vec{A} - \gamma\vec{B})$$
$$= \vec{A}\cdot\vec{A} - \gamma\vec{A}\cdot\vec{B} - \gamma\vec{B}\cdot\vec{A} + \gamma^2\vec{B}\cdot\vec{B}$$
$$= \|\vec{A}\|^2 - 2\gamma\vec{A}\cdot\vec{B} + \gamma^2\|\vec{B}\|^2$$

$\Rightarrow \quad 2\gamma\vec{A}\cdot\vec{B} \leq \|\vec{A}\|^2 + \gamma^2\|\vec{B}\|^2$ now suppose $\gamma = \dfrac{\vec{A}\cdot\vec{B}}{\vec{B}\cdot\vec{B}} = \dfrac{\vec{A}\cdot\vec{B}}{\|\vec{B}\|^2}$

then $2\dfrac{(\vec{A}\cdot\vec{B})^2}{\|\vec{B}\|^2} \leq \|\vec{A}\|^2 + \dfrac{(\vec{A}\cdot\vec{B})^2}{\|\vec{B}\|^4}\|\vec{B}\|^2$

$$2\frac{(\vec{A}\cdot\vec{B})^2}{\|\vec{B}\|^2} \leq \|\vec{A}\|^2 + (\vec{A}\cdot\vec{B})^2\|\vec{B}\|^{-2}$$

$$2(\vec{A}\cdot\vec{B})^2 \leq \|\vec{A}\|^2\|\vec{B}\|^2 + (\vec{A}\cdot\vec{B})^2$$

$$(\vec{A}\cdot\vec{B})^2 \leq \|\vec{A}\|^2\|\vec{B}\|^2$$

Schwarz's inequality.

$$\sqrt{(\vec{A}\cdot\vec{B})^2} \leq \|A\|\,\|B\| \longrightarrow \boxed{|\vec{A}\cdot\vec{B}| \leq \|\vec{A}\|\|\vec{B}\|}$$

b) Since $\vec{A}\cdot\vec{B} = \sum_{n=1}^{\infty} a_n b_n$ where $\vec{A}$ and $\vec{B}$ are infinite dimensional vectors $\vec{A} = \langle a_1, a_2, \ldots\rangle$ and $\vec{B} = \langle b_1, b_2, \ldots\rangle$

From part **a)** we know $|\vec{A}\cdot\vec{B}| \leq \|\vec{A}\|\,\|\vec{B}\|$ for any finite-dimensional vectors. Written component-wise this becomes

$$\left|\sum_{n=1}^{N} a_n b_n\right| \leq \left[\sum_{n=1}^{N} a_n^2\right]\left[\sum_{n=1}^{N} b_n^2\right]$$

taking the limit of both sides as $N \to \infty$ yields:

$$\left|\sum_{n=1}^{\infty} a_n b_n\right| \leq \left[\sum_{n=1}^{\infty} a_n^2\right]\left[\sum_{n=1}^{\infty} b_n^2\right]$$

c) Since $\vec{A}\cdot\vec{B} = \int_0^L A(x)B(x)\,dx$ in terms of functions the cauchy-schwarz inequality becomes

$$\left|\int_0^L A(x)B(x)\,dx\right| \leq \left[\int_0^L A^2(x)\,dx\right]^{1/2}\left[\int_0^L B^2(x)\,dx\right]^{1/2}$$

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2.3.11 $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $\quad u(0,y) = g(y)$, $u(L,y) = 0$ $\quad u(x,0) = 0$, $u(x,H) = 0$

Let $u(x,y) = h(x)\phi(y) \rightarrow \phi(y)\frac{d^2h}{dx^2} + h(x)\frac{d^2\phi}{dy^2} = 0$

dividying by $h\phi$

$$\underbrace{\frac{1}{h}\frac{d^2h}{dx^2}}_{\lambda} = \underbrace{-\frac{1}{\phi}\frac{d^2\phi}{dy^2}}_{\lambda},$$

$$\Rightarrow \frac{d^2h}{dx^2} = \lambda h(x) \quad \begin{matrix} h(0) = g(y) \\ h(L) = 0 \end{matrix} \quad \text{and} \quad \boxed{\frac{d^2\phi}{dy^2} = -\lambda\phi(y) \quad \begin{matrix}\phi(0) = 0 \\ \phi(H) = 0\end{matrix}}$$

$\underline{\lambda = 0}$: $\frac{d^2\phi}{dy} = 0 \rightarrow \phi(y) = C_1 y + C_2$, $\phi(0) = 0 = C_1$, $\phi(H) = 0 = C_2$ $\Big\}$ $\phi(y) = 0$ trivial solut

$\underline{\lambda > 0}$: $\phi(y) = A\cos(\sqrt{\lambda}y) + B\sin(\sqrt{\lambda}y)$ $\quad n = 1,2,3,\ldots$

$\phi(0) = 0 = A$

$\phi(H) = 0 = B\sin(\sqrt{\lambda}H) \rightarrow \sqrt{\lambda}H = n\pi \rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{H}\right)^2}$

$$\Rightarrow \boxed{\phi_n(y) = B_n \sin\left(\frac{n\pi y}{H}\right)} \quad n = 1,2,3,\ldots$$

$\underline{\lambda < 0}$: $\phi(y) = A\cosh(\sqrt{\lambda}y) + B\sinh(\sqrt{\lambda}y)$

$\phi(0) = 0 = A$

$\phi(H) = 0 = B\sinh(\sqrt{\lambda}H) \rightarrow B = 0$ $\Big\}$ $\phi(y) = 0$ trivial solution.

$\Rightarrow \frac{d^2h}{dx^2} = \lambda_n h(x) \rightarrow h(x) \sim e^{\pm\sqrt{\lambda_n}x}$ because $\lambda_n > 0$.

$h(x) = C_1\cosh(\lambda_n x) + C_2\sinh(\lambda_n x)$

$h(L) = 0 = C_1\cosh(\lambda_n L) + C_2\sinh(\lambda_n L)$

$\rightarrow C_1 = -C_2\tanh\lambda_n L \rightarrow h(x) = C_2\{-\tanh(\lambda_n L)\cosh\lambda_n x + \sinh\lambda_n x\}$

In particular any other $h_p = (cte)h(x)$ also is solution, then by convenience let

$h_p(x) = \cosh(\lambda_n L)h(x) = C_2\{-\sinh(\lambda_n L)\cosh(\lambda_n x) + \sinh(\lambda_n x)\cosh(\lambda_n L)\}$

$$\boxed{h_p(x) = C_2\sinh\lambda_n(x-L)}$$

(It's more easy if we beggining with $h(x) \sim e^{\pm\sqrt{\lambda_n}(x-L)}$

$h(x) = C_1\cos\lambda_n(x-L) + C_2\sinh\lambda_n(x-L)$

$h(L) = 0 = C_1 \rightarrow \boxed{h(x) = C_2\sinh\lambda_n(x-L)} \rightarrow$)

$$\Rightarrow \boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) \sinh\left[\left(\frac{n\pi}{H}\right)(x-L)\right]}$$

If $x=0$:

$$u(0,y) = g(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) \sinh\left[\frac{n\pi}{H}(-L)\right]$$

multiplying by $\sin\left(\frac{m\pi y}{H}\right)$ and integrating:

$$\int_0^H g(y) \sin\left(\frac{m\pi y}{H}\right) dy = \sum_{n=1}^{\infty} A_n \sinh\left[\frac{n\pi}{H}(-L)\right] \underbrace{\int_0^H \sin\left(\frac{n\pi y}{H}\right) \sin\left(\frac{m\pi y}{L}\right) dy}_{\frac{H}{2}\delta_{nm}}$$

$$= \frac{H}{2} A_m \sinh\left[\left(\frac{n\pi}{H}\right)(-L)\right]$$

$$\Rightarrow \boxed{A_n = \frac{2}{H \sin[n\pi(-L)/H]} \int_0^H g(y) \sin\left(\frac{n\pi y}{H}\right) dy}$$

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ends insulates!!

* **2.4.1** $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad \frac{\partial u}{\partial x}(0,t)=0 \quad \frac{\partial u}{\partial x}(L,t)=0$

let $u(x,t)=G(t)\phi(x) \rightarrow \phi\frac{\partial G}{\partial t} = kG\frac{\partial^2\phi}{\partial x^2}$ dividying by $kG\phi$

$$\underbrace{\frac{1}{kG}\frac{\partial G}{\partial t}}_{-\lambda} = \underbrace{\frac{1}{\phi}\frac{\partial^2\phi}{\partial x^2}}_{-\lambda} \rightarrow \frac{dG}{dt} = -\lambda k G(t) \rightarrow \boxed{G(t) = Ce^{-\lambda k t}}$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi(x) \qquad \frac{d\phi}{dx}(0)=0 \qquad \frac{d\phi}{dx}(L)=0$$

$\underline{\lambda=0} \rightarrow$
$$\left.\begin{array}{l} \frac{d\phi}{dx} = C_1 \rightarrow \phi(x) = C_1 x + C_0 \\ \frac{d\phi}{dx}(0) = 0 = C_1 \quad \frac{d\phi}{dx}(L) = 0 = C_1 \end{array}\right\} \boxed{\phi_0(x) = C_0}$$

$\underline{\lambda>0} \rightarrow$
$$\phi(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$
$$\phi'(x) = A\sqrt{\lambda}\cos(\sqrt{\lambda}x) - B\sqrt{\lambda}\sin(\sqrt{\lambda}x) \qquad n=1,2,3,\ldots$$
$$\phi'(0) = 0 = A\sqrt{\lambda} \rightarrow A=0$$
$$\phi'(L) = -B\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0 \rightarrow \sqrt{\lambda}L = n\pi \rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$$
$$\boxed{\phi_n(x) = B_n\cos\left(\frac{n\pi x}{L}\right)}$$

$\underline{\lambda<0} \rightarrow$
$$\left.\begin{array}{l} \phi(x) = A\sinh(\sqrt{-\lambda}x) + B\cosh(\sqrt{-\lambda}x) \\ \phi'(x) = A\sqrt{-\lambda}\cosh(\sqrt{-\lambda}x) + B\sqrt{-\lambda}\sinh(\sqrt{-\lambda}x) \\ \phi'(0) = 0 = A\sqrt{-\lambda} \rightarrow A=0 \\ \phi'(L) = B\sqrt{-\lambda}\sinh(\sqrt{-\lambda}L) = 0 \rightarrow B=0 \end{array}\right\} \begin{array}{l}\phi(x)=0 \\ \text{trivial solution}\end{array}$$

$$\Rightarrow \boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)e^{-k(n\pi/L)^2 t}}$$

If $f(x) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)$ multiplyin by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \int_0^L A_0\cos\left(\frac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\delta_{nm}}$$
$$= A_0\int_0^L \cos\left(\frac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty} A_n\frac{L}{2}\delta_{nm}$$

$\underline{\text{if } m=0} \rightarrow \int_0^L f(x)\,dx = A_0\underbrace{\int_0^L dx}_{L} + \sum_{n=1}^{\infty} A_n\frac{L}{2}\cancel{\delta_{n0}}^{\,0} = A_0 L$

$$\rightarrow \boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

If $m \neq 0 \rightarrow \int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \cancel{\frac{A_0 L}{m\pi}\sin\left(\frac{m\pi x}{L}\right)\Big|_0^L}^{\,0} + \underbrace{\sum_{n=1}^{\infty} A_n \frac{L}{2}\delta_{nm}}_{A_m \frac{L}{2}}$

$$\rightarrow \boxed{A_n = \frac{L}{2}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

a) $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases} \rightarrow A_0 = \frac{1}{L}\left\{\cancel{\int_0^{L/2} 0\cdot dx}^{\,0} + \overbrace{\int_{L/2}^{L} 1\cdot dx}^{L/2}\right\} = \frac{1}{2}$

$$A_n = \frac{2}{L}\left\{\cancel{\int_0^{L/2} 0\cdot\cos\left(\frac{m\pi x}{L}\right)dx}^{\,0} + \int_L^{L/2} 1\cdot\cos\left(\frac{m\pi x}{L}\right)dx\right\}$$

$$= \frac{2}{L}\,\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L} = \frac{2}{n\pi}\left[\cancel{\sin(n\pi)}^{\,0} - \underbrace{\sin\left(\frac{n\pi}{2}\right)}_{\begin{cases} 0 & \text{if } n \text{ is even} \\ -1 & \text{if } n = 1,5,9,\ldots \\ +1 & \text{if } n = 3,7,11,\ldots \end{cases}}\right]$$

$$\Rightarrow u(x,t) = \frac{1}{2} + 2\sum_{n \text{ odd}}^{\infty}\left(\frac{-\sin(n\pi/2)}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = \frac{1}{2} + 2\sum_{q=1}^{\infty}\frac{(-1)^q}{(2q-1)\pi}\cos\left[\frac{(2q-1)\pi x}{L}\right]e^{-k\left[\frac{(2q-1)\pi}{L}\right]^2 t}$$

b) $f(x) = 6 + 4\cos\left(\frac{3\pi x}{L}\right) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)$

$\rightarrow A_0 = 6,\ A_3 = 4,\ A_n = 0$ if $n \neq 0, 3$

$$\Rightarrow u(x,t) = 6 + 4\cos\left(\frac{3\pi x}{L}\right)e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$

c) $f(x) = -2\sin\left(\frac{\pi x}{L}\right) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)$

it's not easy equal terms here, because $f(x)$ is a Sine function but at right there is a Cosine series.

$$\Rightarrow A_0 = \frac{1}{L}\int_0^L -2\sin\left(\frac{\pi x}{L}\right)dx = \frac{-2}{L}\,\frac{L}{\pi}\left(-\cos\frac{\pi x}{L}\right)\Big|_0^L = -\frac{4}{L}$$

$$A_n = \frac{2}{L}(-2)\underbrace{\int_0^L \sin\left(\frac{\pi x}{L}\right)\cdot\cos\left(\frac{n\pi x}{L}\right)dx}_{\frac{L(1+\cos n\pi)}{(1-n^2)\pi}} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{8}{(1-n^2)\pi} & n = 2,4,6,\ldots \end{cases}$$

See integration in the next page

Note that $\sin\frac{n\pi x}{L}$, $\cos\frac{m\pi x}{L}$ are not orthogonals in the interval $(0,L)$, **be careful!!**

$$u(x,t) = -\frac{4}{\pi}\left\{1 + \sum_{n=2,4,6}\frac{2}{(1-n^2)}\cos\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right\}$$

$$u(x,t) = -\frac{4}{\pi}\left\{1 + \sum_{s=1}^{\infty}\frac{2}{(1-4s^2)}\cos\left(\frac{2s\pi x}{L}\right)e^{-k\left[\frac{2s\pi}{L}\right]^2 t}\right\}$$

In this case the initial condition is a finite linear combination of eigenfunctions of the boundary value problem and thus we may simply equate coefficients.

d) $f(x) = -3\cos\left(\frac{8\pi x}{L}\right) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$

$A_8 = -3 \qquad A_n = 0 \quad \text{if } n \neq 8$

$\Rightarrow \quad u(x,t) = -3\cos\left(\frac{8\pi x}{L}\right) e^{-k\left(\frac{8\pi}{L}\right)^2 t}$

Note that the result above is the same if:

$A_0 = \frac{1}{L}\int_0^L -3\cos\left(\frac{8\pi x}{L}\right)dx = -\frac{3}{L}\frac{L}{8\pi}\cancel{\sin\left(\frac{8\pi x}{L}\right)\Big|_0^L} \to 0 \longrightarrow A_0 = 0$

$A_n = -\frac{3L}{2}\underbrace{\int_0^L \cos\left(\frac{8\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx}_{\frac{L}{2}\delta_{nm}} \longrightarrow A_8 = -3$

$\underline{A_n = 0 \quad \text{if } n \neq 8}$

Integration for item 2.4.1c: $\quad I_n = \int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$

$\sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha-\beta) + \sin(\alpha+\beta)\}$

$\Rightarrow I_n = \frac{1}{2}\int_0^L \sin\left(\frac{\pi}{L}(1+n)x\right) + \sin\left(\frac{\pi}{L}(1-n)x\right)dx$

When $n=1 \Rightarrow \quad I_1 = \frac{1}{2}\int_0^L \sin\left(\frac{2\pi x}{L}\right)dx = \frac{1}{2}\frac{(-L)}{2\pi}\cancel{\cos\left(\frac{2\pi x}{L}\right)\Big|_0^L} = 0$

When $n>1 \Rightarrow \quad I_n = \frac{1}{2}\left\{\frac{(-L)}{(1+n)\pi}\cos\left(\frac{\pi(1+n)x}{L}\right) + \frac{(-L)}{(1-n)\pi}\cos\left(\frac{\pi(1-n)x}{L}\right)\right\}\Big|_0^L$

$\Rightarrow I_n = -\frac{L}{2\pi}\left\{\frac{1}{(1+n)}\left[\cos(1+n)\pi - 1\right] + \frac{1}{(1-n)}\left[\cos(1-n)\pi - 1\right]\right\}$

$= -\frac{L}{2\pi}\left\{\frac{1}{(1+n)}\left[(-1)^{1+n} - 1\right] + \frac{1}{(1-n)}\left[(-1)^{1-n} - 1\right]\right\}$

$= \frac{-L}{2\pi} * \begin{cases} 0 & \text{if } n \text{ is odd natural number} \\ \underbrace{\left[\frac{-2}{(1+n)} + \frac{-2}{(1-n)}\right]}_{\frac{-2(1-n)-2(1+n)}{(1-n^2)} = -\frac{4}{(1-n^2)}} & \text{if } n \text{ is even} \end{cases}$

$$\Rightarrow \boxed{I_n = \int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{2L}{\pi(1-n^2)} & \text{if } n \text{ is even} \end{cases}}$$

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2.4.2 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad \frac{\partial u}{\partial x}(0,t) = 0 \quad u(L,t) = 0 \quad u(x,0) = f(x)$

Let $u(x,t) = h(t)\phi(x) \rightarrow \phi\frac{\partial h}{\partial t} = kh\frac{\partial \phi}{\partial x}$ dividying by $kh\phi$

$$\underbrace{\frac{1}{kh}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{-\lambda} \rightarrow \frac{dh}{dt} = -k\lambda h(t) \rightarrow \boxed{h(t) = ce^{-k\lambda t}}$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi(x) \qquad \phi'(0) = 0 \quad \phi(L) = 0$$

$\underline{\lambda = 0} \rightarrow \phi'(x) = c_1 \rightarrow \phi(x) = c_1 x + c_2$ } trivial solution
$\phi'(0) = 0 = c_1 \quad \phi(L) = 0 = 0$

$\underline{\lambda > 0} \rightarrow \phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$
$\phi'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$
$\phi'(0) = 0 = B\sqrt{\lambda} \rightarrow B = 0$
$\phi(L) = A\cos\sqrt{\lambda}L = 0 \rightarrow \sqrt{\lambda}L = (2n-1)\frac{\pi}{2} \quad n = 1, 2, 3, \ldots$

$$\Rightarrow \boxed{\lambda_n = \left[\frac{(2n-1)}{L}\frac{\pi}{2}\right]^2} \rightarrow \boxed{\phi_n(x) = A_n\cos\left[\frac{(2n-1)\pi x}{2L}\right]}$$

$\underline{\lambda < 0} \rightarrow h(t) \sim e^{-k\lambda t}$ then solution will grow in time contradicting one of the assumptions of the excercise.

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\left[\frac{(2n-1)\pi}{2L}\right]^2 t}\cos\left[\frac{(2n-1)\pi x}{2L}\right]}$$

To satisfy the initial condition we must have

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n\cos\left[\frac{(2n-1)\pi x}{2L}\right]$$

Multiplyin by $\cos\left[\frac{(2m-1)\pi x}{2L}\right]$ and integrating

$$\int_0^L f(x)\cos\left[\frac{(2m-1)\pi x}{2L}\right]dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos\left[\frac{(2n-1)\pi x}{2L}\right]\cos\left[\frac{(2m-1)\pi x}{2L}\right]dx}_{(L/2)\delta_{nm}}$$

$$= A_m\frac{L}{2}$$

$$\Rightarrow \boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\left[\frac{(2n-1)\pi x}{2L}\right]dx}$$

$$\phi(x) = c_1(\cos nx + i\sin nx) + c_2(\cos nx - i\sin nx)$$
$$= (c_1 + c_2)\cos nx + i\underbrace{(c_1 - c_2)}_{0}\sin nx$$

because we need real solutions (Assuming c_1, c_2 reals)

$\rightarrow c_2 = c_1$

$$\Rightarrow \phi(x) = c_1(e^{inx} + e^{-inx}) = c_1(2\cos nx)$$

$$\boxed{\phi(x) = A\cos nx}$$ first real solution $\lambda > 0$

Also we can choose imaginary coefficients in the complex solution, so ϕ can be

$$\phi(x) = ic_1 e^{inx} + ic_2 e^{-inx} \qquad c_1, c_2 \in \mathbb{R}.$$

in this case then $\phi(x) = i\underbrace{(c_1 + c_2)}_{0}\cos(nx) - (c_1 - c_2)\sin(nx)$ to obtain real solutions

$$\rightarrow c_2 = -c_1 \Rightarrow \phi(x) = ic_1(e^{inx} + e^{-inx}) = ic_1(2i\sin nx)$$

$$\boxed{\phi(x) = B\sin(nx)}$$ second real solution $\lambda > 0$

Note that $\lambda_n = n^2$ satisfy ① and ② above and of course

$$\boxed{\phi_n(x) = A\cos(nx) + B\sin nx} \qquad n = 1, 2, 3, \ldots$$

$\underline{\lambda < 0} \rightarrow \phi = ce^{\pm\sqrt{-\lambda}\,x} \Rightarrow \phi(0) = \phi(2\pi) \rightarrow \cancel{c}e^{\pm\sqrt{-\lambda}\,0} = \cancel{c}e^{\pm\sqrt{-\lambda}(2\pi)}$

$$1 = e^{\pm\sqrt{-\lambda}(2\pi)}$$

$$\frac{1}{2\pi}\underbrace{\ln 1}_{0} = \pm\sqrt{-\lambda}$$

$\rightarrow \lambda = 0$, but $\lambda > 0 \Rightarrow$ No eigenvalues for $\lambda > 0$.

$\rightarrow \phi = A\cosh\sqrt{-\lambda}\,x + B\sinh\sqrt{-\lambda}\,x$ is not a solution because Cosh, Sinh are not periodic functions!!

$\Rightarrow$ total solution is:

$$\phi(x) = c_0 + \sum_{n=1}^{\infty} A_n\cos(nx) + B_n\sin(nx)$$

$$\boxed{\phi(x) = \sum_{n=0}^{\infty} A_n\cos(nx) + B_n\sin(nx)}$$

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2.4.4 $\frac{d^2\phi}{dx^2} = -\lambda\phi \qquad \frac{d\phi}{dx}(0) = 0 \qquad \frac{d\phi}{dx}(L) = 0$

$\lambda < 0 \rightarrow$ suppose $\lambda = -\alpha^2$ with $\alpha \in \mathbb{R}$ and $\alpha \neq 0$:

$$\Rightarrow \frac{d^2\phi}{dx^2} = \alpha^2\phi \rightarrow \phi \sim e^{\pm\alpha x} \rightarrow \phi(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$$

$$\phi'(x) = \alpha\left(C_1 e^{\alpha x} - C_2 e^{-\alpha x}\right)$$

$$\phi'(0) = 0 = \alpha(C_1 - C_2) \rightarrow C_2 = C_1$$

$$\phi'(L) = 0 = \alpha\left(C_1 e^{\alpha L} - C_2 e^{-\alpha L}\right)$$

$$\phi'(L) = 0 = \alpha\, 2C_1 \sinh(\alpha L)$$

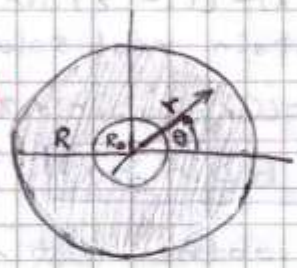
$$\Rightarrow \phi(x) = C_1\left(e^{\alpha x} + e^{-\alpha x}\right) = 2C_1 \cosh \alpha x$$

$$\phi'(L) = 2C_1\alpha \cos \alpha L = 0 \rightarrow C_1 = 0 \rightarrow \phi(x) = 0 \text{ trivial solution}$$

$\hookrightarrow \alpha \neq 0$

Thus there are no nontrivial solution when $\lambda < 0$.

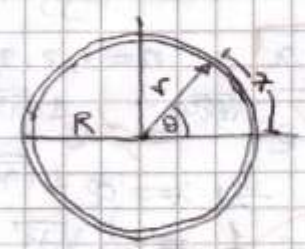
2.4.5



$$\frac{\partial u}{\partial t} = k\nabla^2 u \rightarrow \text{Heat Eq } (1.5.11)$$

$$= k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\cancel{\frac{\partial u}{\partial r}}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right]$$

0 if temperature doesn't depend on r.



If the wire is very thin $r \rightarrow R = cte$

$$\frac{\partial u}{\partial t} = \frac{k}{R^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{k}{R^2}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \frac{k}{R^2}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}\right)$$

$$x = R\theta \rightarrow \left.\begin{aligned}\frac{\partial x}{\partial \theta} &= R \\ \frac{\partial^2 x}{\partial \theta^2} &= 0\end{aligned}\right\}$$

$$= \frac{k}{R^2}\left[\frac{\partial^2 u}{\partial x^2}\underbrace{\left(\frac{\partial x}{\partial \theta}\right)^2}_{R^2} + \frac{\partial u}{\partial x}\cancel{\left(\frac{\partial^2 x}{\partial \theta^2}\right)}\frac{\partial x}{\partial \theta}\right]$$

0

$$= \frac{k}{\cancel{R^2}}\frac{\partial^2 u}{\partial x^2}\cancel{\left(\frac{\partial x}{\partial \theta}\right)^2} \rightarrow \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}}$$

2.4.6 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad -L < x < L, \qquad u(-L,t) = u(L,t) \qquad u(x,0) = f(x)$

$$\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t);$$

a) At equilibrium $\frac{\partial u}{\partial t} = 0 \rightarrow u(x) = Ax + B$

$$u(-L) = u(L)$$

$$A(-L) + \cancel{B} = A(L) + \cancel{B} \rightarrow A = 0$$

$$\Rightarrow \boxed{u(x) = B} \qquad u'(-L) = 0 = u'(L)$$

b) The time-dependent problem is:

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \qquad -L<x<L, \qquad u(-L,t)=u(L,t) \;;\; \frac{\partial u}{\partial x}(-L,t)=\frac{\partial u}{\partial x}(L,t) \qquad u(x,0)=f(x)$$

let $u(x,t)=\phi(x)G(t) \rightarrow \phi\frac{\partial G}{\partial t} = KG\frac{\partial^2\phi}{\partial x^2}$ dividying by $K\phi G$

$$\underbrace{\frac{1}{KG}\frac{\partial G}{\partial t}}_{-\lambda} = \underbrace{\frac{1}{\phi}\frac{\partial^2\phi}{\partial x^2}}_{-\lambda} \Rightarrow \frac{\partial G}{\partial t} = -\lambda K G(t) \rightarrow G(t) = Ce^{-\lambda K t}$$

$$\boxed{\frac{d^2\phi}{dx^2} = -\lambda\phi} \qquad \begin{matrix}\phi(-L)=\phi(L)\\ \phi'(-L)=\phi'(L)\end{matrix} \qquad -L<x<L$$

$\lambda=0$: $\phi'(x)=C_1 \longrightarrow \phi(x)=C_1x+C_0$

$\phi'(-L)=C_1=\phi'(L) \quad \phi(-L)=C_1(-L)+\cancel{C_0}=\phi(L)=C_1L+\cancel{C_0} \rightarrow C_1=0$

$\Rightarrow \boxed{\phi(x)=C_0}$ any cte is an eigenfunction corresponding to the eigenvalue zero

$\lambda>0$: $\phi(x)=C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$

$\phi(-L)=C_1\cos\sqrt{\lambda}(-L)+C_2\sin\sqrt{\lambda}(-L)=\phi(L)=C_1\cos\sqrt{\lambda}L+C_2\sin\sqrt{\lambda}L$

$\Rightarrow C_1[\cancel{\cos\sqrt{\lambda}(-L)}-\cancel{\cos\sqrt{\lambda}L}]+C_2[\sin\sqrt{\lambda}(-L)-\sin\sqrt{\lambda}L]=0$

$C_2[-2\sin\sqrt{\lambda}L]=0$ ①

$\rightarrow \phi'(x)=-C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

applying 2ª BC: $\phi'(-L) = \phi'(L)$

$-C_1\sqrt{\lambda}\sin\sqrt{\lambda}(-L)+C_2\cancel{\sqrt{\lambda}\cos\sqrt{\lambda}(-L)}=-C_1\sqrt{\lambda}\sin\sqrt{\lambda}L+\cancel{C_2\sqrt{\lambda}\cos\sqrt{\lambda}L}$

$\rightarrow C_1\sqrt{\lambda}[2\sin\sqrt{\lambda}L]=0$ ②

From ① and ②, if $\sin\sqrt{\lambda}L\neq 0 \rightarrow C_1,C_2=0$, wich is just the trivial solution. Thus, for nontrivial solutions:

$$\sin\sqrt{\lambda}L=0 \rightarrow \sqrt{\lambda}L=n\pi \rightarrow \lambda_n=\left(\frac{n\pi}{L}\right)^2 \qquad n=1,2,3,\ldots$$

$$\Rightarrow \boxed{\phi(x)=C_1\cos\left(\frac{n\pi x}{L}\right)+C_2\sin\left(\frac{n\pi x}{L}\right)}$$

$\lambda<0$: $\phi(x)=A\cosh\sqrt{-\lambda}x+B\sinh\sqrt{-\lambda}x$

$\phi(-L)=A\cosh\sqrt{\lambda}(-L)+B\sinh\sqrt{-\lambda}(-L)=\phi(L)=A\cosh\sqrt{-\lambda}L+B\sinh\sqrt{-\lambda}L$

$\Rightarrow A[\cancel{\cosh\sqrt{\lambda}(-L)}-\cancel{\cosh\sqrt{\lambda}L}]+B[\sinh\sqrt{-\lambda}(-L)-\sinh\sqrt{\lambda}L]=0$

$B[-2\sinh\sqrt{-\lambda}L]=0$ ③

$\sinh\sqrt{-\lambda}L=0$ only if $\sqrt{-\lambda}L=0$, but $\lambda<0 \rightarrow \sinh\sqrt{-\lambda}L\neq 0$

$\Rightarrow \boxed{B=0} \longrightarrow \phi(x)=A\cosh\sqrt{-\lambda}x$

$\phi'(x)=A\sqrt{-\lambda}\sinh\sqrt{-\lambda}x \longrightarrow$

by the 2° BC $\quad \phi'(-L) = \phi'(L)$

$$A\sqrt{-\lambda}\,\sinh\sqrt{-\lambda}(-L) = A\sqrt{-\lambda}\,\sinh\sqrt{-\lambda}L$$

trivial solution

$$A\sqrt{-\lambda}\,[-2\sinh\sqrt{-\lambda}L] = 0 \qquad \sqrt{-\lambda}L \neq 0 \rightarrow \boxed{A=0}$$

$$\Rightarrow \boxed{u(x,t) = C_0 + \sum_{n=1}^{\infty}\left[C_{1n}\cos\frac{n\pi x}{L} + C_{2n}\sin\left(\frac{n\pi x}{L}\right)\right]e^{-k(n\pi/L)^2 t}}$$

Determining the equilibrium temperature by computing the limit $t\to\infty$:

$$\boxed{\lim_{t\to\infty} u(x,t) = C_0} \rightarrow$$ Equilibrium temperature etc, similar to part a)

If $f(x) = u(x,0) = C_0 + \sum_{n=1}^{\infty}\left[C_{1n}\cos\left(\frac{n\pi x}{L}\right) + C_{2n}\sin\left(\frac{n\pi x}{L}\right)\right]$

integrating $\int_{-L}^{L} f(x)\,dx = \underbrace{\int_{-L}^{L} C_0\,dx}_{C_0(2L)} + \sum_{n=1}^{\infty}\left[C_1\cancel{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx}^{\,0} + C_{2n}\cancel{\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)dx}^{\,0}\right]$

$$\rightarrow \boxed{C_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx}$$

Equilibrium temperature

2.4.7 $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0 \qquad u(a,\theta) = f(\theta)$

We say that the temperature is continuous in $\theta = \pm\pi$ and the heat flow in the θ-direction, wich imply:

$$u(r,-\pi) = u(r,\pi) \quad \text{and} \quad \frac{\partial u}{\partial\theta}(r,-\pi) = \frac{\partial u}{\partial\theta}(r,\pi).$$

as though the two regions were in perfect thermal contact there. This equations are equivalent to

periodicity conditions $\rightarrow u(r,\theta) = u(r,\theta+2\pi)$

Let $u(r,\theta) = \phi(\theta)G(r) \rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}G(r)\frac{d^2\phi}{d\theta^2} = 0$

dividing by $\frac{G\phi}{r^2} \Rightarrow \underbrace{\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right)}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}}_{-\lambda} = 0$

$$\Rightarrow \boxed{\frac{d^2\phi}{d\theta^2} = -\lambda\phi} \quad (1) \quad \text{with} \quad \begin{matrix}\phi(-\pi) = \phi(\pi)\\ \phi'(-\pi) = \phi'(\pi)\end{matrix} \quad \text{and}$$

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda G \rightarrow r\left(r\frac{d^2G}{dr^2} + \frac{dG}{dr}\right) = \lambda G \rightarrow \boxed{r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} = \lambda G} \quad (2)$$

if $\lambda = 0 \Rightarrow r^2 \frac{d^2G}{dr^2} + r\frac{dG}{dr} = 0 \rightarrow r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0$

$$\rightarrow r\frac{dG}{dr} = b_1 \rightarrow \frac{dG}{dr} = \frac{b_1}{r} \rightarrow \boxed{G_0 = b_1 \ln r + b_0}$$

if $|G(0)| < \infty \rightarrow b_1 = 0 \Rightarrow G_0(r) = b_0$

Product solutions and by principle of superposition:

$$u(r,\theta) = C_0 b_0 + \sum_{n=1}^{\infty} a_{1n} r^n \left[C_{1n}\cos n\theta + C_{2n}\sin n\theta\right]$$

$$\boxed{u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)\right]} \qquad \begin{array}{l} 0 < r < a \\ -\pi < \theta < \pi \end{array}$$

It determines the steady-state temperature distribution

Solution → inside a circle.

In order to solve the nonhomogeneous condition $u(a,\theta) = f(\theta)$

$$f(\theta) = u(a,\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n a^n \cos(n\theta) + B_n a^n \sin(n\theta)\right] \qquad -\pi < \theta < \pi$$

integrating from $-\pi$ to π in θ:

$$\int_{-\pi}^{\pi} f(\theta)\,d\theta = A_0 \underbrace{\int_{-\pi}^{\pi} d\theta}_{2\pi} + \sum_{n=1}^{\infty} a^n \left[A_n \cancel{\int_{-\pi}^{\pi}\cos n\theta\, d\theta}^{\,0} + B_n \cancel{\int_{-\pi}^{\pi}\sin n\theta\, d\theta}^{\,0}\right]$$

$$\Rightarrow \boxed{A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta}$$

Multiplying by $\cos(m\theta)$ and integrating:

$$\int_{-\pi}^{\pi} f(\theta)\cos(m\theta)\,d\theta = A_0 \cancel{\int_{-\pi}^{\pi}\cos(m\theta)\,d\theta}^{\,0} + \sum_{n=1}^{\infty} a^n \left[A_n \underbrace{\int_{-\pi}^{\pi}\cos(m\theta)\cos(n\theta)\,d\theta}_{\pi\delta_{nm}} + B_n \cancel{\int_{-\pi}^{\pi}\cos(m\theta)\sin(n\theta)\,d\theta}^{\,0}\right]$$

$$= a^m A_m \pi$$

$$\Rightarrow \boxed{A_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta} \qquad n \geq 1$$

Similary

$$\int_{-\pi}^{\pi} f(\theta)\sin(m\theta)\,d\theta = A_0 \int_{-\pi}^{\pi}\sin(m\theta)\,d\theta + \sum_{n=1}^{\infty} a^n \left[A_n \cancel{\int_{-\pi}^{\pi}\sin(m\theta)\cos(n\theta)\,d\theta}^{\,0} + B_n \underbrace{\int_{-\pi}^{\pi}\sin(m\theta)\sin(n\theta)\,d\theta}_{\pi\delta_{nm}}\right]$$

$$= a^m B_m \pi$$

$$\Rightarrow \boxed{B_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta} \qquad n \geq 1$$

Since $a^n \neq 0$, the coefficients A_n and B_n can be uniquely solved!!

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$0 \le x \le L \qquad 0 \le y \le H$

2.5.1 $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial u}{\partial x}(0,y) = 0 \quad \frac{\partial u}{\partial x}(L,y) = 0 \quad u(x,0) = 0 \quad u(x,H) = f(x)$

* a) $u(x,y) = h(x)\phi(y) \rightarrow \phi \frac{d^2 h}{dx^2} + h \frac{d^2\phi}{dy^2} = 0$ dividing by $h(x)\phi(x)$

$$\Rightarrow \underbrace{\frac{1}{h}\frac{d^2h}{dx^2}}_{-\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dy^2}}_{\lambda} = 0 \rightarrow \begin{array}{ll} \frac{d^2h}{dx^2} = -\lambda h(x) & h'(0) = 0 \quad h'(L) = 0 \\ \frac{d^2\phi}{dy^2} = \lambda \phi(y) & \phi(0) = 0 \end{array}$$

$\underline{\lambda = 0}$: $h'(x) = C_1 \rightarrow h(x) = C_1 x + C_0$

$h'(0) = C_1 = 0$
$h'(L) = C_1 = 0 \rightarrow \boxed{h_0(x) = C_0}$ any constant is eigenfunction for the eigenvalue zero.

$\underline{\lambda > 0}$: $h(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$h'(x) = -\sqrt{\lambda}C_1 \sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$h'(0) = 0 = C_2\sqrt{\lambda} \rightarrow C_2 = 0$

$h'(L) = 0 = -\sqrt{\lambda}C_1 \sin\sqrt{\lambda}L \rightarrow \sqrt{\lambda}L = n\pi \rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$ $n = 1, 2, 3, \ldots$

$\rightarrow \boxed{h_n(x) = C_{1n}\cos\left(\frac{n\pi x}{L}\right)}$

$\underline{\lambda < 0}$: $h(x) = A\cosh\sqrt{-\lambda}x + B\sinh\sqrt{-\lambda}x$

$h'(x) = A\sqrt{-\lambda}\sinh\sqrt{-\lambda}x + B\sqrt{-\lambda}\cosh\sqrt{-\lambda}x$

$h'(0) = 0 = B\sqrt{-\lambda} \rightarrow B = 0$

$h'(L) = 0 = A\sqrt{-\lambda}\sinh\sqrt{-\lambda}L \rightarrow A = 0$ No eigenvalues for $\lambda < 0$.

$\underline{\lambda = 0} \rightarrow \frac{d^2\phi}{dy^2} = 0 \rightarrow \phi(y) = \alpha_1 y + \alpha_0$

$\phi(0) = 0 = \alpha_0 \rightarrow \boxed{\phi_0(y) = \alpha_1 y}$

$\underline{\lambda > 0}$: $\frac{d^2\phi}{dy^2} = \lambda\phi = \left(\frac{n\pi}{L}\right)^2\phi(y) \rightarrow \phi(y) = A\cosh\left(\frac{n\pi y}{L}\right) + B\sinh\left(\frac{n\pi y}{L}\right)$

$\phi(0) = 0 = A \rightarrow \boxed{\phi_n(y) = B_n \sinh\left(\frac{n\pi y}{L}\right)}$

$$\Rightarrow \boxed{u(x,y) = C_0 y + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)}$$

$$f(x) = u(x,H) = C_0 H + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi H}{L}\right)$$

$$\int_0^L f(x)\,dx = C_0 H \underbrace{\int_0^L dx}_{L} + \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi H}{L}\right)\cancelto{0}{\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx} \Rightarrow \boxed{C_0 = \frac{1}{HL}\int_0^L f(x)\,dx}$$

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = C_0 H \cancelto{0}{\int_0^L \cos\left(\frac{m\pi x}{L}\right)dx} + \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi H}{L}\right)\underbrace{\int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx}_{(L/2)\delta_{nm}}$$

$$= C_m \frac{L}{2}\sinh\left(\frac{m\pi H}{L}\right)$$

$$\rightarrow \boxed{A_n = \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

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$0 \le x \le L \qquad 0 \le y \le H$

b) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial u}{\partial x}(0,y) = g(y) \quad \frac{\partial u}{\partial x}(L,y) = 0 \quad u(x,0) = 0 \quad u(x,H) = 0$

let $u(x,y) = h(x)\phi(y) \rightarrow \phi \frac{d^2 h}{dx^2} + h \frac{d^2\phi}{dy^2} = 0$ dividing by ϕh

$$\underbrace{\frac{1}{h}\frac{d^2 h}{dx^2}}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dy^2}}_{-\lambda} = 0 \rightarrow \frac{d^2 h}{dx^2} = \lambda h(x) \qquad h'(L) = 0$$

$$\frac{d^2\phi}{dy^2} = -\lambda\phi(y) \qquad \phi(0) = 0 \qquad \phi(H) = 0$$

$\underline{\lambda = 0}$: $\phi(y) = C_1 x + C_0 \rightarrow \phi(0) = 0 = C_0$ No eigenvalues
$\phi(H) = 0 = C_1 H \rightarrow C_1 = 0$ for $\lambda = 0$.

$\underline{\lambda > 0}$: $\phi(y) = C_1 \cos\sqrt{\lambda}\, y + C_2 \sin\sqrt{\lambda}\, y \rightarrow \phi(0) = 0 = C_1$
$\phi(H) = 0 = C_2 \sin\sqrt{\lambda}\, H$

$\rightarrow \sqrt{\lambda} H = n\pi \rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{H}\right)^2} \rightarrow \boxed{\phi_n(y) = C_{2n} \sin\left(\frac{n\pi y}{H}\right)}$

$\underline{\lambda < 0}$: $\phi(y) = A\cosh\sqrt{-\lambda}\, y + B\sinh\sqrt{-\lambda}\, y \rightarrow \phi(0) = 0 = A$
$\phi(H) = 0 = B\sin(\sqrt{-\lambda} H) \rightarrow B = 0$

No eigenvalues for $\lambda < 0$.

$\underline{\lambda > 0}$: $\frac{d^2 h}{dx^2} = \lambda_n h(x) = \left(\frac{n\pi}{H}\right)^2 h(x) \rightarrow h(x) = A\cosh\frac{n\pi}{H}(x-L) + B\sinh\frac{n\pi}{H}(x-L)$

$h'(x) = A\frac{H}{n\pi}\sinh\frac{n\pi}{H}(x-L) + B\frac{H}{n\pi}\cosh\frac{n\pi}{H}(x-L)$

$h'(L) = 0 = B \rightarrow \boxed{h(x) = A\cosh\frac{n\pi}{H}(x-L)}$

$$\rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \cosh\left[\frac{n\pi}{H}(x-L)\right] \cdot \sin\left(\frac{n\pi y}{H}\right)}$$

$g(y) = \frac{\partial u}{\partial x}(0,y) = \sum_{n=1}^{\infty} A_n \frac{m\pi}{H}\sinh\frac{n\pi}{H}(-L)\sin\left(\frac{n\pi y}{H}\right)$ multiplying by $\sin(m\pi y/H)$ and integrating

$$\int_0^H g(y)\sin\left(\frac{m\pi y}{H}\right)dy = \sum_{n=1}^{\infty} A_n \frac{m\pi}{H}\sinh\frac{n\pi}{H}(-L)\underbrace{\int_0^H \sin\left(\frac{n\pi y}{H}\right)\sin\left(\frac{m\pi y}{H}\right)dy}_{(H/2)\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} A_n \frac{m\pi}{\cancel{H}}\sinh\left[\frac{n\pi}{H}(-L)\right] \cdot \frac{\cancel{H}}{2}\delta_{nm}$$

$$= A_m \frac{m\pi}{2}\sinh\left[\frac{m\pi}{H}(-L)\right]$$

$$\rightarrow \boxed{A_n = \frac{-2}{n\pi\sinh\left(\frac{n\pi L}{H}\right)}\int_0^H g(y)\sin\left(\frac{n\pi y}{H}\right)dy}$$

$0 \le x \le L \qquad 0 \le y \le H$

c) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0 \qquad \frac{\partial u}{\partial x}(0,y) = 0 \quad u(L,y) = g(y) \quad u(x,0) = 0 \quad u(x,H) = 0$

let $u(x,y) = h(x)\phi(y) \rightarrow \phi \frac{d^2h}{dx^2} + h\frac{d^2\phi}{dy^2} = 0$ dividying by ϕh.

$$\Rightarrow \underbrace{\frac{1}{h}\frac{d^2h}{dx^2}}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dy^2}}_{-\lambda} = 0 \rightarrow \frac{d^2h}{dx^2} = \lambda h(x) \qquad h'(0) = 0$$

$$\frac{d^2\phi}{dy^2} = -\lambda\phi(y) \qquad \phi(0) = 0 \qquad \phi(H) = 0$$

$\underline{\lambda = 0}$: $\phi(y) = C_1 y + C_0 \rightarrow \phi(0) = 0 = C_0$
$\phi(H) = 0 = C_1 H \rightarrow C_1 = 0$ No eigenvalues for $\lambda = 0$.

$\underline{\lambda > 0}$: $\phi(y) = C_1 \cos\sqrt{\lambda}\, y + C_2 \sin\sqrt{\lambda}\, y \rightarrow \phi(0) = 0 = C_1$
$\phi(H) = 0 = C_2 \sin\sqrt{\lambda} H \rightarrow \sqrt{\lambda} H = n\pi$

$$\rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{H}\right)^2} \rightarrow \boxed{\phi_n(y) = C_{2n} \sin\left(\frac{n\pi y}{H}\right)}$$

$\underline{\lambda < 0}$: $\phi(y) = A\cosh\sqrt{-\lambda}\, y + B\sinh\sqrt{-\lambda}\, y \rightarrow \phi(0) = 0 = A$
$\phi(H) = 0 = B\sinh(\sqrt{-\lambda} H) \rightarrow B = 0$

No eigenvalues for $\lambda < 0$.

$\underline{\lambda > 0}$: $\frac{d^2h}{dx^2} = \lambda_n h = \left(\frac{n\pi}{H}\right)^2 h(x) \rightarrow h(x) = A\cosh\frac{n\pi}{H}x + B\sinh\frac{n\pi x}{H}$

$$h'(x) = A\frac{n\pi}{H}\sinh\left(\frac{n\pi x}{H}\right) + B\frac{n\pi}{H}\cosh\left(\frac{n\pi x}{H}\right)$$

$$h'(0) = 0 = B\frac{n\pi}{H} \rightarrow B = 0$$

$$\Rightarrow \boxed{h(x) = A\cosh\frac{n\pi x}{H}}$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi x}{H}\right)\sin\left(\frac{n\pi y}{H}\right)}$$

$g(y) = u(L,y) = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi L}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$ multiplying by $\sin\left(\frac{m\pi y}{H}\right)$ and integrating.

$$\int_0^H g(y)\sin\left(\frac{m\pi y}{H}\right)dy = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi L}{H}\right)\underbrace{\int_0^H \sin\left(\frac{n\pi y}{H}\right)\sin\left(\frac{m\pi y}{H}\right)dy}_{(H/2)\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} A_n \frac{H}{2}\delta_{nm}\cosh\left(\frac{n\pi L}{H}\right)$$

$$= A_m \frac{H}{2}\cosh\left(\frac{m\pi L}{H}\right)$$

$$\rightarrow \boxed{A_n = \frac{2}{H\cosh\left(\frac{n\pi L}{H}\right)}\int_0^H g(y)\sin\left(\frac{n\pi y}{H}\right)dy}$$

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$$0 \le x \le L \qquad 0 \le y \le H$$

d) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(0,y) = g(y) \quad u(L,y) = 0 \quad \frac{\partial u}{\partial y}(x,0) = 0 \quad u(x,H) = 0$

let $u(x,y) = h(x)\phi(y) \rightarrow \phi \frac{\partial^2 h}{\partial x^2} + h \frac{\partial^2 \phi}{\partial y^2} = 0$ dividying by $h\phi$

$$\underbrace{\frac{1}{h}\frac{\partial^2 h}{\partial x^2}}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{\partial^2 \phi}{\partial y^2}}_{-\lambda} = 0 \rightarrow \frac{d^2 h}{dx^2} = \lambda h(x) \qquad h(L) = 0$$

$$\frac{d^2\phi}{dy^2} = -\lambda \phi(y) \qquad \phi'(0) = 0 \qquad \phi(H) = 0$$

$\lambda = 0$: $\phi'(y) = C_1 \rightarrow \phi(y) = C_1 y + C_0$ } No eigenvalues for $\lambda = 0$.
$\phi'(0) = 0 = C_1 \quad \phi(H) = 0 = C_0$

$\lambda > 0$: $\phi(y) = C_1 \cos\sqrt{\lambda}\, y + C_2 \sin\sqrt{\lambda}\, y$
$\phi'(y) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\, y + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\, y \qquad \phi'(0) = 0 = C_2\sqrt{\lambda} \rightarrow C_2 = 0$
$\phi(H) = 0 = C_1 \cos\sqrt{\lambda}\, H$

$\rightarrow \sqrt{\lambda}\, H = (2n-1)\frac{\pi}{2} \rightarrow \boxed{\lambda_n = \left(\frac{(2n-1)\pi}{2H}\right)^2} \rightarrow \boxed{\phi(y) = C_1 \cos\left[\frac{(2n-1)\pi y}{2H}\right]}$

$n = 1, 2, 3, \ldots$

$\lambda < 0$: $\phi(y) = A\cosh\sqrt{-\lambda}\, y + B\sinh\sqrt{-\lambda}\, y$
$\phi'(y) = A\sqrt{-\lambda}\sinh\sqrt{-\lambda}\, y + B\sqrt{-\lambda}\cosh\sqrt{-\lambda}\, y \rightarrow \phi'(0) = 0 = B\sqrt{-\lambda} \rightarrow B = 0$
$\phi(H) = 0 = A\cosh\sqrt{-\lambda}\, H$
$\rightarrow A = 0$

No eigenvalues for $\lambda < 0$.

$\lambda > 0$: $\frac{d^2 h}{dx^2} = \lambda_n h = \underbrace{\left[\frac{(2n-1)\pi}{2H}\right]^2}_{\lambda_n} h(x) \rightarrow h(x) = A\cosh[\lambda_n(x-L)] + B\sinh[\lambda_n(x-L)]$
$h(L) = 0 = A$
$\rightarrow \boxed{h(x) = B\sinh[\lambda_n(x-L)]}$

$$\Rightarrow \boxed{u(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left[\frac{(2n-1)\pi}{2H}(x-L)\right]\cos\left[\frac{(2n-1)\pi y}{2H}\right]}$$

$$g(y) = u(0,y) = \sum_{n=1}^{\infty} B_n \sinh\left[\frac{(2n-1)\pi}{2H}(-L)\right]\cos\left[\frac{(2n-1)\pi y}{2H}\right]$$

Multiply by $\cos\left[\frac{(2m-1)\pi y}{2H}\right]$ and integrate:

$$\int_0^H g(y)\cos\left[\frac{(2m-1)\pi y}{2H}\right]dy = \sum_{n=1}^{\infty} B_n \sinh\left[\frac{(2n-1)\pi}{2H}(-L)\right]\underbrace{\int_0^H \cos\left[\frac{(2n-1)\pi y}{2H}\right]\cos\left[\frac{(2m-1)\pi y}{2H}\right]dy}_{(H/2)\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} B_n \frac{H}{2}\delta_{nm} \sinh\left[\frac{(2n-1)\pi}{2H}(-L)\right]$$

$$= B_m \frac{H}{2}\sinh\left[\frac{(2m-1)\pi}{2H}(-L)\right]$$

$$\Rightarrow \boxed{B_n = \frac{-2}{H\sinh\left[\frac{(2n-1)\pi L}{2H}\right]}\int_0^H g(y)\cos\left[\frac{(2n-1)\pi y}{2H}\right]dy}$$

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$0 \le x \le L \qquad 0 \le y \le H \qquad u(x,H) = f(x)$

*e) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(0,y)=0 \qquad u(L,y)=0 \qquad u(x,0) - \frac{\partial u}{\partial y}(x,0) = 0$

Let $u(x,y) = h(x)\phi(y) \rightarrow \phi \frac{d^2 h}{dx^2} + h \frac{d^2\phi}{dy^2} = 0$ dividying by ϕh

$$\underbrace{\frac{1}{h}\frac{d^2h}{dx^2}}_{-\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dy^2}}_{\lambda} = 0 \longrightarrow \begin{array}{ll} \frac{d^2h}{dx^2} = -\lambda h(x) & h(0)=0 \qquad h(L)=0 \\ \frac{d^2\phi}{dy^2} = \lambda\phi(y) & \phi(0) - \phi'(0) = 0 \end{array}$$

$\underline{\lambda = 0}$: $h(x) = C_1 x + C_0 \rightarrow \left.\begin{array}{l} h(0) = 0 = C_0 \\ h(L) = 0 = C_1 L \end{array}\right\}$ No eigenvalues for $\lambda = 0$.

$\underline{\lambda > 0}$: $h(x) = C_1 \cos\sqrt{\lambda}\,x + C_2 \sin\sqrt{\lambda}\,x \rightarrow h(0) = 0 = C_1$

$h(L) = 0 = C_2 \sin(\sqrt{\lambda}L)$

$\rightarrow \sqrt{\lambda}L = n\pi \rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2} \rightarrow \boxed{h_n(x) = C_{2n} \sin\left(\frac{n\pi x}{L}\right)}$

$\lambda < 0$: $h(x) = A\cosh\sqrt{-\lambda}\,x + B\sinh\sqrt{-\lambda}\,x \rightarrow h(0) = 0 = A$

$h(L) = 0 = B\sinh(\sqrt{-\lambda}L) \rightarrow B = 0$

$\rightarrow$ No eigenvalues for $\lambda < 0$.

$\underline{\lambda > 0}$: $\frac{d^2\phi}{dy^2} = \lambda_n \phi = \left(\frac{n\pi}{L}\right)^2 \phi(y) \longrightarrow \phi(y) = A\cosh\left(\frac{n\pi y}{L}\right) + B\sinh\left(\frac{n\pi y}{L}\right)$

$\phi'(y) = \left(\frac{n\pi}{L}\right)\left\{A\sinh\left(\frac{n\pi y}{L}\right) + B\cosh\left(\frac{n\pi y}{L}\right)\right\}$

$\Rightarrow \phi(0) - \phi'(0) = 0 = A - \left(\frac{n\pi}{L}\right)B \rightarrow A = \left(\frac{n\pi}{L}\right)B$

$\Rightarrow \boxed{\phi_n(y) = B_n\left[\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi y}{L}\right) + \sinh\left(\frac{n\pi y}{L}\right)\right]}$

$$\Rightarrow \boxed{u(x,y) = \sum_{n=1}^{\infty} B_n\left[\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi y}{L}\right) + \sinh\left(\frac{n\pi y}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)}$$

$$0 \le x \le L \qquad 0 \le y \le H$$

f) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ $\quad u(0,y) = f(y) \quad u(L,y) = 0 \quad \frac{\partial u}{\partial y}(x,0) = 0 \quad \frac{\partial u}{\partial y}(x,H) = 0$

Let $u(x,y) = h(x)\phi(y) \rightarrow \phi \frac{\partial^2 h}{\partial x^2} + h \frac{\partial^2 \phi}{\partial y^2} = 0$ dividying by $h\phi$

$$\underbrace{\frac{1}{h}\frac{\partial^2 h}{\partial x^2}}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{\partial^2 \phi}{\partial y^2}}_{-\lambda} = 0 \rightarrow \frac{d^2 h}{dx^2} = \lambda h(x) \qquad h(L) = 0$$

$$\frac{d^2\phi}{dy^2} = -\lambda \phi(y) \qquad \phi'(0) = 0 \qquad \phi'(H) = 0$$

$\underline{\lambda = 0}$: $\phi'(y) = C_1 \rightarrow \phi(y) = C_1 y + C_0$, $\phi'(0) = 0 = C_1$, $\phi'(H) = C_1 = 0$ $\rightarrow \boxed{\phi_0(y) = C_0}$

$\underline{\lambda > 0}$: $\phi(y) = C_1 \cos\sqrt{\lambda}\, y + C_2 \sin\sqrt{\lambda}\, y$

$\phi'(y) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}\, y) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}\, y) \rightarrow \phi'(0) = 0 = C_2\sqrt{\lambda} \rightarrow C_2 = 0$

$\phi'(H) = 0 = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda} H)$

$\rightarrow \sqrt{\lambda} H = n\pi \rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{H}\right)^2}$ $\quad n = 1, 2, 3 \ldots$

$\Rightarrow \boxed{\phi_n(y) = C_{1n} \cos\left(\frac{n\pi y}{H}\right)}$

$\underline{\lambda < 0}$: $\phi(y) = A\cosh\sqrt{-\lambda}\, y + B\sinh\sqrt{-\lambda}\, y$

$\phi'(y) = A\sqrt{-\lambda}\sinh(\sqrt{-\lambda}\, y) + B\sqrt{-\lambda}\cosh(\sqrt{-\lambda}\, y) \rightarrow \phi'(0) = 0 = B$

$\phi'(H) = A\sqrt{-\lambda}\sinh(\sqrt{-\lambda} H) \rightarrow \underline{H = 0}$

No eigenvalues for $\lambda < 0$.

$\underline{\lambda = 0} \rightarrow \frac{d^2 h}{dx^2} = 0 \rightarrow h(x) = a_1 x + a_0$

$h(L) = a_1 L + a_0 = 0 \rightarrow a_1 = -\frac{a_0}{L} \Rightarrow \boxed{h_0(x) = a_0\left(1 - \frac{x}{L}\right)}$

$\underline{\lambda > 0} \rightarrow \frac{d^2 h}{dx^2} = \lambda h = \left(\frac{n\pi}{H}\right)^2 h(x) \rightarrow h(x) = A\cosh\left[\frac{n\pi(x-L)}{H}\right] + B\sinh\left[\frac{n\pi(x-L)}{H}\right]$

$h(L) = 0 = A \rightarrow \boxed{h_n(x) = B\sinh\left[\frac{n\pi(x-L)}{H}\right]}$

$\Rightarrow u_0 = h_0\phi_0 = a_0 c_0\left(1 - \frac{x}{L}\right) = B_0(L - x)$

$u_n = h_n\phi_n$ similarly

$$\rightarrow \boxed{u(x,y) = B_0(L-x) + \sum_{n=1}^{\infty} B_n \sinh\left[\frac{n\pi(x-L)}{H}\right]\cos\left(\frac{n\pi y}{H}\right)}$$

$$f(y) = u(0,y) = B_0 L + \sum_{n=1}^{\infty} B_n \sinh\left[\frac{n\pi(-L)}{H}\right]\cos\left(\frac{n\pi y}{H}\right)$$

$$\int_0^H f(y)\,dy = B_0 L \underbrace{\int_0^H dy}_{H} + \sum_{n=1}^{\infty} B_n \sinh\left[\frac{n\pi(-L)}{H}\right]\cancelto{0}{\int_0^H \cos\left(\frac{n\pi y}{H}\right)dy}$$

$$\rightarrow \boxed{B_0 = \frac{1}{HL}\int_0^H f(y)\,dy}$$

$$\int_0^H f(y)\cos\left(\frac{m\pi y}{H}\right)dy = B_0 L \cancelto{0}{\int_0^H \cos\left(\frac{m\pi y}{H}\right)dy} + \sum_{n=1}^{\infty} B_n \sinh\left[\frac{n\pi(-L)}{H}\right]\underbrace{\int_0^H \cos\left(\frac{n\pi y}{H}\right)\cos\left(\frac{m\pi y}{H}\right)dy}_{(H/2)\delta_{nm}}$$

$$= B_m \frac{H}{2}\sinh\left(\frac{m\pi(-L)}{H}\right)$$

$$\Rightarrow \boxed{B_n = \frac{-2}{H\sinh\left(\frac{n\pi L}{H}\right)}\int_0^H f(y)\cos\left(\frac{n\pi y}{H}\right)dy}$$

$0 \le x \le L \qquad 0 \le y \le H \qquad f(x) = u(x,0) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}$

8) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial u}{\partial x}(0,y) = 0 \qquad \frac{\partial u}{\partial x}(L,y) = 0 \qquad \frac{\partial u}{\partial y}(x,H) = 0$

let $u(x,y) = h(x)\phi(y) \rightarrow \phi \frac{\partial^2 h}{\partial x^2} + h \frac{\partial^2 \phi}{\partial y^2} = 0$ dividying by ϕh:

$$\underbrace{\frac{1}{h}\frac{\partial^2 h}{\partial x^2}}_{-\lambda} + \underbrace{\frac{1}{\phi}\frac{\partial^2 \phi}{\partial y^2}}_{\lambda} = 0 \rightarrow \frac{d^2 h}{dx^2} = -\lambda h(x) \qquad h'(0) = 0 \qquad h'(L) = 0$$

$$\frac{d^2 \phi}{dy^2} = \lambda \phi(y) \qquad \phi'(H) = 0$$

$\underline{\lambda = 0}$: $\left.\begin{array}{l} h'(x) = C_1 \rightarrow h(x) = C_1 x + C_0 \\ h'(0) = 0 = C_1 \quad h'(L) = 0 = C_1 \end{array}\right\} \boxed{h_0(x) = C_0}$

$\underline{\lambda > 0}$: $h(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$h'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x \rightarrow h'(0) = 0 = C_2\sqrt{\lambda} \rightarrow C_2 = 0$

$h'(L) = 0 = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L)$

$\rightarrow \sqrt{\lambda}L = n\pi \rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2} \quad n = 1,2,3\ldots$

$\Rightarrow \boxed{h_n(x) = C_{1n}\cos\left(\frac{n\pi x}{L}\right)}$

$\underline{\lambda < 0}$: $h(x) = A\cosh\sqrt{-\lambda}x + B\sinh\sqrt{-\lambda}x$

$h'(x) = A\sqrt{-\lambda}\sinh\sqrt{-\lambda}x + B\sqrt{-\lambda}\cosh\sqrt{-\lambda}x \rightarrow h'(0) = 0 = B\sqrt{-\lambda} \rightarrow B = 0$

$h'(L) = 0 = A\sqrt{-\lambda}\sin\sqrt{-\lambda}L = 0$

$\rightarrow A = 0 \rightarrow$ No eigenvalues for $\lambda < 0$.

$\underline{\lambda = 0}$: $\frac{d^2\phi}{dy^2} = 0 \rightarrow \left.\begin{array}{l} \phi(y) = a_1 y + a_0 \\ \phi'(H) = 0 = a_1 \end{array}\right\} \boxed{\phi_0(y) = a_0}$

$\underline{\lambda > 0}$: $\frac{d^2\phi}{dy^2} = \lambda_n \phi = \left(\frac{n\pi}{L}\right)^2 \phi(y) \rightarrow \phi(y) = A\cosh\left[\frac{n\pi}{L}(y-H)\right] + B\sinh\left[\left(\frac{n\pi}{L}\right)(y-H)\right]$

$\phi'(y) = \left(\frac{n\pi}{L}\right)\left\{A\sinh\left[\left(\frac{n\pi}{L}\right)(y-H)\right] + B\cosh\left[\left(\frac{n\pi}{L}\right)(y-H)\right]\right\}$

$\phi'(H) = 0 = \left(\frac{n\pi}{L}\right)B \rightarrow B = 0 \rightarrow \boxed{\phi_n(y) = A_n\cosh\left[\left(\frac{n\pi}{L}\right)(y-H)\right]}$

$$\Rightarrow \boxed{u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh\left[\left(\frac{n\pi}{L}\right)(y-H)\right]\cos\left(\frac{n\pi x}{L}\right)}$$

$$f(x) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cosh\left[\left(\frac{n\pi}{L}\right)(-H)\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$\int_0^L f(x)\,dx = A_0 \underbrace{\int_0^L dx}_{L} + \sum_{n=1}^{\infty} A_n \cosh\left[\left(\frac{n\pi}{L}\right)(-H)\right]\cancel{\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx}^{\,0}$$

$$\rightarrow \boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = A_0 \cancel{\int_0^L \cos\left(\frac{m\pi x}{L}\right)dx}^{\,0} + \sum_{n=1}^{\infty} A_n \cosh\left[\quad\right] \underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx}_{\frac{L}{2}\delta_{nm}}$$

$$= A_m \frac{L}{2}\cosh\left[\frac{m\pi}{L}(-H)\right]$$

$$\boxed{A_n = \frac{2}{L\cosh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

Since $f(x) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases} \Rightarrow A_0 = \frac{1}{L}\left\{\underbrace{\int_0^{L/2} dx}_{L/2} + \cancel{\int_{L/2}^{L} 0\cdot dx}^{\,0}\right\}$

$$\boxed{A_0 = \frac{1}{2}}$$

$$A_n = \frac{2}{L\cosh\left(\frac{n\pi H}{L}\right)}\left\{\int_0^{L/2}\cos\left(\frac{n\pi x}{L}\right)dx + \cancel{\int_{L/2}^{L} 0\cdot\cos\left(\frac{n\pi x}{L}\right)dx}^{\,0}\right\}$$

$$= \frac{2}{\cancel{L}\cosh\left(\frac{n\pi H}{L}\right)}\left[\frac{\cancel{L}}{(n\pi)}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2}\right] = \frac{2}{(n\pi)\cosh\left(\frac{n\pi H}{L}\right)}\left[\underbrace{\sin\frac{n\pi}{2}}_{\begin{cases}1 & \text{if } n=1,5,9\ldots\\ -1 & \text{if } n=3,7,11,\ldots\\ 0 & \text{if } n \text{ is even}\end{cases}} - \cancel{\sin 0}^{\,0}\right]$$

$$\boxed{A_n = \frac{2\sin(n\pi/2)}{n\pi\cosh(n\pi H/L)}}$$

$$\Rightarrow u(x,y) = \frac{1}{2} + \sum_{n\ \text{odd}} \frac{2\sin(n\pi/2)}{(n\pi)\cosh(n\pi H/L)}\cosh\left[\frac{n\pi}{L}(y-H)\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$\boxed{u(x,y) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\cosh[n\pi(y-H)/L]}{(2n-1)\cosh[(2n-1)\pi H/L]}\cos\left(\frac{n\pi x}{L}\right)}$$

2.5.2 $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \frac{\partial u}{\partial x}(0,y) = 0 \quad \frac{\partial u}{\partial x}(L,y) = 0 \quad \frac{\partial u}{\partial y}(x,0) = 0$

$0 \le x \le L \quad 0 \le y \le H \quad \frac{\partial u}{\partial y}(x,H) = f(x)$

* **a)** From the first three boundary conditions, the $x=0$, $x=L$ and $y=0$ sides of the rectangle are insulated (no heat flow). From the last boundary condition, $f(x)$ is the prescribed heat flow at x and $y=H$. Thus $\int_0^L f(x)\,dx$ gives the total net prescribed heat flow. For there to exist an equilibrium temperature distribution, there must be zero net heat flow. Otherwise the temperature will never equilibrate. Thus for there to be a solution to this problem, we must have:

$\int_0^L f(x)\,dx = 0 \quad \left(\text{Remember } \begin{bmatrix}\text{Heat flow}\\ \text{at } y=L\\ \text{per unit area}\end{bmatrix} = -K_0\frac{\partial u}{\partial y}(x,H) = -K_0 f(x)\right)$

b) $u(x,y) = h(x)\phi(y) \rightarrow \phi\frac{d^2h}{dx^2} + h\frac{d^2\phi}{dy^2} = 0$ dividying by ϕh then:

$$\underbrace{\frac{1}{h}\frac{d^2h}{dx^2}}_{-\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dy^2}}_{\lambda} = 0 \rightarrow \frac{d^2h}{dx^2} = -\lambda h(x) \qquad h'(0) = 0 \qquad h'(L) = 0$$

$$\frac{d^2\phi}{dy^2} = \lambda\phi(y) \qquad \phi'(0) = 0$$

$\underline{\lambda=0}$: $h'(x) = C_1 \longrightarrow h(x) = C_1 x + C_0$ $\Big\}$ $\boxed{h_0(x) = C_0}$

$h'(0) = 0 = C_1$ and $h'(L) = 0 = C_1$

$\underline{\lambda>0}$: $h(x) = C_1 \cos\sqrt{\lambda}\,x + C_2 \sin\sqrt{\lambda}\,x$

$h'(x) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,x) \rightarrow h'(0) = 0 = C_2\sqrt{\lambda} \rightarrow C_2 = 0$

$\Rightarrow h'(L) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,L) = 0 \rightarrow \sqrt{\lambda}\,L = n\pi \rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$ $n = 1, 2, 3, \ldots$

$\Rightarrow \boxed{h_n(x) = C_{1n}\cos\left(\frac{n\pi x}{L}\right)}$

$\underline{\lambda<0}$: $h(x) = A\cosh\sqrt{-\lambda}\,x + B\sinh\sqrt{-\lambda}\,x$

$h'(x) = A\sqrt{-\lambda}\sinh\sqrt{-\lambda}\,x + B\sqrt{-\lambda}\cosh\sqrt{-\lambda}\,x \rightarrow h'(0) = 0 = B\sqrt{-\lambda} \rightarrow B = 0$

$h'(L) = A\sqrt{-\lambda}\sinh(\sqrt{-\lambda}\,x) = 0 \rightarrow A = 0$ No eigenvalues for $\lambda<0$.

$\underline{\lambda=0}$: $\frac{d^2\phi}{dy^2} = 0 \rightarrow \phi'(y) = a_1 \rightarrow \phi(y) = a_1 y + a_0$ $\Big\}$ $\boxed{\phi_0(y) = a_0}$

$\phi'(0) = 0 = a_1$

$\underline{\lambda>0}$: $\frac{d^2\phi}{dy^2} = \lambda_n\phi = \left(\frac{n\pi}{L}\right)^2\phi(y) \rightarrow \phi(y) = A\cosh\left(\frac{n\pi y}{L}\right) + B\sinh\left(\frac{n\pi y}{L}\right)$

$\phi'(y) = \left(\frac{n\pi}{L}\right)\left[A\sinh\left(\frac{n\pi y}{L}\right) + B\cosh\left(\frac{n\pi y}{L}\right)\right]$

$\phi'(0) = 0 = \left(\frac{n\pi}{L}\right)B \rightarrow B = 0 \rightarrow \boxed{\phi_n(y) = A_n\cosh\left(\frac{n\pi y}{L}\right)}$

$$\Rightarrow \boxed{u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n\cosh\left(\frac{n\pi y}{L}\right)\cos\left(\frac{n\pi x}{L}\right)}$$

$$\frac{\partial u}{\partial y} = \sum_{n=1}^{\infty} A_n\left(\frac{n\pi}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{\partial u}{\partial y}(x,H) = \sum_{n=1}^{\infty} A_n\left(\frac{n\pi}{L}\right)\sinh\left(\frac{n\pi H}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

multiplying by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating

$$\int_0^H f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} A_n\left(\frac{n\pi}{L}\right)\sinh\left(\frac{n\pi H}{L}\right)\underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\delta_{nm}}$$

$$= A_m\left(\frac{m\pi}{\cancel{L}}\right)\sinh\left(\frac{m\pi H}{L}\right)\cdot\frac{\cancel{L}}{2}$$

$$\Rightarrow \boxed{A_n = \frac{2}{(n\pi)\sinh(n\pi H/L)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

Note that A_n will be determined by this condition, but A_0 is an arbitrary constant. By other hand note that

$$\int_0^L f(x)\,dx = \sum_{n=1}^{\infty} A_n\left(\frac{n\pi}{L}\right)\sinh\left(\frac{n\pi H}{L}\right)\cancel{\int_0^L \cos\left(\frac{n\pi H}{L}\right)dx} \rightarrow 0$$

$\rightarrow \boxed{\int_0^L f(x)\,dx = 0}$ Just the condition from part a), thus the metod works only under these conditions

c) $\frac{\partial u}{\partial t} = k\nabla^2 u \qquad u(x,y,t=0) = g(x,y)$

The steady-solution is when $\frac{\partial u}{\partial t} = 0 \rightarrow \nabla^2 u = 0$ wit BCs

The solution for Laplace equation was obtained above for the equilibrium. In steady-state OR equilibrium there is zero net heat flow into the rectangle. This means the total heat energy $\int_0^L\int_0^H e\,dy\,dx = \int_0^L\int_0^H c\rho \underbrace{u(x,y,t=0)}_{g(x,y)}\,dy\,dx = cte$

$$\Rightarrow \int_0^L\int_0^H c\rho\, g(x,y)\,dy\,dx = \int_0^L\int_0^H c\rho\left[A_0 + \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi y}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\right]dy\,dx$$

$$= c\rho A_0 \int_0^L\int_0^H dy\,dx + c\rho\sum_{n=1}^{\infty} A_n \int_0^H \cosh\left(\frac{n\pi y}{L}\right)dy \cancel{\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx} \rightarrow 0$$

$$\cancel{c\rho}\int_0^L\int_0^H g(x,y)\,dy\,dx = \cancel{c\rho} A_0 LH \rightarrow \boxed{A_0 = \frac{1}{LH}\int_0^L\int_0^H g(x,y)\,dy\,dx}$$

* 2.5.3 The general solution to the Laplace's equation in polar coordinates is

$$u(r,\theta) = \phi(\theta)G(r)$$

with $\phi(\theta) = C_0 + [C_1\cos(n\theta) + C_2\sin n\theta]$

$G(r) = b_0 + b_1 \ln r + [a_1 r^n + a_2 r^{-n}]$

See solution to Problem 2.4.7 or Sec 2.5.2 from book

If $r \to \infty \Rightarrow |u(r,\theta)| < \infty \Rightarrow b_1 = 0$ and $a_1 = 0$.

$$\Rightarrow \boxed{u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n r^{-n}\cos(n\theta) + B_n r^{-n}\sin(n\theta)\right]} \qquad a < r < \infty,\ -\pi < \theta < \pi$$

Solution outside a circle

a) $u(a,\theta) = \ln 2 + 4\cos 3\theta = A_0 + \sum_{n=1}^{\infty}\left[A_n a^{-n}\cos(n\theta) + B_n a^{-n}\sin(n\theta)\right]$

$\Rightarrow \boxed{A_0 = \ln 2} \quad 4 = A_3 a^{-3} \rightarrow \boxed{A_3 = 4a^3}$ Since there are not sine terms $\Rightarrow \boxed{B_n = 0}$

and $A_n = 0$ if $n \neq 0, 3$.

$$\Rightarrow \boxed{u(r,\theta) = \ln 2 + 4\left(\frac{a}{r}\right)^3\cos(3\theta)}$$

b) $f(\theta) = u(a,\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n a^{-n}\cos(n\theta) + B_n a^{-n}\sin(n\theta)\right]$

$$\int_{-\pi}^{\pi} f(\theta)\,d\theta = A_0 \underbrace{\int_{-\pi}^{\pi} d\theta}_{2\pi} + \sum_{n=1}^{\infty}\left\{A_n a^{-n}\cancelto{0}{\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta} + B_n a^{-n}\cancelto{0}{\int_{-\pi}^{\pi}\sin(n\theta)\,d\theta}\right\}$$

$$\Rightarrow \boxed{A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta}$$

$$\int_{-\pi}^{\pi} f(\theta)\cos(m\theta)\,d\theta = A_0\cancelto{0}{\int_{-\pi}^{\pi}\cos(m\theta)\,d\theta} + \sum_{n=1}^{\infty} a^{-n}\left\{A_n \overbrace{\int_{-\pi}^{\pi}\cos(m\theta)\cos(n\theta)\,d\theta}^{\pi\delta_{nm}} + B_n\cancelto{0}{\int_{-\pi}^{\pi}\cos(m\theta)\sin(n\theta)\,d\theta}\right\}$$

$$= \sum_{n=1}^{\infty} a^{-n} A_n \pi \delta_{nm}$$

$$= a^{-m} A_m \pi \Rightarrow \boxed{A_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta}$$

$$\int_{-\pi}^{\pi} f(\theta)\sin(m\theta)\,d\theta = A_0\cancelto{0}{\int_{-\pi}^{\pi}\sin(m\theta)\,d\theta} + \sum_{n=1}^{\infty} a^{-n}\left\{A_n\cancelto{0}{\int_{-\pi}^{\pi}\sin(m\theta)\cos(n\theta)\,d\theta} + B_n \underbrace{\int_{-\pi}^{\pi}\sin(m\theta)\sin(n\theta)\,d\theta}_{\pi\delta_{nm}}\right\}$$

$$= \sum_{n=1}^{\infty} a^{-n} B_n \pi \delta_{nm}$$

$$= a^{-m} B_m \pi \Rightarrow \boxed{B_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta}$$

* 2.5.4 $u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right] r^n$ where:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\bar\theta)\,d\bar\theta \qquad A_n = \frac{a^{-n}}{\pi}\int_{-\pi}^{\pi} f(\bar\theta)\cos(n\bar\theta)\,d\bar\theta \qquad B_n = \frac{a^{-n}}{\pi}\int_{-\pi}^{\pi} f(\bar\theta)\sin(n\bar\theta)\,d\bar\theta$$

$$\Rightarrow u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\bar\theta)\,d\bar\theta + \sum_{n=1}^{\infty}\left[\frac{a^{-n}}{\pi}\int_{-\pi}^{\pi} f(\bar\theta)\cos(n\bar\theta)\,d\bar\theta\cos(n\theta) + \frac{a^{-n}}{\pi}\int_{-\pi}^{\pi} f(\bar\theta)\sin(n\bar\theta)\,d\bar\theta\sin(n\theta)\right] r^n$$

$$= \int_{-\pi}^{\pi}\left[\frac{1}{2\pi}f(\bar\theta) + \sum_{n=1}^{\infty}\frac{1}{\pi}f(\bar\theta)\left[\cos(n\bar\theta)\cos(n\theta) + \sin(n\bar\theta)\sin(n\theta)\right]\left(\frac{r}{a}\right)^n\right]d\bar\theta$$

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar\theta)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\bar\theta)\right]d\bar\theta$$

Now for $n=0$ $\left(\frac{r}{a}\right)^n\cos(\theta-\bar\theta) = 1\cdot\cos 0 = 1$. Thus we can start the sum at $n=0$ if subtract 1. This gives us:

$$\boxed{u(r,\theta)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar\theta)\left[-\frac{1}{2}+\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\bar\theta)\right]d\bar\theta}$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar\theta)\left[-\frac{1}{2}+\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n \mathrm{Re}\, e^{in(\theta-\bar\theta)}\right]d\bar\theta$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar\theta)\left[-\frac{1}{2}+\mathrm{Re}\left\{\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\bar\theta)}\right\}\right]d\bar\theta$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar\theta)\left[-\frac{1}{2}+\mathrm{Re}\left\{\underbrace{\sum_{n=0}^{\infty}\left(\frac{r}{a}e^{i(\theta-\bar\theta)}\right)^n}_{\text{This is a geometric series which converges to:}}\right\}\right]d\bar\theta$$

$$\sum_{n=0}^{\infty}\left(\frac{r}{a}e^{i(\theta-\bar\theta)}\right)^n=\frac{1}{1-\frac{r}{a}e^{i(\theta-\bar\theta)}}=\frac{a}{a-re^{i(\theta-\bar\theta)}}\left(\frac{a-re^{-i(\theta-\bar\theta)}}{a-re^{-i(\theta-\bar\theta)}}\right)$$

$$=\frac{a^2-ar\,e^{i(\theta-\bar\theta)}}{a^2-ar\left(e^{i(\theta-\bar\theta)}+e^{-i(\theta-\bar\theta)}\right)+r^2}=\frac{a^2-ar\,e^{-i(\theta-\bar\theta)}}{a^2+r^2-2ar\cos(\theta-\bar\theta)}$$

Thus

$$\mathrm{Re}\left[\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\bar\theta)}\right]=\frac{a^2-ar\cos(\theta-\bar\theta)}{a^2+r^2-2ar\cos(\theta-\bar\theta)}\quad\text{and}$$

$$-\frac{1}{2}+\mathrm{Re}\left[\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\bar\theta)}\right]=\frac{-[a^2+r^2-\cancel{2ar\cos(\theta-\bar\theta)}]+2(a^2-\cancel{2ar\cos(\theta-\bar\theta)})}{2[a^2+r^2-2ar\cos(\theta-\bar\theta)]}$$

$$=\frac{a^2-r^2}{2[a^2+r^2-2ar\cos(\theta-\bar\theta)]}$$

$$\Rightarrow\boxed{u(r,\theta)=\frac{a^2-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(\bar\theta)\,d\bar\theta}{a^2+r^2-2ar\cos(\theta-\bar\theta)}}$$

Poisson's integral formula!!

2.5.5 $\nabla^2 u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0 \qquad 0\le\theta\le\pi/2 \qquad 0\le r\le 1$

Let $u(r,\theta)=\phi(\theta)G(r)\longrightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta)+\frac{1}{r^2}G(r)\frac{d^2\phi}{d\theta^2}=0$

dividyng by $\frac{G\phi}{r^2}\Rightarrow \underbrace{\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right)}_{\lambda}+\underbrace{\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}}_{-\lambda}=0 \quad\searrow\boxed{\frac{d^2\phi}{d\theta^2}=-\lambda\phi}$

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right)=r\left(r\frac{d^2G}{dr^2}+\frac{dG}{dr}\right)=\lambda G\longrightarrow\boxed{r^2\frac{d^2G}{dr^2}+r\frac{dG}{dr}=\lambda G}$$

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* a) $u(r,\theta)=\phi(\theta)G(r)$ $\quad 0\le\theta\le\pi/2$ $\quad 0\le r\le 1$ $\quad u(1,\theta)=f(\theta)$

$$\frac{d^2\phi}{d\theta^2}=-\lambda\phi \qquad \phi'(0)=0,\ \phi(\pi/2)=0 \qquad r\frac{d}{dr}\left(r\frac{dG}{dr}\right)=r^2\frac{d^2G}{dr^2}+r\frac{dG}{dr}=\lambda G$$

$\underline{\lambda=0}$: $\phi'(\theta)=C_1 \rightarrow \phi(\theta)=C_1\theta+C_0$, $\phi'(0)=0=C_1$, $\phi(\pi/2)=0=C_0$ $\Big\}$ No eigenvalues for $\lambda=0$

$\underline{\lambda>0}$: $\phi(\theta)=C_1\cos\sqrt{\lambda}\theta+C_2\sin\sqrt{\lambda}\theta$

$\phi'(\theta)=-C_1\sqrt{\lambda}\sin(\sqrt{\lambda}\theta)+C_2\sqrt{\lambda}\cos(\sqrt{\lambda}\theta)$ $\qquad \phi'(0)=0=C_2\sqrt{\lambda}\rightarrow C_2=0$

$\phi(\pi/2)=C_1\cos\sqrt{\lambda}(\pi/2)=0 \rightarrow \sqrt{\lambda}\,\cancel{\pi/2}=(2n-1)\cancel{\pi/2}$ $\quad n=1,2,3,\ldots$

$\boxed{\lambda_n=(2n-1)^2}$

$\Rightarrow \boxed{\phi_n(\theta)=C_1\cos(2n-1)\theta}$

$\underline{\lambda<0}$: $\phi(\theta)=A\cosh\sqrt{-\lambda}\theta+B\sinh\sqrt{-\lambda}\theta$

$\phi'(\theta)=A\sqrt{-\lambda}\sinh\sqrt{-\lambda}\theta+B\sqrt{-\lambda}\cosh\sqrt{-\lambda}\theta \rightarrow \phi'(0)=0=B\sqrt{-\lambda}\rightarrow B=0$

$\rightarrow \phi(\pi/2)=A\cosh\sqrt{-\lambda}\pi/2=0 \rightarrow A=0$ No eigenvalues for $\lambda<0$.

$\underline{\lambda>0}$: $\left(r^2\frac{d^2G}{dr^2}+r\frac{dG}{dr}\right)=\lambda_n G=(2n-1)^2G(r)$ $\qquad$ see solution to problem 2.4.7 OR sec 2.5.2 from book

Solution for this eq is: $\boxed{G(r)=a_1r^{2n-1}+a_2r^{-(2n-1)}}$

if $|G(0)|<\infty \rightarrow a_2=0 \rightarrow G(r)=a_1r^{2n-1}$

$$\Rightarrow \boxed{u(r,\theta)=\sum_{n=1}^{\infty}A_nr^{2n-1}\cos(2n-1)\theta}$$

$f(\theta)=u(1,\theta)=\sum_{n=1}^{\infty}A_n\cos(2n-1)\theta$ $\qquad$ multiplying by $\cos(2m-1)\theta$ and integrating.

$$\int_0^{\pi/2}f(\theta)\cos(2m-1)\theta\,d\theta=\sum_{n=1}^{\infty}A_n\underbrace{\int_0^{\pi/2}\cos(2m-1)\theta\cos(2n-1)\theta\,d\theta}$$

$\cos a\cos b=\frac{1}{2}[\cos(a+b)+\cos(a-b)]$

$$=\sum_{n=1}^{\infty}A_n\underbrace{\frac{1}{2}\int_0^{\pi/2}\left[\cos 2(m+n-1)\theta+\cos 2(m-n)\theta\right]d\theta}_{I}$$

$\underline{\text{if } n=m}$: $I=\frac{1}{2}\int_0^{\pi/2}[\cos(2(2m-1)\theta)+1]\,d\theta=\frac{1}{2}\left[\cancel{\frac{\sin 2(2m-1)\theta}{2(2m-1)}\Big|_0^{\pi/2}}^{\,0}+\frac{\pi}{2}\right]=\frac{\pi}{4}$

$\underline{\text{if } n\neq m}$: $I=\frac{1}{2}\left[\underbrace{\frac{\sin 2(m+n-1)\theta}{2(m+n-1)}\Big|_0^{\pi/2}}_{0}+\underbrace{\frac{\sin 2(m-n)}{2(m-n)}\Big|_0^{\pi/2}}_{0}\right]=0$

Because $2(m+n-1)$ OR $2(m-1)$ are even numbers and $\sin\left[(\text{even number})\frac{\pi}{2}\right]=0$. ✓

$$\Rightarrow I = \boxed{\int_0^{\pi/2} \cos(2m-1)\theta \cos(2n-1)\theta \, d\theta = \frac{\pi/2}{2}\delta_{nm}}$$

$$\Rightarrow \int_0^{2\pi} f(\theta)\cos(2m-1)\theta \, d\theta = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{\pi/2} \cos(2n-1)\theta \cos(2m-1)\theta \, d\theta}_{(\pi/4)\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} A_n \frac{\pi}{4}\delta_{nm}$$

$$= A_n \frac{\pi}{4} \longrightarrow \boxed{A_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\cos(2n-1)\theta \, d\theta}$$

b) $u(r,\theta) = \phi(\theta)G(r) \qquad 0 \le \theta \le \pi/2 \qquad 0 \le r \le 1 \qquad u(1,\theta) = f(\theta)$

$$\frac{d^2\phi}{d\theta^2} = -\lambda\phi \qquad \begin{matrix}\phi'(0) = 0 \\ \phi'(\pi/2) = 0\end{matrix} \qquad r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} = \lambda G$$

$\underline{\lambda = 0}$: $\phi'(\theta) = C_1 \longrightarrow \phi(\theta) = C_1\theta + C_0$, $\phi'(0) = 0 = C_1$, $\phi'(\pi/2) = 0 = C_1$ $\Big\}$ $\boxed{\phi_0(\theta) = C_0}$

$\underline{\lambda > 0}$: $\phi(\theta) = C_1\cos\sqrt{\lambda}\theta + C_2\sin\sqrt{\lambda}\theta$

$\phi'(\theta) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\theta + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\theta \rightarrow \phi'(0) = 0 = C_2\sqrt{\lambda} \rightarrow C_2 = 0$

$\phi'(\pi/2) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi/2 = 0 \rightarrow \sqrt{\lambda}\cancel{\pi}/2 = \cancel{n}\pi \rightarrow \boxed{\lambda_n = (2n)^2}$ $n = 1, 2, 3, \ldots$

$\Rightarrow \boxed{\phi(\theta) = C_1\cos(2n\theta)}$

$\underline{\lambda < 0}$: $\phi(\theta) = A\cosh\sqrt{-\lambda}\theta + B\sinh\sqrt{-\lambda}\theta \rightarrow$ It's not solution because no oscillates.

$\underline{\lambda = 0}$: $r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \rightarrow r\frac{dG}{dr} = b_1 \rightarrow \frac{dG}{dr} = \frac{b_1}{r} \rightarrow \underline{G_0(r) = b_1\ln r + b_0}$

We don't want singularities in $r \to 0 \Rightarrow b_1 = 0 \rightarrow \boxed{G_0(r) = b_0}$

$\underline{\lambda > 0}$: $\left(r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr}\right) = \lambda_n G = (2n)^2 G(r) \rightarrow G_n(r) = a_1 r^{2n} + a_2 r^{-2n}$

we don't want singularities in $r \to 0 \Rightarrow a_2 = 0 \rightarrow \boxed{G_n(r) = a_1 r^{2n}}$

$$\Rightarrow \boxed{u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^{2n}\cos(2n\theta)}$$

$$f(\theta) = u(1,\theta) = A_0 + \sum_{n=1}^{\infty} A_n\cos(2n\theta) \rightarrow \int_0^{\pi/2} f(\theta)\,d\theta = A_0\overbrace{\int_0^{\pi/2} d\theta}^{\pi/2} + \sum_{n=1}^{\infty} A_n \cancel{\int_0^{\pi/2}\cos(2n\theta)\,d\theta}^{\,0}$$

$$\Rightarrow \boxed{A_0 = \frac{2}{\pi}\int_0^{\pi/2} f(\theta)\,d\theta}$$

Multiplying by $\cos(2m\theta)$ and integrating $\Rightarrow$

$$\int_0^{\pi/2} f(\theta)\cos(2m\theta)\,d\theta = A_0 \cancel{\int_0^{\pi/2}\cos(2m\theta)\,d\theta}^{\,0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{\pi/2}\cos 2m\theta \cos 2n\theta\, d\theta}_{\frac{(\pi/2)}{2}\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} A_n \frac{\pi}{4}\delta_{nm}$$

$$= A_m \frac{\pi}{4} \longrightarrow \boxed{A_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\cos(2m\theta)\,d\theta}$$

* c) $u(r,\theta) = \phi(\theta)G(r) \qquad 0 \le \theta \le \pi/2 \qquad 0 \le r \le 1 \qquad \frac{\partial u}{\partial r}(1,\theta) = f(\theta)$

$$\frac{d^2\phi}{d\theta^2} = -\lambda\phi \qquad \begin{matrix}\phi(0)=0\\ \phi(\pi/2)=0\end{matrix} \qquad r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} = \lambda G$$

$\underline{\lambda = 0}$: $\phi(\theta) = C_1\theta + C_0 \longrightarrow \left.\begin{matrix}\phi(0) = 0 = C_0\\ \phi(\pi/2) = 0 = C_1(\pi/2) \rightarrow C_1 = 0\end{matrix}\right\}$ No eigenvalues for $\lambda = 0$.

$\underline{\lambda > 0}$: $\phi(\theta) = C_1\cos\sqrt{\lambda}\,\theta + C_2\sin\sqrt{\lambda}\,\theta \longrightarrow \begin{matrix}\phi(0) = 0 = C_1\\ \phi(\pi/2) = 0 = C_2\sin\sqrt{\lambda}(\pi/2)\end{matrix}$

$$\rightarrow \sqrt{\lambda}\frac{\cancel{\pi}}{2} = n\cancel{\pi} \rightarrow \boxed{\lambda_n = (2n)^2} \rightarrow \boxed{\phi_n(\theta) = C_n\sin(2n\theta)}$$

$\lambda < 0$: $\phi(\theta) = A\cosh\sqrt{-\lambda}\,\theta + B\sinh\sqrt{-\lambda}\,\theta \longrightarrow$ It's not solution because no oscillates.

$\underline{\lambda > 0}$: $r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} = \lambda_n G = (2n)^2 G(r) \longrightarrow G_n(r) = a_1 r^{2n} + a_2 r^{-2n}$

we don't want singularities in $r \to 0 \Rightarrow a_2 = 0 \rightarrow \boxed{G_n(r) = a_1 r^{2n}}$

$$\Rightarrow \boxed{u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n}\sin(2n\theta)}$$

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} A_n(2n)r^{2n-1}\sin(2n\theta)$$

$$f(\theta) = \frac{\partial u}{\partial r}(1,\theta) = \sum_{n=1}^{\infty} A_n(2n)\sin(2n\theta)$$

Multiplying by $\sin(2m\theta)$ and integrating:

$$\int_0^{\pi/2} f(\theta)\sin(2m\theta)\,d\theta = \sum_{n=1}^{\infty} A_n(2n)\underbrace{\int_0^{\pi/2}\sin(2n\theta)\sin(2m\theta)\,d\theta}_{\frac{(\pi/2)}{2}\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} A_n \frac{n\pi}{2}\delta_{nm}$$

$$= A_m\frac{m\pi}{2} \longrightarrow \boxed{A_n = \frac{2}{n\pi}\int_0^{\pi/2} f(\theta)\sin(2n\theta)\,d\theta}$$

d) $u(r,\theta)=\phi(\theta)G(r)$ $\quad 0\le\theta\le\pi/2 \quad 0\le r\le 1 \quad \frac{\partial u}{\partial r}(1,\theta)=g(\theta)$

$$\frac{d^2\phi}{d\theta^2}=-\lambda\phi \qquad \begin{matrix}\phi'(0)=0\\ \phi'(\pi/2)=0\end{matrix} \qquad r\frac{d}{dr}\left(r\frac{dG}{dr}\right)=r^2\frac{d^2G}{dr^2}+r\frac{dG}{dr}=\lambda G$$

This is the same problem that **2.5.1**, except the condition outside of this box.

$\Rightarrow$ $\boxed{u(r,\theta)=A_0+\sum_{n=1}^{\infty}A_n r^{2n}\cos(2n\theta)}$ and computating $\frac{\partial u}{\partial r}$:

$$\frac{\partial u}{\partial r}=\sum_{n=1}^{\infty}A_n(2n)r^{2n-1}\cos(2n\theta)$$

$g(\theta)=\frac{\partial u}{\partial r}(1,\theta)=\sum_{n=1}^{\infty}A_n(2n)\cos(2n\theta)$ multiplying by $\cos(2m\theta)$

$$\int_0^{\pi/2}g(\theta)\cos(2m\theta)=\sum_{n=1}^{\infty}A_n(2n)\underbrace{\int_0^{\pi/2}\cos(2n\theta)\cos(2m\theta)\,d\theta}_{\frac{(\pi/2)}{2}\delta_{nm}}$$

$$=\sum_{n=1}^{\infty}A_n\frac{n\pi}{2}\delta_{nm}$$

$$=A_n\frac{m\pi}{2}\longrightarrow\boxed{A_n=\frac{2}{n\pi}\int_0^{\pi/2}g(\theta)\cos(2n\theta)\,d\theta}$$

Note that A_n will be determined by this condition, but A_0 is an arbitrary constant. By other hand if we compute:

$$\int_0^{\pi/2}g(\theta)\,d\theta=\int_0^{\pi/2}\left(\sum_{n=1}^{\infty}A_n(2n)\cos(2n\theta)\right)d\theta=\sum_{n=1}^{\infty}A_n(2n)\cancel{\int_0^{\pi/2}\cos(2n\theta)\,d\theta}\to 0$$

$\Rightarrow$ $\boxed{\int_0^{\pi/2}g(\theta)\,d\theta=0}$ Remembering $\left[\begin{matrix}\text{Heat flow}\\ \text{at } r=1\\ \text{per unit area}\end{matrix}\right]=-K_0\frac{\partial u}{\partial r}(1,\theta)$

$\Rightarrow$ This integral says that heat flow trough boundary at $r=1$ is zero. In addition $\frac{\partial u}{\partial y}(x,0)=0$ and $\frac{\partial u}{\partial y}(x,H)=0$ say that the boundaries in $\theta=0$ and $\theta=\pi/2$ are insulated (no heat flow), then the net heat flow is zero. It is consistent if $u(r,\theta)$ is really the solution for the equilibrium temperature distribution because $\nabla^2u=0$ is satisfied.

Heat Eq $\rightarrow \frac{\partial u}{\partial t}=-K\nabla^2u \Rightarrow \nabla^2u=0 \rightarrow \frac{\partial u}{\partial t}=0$ condition for equilibrium.

2.5.6 $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ $\quad 0 \le r \le a \quad 0 \le \theta \le \pi$

Let $u(r,\theta) = \phi(\theta)G(r) \rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)\phi + \frac{1}{r^2}G\frac{d^2\phi}{d\theta^2} = 0$

dividing by $\frac{\phi G}{r^2} \rightarrow \underbrace{\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right)}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}}_{-\lambda} = 0$

$$\boxed{\frac{d^2\phi}{d\theta^2} = -\lambda\phi}$$

$$\boxed{r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda G}$$

*a) $u = 0$ on the diameter $\rightarrow u(r,0) = 0 = u(r,\pi)$. // $u(a,\theta) = g(\theta)$

Since $u(r,\theta) = \phi(\theta)G(r) \rightarrow \phi(0) = 0 = \phi(\pi)$

$\underline{\lambda = 0}$: $\phi(\theta) = C_1\theta + C_0 \rightarrow \phi(0) = 0 = C_0$, $\phi(\pi) = 0 = C_1\pi$ } No eigenvalues for $\lambda = 0$.

$\underline{\lambda > 0}$: $\phi(\theta) = C_1\cos\sqrt{\lambda}\theta + C_2\sin\sqrt{\lambda}\theta \rightarrow \phi(0) = 0 = C_1$

$\phi(\pi) = 0 = C_2\sin\sqrt{\lambda}\pi$

$\rightarrow \sqrt{\lambda}\cancel{\pi} = n\cancel{\pi} \rightarrow \boxed{\lambda_n = n^2}$ $\rightarrow \boxed{\phi_n(\theta) = C_n\sin(n\theta)}$

$\underline{\lambda > 0}$: $r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda_n G = n^2 G(r) \rightarrow G(r) = a_1 r^n + a_2 r^{-n}$

We don't want singularities in $r \to 0 \Rightarrow a_2 = 0 \rightarrow \boxed{G_n(r) = a_1 r^n}$

$$\Rightarrow \boxed{u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)}$$

To solve the nonhomogeneus condition:

$g(\theta) = u(a,\theta) = \sum_{n=1}^{\infty} A_n a^n \sin(n\theta)$ Multiplying by $\sin(m\theta)$ and integrating.

$$\int_0^{\pi} g(\theta)\sin(m\theta)\,d\theta = \sum_{n=1}^{\infty} A_n a^n \underbrace{\int_0^{\pi}\sin(n\theta)\sin(m\theta)\,d\theta}_{(\pi/2)\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} A_n a^n \frac{\pi}{2}\delta_{nm}$$

$$= A_m a^m \frac{\pi}{2} \rightarrow \boxed{A_n = \frac{2}{\pi a^n}\int_0^{\pi} g(\theta)\sin(n\theta)\,d\theta}$$

$0 < r < a \qquad 0 < \theta < \pi \qquad u(a,\theta) = g(\theta)$

b) If the diameter is insulated $\rightarrow \frac{\partial u}{\partial \theta}(r,0) = 0 = \frac{\partial u}{\partial \theta}(r,\pi)$

Since $u(r,\theta) = \phi(\theta)G(r) \rightarrow \quad \phi'(0) = 0 = \phi'(\pi)$

$\lambda = 0$: $\phi'(\theta) = C_1 \longrightarrow \phi(\theta) = C_1\theta + C_0$
$\phi'(0) = 0 = C_1 \qquad \phi'(\pi) = 0 = C_1\pi \rightarrow C_1 = 0$ $\Big\} \rightarrow \boxed{\phi_0(\theta) = C_0}$

$\lambda > 0$: $\phi(\theta) = C_1 \cos\sqrt{\lambda}\,\theta + C_2 \sin\sqrt{\lambda}\,\theta$
$\phi'(\theta) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\theta + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,\theta \rightarrow \phi'(0) = 0 = C_2\sqrt{\lambda} \rightarrow C_2 = 0$
$\phi'(\pi) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi = 0$

$\rightarrow \sqrt{\lambda}\,\pi = n\pi \rightarrow \boxed{\lambda_n = n^2}$
$\rightarrow \boxed{\phi_n(x) = C_n \cos(n\theta)}$

$\lambda < 0$: $\phi(\theta) = A\cosh\sqrt{-\lambda}\,\theta + B\sinh\sqrt{-\lambda}\,\theta \rightarrow$ It's not solution because no oscillates

$\lambda = 0$: $r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \rightarrow r\frac{dG}{dr} = b_1 \rightarrow \frac{dG}{dr} = \frac{b_1}{r} \rightarrow \underline{G_0(r) = b_1 \ln r + b_0}$

We don't want singularities in $r \to 0 \Rightarrow b_1 = 0 \rightarrow \boxed{G_0(r) = b_0}$

$\lambda > 0$: $r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = n^2 G(r) \longrightarrow G(r) = a_1 r^n + a_2 r^{-n}$

For no singularities in $r \to 0 \Rightarrow a_2 = 0 \rightarrow \boxed{G_n(r) = a_n r^n}$

$$\Rightarrow \boxed{u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta)}$$

$$g(\theta) = u(a,\theta) = A_0 + \sum_{n=1}^{\infty} A_n a^n \cos(n\theta)$$

Integratin from 0 to π in θ:

$$\int_0^{\pi} g(\theta)\,d\theta = A_0\int_0^{\pi} d\theta + \sum_{n=1}^{\infty} A_n a^n \cancel{\int_0^{\pi} \cos(n\theta)\,d\theta}^{\,0}$$

$$= A_0\pi \longrightarrow \boxed{A_0 = \frac{1}{\pi}\int_0^{\pi} g(\theta)\,d\theta}$$

Multiplying by $\cos(m\theta)$ and integrating:

$$\int_0^{\pi} g(\theta)\cos(m\theta)\,d\theta = A_0\cancel{\int_0^{\pi}\cos(m\theta)\,d\theta}^{\,0} + \sum_{n=1}^{\infty} A_n a^n \underbrace{\int_0^{\pi}\cos(m\theta)\cos(n\theta)\,d\theta}_{(\pi/2)\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} A_n a^n \frac{\pi}{2}\delta_{nm}$$

$$= A_m a^m \frac{\pi}{2}$$

$$\Rightarrow \boxed{A_n = \frac{2}{\pi a^n}\int_0^{\pi} f(\theta)\cos(n\theta)\,d\theta}$$

2.5.7 $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ $\quad 0 \le r \le a \quad 0 \le \theta \le \pi/3$

Let $u(r,\theta) = \phi(\theta)G(r) \rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi + \frac{G}{r^2}\frac{d^2\phi}{d\theta^2} = 0$

dividying by $\frac{\phi G}{r} \rightarrow \underbrace{\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right)}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}}_{-\lambda} = 0$ $\rightarrow \boxed{\frac{d^2\phi}{d\theta^2} = -\lambda\phi}$

$$\boxed{r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda G}$$

a) $u(r,0) = 0 = u(r,\pi/3) \rightarrow \phi(0) = 0 = \phi(\pi/3)$ / $u(a,\theta) = f(\theta)$

$\underline{\lambda = 0}$: $\phi(\theta) = C_1\theta + C_2 \rightarrow \phi(0) = 0 = C_2$, $\phi(\pi/3) = 0 = C_1\pi/3 \rightarrow C_1 = 0$ } No eigenvalues for $\lambda = 0$.

$\underline{\lambda > 0}$: $\phi(\theta) = C_1\cos\sqrt{\lambda}\theta + C_2\sin\sqrt{\lambda}\theta \rightarrow \phi(0) = 0 = C_1$, $\phi(\pi/3) = C_2\sin\sqrt{\lambda}\frac{\pi}{3} = 0$

$\rightarrow \sqrt{\lambda}\frac{\cancel{\pi}}{3} = n\cancel{\pi} \rightarrow \boxed{\lambda_n = (3n)^2}$ $\quad n = 1,2,3\ldots$ $\rightarrow \boxed{\phi_n(\theta) = C_n\sin(3n\theta)}$

$\lambda < 0$: $\phi(\theta) = A\cosh\sqrt{-\lambda}\theta + B\sinh\sqrt{-\lambda}\theta \rightarrow$ It's not solution because no oscillates.

$\underline{\lambda > 0}$: $r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = (3n)^2 G(r) \rightarrow G(r) = a_1 r^{3n} + a_2 r^{-3n}$

we don't want singularities in $r \rightarrow 0 \Rightarrow a_2 = 0 \rightarrow \boxed{G_n = A_n r^{3n}}$

$$\Rightarrow \boxed{u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{3n}\sin(3n\theta)}$$

$f(\theta) = u(a,\theta) = \sum_{n=1}^{\infty} A_n a^{3n}\sin(3n\theta)$. Multiplying by $\sin(3m\theta)$ and integrating.

$$\int_0^{\pi/3} f(\theta)\sin(3m\theta) = \sum_{n=1}^{\infty} A_n a^{3n}\underbrace{\int_0^{\pi/3}\sin(3m\theta)\sin(3n\theta)\,d\theta}_{\left(\frac{\pi/3}{2}\right)\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} A_n a^{3n}\frac{\pi}{6}\delta_{nm}$$

$$= A_m a^{3m}\frac{\pi}{6}$$

$$\Rightarrow \boxed{A_n = \frac{6}{\pi a^{3n}}\int_0^{\pi/3} f(\theta)\sin(3n\theta)\,d\theta}$$

b) $\frac{\partial u}{\partial \theta}(r,0) = 0 = \frac{\partial u}{\partial \theta}(r,\pi/3) \rightarrow \phi'(0) = 0 = \phi'(\pi/3)$ / $u(a,\theta) = f(\theta)$

$\underline{\lambda = 0}$: $\phi'(\theta) = C_1 \longrightarrow \phi(\theta) = C_1\theta + C_0$
$\phi'(0) = 0 = C_1 \qquad \phi'(\pi/3) = C_1 \pi/3 = 0 \rightarrow C_1 = 0$ $\Big\}$ $\boxed{\phi_0(\theta) = C_0}$

$\underline{\lambda > 0}$: $\phi(\theta) = C_1 \cos\sqrt{\lambda}\,\theta + C_2 \sin\sqrt{\lambda}\,\theta$
$\phi'(\theta) = -\sqrt{\lambda}\,C_1 \sin\sqrt{\lambda}\,\theta + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,\theta \rightarrow \phi'(0) = 0 = C_2\sqrt{\lambda} \rightarrow C_2 = 0$
$\phi'(\pi/3) = -\sqrt{\lambda}\,C_1 \sin\left(\sqrt{\lambda}\frac{\pi}{3}\right) = 0$

$\rightarrow \sqrt{\lambda}\frac{\pi}{3} = n\pi \rightarrow \boxed{\lambda_n = (3n)^2} \rightarrow \boxed{\phi_n(\theta) = C_n \cos(3n\theta)}$

$\underline{\lambda < 0}$: $\phi(\theta) = A\cosh\sqrt{-\lambda}\,\theta + B\sinh\sqrt{-\lambda}\,\theta \rightarrow$ It's not solutions because no oscillates.

$\underline{\lambda = 0}$: $r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \rightarrow r\frac{dG}{dr} = b_1 \rightarrow \frac{dG}{dr} = \frac{b_1}{r} \rightarrow \underline{G(r) = b_1 \ln r + b_0}$
for no singularities in $r \rightarrow 0 \Rightarrow b_1 = 0 \rightarrow \boxed{G_0(r) = b_0}$

$\underline{\lambda > 0}$: $r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = (3n)^2 G(r) \rightarrow G(r) = b_1 r^{3n} + b_2 r^{-3n}$
for no singularities in $r \rightarrow 0 \Rightarrow b_2 = 0 \rightarrow \boxed{G_n(r) = b_1 r^{3n}}$

$$\Rightarrow \boxed{u(r,\theta) = B_0 + \sum_{n=1}^{\infty} B_n r^{3n} \cos(3n\theta)}$$

$f(\theta) = u(a,\theta) = B_0 + \sum_{n=1}^{\infty} B_n a^n \cos(3n\theta)$ an integrating:

$$\int_0^{\pi/3} f(\theta)\,d\theta = B_0 \int_0^{\pi/3} d\theta + \sum_{n=1}^{\infty} B_n a^n \cancel{\int_0^{\pi/3} \cos(3n\theta)\,d\theta} \rightarrow 0$$

$$= B_0 \frac{\pi}{3} \longrightarrow \boxed{A_0 = \frac{3}{\pi}\int_0^{\pi/3} g(\theta)\,d\theta}$$

Similarity, multiplying by $\cos(3m\theta)$ and integrating:

$$\int_0^{\pi/3} f(\theta)\cos(3m\theta)\,d\theta = B_0 \int_0^{\pi/3} \cos(3m\theta)\,d\theta + \sum_{n=1}^{\infty} B_n a^n \underbrace{\int_0^{\pi/3} \cos(3m\theta)\cos(3n\theta)\,d\theta}_{\frac{(\pi/3)}{2}\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} B_n a^n \frac{\pi}{6}\delta_{nm}$$

$$= B_m a^m \frac{\pi}{6} \rightarrow \boxed{A_n = \frac{6}{\pi a^{3n}}\int_0^{\pi/3} f(\theta)\cos(3m\theta)\,d\theta}$$

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2.5.8 $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad a \le r \le b \qquad -\pi \le \theta \le \pi$

let $u(r,\theta) = \phi(\theta)G(r) \rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi + \frac{1}{r^2}G\frac{d^2\phi}{d\theta^2} = 0$

dividing by $\frac{\phi G}{r^2} \rightarrow \underbrace{\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right)}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}}_{-\lambda} = 0$

$$\boxed{r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda G} \qquad \boxed{\frac{d^2\phi}{d\theta^2} = -\lambda\phi}$$

We say that the temperature is continous in $\theta = \pm\pi$ and the heat flow in the θ-direction, wich imply:

$u(r,-\pi) = u(r,\pi) \longrightarrow \phi(-\pi) = \phi(\pi)$

$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \longrightarrow \phi'(-\pi) = \phi'(\pi)$

$\underline{\lambda = 0}$: $\phi'(\theta) = C_1 \longrightarrow \phi(\theta) = C_1\theta + C_0$

$\phi'(-\pi) = C_1 = \phi'(\pi)$

$\phi(-\pi) = \phi(\pi) \longrightarrow C_1(-\pi) + \cancel{C_0} = C_1\pi + \cancel{C_0} \rightarrow C_1 = 0$ $\Big\}$ $\boxed{\phi_0(\theta) = C_0}$

$\underline{\lambda > 0}$: $\phi(\theta) = C_1\cos\sqrt{\lambda}\theta + C_2\sin\sqrt{\lambda}\theta$

$\phi(-\pi) = \phi(\pi) \rightarrow \cancel{C_1\cos\sqrt{\lambda}(-\pi)} + C_2\underbrace{\sin\sqrt{\lambda}(-\pi)}_{-\sin\sqrt{\lambda}\pi} = \cancel{C_1\cos\sqrt{\lambda}(\pi)} + C_2\sin\sqrt{\lambda}\pi$

$\Rightarrow C_2[-2\sin\sqrt{\lambda}\pi] = 0$ ①

$\phi'(\theta) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\theta + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\theta$

$\phi'(-\pi) = \phi'(\pi) \rightarrow -C_1\sqrt{\lambda}\underbrace{\sin\sqrt{\lambda}(-\pi)}_{-\sin\sqrt{\lambda}\pi} + \cancel{C_2\sqrt{\lambda}\cos\sqrt{\lambda}(-\pi)} = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi + \cancel{C_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)}$

$\Rightarrow C_1\sqrt{\lambda}[-2\sin\sqrt{\lambda}\pi] = 0$ ②

from ① and ② if $\sin\sqrt{\lambda}\pi \neq 0 \rightarrow C_1, C_2 = 0$ wich is just the trivial solution. Thus, for nontrivial solutions:

$$\sin\sqrt{\lambda}\pi = 0 \rightarrow \sqrt{\lambda}\pi = n\pi \rightarrow \boxed{\lambda_n = n^2}$$

$$\Rightarrow \boxed{\phi_n(\theta) = C_{1n}\cos(n\theta) + C_{2n}\cos(n\theta)}$$

$\underline{\lambda = 0}$: $r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \rightarrow r\frac{dG}{dr} = b_1 \rightarrow \frac{dG}{dr} = \frac{b_1}{r} \rightarrow \boxed{G_0(r) = b_1\ln r + b_0}$

$\underline{\lambda > 0}$: $r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = n^2 G(r) \rightarrow$ On susbstituing $G = ar^p$ we can determine that $p = \pm n$

$$\Rightarrow \boxed{G_n(r) = a_{1n}r^n + a_{2n}r^{-n}}$$

$\underline{\lambda < 0}$: $\phi(\theta) = A\cosh\sqrt{-\lambda}\theta + B\sinh\sqrt{-\lambda}\theta \Rightarrow \phi(-\pi) = \phi(\pi) \rightarrow B = 0$

$\phi'(-\pi) = \phi'(\pi) \rightarrow A = 0$

<u>No eigenvalues for $\lambda < 0$</u>

Thus, the general solution for Laplace's equation in polar coordinates when $-\pi \le \theta \le \pi$ is:

$$\boxed{u(r,\theta) = (A_0 + B_0 \ln r) + \sum_{n=1}^{\infty} \left[A_n r^n + B_n r^{-n}\right] \cos(n\theta) + \sum_{n=1}^{\infty} \left[C_n r^n + D_n r^{-n}\right] \sin(n\theta)}$$

It's a good idea if you can remember this!!

*a) $u(a,\theta) = f(\theta)$ and $u(b,\theta) = g(\theta)$

$$f(\theta) = u(a,\theta) = A_0 + B_0 \ln a + \sum_{n=1}^{\infty} \left[A_n a^n + B_n a^{-n}\right] \cos(n\theta) + \sum_{n=1}^{\infty} \left[C_n a^n + D_n a^{-n}\right] \sin(n\theta) \quad (*)$$

$$g(\theta) = u(b,\theta) = A_0 + B_0 \ln b + \sum_{n=1}^{\infty} \left[A_n b^n + B_n b^{-n}\right] \cos(n\theta) + \sum_{n=1}^{\infty} \left[C_n b^n + D_n b^{-n}\right] \sin n\theta \quad (**)$$

Integration (*) and (**) from $\theta = -\pi$ to $\theta = \pi$

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = (A_0 + B_0 \ln a) 2\pi$$

$$\int_{-\pi}^{\pi} g(\theta)\, d\theta = (A_0 + B_0 \ln b) 2\pi$$

$$\boxed{\begin{aligned} B_0 &= \frac{1}{2\pi \ln(a/b)} \int_{-\pi}^{\pi} \left[f(\theta) - g(\theta)\right] d\theta \\ A_0 &= \frac{1}{2\pi \ln(b/a)} \int_{-\pi}^{\pi} \left[f(\theta) \ln b - g(\theta) \ln a\right] d\theta \end{aligned}}$$

Multiplying (*) and (**) by $\cos(m\theta)$ and integrating using orthogonality

$$\int_{-\pi}^{\pi} f(\theta) \cos(m\theta)\, d\theta = \left[A_m a^m + B_m a^{-m}\right] \pi$$

$$\int_{-\pi}^{\pi} g(\theta) \cos(m\theta)\, d\theta = \left[A_m b^m + B_m b^{-m}\right] \pi$$

$$\Rightarrow \boxed{\begin{aligned} B_n &= \frac{1}{\pi(a^{-2n} - b^{-2n})} \int_{-\pi}^{\pi} \left[a^{-n} f(\theta) - b^{-n} g(\theta)\right] \cos(n\theta)\, d\theta \\ A_n &= \frac{1}{\pi(a^{2n} - b^{2n})} \int_{-\pi}^{\pi} \left[a^{n} f(\theta) - b^{n} g(\theta)\right] \cos(n\theta)\, d\theta \end{aligned}}$$

Now multiplying (*) and (**) by $\sin(m\theta)$ and integrating by use of orthogonality of $\sin(m\theta)$ and $\cos(m\theta)$:

$$\int_{-\pi}^{\pi} f(\theta)\sin(m\theta)\,d\theta = \left[C_m a^m + D_m a^{-m}\right]\pi$$

$$\int_{-\pi}^{\pi} g(\theta)\sin(m\theta)\,d\theta = \left[C_m b^m + D_m a^{-m}\right]\pi$$

$$\Rightarrow \boxed{\begin{aligned} D_n &= \frac{1}{\pi(a^{-2n}-b^{-2n})}\int_{-\pi}^{\pi}\left[a^{-n}f(\theta)-b^{-n}g(\theta)\right]\sin(n\theta)\,d\theta \\ C_n &= \frac{1}{\pi(a^{2n}-b^{2n})}\int_{-\pi}^{\pi}\left[a^{n}f(\theta)-b^{n}g(\theta)\right]\sin(n\theta)\,d\theta \end{aligned}}$$

Some particular case are:

if $f(\theta)=g(\theta)=0 \rightarrow A_0, B_0, A_n, B_n, C_n, D_n = 0$ all coeffitients are zero.

$\rightarrow u(r,\theta)=0$ for the equilibrium solution!!

if $g(\theta)=0 \rightarrow \underline{A_0 = -B_0 \ln b}$,

The physical condition is that the net heat flow is zero

$$\Rightarrow \int_{-\pi}^{\pi}\frac{\partial u}{\partial r}(b,\theta)(b\,d\theta) - \int_{-\pi}^{\pi}\frac{\partial u}{\partial r}(a,\theta)(a\,d\theta) = 0$$

Remember that $\Phi = -k\left.\frac{\partial u}{\partial r}\right|_{a,b}$ Fourier's law.

b) $$\frac{\partial u}{\partial r}(a,\theta) = 0 = \frac{B_0}{a} + \sum_{n=1}^{\infty}\left[nA_n a^{n-1} - nB_n a^{-n-1}\right]\cos(n\theta) + \sum_{n=1}^{\infty}\left[nC_n a^{n-1} - nD_n a^{-n-1}\right]\sin(n\theta)$$

$\Rightarrow B_0 = 0$, $B_n = A_n a^{2n}$ and $D_n = C_n a^{2n}$

$$\Rightarrow u(r,\theta) = \sum_{n=1}^{\infty} A_n\left[r^n + a^{2n}r^{-n}\right]\cos n\theta + C_n\left[r^n + a^{2n}r^{-n}\right]\sin(n\theta)$$

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty}\left[r^n + a^{2n}r^{-n}\right]\left[A_n\cos(n\theta) + C_n\sin(n\theta)\right]}$$

$$g(\theta) = u(b,\theta) = \sum_{n=1}^{\infty}\left[b^n + a^{2n}b^{-n}\right]\left\{A_n\cos(n\theta) + C_n\sin(n\theta)\right\}$$

Multiplying by $\cos(m\theta)$ OR $\sin(m\theta)$ an using orthogonality after of integrate:

$$\boxed{\begin{aligned} A_n &= \frac{1}{\pi\left[b^n + a^{2n}b^{-n}\right]}\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\,d\theta \\ B_n &= \frac{1}{\pi\left[b^n + a^{2n}b^{-n}\right]}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta \end{aligned}}$$

c) $\frac{\partial u}{\partial r}(a,\theta) = f(\theta)$ and $\frac{\partial u}{\partial r}(b,\theta) = g(\theta)$

$$\frac{\partial u}{\partial r} = \frac{B_0}{r} + \sum_{n=1}^{\infty} n\left[A_n r^{n-1} - B_n r^{-n-1}\right]\cos(n\theta) + \sum_{n=1}^{\infty} n\left[C_n r^{n-1} - D_n r^{-n-1}\right]\sin(n\theta)$$

$$f(\theta) = \frac{\partial u}{\partial r}(a,\theta) = \frac{B_0}{a} + \sum_{n=1}^{\infty} n\left[A_n a^{n-1} - B_n a^{-n-1}\right]\cos(n\theta) + \sum_{n=1}^{\infty} n\left[C_n a^{n-1} - D_n a^{-n-1}\right]\sin(n\theta) \quad (*)$$

$$g(\theta) = \frac{\partial u}{\partial r}(b,\theta) = \frac{B_0}{b} + \sum_{n=1}^{\infty} n\left[A_n b^{n-1} - B_n b^{-n-1}\right]\cos(n\theta) + \sum_{n=1}^{\infty} n\left[C_n b^{n-1} - D_n b^{-n-1}\right]\sin(n\theta) \quad (**)$$

Integrating (*) an (**) from $\theta = -\pi$ to $\theta = \pi$:

$$\left.\begin{aligned}\int_{-\pi}^{\pi} f(\theta)\,d\theta &= \frac{B_0}{a}(2\pi)\\ \int_{-\pi}^{\pi} g(\theta)\,d\theta &= \frac{B_0}{b}(2\pi)\end{aligned}\right\} \quad \boxed{B_0 = \frac{a}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{b}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta}$$

Note that from this:

$$\left.\begin{aligned}a\int_{-\pi}^{\pi} f(\theta)\,d\theta - b\int_{-\pi}^{\pi} g(\theta)\,d\theta &= 0\\ \int_{-\pi}^{\pi} \frac{\partial u}{\partial r}(a,\theta)(a\,d\theta) - \int_{-\pi}^{\pi} \frac{\partial u}{\partial r}(b,\theta)(b\,d\theta) &= 0\end{aligned}\right\}$$

This is just condition for no net heat flow trough boundaries

Remember: $\Phi = -K_0 \frac{\partial u}{\partial r}\Big|_{a,b}$ → Fourier's Law

Multiplying (*) and (**) by $\cos(m\theta)$ and integrate using orthogonality:

$$\int_{-\pi}^{\pi} f(\theta)\cos(m\theta)\,d\theta = m\left[A_m a^{m-1} - B_n a^{-m-1}\right]\pi$$

$$\int_{-\pi}^{\pi} g(\theta)\cos(m\theta)\,d\theta = m\left[A_m b^{m-1} - B_n b^{-m-1}\right]\pi$$

$$\Rightarrow \boxed{\begin{aligned}B_n &= \frac{1}{n\pi\left(b^{-2n} - a^{-2n}\right)}\int_{-\pi}^{\pi}\left[a^{-n+1} f(\theta) - b^{-n+1} g(\theta)\right]\cos(n\theta)\,d\theta\\ A_n &= \frac{1}{n\pi\left(a^{2n} - b^{2n}\right)}\int_{-\pi}^{\pi}\left[a^{n+1} f(\theta) - b^{n+1} g(\theta)\right]\sin(n\theta)\,d\theta\end{aligned}}$$

Similarity multiplying (*) and (**) by $\sin(m\theta)$ and integrating (use orthogonality):

$$\int_{-\pi}^{\pi} f(\theta)\sin(m\theta)\,d\theta = m\left[C_m a^{m-1} - D_n a^{-m-1}\right]\pi$$

$$\int_{-\pi}^{\pi} g(\theta)\sin(m\theta)\,d\theta = m\left[C_m b^{m-1} - D_n b^{-m-1}\right]\pi$$

$$\Rightarrow \boxed{\begin{aligned} D_n &= \frac{1}{n\pi(b^{-2n}-a^{-2n})}\int_{-\pi}^{\pi}\left[a^{-n+1}f(\theta) - b^{-n+1}g(\theta)\right]\sin(n\theta)\,d\theta \\ C_n &= \frac{1}{n\pi(a^{2n}-b^{2n})}\int_{-\pi}^{\pi}\left[a^{n+1}f(\theta) - b^{n+1}g(\theta)\right]\sin(n\theta)\,d\theta \end{aligned}}$$

Note that for any boundary conditions at $r = a, b$; the general solution for Laplace's equation satisfy that the net heat flow is zero:

$$\int_{-\pi}^{\pi} \frac{\partial u}{\partial r}(b,\theta)\,d\theta = \frac{B_0}{b}\int_{-\pi}^{\pi} d\theta + \sum_{n=1}^{\infty} n\left[A_n b^{n-1} - B_n b^{-n-1}\right]\cancel{\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta}^{\,0}$$

$$\sum_{n=1}^{\infty} n\left[C_n b^{n-1} - D_n b^{-n-1}\right]\cancel{\int_{-\pi}^{\pi}\sin(n\theta)\,d\theta}^{\,0}$$

$$\int_{-\pi}^{\pi} \frac{\partial u}{\partial r}(b,\theta)\,d\theta = \frac{B}{b}(\pi)$$

Similarly for $\left.\frac{\partial u}{\partial r}\right|_{r=a}$:

$$\int_{-\pi}^{\pi} \frac{\partial u}{\partial r}(a,\theta)\,d\theta = \frac{B}{a}(\pi)$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} \frac{\partial u}{\partial r}(b,\theta)\,b\,d\theta - \int_{-\pi}^{\pi} \frac{\partial u}{\partial r}(a,\theta)\,a\,d\theta = 0}$$

Physical condition that say the net flow trough the boundaries is zero!!
It is an automatic condition of the solution for Laplace's eq. because it's the equilibrium solution (steady-state) for heat equation.

* 2.5.9 $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ $\quad a \le r \le b \quad 0 \le \theta \le \pi/2$

let $u(r,\theta) = \phi(\theta)U(r) \longrightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi + \frac{G}{r^2}\frac{d^2\phi}{d\theta^2} = 0$

dividing by $\frac{\phi G}{r^2} \longrightarrow \underbrace{\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right)}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}}_{-\lambda} = 0$

$$\boxed{r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda G} \qquad \boxed{\frac{d^2\phi}{d\theta^2} = -\lambda\phi}$$

We need solve these equations $\Longrightarrow$

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$a \le r \le b \qquad 0 \le \theta \le \pi/2 \qquad u(b,\theta) = f(\theta)$

a) $u(r,0) = 0 = u(r,\pi/2) \rightarrow \phi(0) = 0$, $\phi(\pi/2) = 0$ $\qquad u(a,\theta) = 0 \rightarrow G(a) = 0$

$\underline{\lambda = 0}$: $\phi(\theta) = C_1\theta + C_0 \rightarrow \phi(0) = 0 = C_0$, $\phi(\pi/2) = C_1(\pi/2) = 0 \rightarrow C_1 = 0$ } No eigenvalues for $\lambda = 0$.

$\underline{\lambda > 0}$: $\phi(\theta) = C_1 \cos\sqrt{\lambda}\,\theta + C_2 \sin\sqrt{\lambda}\,\theta \rightarrow \phi(0) = 0 = C_1$, $\phi(\pi/2) = C_2 \sin\sqrt{\lambda}(\pi/2) = 0$

$$\rightarrow \sqrt{\lambda}\,\frac{\pi}{2} = n\pi \rightarrow \boxed{\lambda_n = (2n)^2} \Rightarrow \boxed{\phi_n(\theta) = C_n \sin(2n\theta)}$$

$\underline{\lambda < 0}$: $\phi(\theta) \sim \left\{ \begin{matrix} \cosh\sqrt{-\lambda}\,\theta \\ \sinh\sqrt{-\lambda}\,\theta \end{matrix} \right\} \rightarrow$ They are not solution because no oscillate.

$\underline{\lambda > 0}$: $r\dfrac{d}{dr}\left(r\dfrac{dG}{dr}\right) = (2n)^2 G(r) \rightarrow G(r) = b_1 r^{2n} + b_2 r^{-2n}$

$$G(a) = 0 = b_1 a^{2n} + b_2 a^{-2n} \rightarrow b_2 = -b_1 a^{4n}$$

$$\Rightarrow G_n(r) = B_n\left[r^{2n} - \frac{a^{4n}}{r^{2n}}\right] \text{ OR } \boxed{G_n(r) = B_n\left[\left(\frac{r}{a}\right)^{2n} - \left(\frac{a}{r}\right)^{2n}\right]}$$

$$\Rightarrow \boxed{u(r,\theta) = \sum_{n=1}^{\infty} B_n\left[\left(\frac{r}{a}\right)^{2n} - \left(\frac{a}{r}\right)^{2n}\right]\sin(2n\theta)}$$

$$f(\theta) = u(b,\theta) = \sum_{n=1}^{\infty} B_n\left[\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right]\sin(2n\theta)$$

Multiplying by $\sin(2m\theta)$ and integrating:

$$\int_0^{\pi/2} f(\theta)\sin(2m\theta)\,d\theta = \sum_{n=1}^{\infty} B_n\left[\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right]\underbrace{\int_0^{\pi/2}\sin(2n\theta)\sin(2m\theta)\,d\theta}_{\frac{(\pi/2)}{2}\delta_{mn}}$$

$$= B_m\left[\left(\frac{b}{a}\right)^{2m} - \left(\frac{a}{b}\right)^{2m}\right]\frac{\pi}{4}$$

$$\Rightarrow \boxed{B_n = \frac{4}{\pi\left[\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right]}\int_0^{\pi/2} f(\theta)\sin(2m\theta)\,d\theta}$$

b) $u(a,\theta) = 0 = u(b,\theta) \rightarrow \boxed{\begin{matrix} G(a) = 0 \\ G(b) = 0 \end{matrix}}$ $\qquad u(r,0) = 0 \rightarrow \phi(0) = 0$, $\boxed{u(r,\pi/2) = f(r)}$

Be careful!! The two homogeneus boundary conditions ar in r, and hence $G(r)$ must be an eingenvalue problem. $\Longrightarrow$

This election was made because we hope that solution in r oscillates!!

$$\overbrace{\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right)}^{-\lambda} + \overbrace{\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}}^{\lambda} = 0 \Rightarrow \boxed{\frac{d^2\phi}{d\theta^2} = \lambda\phi(\theta)} \quad \text{and}$$

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = r\left(r\frac{d^2G}{dr^2} + \frac{dG}{dr}\right) = -\lambda G \rightarrow \boxed{r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} = -\lambda G}$$

Eigenvalue problem with BCs: $G(a)=0 \quad G(b)=0$

$\underline{\lambda=0}$: $\frac{d}{dr}\left(r\frac{dG}{dr}\right)=0 \rightarrow r\frac{dG}{dr}=b_1 \rightarrow \frac{dG}{dr}=\frac{b_1}{r} \rightarrow G(r)=b_1\ln r + b_0$

$G(a)=0=b_1\ln a + b_0 \rightarrow b_0 = -b_1\ln a$

$G(b)=0=b_1\ln b - b_1\ln a$

$0=b_1(\ln b - \ln a) \rightarrow$ Since $b\neq a \rightarrow b_1=0$

$\Rightarrow b_0=0$

No eigenvalues for $\lambda=0$

$\underline{\lambda>0}$: $r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} + \lambda G = 0$ $\quad$ Suposse $G\propto r^\rho$

$\frac{dG}{d\rho}\propto \rho r^{\rho-1}$

$\frac{d^2G}{d\rho^2}\propto \rho(\rho-1)r^{\rho-2}$

$\Rightarrow \cancel{r^2}(\rho(\rho-1)r^{\rho\cancel{-2}}) + \cancel{r}(\rho r^{\rho\cancel{-1}}) + \lambda r^\rho = 0$

$[(\rho^2-\cancel{\rho})+\cancel{\rho}+\lambda]r^\rho = 0$

$\Rightarrow \rho^2=-\lambda \rightarrow \rho=\pm\sqrt{-\lambda}$ $\quad$ if $\lambda>0 \rightarrow \rho=\pm i\sqrt{\lambda}$

$\Rightarrow G(r)\propto r^{\pm i\sqrt{\lambda}}$ $\quad$ use here $r=e^{\ln r}$

$G(r)\propto e^{\pm i\sqrt{\lambda}\ln r} \rightarrow \boxed{G(r)=C_1\cos(\sqrt{\lambda}\ln r)+C_2\sin(\sqrt{\lambda}\ln r)}$

Remember that Cosine and Sine can be construct by superposition of complex exponentials.

This form is hard for apply boundary conditions. We can try obtain other combinations of solutions is choose C_1 and C_2 as follow

$$G(r) = [\cos(\sqrt{\lambda}\ln a) - \sin(\sqrt{\lambda}\ln a)]\cos(\sqrt{\lambda}\ln r) + [\cos(\sqrt{\lambda}\ln a)+\sin(\sqrt{\lambda}\ln a)]\sin(\sqrt{\lambda}\ln r)$$

$$= \cos(\sqrt{\lambda}\ln a)\cos(\sqrt{\lambda}\ln r) + \sin(\sqrt{\lambda}\ln a)\sin(\sqrt{\lambda}\ln r) + \cos(\sqrt{\lambda}\ln a)\sin(\sqrt{\lambda}\ln r) - \sin(\sqrt{\lambda}\ln a)\cos(\sqrt{\lambda}\ln r)$$

$$= \cos[\sqrt{\lambda}(\ln r - \ln a)] + \sin[\sqrt{\lambda}(\ln r - \ln a)]$$

$$= \cos[\sqrt{\lambda}\ln(r/a)] + \sin[\sqrt{\lambda}\ln(r/a)]$$

Then additionating coefficients in the superposition $\rightarrow$ $\boxed{G(r) = C_3\cos[\sqrt{\lambda}\ln(r/a)] + C_4\sin[\sqrt{\lambda}\ln(r/a)]}$

Now we can apply Boundary conditions:

$G(a) = 0 = C_3 \Rightarrow G(b) = C_4 \sin[\sqrt{\lambda}\ln(b/a)] = 0$

$\Rightarrow \sqrt{\lambda}\ln(b/a) = n\pi \quad n = 1, 2, \ldots$

$\Rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{\ln(b/a)}\right)^2} \rightarrow$ eigenvalues for $\lambda > 0$

$$\boxed{G_n(r) = C_n \sin\left[\frac{n\pi \ln(r/a)}{\ln(b/a)}\right]}$$

$\underline{\lambda < 0}$ $\quad r^2 \frac{d^2G}{dr^2} + r\frac{dG}{dr} + \lambda G = 0$ supossing $G \propto r^\rho$ and by similar analysis

$\rho = \pm\sqrt{-\lambda}$ but $\lambda < 0 \Rightarrow \sqrt{-\lambda}$ is real, then:

$$G(r) \propto r^{\pm\sqrt{-\lambda}} = e^{\pm\sqrt{-\lambda}\ln r}$$

by similar way than for $\lambda > 0$ we can construct solution as superposition of hyperbolics sines an cosines:

$$G(r) = b_3 \cosh[\sqrt{-\lambda}\ln(r/a)] + b_4 \sinh[\sqrt{-\lambda}\ln(r/a)]$$

$G(a) = 0 = b_3 \rightarrow G(b) = b_4 \sinh[\sqrt{-\lambda}\ln(b/a)] = 0$

$\hookrightarrow b_4 = 0$ because $b \neq a$.

then, there are <u>not eigenvalues for $\lambda < 0$</u>.

It's reasonable because hyperbolics functions no oscillate!!

Now, we nee solutinate the other differential equation for $\phi(\theta)$ with the BC: $\phi(0) = 0$ ~~and~~ ~~[illegible]~~

$\underline{\lambda > 0}$: $\frac{d^2\phi}{d\theta^2} = \lambda\phi(\theta) \rightarrow \phi \propto e^{\pm\sqrt{\lambda}\theta} \Rightarrow \phi(\theta) = A\cosh(\sqrt{\lambda}\theta) + B\sinh(\sqrt{\lambda}\theta)$

$\phi(0) = 0 = A \rightarrow \phi = B\sinh(\sqrt{\lambda}\theta)$

OR $\boxed{\phi_n(\theta) = B_n \sinh\left(\frac{n\pi\theta}{\ln(b/a)}\right)}$

$$\Rightarrow \boxed{U(r,\theta) = \sum_{n=1}^{\infty} B_n \sinh\left[\frac{n\pi\theta}{\ln(b/a)}\right] \sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]} \qquad \begin{array}{l} a \leq r \leq b \\ 0 \leq \theta \leq \pi/2 \end{array}$$

$$f(r) = U(r, \pi/2) = \sum_{n=1}^{\infty} B_n \sinh\left[\frac{n\pi^2}{2\ln(b/a)}\right] \sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]$$

$$\rightarrow dz = \frac{1}{\ln(b/a)} \cdot \frac{1}{r}\, dr$$

Let $z = \dfrac{\ln(r/a)}{\ln(b/a)}$ if $a < r < b \rightarrow 0 < z < 1$

$$\Rightarrow f(r) = \sum_{n=1}^{\infty} B_n \sinh\left[\frac{n\pi^2}{2\ln(b/a)}\right] \sin(n\pi z)$$

multiplying by $\sin(m\pi z)$ and integrating:

$$\int_0^1 f(r)\sin(m\pi z)\, dz = \sum_{n=1}^{\infty} B_n \sinh\left[\frac{n\pi^2}{2\ln(b/a)}\right] \underbrace{\int_0^1 \sin(n\pi z)\sin(m\pi z)\, dz}_{(1/2)\delta_{nm}}$$

$$\int_0^1 f(r)\sin(m\pi z)\left[\frac{dr}{r\ln(b/a)}\right] = \frac{1}{2} B_m \sinh\left[\frac{m\pi^2}{2\ln(b/a)}\right]$$

$$\Rightarrow \boxed{B_n = \frac{2}{\ln\left(\frac{b}{a}\right)\cdot \sinh\left[\frac{n\pi^2}{2\ln(b/a)}\right]} \int_a^b \frac{f(r)}{r} \sin\left[m\pi \frac{\ln(r/a)}{\ln(b/a)}\right] dr}$$

2.5.10 Poisson's equation $\nabla^2 u = g(x)$, subject to $u = f(x)$ on the boundary.

Assume that there are two solutions u an v:

$\Rightarrow$ $\nabla^2 u = g(x)$ and $u = f(x)$ on the boundary.
$\nabla^2 v = g(x)$ and $v = f(x)$ on the boundary.

Let $w = u - v \rightarrow \nabla^2 w = \nabla^2(u - v) = \nabla^2 u - \nabla^2 v$
$= g(x) - g(x) = 0$

$\Rightarrow \boxed{\nabla^2 w = 0} \rightarrow w$ satisfy Laplace's Equation

w(on the boundary) $= \underbrace{u(\text{On the boundary})}_{f(x)} - \underbrace{v(\text{On the boundary})}_{f(x)}$

$\Rightarrow \boxed{w(\text{On the boundary}) = 0}$

Solution for Laplace's Equation is solution for steady state

Since w satisfies Laplace's equation, we know that it must obtain its maximum and minimum on the boundary (Remember in steady state the maximum and minimum temperatures occur on the boundary). Then since $w = 0$ on the boundary, w must be zero everywhere.

$w = 0 \rightarrow \boxed{u = v}$ thus the solution u is unique !!

2.5.11 If Laplace's equation is satisfied in three dimensions:

$$\nabla^2 u = 0 \rightarrow \iiint_R \nabla^2 u \, dv = 0$$

$$\nabla^2 u = \vec{\nabla}\cdot(\vec{\nabla} u) \rightarrow \iiint_R \vec{\nabla}\cdot(\vec{\nabla} u)\, dv = \boxed{\oint_S (\vec{\nabla} u)\cdot\hat{n}\, ds = 0}$$

by theorem divergence

$$\vec{\phi} = -K_0 \vec{\nabla} u \rightarrow \text{ using this } \rightarrow \boxed{\oint_S \vec{\phi}\cdot\hat{n}\, ds = 0}$$

$\Rightarrow$ If Laplace's equation is satisfied, it means that the heat energy flow out the closed surface is null.

2.5.12 $\int_V \nabla\cdot\vec{A}\, dv = \oint_S \vec{A}\cdot\hat{n}\, ds \rightarrow$ Divergen theorem

a) if $\vec{A} = u\vec{\nabla} u \rightarrow \vec{\nabla}\cdot\vec{A} = \nabla\cdot(u\vec{\nabla} u) = u\nabla^2 u + \vec{\nabla} u\cdot\vec{\nabla} u$

$$\rightarrow \int_V [u\nabla^2 u + \vec{\nabla} u\cdot\vec{\nabla} u]\, dv = \oint_S (u\vec{\nabla} u)\cdot\hat{n}\, ds$$

$$\boxed{\int_V u\nabla^2 u\, dv = \oint_S (u\vec{\nabla} u)\cdot\hat{n}\, ds - \int_V |\vec{\nabla} u|^2\, dv} \quad (*)$$

b) Assume that there are two solutions u and v of Laplace's equation:

$\nabla^2 u = 0$ with u given on boundary
$\nabla^2 v = 0$ with v given on boundary

Obviously u(on boundary) $= v$(on boundary) because they represent fixed temperature on boundary.

Let $w = u - v \rightarrow \nabla^2 w = 0$, and put w in $(*)$

$$\int_V w\cancel{\nabla^2 w}^{\,0}\, dv = \oint_S (w\vec{\nabla} w)\cdot\hat{n}\, ds - \int_V |\vec{\nabla} w|^2\, dv$$

$$0 = \oint_S (\underbrace{w(\text{on boundary})}_{0}\vec{\nabla} w)\cdot\hat{n}\, ds - \int_V |\vec{\nabla} w|^2\, dv$$

w(on boundary) $= 0$

$$\Rightarrow \int_V |\vec{\nabla} w|^2\, dv = 0$$

$\Rightarrow \vec{\nabla} w = 0 \rightarrow w = cte \rightarrow u - v = cte$ OR $\boxed{u = v + cte}$

thus the solution is unique, apart from an unimportant arbitrary additive cte.

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c) Assume u, v two solutions of Laplace's equation:

$\nabla^2 u = 0$ with $\vec{\nabla} u \cdot \hat{n} = 0$ on the boundary
$\nabla^2 v = 0$ with $\vec{\nabla} v \cdot \hat{n} = 0$ on the boundary

let $w = u - v \rightarrow \nabla^2 w = 0$ and $\vec{\nabla} w \cdot \hat{n} = \vec{\nabla}(u-v)\cdot\hat{n}$
$= \vec{\nabla} u \cdot \hat{n} - \vec{\nabla} v \cdot \hat{n}$

$$\Rightarrow \vec{\nabla} w \cdot \hat{n}\Big|_{\text{on the boundary}} = \underbrace{\vec{\nabla} u \cdot \hat{n}\Big|_{\text{boundary}}}_{0} - \underbrace{\vec{\nabla} v \cdot \hat{n}\Big|_{\text{boundary}}}_{0}$$

$\Rightarrow \vec{\nabla} w \cdot \hat{n} = 0$ on the boundary.

Put w in (*) $\Rightarrow \int_V w \underbrace{\nabla^2 w}_{0} dv = \oint_S w \cancel{\vec{\nabla} w \cdot \hat{n}} ds - \int_V |\vec{\nabla} w|^2 dv$

by Laplace's equation $\rightarrow 0$

$$\Rightarrow \int_V |\vec{\nabla} w|^2 dv = 0$$

$\Rightarrow \vec{\nabla} w = 0 \rightarrow w = cte \rightarrow u - v = cte$ OR $\boxed{u = v + cte}$

thus the solution is unique.

d) Assume u, v two solutions:

$\nabla^2 u = 0$ with $\vec{\nabla} u \cdot \hat{n} + h u = 0$ on the boundary
$\nabla^2 v = 0$ with $\vec{\nabla} v \cdot \hat{n} + h v = 0$ on the boundary

let $w = u - v \rightarrow \nabla^2 w = 0$ and $\vec{\nabla} w \cdot \hat{n} = \vec{\nabla} u \cdot \hat{n} - \vec{\nabla} v \cdot \hat{n}$

if $\vec{\nabla} u \cdot \hat{n} = -hu$, $\vec{\nabla} v \cdot \hat{n} = -hv$ on the boundaries $\Rightarrow \vec{\nabla} w \cdot \hat{n} = -h(u-v)$
$= -hw$ on the boundary

Put w in (*): $\int_V w \cancel{\nabla^2 w} dv = \oint_S w \vec{\nabla} w \cdot \hat{n} ds - \int_V |\vec{\nabla} w|^2 dv$ (0 by Laplace's Eq.)

$$0 = \oint_S w(-hw) ds - \int_V |\vec{\nabla} w|^2 dv$$

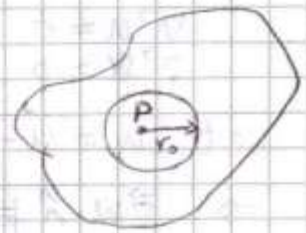
$$0 = -h \oint_S w^2 ds - \int_V |\vec{\nabla} w|^2 dv$$

Integral of square cantities are positive even, then the last equation is valid only if

$w = 0$ OR $\vec{\nabla} w = 0$ both lead to $\boxed{u = v}$ since $w = u - v$ — thus the solution is unique.

2.5.13 Prove that the temperature satisfyin Laplace's equation cannot attain its minimun in the interior.

Suppose the minimum occurs at point P. By the Mean Value Theorem, the value at P must be the average of the points on a circle aroun P.



$$u(P,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

This is impossible if $u(P)$ is less than the value of u at all the points on the circle. Thus, by contradiction, the minimum must be on the boundary.

2.5.14 $\frac{\partial u}{\partial t} = -k\frac{\partial^2 u}{\partial x^2} \rightarrow$ "backward" heat Equation $\quad u(0,t)=0 \quad u(L,t)=0 \quad u(x,0)=f(x)$

Let $u(x,t) = \phi(x)h(t) \rightarrow \phi\frac{\partial h}{\partial t} = -kh\frac{\partial^2\phi}{\partial x^2}$ dividing by $kh\phi$

$$\underbrace{\frac{1}{kh}\frac{\partial h}{\partial t}}_{\lambda} = \underbrace{-\frac{1}{\phi}\frac{\partial^2\phi}{\partial x^2}}_{\lambda} \rightarrow \frac{dh}{dt} = \lambda k h(t) \rightarrow \boxed{h(t) = Ce^{\lambda k t}}$$

$$\boxed{\frac{d^2\phi}{dx^2} = -\lambda\phi(x)} \rightarrow \begin{array}{l}\phi(0)=0\\ \phi(L)=0\end{array}$$

$\underline{\lambda=0}$: $\phi(x) = C_1x + C_0 \rightarrow \left.\begin{array}{l}\phi(0)=0=C_0\\ \phi(L)=0=C_1L \rightarrow C_1=0\end{array}\right\}$ No eigenvalues for $\lambda=0$.

$\underline{\lambda>0}$: $\phi(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x) \rightarrow \begin{array}{l}\phi(0)=0=C_1\\ \phi(L)=0=C_2\sin\sqrt{\lambda}L\end{array}$

$\rightarrow \sqrt{\lambda}L = n\pi \rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2} \rightarrow \boxed{\phi_n(x) = C_n\sin\left(\frac{n\pi x}{L}\right)}$ $n=1,2,\ldots$

$\underline{\lambda<0}$: $\phi(x) = A\cosh\sqrt{-\lambda}x + B\sinh\sqrt{-\lambda}x \rightarrow$ It's not eigenfunction because no oscillate.

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} C_n\sin\left(\frac{n\pi x}{L}\right)e^{\left(\frac{n\pi}{L}\right)^2 kt}}$$

Where C_n is determined using $u(x,0)=f(x)$.

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} C_n\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} u_n(x,0)$$

<u>If the problem is well posed</u>, then if the initial datum is

$$u_n(x,0) = f_n(x) = \frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)$$

We would have a unique solution, wich is:

$$u_n(x,t) = \frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)e^{\left(\frac{n\pi}{L}\right)^2 kt} \text{ such that } u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \Rightarrow$$

Morever if the problem is well posed, then changing the initial datum $f(x) \to f(x) + f_n(x)$ the difference between the solutions corresponding to the original problem and the new problem with the last datum, should tend to zero if $n \to \infty$. This is because $f_n(x) \to 0$ uniformly as $n \to \infty$. Nevertheless, by linearity of the problem we have that the difference between these two solution is exactly $-u_n$,

let $u_n^{(I)}(x,t)$ solution when $u(x,0) = f(x)$

and $u_n^{(II)}(x,t)$ solution when $u(x,0) = f(x) + f_n(x)$

$$\Rightarrow \left| u_n^{(I)}(x,t) - u_n^{(II)}(x,t) \right| = \left| -u_n(x,t) \right|$$

since $u_n(x,t)$ is the solution when $u(x,0) = f_n(x)$.

$$u_n(x,t) = \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) e^{\left(\frac{n\pi}{L}\right)^2 K t}.$$

If $n \to \infty \Rightarrow f_n(x) = \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) \to 0$

then, <u>if the problem is well posed</u> for $n \to \infty$ the diference $|u_n^{(I)} - u_n^{(II)}| \to 0$, however for $t > 0$

$$\sup_{x \in [0,L]} \left| u_n^{(I)} - u_n^{(II)} \right| \geq u_n\left(\frac{L}{2n}, t\right) = \frac{1}{n} e \sin\left(\frac{\pi}{2}\right) e^{\left(\frac{n\pi}{L}\right)^2 K t} \longrightarrow \infty$$

as $n \to \infty$

hence <u>by contradiction</u>, we can conclude that the <u>problem is not well posed</u>.

2.15 $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 \leq x < \infty \quad 0 \leq y \leq H$ <u>Semi-infinite strip</u>

let $u(x,y) = h(y)\phi(x) \to h \frac{\partial^2 \phi}{\partial x^2} + \phi \frac{\partial^2 h}{\partial y^2} = 0$ dividing by $h\phi$

$$\underbrace{\frac{1}{\phi} \frac{d^2\phi}{dx^2}}_{\lambda} + \underbrace{\frac{1}{h} \frac{d^2 h}{dy^2}}_{-\lambda} = 0 \longrightarrow \frac{d^2\phi}{dx^2} = \lambda \phi(x)$$

$$\frac{d^2 h}{dy^2} = -\lambda h(y) \qquad \Rightarrow$$

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a) $\left.\begin{array}{l}\frac{\partial u}{\partial y}(x,0)=0\\ \frac{\partial u}{\partial y}(x,H)=0\end{array}\right\} \rightarrow \begin{array}{l} h'(0)=0\\ h'(H)=0\end{array}$ $\qquad u(0,y)=f(y)$

$\underline{\lambda=0}$: $h'(y)=C_1 \longrightarrow h(y)=C_1y+C_0$, $h'(0)=0=C_1$, $h'(H)=0=C_1$ $\Big\}$ $\boxed{h_0(y)=C_0}$

$\underline{\lambda>0}$: $h(y)=C_1\cos\sqrt{\lambda}\,y+C_2\sin\sqrt{\lambda}\,y$

$h'(y)=-C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,y+C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,y \rightarrow h'(0)=0=C_2\sqrt{\lambda}\rightarrow C_2=0$

$h'(H)=-C_1\sqrt{\lambda}\sin\sqrt{\lambda}H=0$

$\rightarrow \sqrt{\lambda}H=n\pi \rightarrow \boxed{\lambda_n=\left(\frac{n\pi}{H}\right)^2} \rightarrow \boxed{h(y)=C\cos\left(\frac{n\pi y}{H}\right)}$

$n=1,2,3,\ldots$

$\underline{\lambda<0}$: $h(y)=A\cosh\sqrt{-\lambda}\,y+B\sinh\sqrt{-\lambda}\,y$

↳ It's not solution because no oscillate.

$\underline{\lambda=0}$: $\frac{d^2\phi}{dx^2}=0 \rightarrow \phi(x)=a_1x+a_0$

We don't want singularities when $x\rightarrow\infty \Rightarrow a_1=0$.

$\Rightarrow \boxed{\phi_0(x)=a_0}$

$\underline{\lambda>0}$: $\frac{d^2\phi}{dx^2}=\lambda_n\phi=\left(\frac{n\pi}{H}\right)^2\phi(x) \rightarrow \phi\propto e^{\pm\left(\frac{n\pi}{H}\right)x}$

we don't want singularities when $x\rightarrow\infty$, the we reject $e^{+\sqrt{\lambda_n}}$, thus $\boxed{\phi_n(x)=a_n e^{-\left(\frac{n\pi}{H}\right)x}}$

$$\Rightarrow \boxed{u(x,y)=A_0+\sum_{n=1}^{\infty}A_n e^{-\left(\frac{n\pi}{H}\right)x}\cos\left(\frac{n\pi y}{H}\right)}$$

$f(y)=u(0,y)=A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi y}{H}\right)$ integrating:

$$\int_0^H f(y)\,dy=A_0\underbrace{\int_0^H dy}_{H}+\sum_{n=1}^{\infty}A_n\cancel{\int_0^H\cos\left(\frac{n\pi y}{H}\right)dy}^{\,0}$$

$$\rightarrow \boxed{A_0=\frac{1}{H}\int_0^H f(y)\,dy}$$

Multiplying by $\cos\left(\frac{m\pi y}{H}\right)$ and integrating:

$$\int_0^H f(y)\cos\left(\frac{m\pi y}{H}\right)dy=A_0\cancel{\int_0^H\cos\left(\frac{m\pi y}{H}\right)dy}^{\,0}+\sum_{n=1}^{\infty}A_n\underbrace{\int_0^H\cos\left(\frac{n\pi y}{H}\right)\cos\left(\frac{m\pi y}{H}\right)dy}_{(H/2)\delta_{nm}}$$

$$=\sum_{n=1}^{\infty}A_n\frac{H}{2}\delta_{nm}$$

$$=A_m\frac{H}{2} \rightarrow \boxed{A_n=\frac{2}{H}\int_0^H f(y)\cos\left(\frac{n\pi y}{H}\right)dy}$$

b) $u(x,0) = 0 = u(x,H) \rightarrow \begin{array}{l} h(0)=0 \\ h(H)=0 \end{array}$ $\qquad u(0,y) = f(y)$

$\underline{\lambda = 0}$: $h(y) = C_1 y + C_0 \rightarrow \left.\begin{array}{l} h(0) = 0 = C_0 \\ h(H) = 0 = C_1 H \rightarrow C_1 = 0 \end{array}\right\}$ No eigenvalues

$\underline{\lambda > 0}$: $h(y) = C_1 \cos\sqrt{\lambda}\, y + C_2 \sin\sqrt{\lambda}\, y \rightarrow \begin{array}{l} h(0) = 0 = C_1 \\ h(H) = C_2 \sin(\sqrt{\lambda} H) = 0 \end{array}$

$\rightarrow \sqrt{\lambda} H = n\pi \rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{H}\right)^2} \rightarrow \boxed{h(y) = C_2 \sin\left(\frac{n\pi y}{H}\right)}$

$\underline{\lambda < 0}$: $h(y) = A\cosh\sqrt{-\lambda}\, y + B \sinh\sqrt{-\lambda}\, y \rightarrow$ It's not solution because no oscillate.

$\underline{\lambda > 0}$: $\frac{d^2\phi}{dx^2} = \lambda_n \phi = \left(\frac{n\pi}{H}\right)^2 \phi(x) \rightarrow \phi(x) \propto e^{\pm\left(\frac{n\pi}{H}\right)x}$

we $|u(x,t)|$ finite if $x \to \infty \Rightarrow$ reject $e^{+\lambda_n x}$:

$\Rightarrow \boxed{\phi_n(x) = a_n e^{-\left(\frac{n\pi}{H}\right)x}}$ $\qquad n = 1, 2, 3, \ldots$

$$\Rightarrow \boxed{u(x,y) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/H)x} \sin\left(\frac{n\pi y}{H}\right)}$$

$f(y) = u(0,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right)$ $\qquad$ multiplying by $\sin\left(\frac{m\pi y}{H}\right)$ and integrating:

$$\int_0^H f(y) \sin\left(\frac{m\pi y}{H}\right) dy = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^H \sin\left(\frac{n\pi y}{H}\right) \sin\left(\frac{m\pi y}{H}\right) dy}_{(H/2)\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} A_n \frac{H}{2} \delta_{nm}$$

$$= A_m \frac{H}{2} \longrightarrow \boxed{A_n = \frac{2}{H} \int_0^H f(y) \sin\left(\frac{n\pi y}{H}\right) dy}$$

c) $u(x,0) = 0 = u(x,H)$ $\qquad \frac{\partial u}{\partial x}(0,y) = f(y)$

This is the same problem that item b) and its solution is $\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/H)x} \sin\left(\frac{n\pi y}{H}\right)}$

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} A_n \left(-\frac{n\pi}{H}\right) e^{-(n\pi/H)x} \sin\left(\frac{n\pi y}{H}\right) \Rightarrow$$

$$f(y) = \frac{\partial u}{\partial x}(0,y) = \sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$ Multiplyin by $\cos\left(\frac{m\pi y}{H}\right)$ and integrating

$$\int_0^H f(y)\sin\left(\frac{m\pi y}{H}\right)dy = \sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{H}\right)\underbrace{\int_0^H \sin\left(\frac{n\pi y}{H}\right)\sin\left(\frac{m\pi y}{H}\right)dy}_{(H/2)\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} A_n\left(-\frac{m\pi}{H}\right)\frac{H}{2}\delta_{nm}$$

$$= A_m\left(-\frac{m\pi}{\cancel{H}}\right)\frac{\cancel{H}}{2} \rightarrow \boxed{A_n = -\frac{2}{n\pi}\int_0^H f(y)\sin\left(\frac{n\pi y}{H}\right)dy}$$

e) $\frac{\partial u}{\partial y}(x,0) = 0 = \frac{\partial u}{\partial y}(x,H)$ and $\frac{\partial u}{\partial x}(0,y) = f(y)$

This problem is the same that item a), only change the last condition. Its solution is:

$$\boxed{u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/H)x}\cos\left(\frac{n\pi y}{H}\right)}$$

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{H}\right)e^{-(n\pi/H)x}\cos\left(\frac{n\pi y}{H}\right)$$

$$f(y) = \frac{\partial u}{\partial x}(0,y) = \sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{H}\right)\cos\left(\frac{n\pi y}{H}\right)$$ Multiplying by $\cos\left(\frac{m\pi y}{H}\right)$ and integrating:

$$\int_0^H f(y)\cos\left(\frac{m\pi y}{H}\right)dy = \sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{H}\right)\underbrace{\int_0^H \cos\left(\frac{n\pi y}{H}\right)\cos\left(\frac{m\pi y}{H}\right)dy}_{(H/2)\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{\cancel{H}}\right)\frac{\cancel{H}}{2}\delta_{nm}$$

$$= A_m\left(-\frac{m\pi}{2}\right) \rightarrow \boxed{A_n = -\frac{2}{n\pi}\int_0^H f(y)\cos\left(\frac{n\pi y}{H}\right)dy}$$

$$\int_0^H f(y)\,dy = \sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{H}\right)\underbrace{\int_0^H \cos\left(\frac{n\pi y}{H}\right)dy}_{0}$$

Then, the solution exists only if $\boxed{\int_0^H f(y)\,dy = 0}$

Note that the two first boundary conditions say that the sides $y=0$ and $y=L$ are insulates. The side $x=0$ is not insulated, but $\int_0^H f(y)\,dy = 0$ say that the net heat flow is zero.

Remember that $\Phi = -K_0\frac{\partial u}{\partial x}$ → Fourier's law.

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$0 \le x \le L \qquad\qquad 0 \le y \le H$

2.5.16 $\frac{\partial u}{\partial x}(0,y) = 0 \quad \frac{\partial u}{\partial x}(L,y) = g(y) \quad \frac{\partial u}{\partial y}(x,0) = 0 \quad \frac{\partial u}{\partial y}(x,H) = f(x)$

a) There are thwo sides insulated at $x=0$ and $y=0$. The others sides are not insulated. The solution for Laplace's equation could represent an equilibrium heat flow: If you remember $\frac{\partial u}{\partial t} = -k_0 \nabla^2 u \rightarrow$ heat Equation

For Equilibrium $\frac{\partial u}{\partial t} = 0 \rightarrow \nabla^2 u = 0 \rightarrow$ Laplace's Eq.

The solvability condition mathematically represents the physical requirement that the heat flux through te boundary is zero, otherwise the system is not at equilibrium $\Rightarrow$ no equilibrium solution.

$$\Rightarrow \oint_C \Phi_{flow}\, ds = 0 \qquad \Phi = -k_0 \nabla u \rightarrow \text{Fourier Law.}$$

$$\Rightarrow -k_0 \left\{ \int_0^L \cancel{\frac{\partial u}{\partial y}(x,0)}^{\,0} dx + \int_0^H \underbrace{\frac{\partial u}{\partial x}(L,y)}_{g(y)} dy + \int_0^L \overbrace{\frac{\partial u}{\partial y}(x,H)}^{f(x)} dx + \int_0^H \cancel{\frac{\partial u}{\partial x}(0,y)}^{\,0} dy \right\} = 0$$

$$\Rightarrow \boxed{\int_0^L f(x)\,dx + \int_0^H g(y)\,dy = 0} \rightarrow$$ Mathematical condition that satisfy requirement that flow trough boundary is zero.

b) Let $f(x) = f_0 = cte$ and $g(x) = g_0 = cte$.

Solvability requires $\int_0^L f_0\,dx + \int_0^H g_0\,dy = 0$ solvability condition (*)

$$f_0 \int_0^L dx + g_0 \int_0^H dy = \boxed{f_0 L + g_0 H = 0}$$

Let $U(x,y) = A(x^2 - y^2)$ and check PDE with BC's

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2A - 2A = 0 \checkmark$$

BC's $\frac{\partial U}{\partial x}(0,y) = 2Ax\big|_{x=0} = 0 \checkmark$

$\frac{\partial U}{\partial y}(x,0) = -2Ay\big|_{y=0} = 0 \checkmark$

$$\left.\begin{aligned} g_0 &= \frac{\partial U}{\partial x}(L,y) = 2Ax\big|_{x=L} = 2AL \rightarrow A = \frac{1}{2L} g_0 \\ f_0 &= \frac{\partial U}{\partial y}(x,H) = -2Ay\big|_{y=H} = -2AH \rightarrow A = -\frac{1}{2H} f_0 \end{aligned}\right\} \Rightarrow$$

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$$\left.\begin{aligned} A &= \frac{1}{2L} g_0 \\ A &= -\frac{1}{2H} f_0 \end{aligned}\right\} \text{ 2 Values for A?!} \rightarrow$$

Remember (*) gives solvability condition:

$$f_0 L + g_0 H = 0 \rightarrow \boxed{f_0 = -\frac{g_0 H}{L}}$$

$$\Rightarrow A = -\frac{1}{2\cancel{H}}\left(-\frac{g_0 \cancel{H}}{L}\right) = \frac{1}{2L} g_0$$

thus only there is one value for A and the solution is unique.

$$\Rightarrow \boxed{U(x,y) = \frac{g_0}{2L}(x^2 - y^2)} \quad \text{OR} \quad U = -\frac{f_0}{2H}(x^2 - y^2)$$

c) To solve non-constant $f(x)$, $g(x)$ problem we would split it into two problems:

$$U_x=0 \; \underset{U_y=0}{\overset{U_y=f(x)}{\boxed{\nabla^2 U_1=0}}} \; U_x=0 \; + \; U_x=0 \; \underset{U_y=0}{\overset{U_y=0}{\boxed{\nabla^2 U_2=0}}} \; U_x=g(y)$$

But it don't garantee solvability because is probably that $\int_0^L f(x)\,dx \neq 0$

The overall solvability require zero net flux between te boundaries, no each independiently. But if we define average fow:

$$\bar{f} = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad \bar{g} = \frac{1}{H}\int_0^H g(y)\,dy$$

and define

$$\tilde{f}(x) = f(x) - \bar{f} \quad \text{and} \quad \tilde{g}(x) = g(y) - \bar{g}$$

then we can split into three problem:

$$U_x=0 \; \underset{U_y=0}{\overset{U_y=\bar{f}}{\boxed{\nabla^2 U=0}}} \; U_x=\bar{g} + \overset{0}{V_x} \; \underset{V_y=0}{\overset{V_y=\tilde{f}}{\boxed{\nabla^2 V=0}}} \; V_x=0 + \overset{0}{W_x} \; \underset{W_x=0}{\overset{W_y=0}{\boxed{\nabla^2 W=0}}} \; W_x=\tilde{g}$$

where the first problem is like **b)** and remaining two problems that satisfy solvability conditions:

$$\int_0^L \tilde{f}(x)\,dx = \int_0^L [f(x) - \bar{f}]\,dx = \underbrace{\int_0^L f(x)\,dx}_{L\cdot\bar{f}} - \bar{f}\cdot\int_0^L dx = 0$$

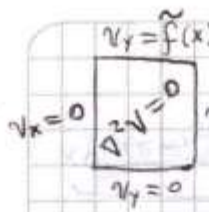
$$\int_0^L \tilde{g}(x)\,dx = 0 \quad \text{similarlly,} \rightarrow \text{satisfy solvability conditions.}$$

For the first problem the solution is $\boxed{\bar{U}(x,y) = \frac{\bar{g}}{2L}(x^2 - y^2)}$ (Just like in item **b)**)

And $U(x,y) = \bar{U}(x,y) + V(x,y) + W(x,y)$.

The "V" and "W" problems will have the sam base eigenfunction in sines and cosines $\rightarrow$

$0 \le x \le L \qquad 0 \le y \le H$



$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{let } v(x,y) = h(y)\,\phi(x)$$

$$0 = h\frac{\partial^2 \phi}{\partial x^2} + \phi\frac{\partial^2 h}{\partial y^2} \quad \text{dividing by } h\phi$$

$$\underbrace{\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{-\lambda} + \underbrace{\frac{1}{h}\frac{d^2h}{dy^2}}_{\lambda} = 0 \rightarrow \boxed{\frac{d^2h}{dy^2} = \lambda h(y)} \quad \text{with } h'(0) = 0$$

$$\boxed{\frac{d^2\phi}{dx^2} = -\lambda\phi(x)} \quad \text{with } \phi'(0) = 0 \text{ and } \phi'(L) = 0$$

$\underline{\lambda = 0}$ $\quad \phi'(x) = C_1 \longrightarrow \phi(x) = C_1 x + C_0$, $\quad \phi'(0) = 0 = C_1$, $\quad \phi'(L) = 0 = C_1$ $\Big\}$ $\boxed{\phi_0(x) = C_0}$

$\underline{\lambda > 0}$: $\quad \phi(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$\phi'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x \rightarrow \phi'(0) = 0 = C_2\sqrt{\lambda} \rightarrow C_2 = 0$

$\phi'(L) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$

$\rightarrow \sqrt{\lambda}L = n\pi \rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 0, 1, 2, \ldots \rightarrow \boxed{\phi_n(x) = C_1\cos\left(\frac{n\pi x}{L}\right)}$

$\underline{\lambda < 0}$: $\quad \phi(x) = A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x)$

↳ It's not solution because no oscillate.

$\underline{\lambda = 0}$: $\quad \frac{d^2h}{dy^2} = 0 \rightarrow h'(y) = b_1 \longrightarrow h(y) = \cancel{b_1}^{\,0}x + b_0 \rightarrow \boxed{h_0(y) = b_0}$

$h'(0) = 0 = b_1 \longrightarrow$

$\underline{\lambda > 0}$: $\quad \frac{d^2h}{dy^2} = \lambda_n h = \left(\frac{n\pi}{L}\right)^2 h(y) \rightarrow h(y) = A\cosh\left(\frac{n\pi y}{L}\right) + B\sinh\left(\frac{n\pi y}{L}\right)$

$h'(y) = A\left(\frac{n\pi}{L}\right)\sinh\left(\frac{n\pi y}{L}\right) + B(\cdot)\cosh(\cdots)$

$h'(0) = 0 = B$

$$\Rightarrow \boxed{h_n(y) = A\cosh\left(\frac{n\pi y}{L}\right)}$$

$$\boxed{v(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right)}$$

$$\tilde{f}(x) = \frac{\partial v}{\partial x}(0,H) = \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right) A_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi H}{L}\right)$$

Multiply by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating

$$\int_0^L \tilde{f}(x)\cos\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}\frac{n\pi A_n}{L}\underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx\,\cos\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\delta_{nm}}\sinh\left(\frac{n\pi H}{L}\right)$$

$$= \frac{m\pi}{\cancel{L}}A_m\frac{\cancel{L}}{2}\sinh\left(\frac{m\pi H}{L}\right)$$

$$\Rightarrow \boxed{A_n = \frac{2}{n\pi\sinh(m\pi H/L)}\int_0^L \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

where $\tilde{f}(x) = f(x) - \bar{f}$

$\longrightarrow$

$$\cancel{\int_0^L \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_0^L \cos^2\left(\frac{n\pi x}{L}\right)dx}$$

$$\cancel{+\sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi H}{L}\right)\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx}$$

$$\cancel{= A_m \cosh\left(\frac{m\pi H}{L}\right)\frac{L}{2}}$$

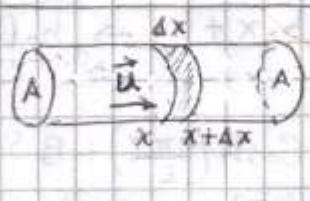
To solve W problem it is similar like we solved v problem. Simply exchange $x \longleftrightarrow y$, $L \longleftrightarrow H$, $\tilde{f} \longleftrightarrow \tilde{g}$

$$\rightarrow \boxed{W(x,y) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi y}{H}\right)\cosh\left(\frac{n\pi x}{L}\right)}$$

With $\boxed{B_n = \frac{2}{n\pi \sinh(n\pi H/L)}\int_0^H \tilde{g}(y)\cos\left(\frac{n\pi y}{H}\right)dy}$

$$\Rightarrow \boxed{U(x,y) = \bar{U}(x,y) + v(x,y) + W(x,y)}$$ done!

2.5.17



$$\frac{\partial}{\partial t}\left(\text{mass in slice}\right) = \left(\begin{matrix}\text{mass}\\ \text{flowing in}\end{matrix}\right) - \left(\begin{matrix}\text{mass}\\ \text{flowing out}\end{matrix}\right)$$

1D:

$$\frac{\partial}{\partial t}(\rho A \Delta x) = \rho(x)u(x)A - \rho(x+\Delta x)u(x+\Delta x)A$$

$$\Rightarrow \frac{\partial \rho}{\partial x} = \frac{\rho(x)u(x) - \rho(x+\Delta x)u(x+\Delta x)}{\Delta x} = -\frac{\partial}{\partial x}(\rho u) \rightarrow \boxed{\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial x}(\rho u) = 0}$$

extension to 3D: $\vec{\nabla}(\rho\vec{u}) = \left[\frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z)\right]$

$$\Rightarrow \frac{\partial}{\partial t}\int_V \rho\, dV =$$

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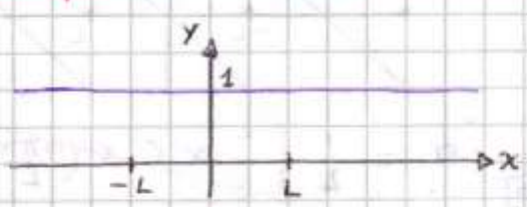
Solución a los problemas de Haberman

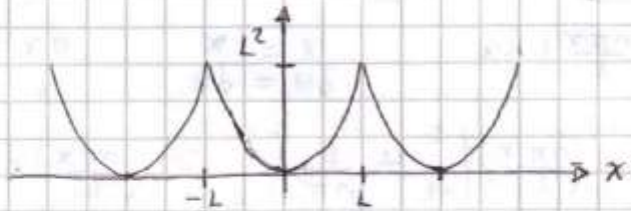
Cap 3

3.2.1 a) $f(x) = 1$

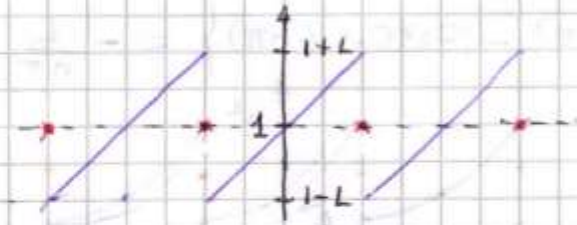


Here the Fourier series for $f(x)$ and $f(x)$ itself have identical graphs.

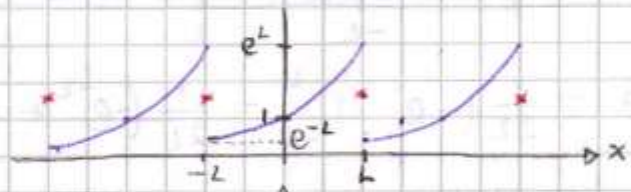
* b) $f(x) = x^2$



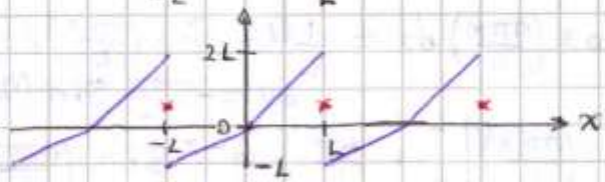
c) $f(x) = 1 + x$



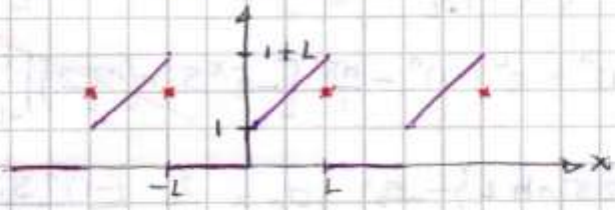
* d) $f(x) = e^x$



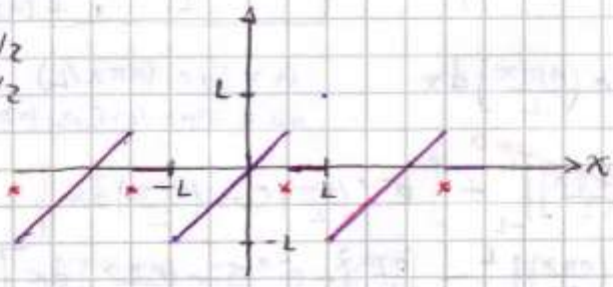
e) $f(x) = \begin{cases} x & \text{if } x < 0 \\ 2x & \text{if } x > 0 \end{cases}$



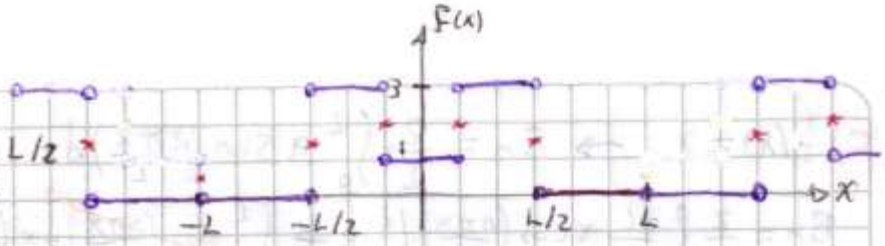
* f) $f(x) = \begin{cases} 0 & x < 0 \\ 1 + x & x > 0 \end{cases}$



g) $f(x) = \begin{cases} x & x < L/2 \\ 0 & x > L/2 \end{cases}$



b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

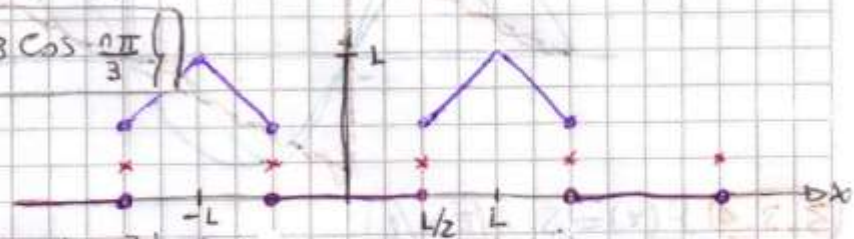


$$A_0 = \frac{1}{L}\left\{\int_0^{L/6} dx + \int_{L/6}^{L/3} 3\,dx\right\} = \frac{1}{L}\left\{\frac{L}{6} + \frac{3L}{3} - \frac{3L}{6}\right\} = \left(1 - \frac{2}{6}\right) = \boxed{\frac{2}{3}}$$

$$A_n = \frac{2}{L}\left\{\int_0^{L/6} \cos\left(\frac{n\pi x}{L}\right)dx + 3\int_{L/6}^{L/3} \cos\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$\boxed{A_n = \frac{2}{n\pi}\left\{-2\cos\left(\frac{n\pi}{6}\right) - 1 + 3\cos\frac{n\pi}{3}\right\}}$$

c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$



$$A_0 = \frac{1}{L}\int_{L/2}^{L} x\,dx = \frac{1}{L}\left(\frac{L^2}{2} - \frac{(L/2)^2}{2}\right) = \frac{L}{2}\left(\frac{3}{4}\right) = \boxed{\frac{3L}{8}}$$

$$A_n = \frac{2}{L}\int_{L/2}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx$$

$u = x$ $\quad dv = \cos(n\pi x/L)$

$du = dx$ $\quad v = \frac{L}{n\pi}\sin(n\pi x/L)$

$$= \frac{2}{L}\left\{\frac{L}{n\pi}x\sin\frac{n\pi x}{L}\Big|_{L/2}^{L} - \frac{L}{n\pi}\int_{L/2}^{L}\sin\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$\boxed{A_n = \frac{2}{n\pi}\left\{-\frac{L}{2}\sin\left(\frac{n\pi}{2}\right) + \frac{L}{n\pi}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right]\right\}}$$

3.3.6 a) $f(x) = x \rightarrow A_0 = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\frac{L^2}{2} = \boxed{\frac{L}{2}}$

$$A_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left\{\cancel{\frac{L}{n\pi}x\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L}^{\,0} - \frac{L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$= \frac{2L}{(n\pi)^2}\left[(-1)^n - 1\right] = \boxed{\begin{cases} 0 & n \text{ even} \\ \frac{-4L}{(n\pi)^2} & n \text{ odd} \end{cases}}$$

$$\boxed{f(x) \simeq \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n\,odd}^{N}\frac{1}{n^2}\cos\left(\frac{n\pi x}{L}\right)}$$

b) $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases} \rightarrow A_0 = \frac{1}{L}\int_{L/2}^{L} dx = \boxed{\frac{1}{2}}$

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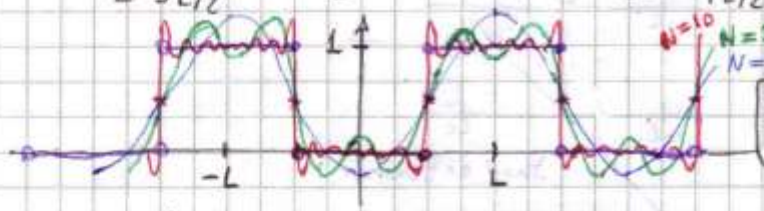
$n = 2k-1 \qquad k = 1, 2, 3, \ldots$

$$A_n = \frac{2}{L}\int_{L/2}^{L} \cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L} = -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \boxed{\frac{2}{(2k-1)\pi}(-1)^k}$$



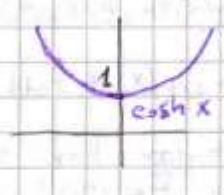
$$\boxed{f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{N}\frac{(-1)^k}{(2k-1)}\cos\frac{(2k-1)\pi x}{L}}$$

c) $g(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases} \Rightarrow g(x) = -f(x) + 1$ where $f(x)$ is defined like in 3.3.6 c.

$$\Rightarrow \boxed{g(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{N}\frac{(-1)^k}{(2k-1)}\cos\frac{(2k-1)\pi x}{L}}$$

3.3.7 $e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})$

$$= \underbrace{\cosh x}_{\text{even function}} + \underbrace{\sinh x}_{\text{odd function}}$$



3.3.8 Suppose $f(x)$ is defined for $x \geq 0$, then the even

a) extensión of $f(x)$ is: $f_{\text{even Ext}}(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x \leq 0 \end{cases}$

and Even part of $f(x)$ is $f_e(x) = \frac{1}{2}\left(f(x) + f(-x)\right)$

b) Similarly $f_{\text{odd Ext}}(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ -f(-x) & \text{if } x < 0 \end{cases}$

and odd part of $f(x)$ is $f_o(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$

c) $f(x) = \begin{cases} x & x > 0 \\ x^2 & x < 0 \end{cases} \rightarrow f_e(x) = \begin{cases} \frac{1}{2}(x + (-x)^2) & \text{if } x > 0 \\ \frac{1}{2}(x^2 + (-x)) & \text{if } x < 0 \end{cases} = \begin{cases} \frac{1}{2}(x + x^2) & x > 0 \\ \frac{1}{2}(x^2 - x) & x < 0 \end{cases}$

$f(-x) = \begin{cases} (-x)^2 & x > 0 \\ -x & x < 0 \end{cases} \Rightarrow f_o(x) = \begin{cases} \frac{1}{2}(x - (-x)^2) & \text{if } x > 0 \\ \frac{1}{2}(x^2 - (-x)) & x < 0 \end{cases} = \begin{cases} \frac{1}{2}(x - x^2) & x > 0 \\ \frac{1}{2}(x^2 + x) & x < 0 \end{cases}$

$+$

$$f_e(x) + f_o(x) = f(x) = \begin{cases} x & \text{if } x > 0 \\ x^2 & \text{if } x < 0. \end{cases}$$

$f_{\text{even ext}} = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

$f_{\text{odd ext}} = \begin{cases} x & x > 0 \\ x & < x 0 \end{cases}$

$\Rightarrow \underbrace{f_{\text{even ext}} + f_{\text{odd ext}}}_{2f(x) \rightarrow \text{ defined in } x > 0} = \begin{cases} 2x & x \geq 0 \\ 0 & x < 0 \end{cases}$

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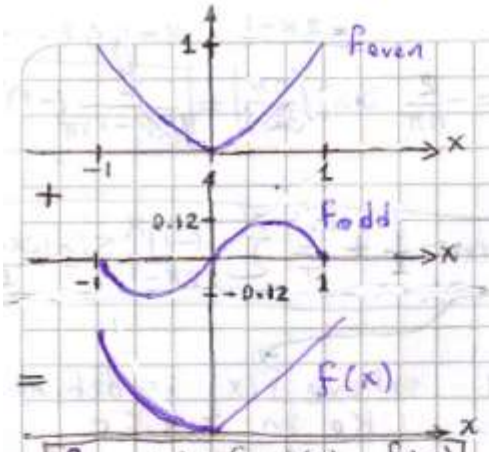
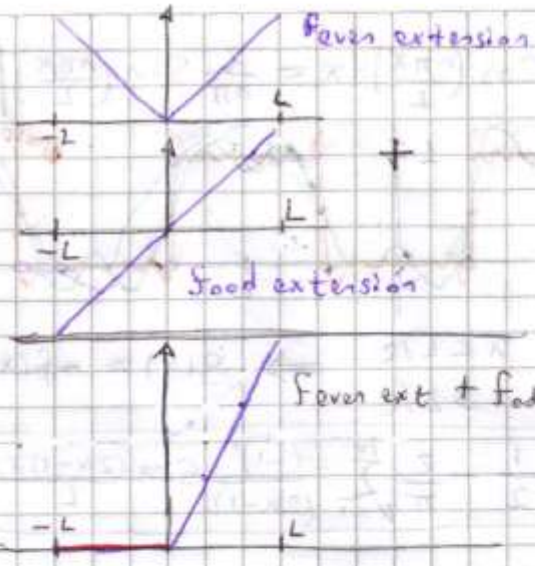


$$\boxed{f_{even}(x) + f_{odd}(x) = f(x)}$$



$$f_{even\ ext} + f_{odd\ ext} = \begin{cases} 2f(x) & x>0 \\ 0 & x<0 \end{cases}$$

(3.3.9) From last question in 3.3.8 c).

$$f_{even\ ext}(x) + f_{odd\ ext}(x) = \begin{cases} 2f(x) & 0<x<L \\ 0 & -L<x<0 \end{cases}$$

It's similar for sum of the Fourier sine series of $f(x)$ and the Fourier cosine series of $f(x)$ with periodic extension. It's valid for any $f(x)$.

Let $f(x)$ be a piecewise continuous function.

Fourier sine series of $f(x)$ is $f_s(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)$

where $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

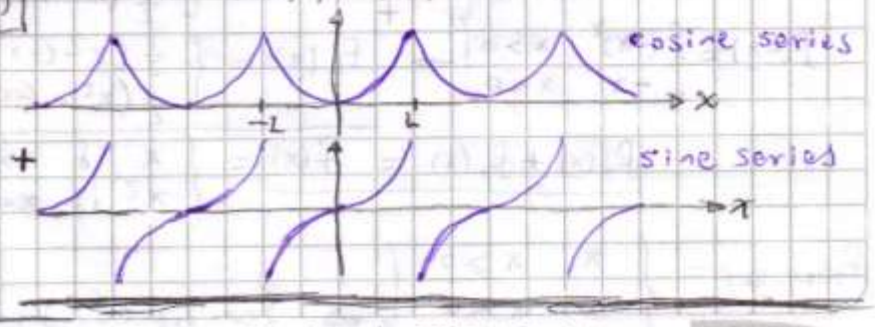
Fourier cosine series of $f(x)$ is $f_c(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/L)$

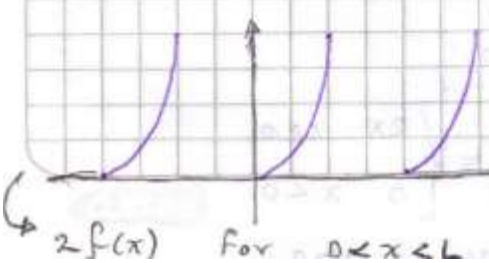
where $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ and $A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$\Rightarrow f_c(x) + f_s(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/L) + \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)$$

$\boxed{f_c(x) + f_s(x) \approx 2f(x)}$ with $f(x) \neq 0$ for $x>0$.

Suppose $f(x) = x^2$ $\longrightarrow$



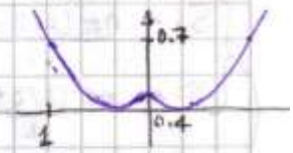


$\leftarrow$ Cosine Series + Sineseries

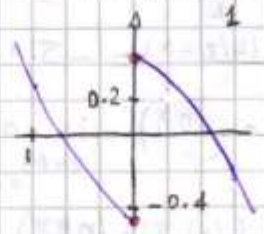
$\hookrightarrow$ $2f(x)$ for $0<x<L$

3.3.10 $f(x)=\begin{cases} x^2 & x<0 \\ e^{-x} & x>0 \end{cases} \Rightarrow f(-x)=\begin{cases} (-x)^2 & x>0 \\ e^{-(-x)} & x<0 \end{cases}=\begin{cases} x^2 & x>0 \\ e^{x} & x<0 \end{cases}$

$$\Rightarrow f_e(x)=\frac{1}{2}\left(f(x)+f(-x)\right)=\frac{1}{2}\begin{cases} x^2+e^{x} & x<0 \\ e^{-x}+x^2 & x>0 \end{cases}$$



$$f_{odd}(x)=\frac{1}{2}\left(f(x)-f(-x)\right)=\frac{1}{2}\begin{cases} x^2-e^{x} & x<0 \\ e^{-x}-x^2 & x>0 \end{cases}$$



3.3.11 The procedure is like in 3.3.8 c) In each point in x axis graph $\frac{1}{2}[f(x)-f(-x)]$ and obtain sketch of f_{odd} and graph $\frac{1}{2}[f(x)+f(-x)]$ and obtain sketch of f_{even}.

3.3.12 $f(x)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right)$ where $B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

a) Let $f_n(x)=B_n\sin\left(\frac{n\pi x}{L}\right)$ the even term (n even) of the Fourier sine series.

$$\begin{cases} f_n(L/2-x)=B_n\sin\left(\frac{n\pi(L/2-x)}{L}\right)=B_n\left\{\cancel{\sin\frac{n\pi}{2}}^{\,0\ n\text{ even}}\cos\frac{n\pi x}{L}-\cos\frac{n\pi}{2}\sin\left(\frac{n\pi x}{L}\right)\right\} \\ f_n(L/2+x)=B_n\sin\left(\frac{n\pi(L/2+x)}{L}\right)=B_n\left\{\cancel{\sin\frac{n\pi}{2}}^{\,0\ n\text{ even}}\cos\frac{n\pi x}{2}+\cos\frac{n\pi}{2}\sin\frac{n\pi x}{2}\right\} \end{cases}$$

$\Rightarrow \boxed{f_n(L/2-x)=f_n(L/2+x)}$ when n is even, then f_n is odd (antisymetric) around $x=L/2$.

b) Let n odd number, then:

$$\begin{cases} \sin\left(\frac{n\pi(L/2-x)}{L}\right)=\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right)-\cancel{\cos\left(\frac{n\pi}{2}\right)}^{\,0\ n\text{ odd}}\sin\left(\frac{n\pi x}{L}\right) \\ \sin\left(\frac{n\pi(L/2+x)}{L}\right)=\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right)+\cancel{\cos\left(\frac{n\pi}{2}\right)}^{\,0\ n\text{ odd}}\sin\left(\frac{n\pi x}{L}\right) \end{cases}$$

$\Rightarrow \sin\left(\frac{n\pi x}{L}\right)$ is even function about $x=L/2$, then $f(x)\sin\left(\frac{n\pi x}{L}\right)$ is odd function around $x=L/2$ when n is odd since $f(x)$ is odd around $x=L/2$ (odd funct x even funct = odd funct)

$$\int_0^{L/2} f(x)\sin\left(\frac{n\pi x}{L}\right)dx=-\int_{L/2}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$\Rightarrow \boxed{B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx=0}$ when n is odd and $f(x)$ is odd function around $x=L/2$.

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3.3.13 Let n even number:

$$\begin{cases} \sin\frac{n\pi(L/2-x)}{L} = \cancel{\sin\frac{n\pi}{2}}^{\,0\ n\text{ even}}\cos\frac{n\pi x}{L} - \cos\frac{n\pi}{2}\sin\frac{n\pi x}{L} \\ \sin\frac{n\pi(L/2+x)}{L} = \cancel{\sin\frac{n\pi}{2}}^{\,0\ n\text{ even}}\cos\frac{n\pi x}{L} + \cos\frac{n\pi}{2}\sin\frac{n\pi x}{L} \end{cases}$$

$\rightarrow \sin\frac{n\pi(L/2+x)}{L} = -\sin\frac{n\pi(L/2-x)}{L} \rightarrow \sin\frac{n\pi x}{L}$ is odd function around $x = L/2$ (since n even)

$\Rightarrow f(x)\sin\left(\frac{n\pi x}{L}\right)$ is odd function around $x = L/2$ since $f(x)$ is even function around $x = L/2$ and n even. (even funct x odd funct = odd f)

$$\Rightarrow \int_0^{L/2} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = -\int_{L/2}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

When n is even and $f(x)$ is an even function around $x = L/2$ $\Rightarrow \boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2}\right)dx = 0}$

3.3.14 Let n odd number:

a)
$$\begin{cases} \cos\frac{n\pi(L/2-x)}{L} = \cancel{\cos\frac{n\pi}{2}}^{\,0\ n\text{ odd}}\cos\frac{n\pi x}{L} + \sin\frac{n\pi}{2}\sin\frac{n\pi x}{L} \\ \cos\frac{n\pi(L/2+x)}{L} = \cancel{\cos\frac{n\pi}{2}}^{\,0\ n\text{ odd}}\cos\frac{n\pi x}{L} - \sin\frac{n\pi}{2}\sin\frac{n\pi x}{L} \end{cases}$$

$\rightarrow \cos\frac{n\pi(L/2-x)}{L} = -\cos\frac{n\pi(L/2+x)}{L} \rightarrow \cos\frac{n\pi x}{2}$ is odd function around $x = L/2$ (since n odd)

$\Rightarrow f(x)\cos\left(\frac{n\pi x}{L}\right)$ is odd function around $x = L/2$, since $f(x)$ is even function (around $x = L/2$) and n odd.

$$\Rightarrow \int_0^{L/2} f(x)\cos\frac{n\pi x}{L}dx = -\int_{L/2}^{L} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$\rightarrow \boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{2}\right)dx = 0}$ when n odd and $f(x)$ even function around $x = L/2$

b) Since $f(x)$ is even around $x = L/2$, then even extension of $f(x)$ on the interval $0 \le x \le L/2$ will be the same as the even extension of $f(x)$ on the interval $0 \le x \le L$. Thus the Fourier cosine series on $0 \le x \le L/2$ will be the same as as that on $0 \le x \le L$. Now the Fourier cosine series on $0 \le x \le L/2$ will be

$$f(x) \sim \sum_{n=0}^{\infty} C_n \cos\left[\frac{n\pi x}{(L/2)}\right] = \sum_{n=0}^{\infty} C_n \cos\left(\frac{2n\pi x}{L}\right)$$

By comparison to the cosine series on $0 \le x \le L$:

$f(x) \sim \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{2}\right)$ we see $A_n = 0$ (n odd) and $A_n = C_{n/2}$ (n even)

3.3.15 Let n even number:

$$\begin{cases} \cos\frac{n\pi(L/2-x)}{L} = \cos\frac{n\pi}{2}\cos\frac{n\pi x}{L} + \cancel{\sin\frac{n\pi}{2}}^{\,0\ n\text{ even}}\sin\frac{n\pi x}{L} \\ \cos\frac{n\pi(L/2+x)}{L} = \cos\frac{n\pi}{2}\cos\frac{n\pi x}{L} - \cancel{\sin\frac{n\pi}{2}}^{\,0\ n\text{ even}}\sin\frac{n\pi x}{L} \end{cases}$$

$\rightarrow \cos\frac{n\pi(L/2-x)}{L} = \cos\frac{n\pi(L/2+x)}{L} \rightarrow \cos\frac{n\pi x}{L}$ is even function around $x=L/2$ (n even)

$\Rightarrow f(x)\cos\left(\frac{n\pi x}{L}\right)$ is odd function around $x=L/2$ since $f(x)$ is odd function around $x=L/2$ and n even.

$$\Rightarrow \int_0^{L/2} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = -\int_{L/2}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$\rightarrow \boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0}$ when n even and $f(x)$ odd function around $x=L/2$.

3.3.16 Notice that the linear transformation $y = \frac{a+b}{2} + \frac{b-a}{2L}x$ transforms the interval $-L \le x \le L$ into the interval $a \le y \le b$.

$$\text{at } x=-L \rightarrow y = \frac{a+b}{2} - \frac{b-a}{2} = a$$

$$\text{at } x=L \rightarrow y = \frac{a+b}{2} + \frac{b-a}{2} = b$$

Solving for x:

$$x = \frac{2L}{b-a}\left[y - \frac{a+b}{2}\right] = \frac{2L}{b-a}y - \frac{a+b}{b-a}L$$

Thus a representation of $g(y)$ using periodic trigonometric functions with period $(b-a)$ is:

$$g(y) \sim a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2n\pi}{b-a}y - \frac{a+b}{b-a}n\pi\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{2n\pi}{b-a}y - \frac{a+b}{b-a}n\pi\right)$$

Notice that these trigonometric functions do have period $b-a$.

The coefficients are given by:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(y)\,dx = \frac{1}{2L}\int_a^b g(y)\frac{2L}{b-a}dy = \boxed{\frac{1}{b-a}\int_a^b g(y)\,dy = a_0}$$

Similarly:

$$a_n = \frac{1}{L}\int_{-L}^{L} g(y)\cos\left(\frac{n\pi x}{L}\right)dx = \boxed{\frac{2}{b-a}\int_a^b g(y)\cos\left(\frac{2n\pi}{b-a}y - \frac{a+b}{b-a}n\pi\right)dy = a_n}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} g(y)\sin\left(\frac{n\pi x}{L}\right)dx = \boxed{\frac{2}{b-a}\int_a^b g(y)\sin\left(\frac{2n\pi}{b-a}y - \frac{a+b}{b-a}n\pi\right)dy = b_n}$$

3.3.17 a) $\int_0^1 \frac{dx}{1+x^2} = \text{Arctg}\,\Big|_0^1 = \underbrace{\text{Arctg}\,1}_{\pi/4} - \cancel{\text{Arctg}\,0}^{\,0} = \boxed{\pi/4}$

b) $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

$$\Rightarrow \int_0^1 \frac{dx}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}dx = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}}\ \cancel{x^{2n+1}\Big|_0^1}^{\,1}$$

c) Equalling a) and b)

$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)} \longrightarrow \pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}$$

$$\boxed{\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)}$$

3.3.18 Assume $f(x)$ is continuous.

a) $f(x)$ equals its Fourier series for all $-L \le x \le L$ if $f(x)$ is piecewise smooth and $f(-L) = f(L)$.

b) $f(x)$ equals its Fourier sine series for all $0 \le x \le L$ if $f(x)$ is piecewise smooth and $f(0) = f(L) = 0$.

c) $f(x)$ equals its Fourier cosine series for all $0 \le x \le L$ if $f(x)$ is piecewise smooth.

3.4.1* a) Integrating by parts will hold within each closed interval where the quantities are continuous.

Thus if $z_1 \in (a,c)$ then: $\int_a^{z_1} u \frac{dv}{dx} dx = uv\Big|_a^{z_1} - \int_a^{z_1} v \frac{du}{dx} dx$

and if $z_2 \in (c,b)$ then: $\int_{z_2}^{b} u \frac{dv}{dx} dx = uv\Big|_{z_2}^{b} - \int_{z_2}^{b} v \frac{du}{dx} dx$

$$\Rightarrow \lim_{z_1 \to c^-} \int_a^{z_1} u \frac{dv}{dx} dx = \lim_{z_1 \to c^-} \left(uv\Big|_a^{z_1} - \int_a^{z_1} v \frac{du}{dx} dx \right)$$

$$\int_a^{c^-} u \frac{dv}{dx} dx = uv\Big|_a^{c^-} - \int_a^{c^-} v \frac{du}{dx} dx$$

$$\lim_{z_2 \to c^+} \int_{z_2}^{b} u \frac{dv}{dx} dx = \lim_{z_2 \to c^+} \left(uv\Big|_{z_2}^{b} - \int_{z_2}^{b} v \frac{du}{dx} dx \right)$$

$$\int_{c^+}^{b} u \frac{dv}{dx} dx = uv\Big|_{c^+}^{b} - \int_{c^+}^{b} v \frac{du}{dx} dx$$

$$\Rightarrow \int_a^{c^-} u \frac{dv}{dx} dx + \int_{c^+}^{b} u \frac{dv}{dx} dx = uv\Big|_a^{c^-} + uv\Big|_{c^+}^{b} - \int_a^{c^-} v \frac{du}{dx} dx - \int_{c^+}^{b} v \frac{du}{dx} dx$$

$$\Rightarrow \int_a^{b} u \frac{dv}{dx} dx = uv\Big|_a^{c^-} + uv\Big|_{c^+}^{b} - \int_a^{b} v \frac{du}{dx} dx$$

$$\boxed{\int_a^{b} u \frac{dv}{dx} dx = uv\Big|_a^{b} + uv\Big|_{c^+}^{c^-} - \int_a^{b} v \frac{du}{dx} dx}$$

b) If u and v are continuous at $x = c$ then:

$uv\Big|_{c^+}^{c^-} = 0$ and thus $\boxed{\int_a^{b} u \frac{dv}{dx} dx = uv\Big|_a^{b} - \int_a^{b} v \frac{du}{dx} dx}$

3.4.2 If $\frac{df}{dx}$ is piecewise smooth then:

$$\frac{df}{dx} \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Where $A_0 = \frac{1}{2L} \int_{-L}^{L} f'(x)\, dx = \frac{1}{2L} \left(f(L) - f(-L) \right)$

If the Fourier series of $f(x)$ is continuous then $f(x)$ is continuous and $2L$-periodic, such that:

$f(L) = f(-L) \Rightarrow \boxed{A_0 = 0}$

$$A_n = \frac{1}{L}\int_{-L}^{L} f'(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad u=\cos\left(\frac{n\pi x}{L}\right) \quad dv = f'(x)dx$$

$$du = -\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)dx \quad v = f(x)$$

$$= \frac{1}{L}\left\{ \cancel{f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}}^{\,0} + \frac{n\pi}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \right\}$$

$$= \frac{n\pi}{L}\cdot \underbrace{\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx}_{\beta_n} = \frac{n\pi}{L}\beta(x)$$

$\beta_n \rightarrow$ a coefficient in the Fourier series of $f(x)$.

$$B_n = \frac{1}{L}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad u=\sin\left(\frac{n\pi x}{L}\right) \quad dv = f'(x)dx$$

$$du = \frac{n\pi}{L}\cos\frac{n\pi x}{L}dx \quad v = f(x)$$

$$= \frac{1}{L}\left\{ f(x)\sin\frac{n\pi x}{L}\Big|_{-L}^{L} - \frac{n\pi}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \right\}$$

$$= -\frac{n\pi}{L}\cdot \underbrace{\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx}_{\alpha_n}$$

$\alpha_n \rightarrow$ a coefficient in the Fourier series of $f(x)$.

So $$\frac{df}{dx} \sim \sum_{n=1}^{\infty}\frac{n\pi}{L}\beta_n\cos\left(\frac{n\pi x}{L}\right) - \sum_{n=1}^{\infty}\frac{n\pi}{L}\alpha_n\sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty}\beta_n\frac{d}{dx}\left(\sin\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty}\alpha_n\frac{d}{dx}\left(\cos\frac{n\pi x}{L}\right)$$

$$= \frac{d}{dx}\left\{ \alpha_0 + \sum_{n=1}^{\infty}\alpha_n\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty}\beta_n\sin\frac{n\pi x}{L} \right\}$$

$$\boxed{\frac{df}{dx} = \frac{d}{dx}\left[\text{Fourier series of } f(x)\right]}$$

Thus the Fourier series for $f(x)$ can be differentiated term by term to obtain the Fourier series for $f'(x)$.

3.4.3* a) We want to determine the sine coefficients of

$$\frac{df}{dx} \sim \sum_{n=1}^{\infty}\beta_n\sin\left(\frac{n\pi x}{L}\right) \rightarrow \beta_n = \frac{2}{L}\int_0^L \frac{df}{dx}\sin\left(\frac{n\pi x}{L}\right)dx$$

where the cosine coefficients of f are given:

$$f \sim \sum_{n=0}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) \rightarrow a_0 = \frac{1}{L}\int_0^L f(x)dx \qquad a_n = \frac{2}{L}\int_0^L f\cos\left(\frac{n\pi x}{L}\right)dx$$

$(n \neq 0)$.

$\Rightarrow$

$$\beta_n = \frac{2}{L}\int_0^L \frac{df}{dx}\sin\left(\frac{n\pi x}{L}\right)dx \qquad u = \sin\left(\frac{n\pi x}{L}\right) \quad dv = \frac{df}{dx}dx$$

$$du = \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)dx \quad v = f(x)$$

$$= \frac{2}{L}\left\{ f(x)\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{x_0^-} + f(x)\sin\left(\frac{n\pi x}{L}\right)\Big|_{x_0^+}^{L} - \frac{n\pi}{2}\underbrace{\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx}_{(L/2)a_n}\right\}$$

$$= \frac{2}{L}\left\{ f(x_0^-)\sin\left(\frac{n\pi x_0^-}{L}\right) - f(x_0^+)\sin\left(\frac{n\pi x_0^+}{L}\right) - \frac{n\pi}{2}a_n\right\}$$

$$= \frac{2}{L}\left\{(\alpha-\beta)\sin\left(\frac{n\pi x_0}{L}\right) - \frac{n\pi}{2}a_n\right\}$$

→ Here we used that $\sin\left(\frac{n\pi x}{L}\right)$ is a continuous function.

$$\sin\left(\frac{n\pi x_0^-}{L}\right) = \sin\left(\frac{n\pi x_0^+}{L}\right) = \sin\left(\frac{n\pi x_0}{L}\right)$$

$$\boxed{\beta_n = \frac{2}{L}(\alpha-\beta)\sin\left(\frac{n\pi x_0}{L}\right) - \frac{n\pi}{L}a_n}$$

$$\hookrightarrow \boxed{f'(x) = \sum_{n=1}^{\infty}\beta_n \sin(n\pi x/L)}$$

b) We want to determine the cosine coefficients of

$$\frac{df}{dx} \sim \sum_{n=0}^{\infty}\alpha_n\cos\left(\frac{n\pi x}{L}\right) \rightarrow \alpha_n = \frac{2}{L}\int_0^L \frac{df}{dx}\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\alpha_0 = \frac{1}{L}\int_0^L \frac{df}{dx}dx = \frac{1}{L}\left\{\int_0^{x_0^-} f'(x)dx + \int_{x_0^+}^{L} f'(x)dx\right\}$$

$$\alpha_0 = \frac{1}{L}\left\{f(x_0^-) - f(0) + f(L) - f(x_0^+)\right\} = \boxed{\frac{1}{L}\left\{(\alpha-\beta) + f(L) - f(0)\right\}}$$

We know the sine series coefficient of $f(x)$ are given:

$$f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow \alpha_n = \frac{2}{L}\int_0^L f'(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad u = \cos(n\pi x/L) \quad dv = f'(x)dx$$

$$du = \frac{-L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) \quad v = f(x)$$

$$= \frac{2}{L}\left\{ f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{x_0^-} + f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_{x_0^+}^{L} + \frac{n\pi}{L}\underbrace{\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}_{(L/2)b_n}\right\}$$

$$= \frac{2}{L}\left\{\underbrace{f(x_0^-)}_{\alpha}\cos\left(\frac{n\pi x_0^-}{L}\right) - f(0) + f(L)(-1)^n - \underbrace{f(x_0^+)}_{\beta}\cos\left(\frac{n\pi x_0^+}{L}\right) + \frac{n\pi}{2}b_n\right\}$$

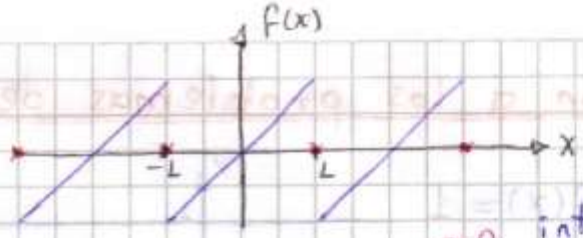
$$= \frac{2}{L}\left\{(\alpha-\beta)\cos\left(\frac{n\pi x_0}{L}\right) + f(L)(-1)^n - f(0) + \frac{n\pi}{2}b_n\right\}$$

$$\boxed{\alpha_n = \frac{2}{L}\left[(\alpha-\beta)\cos\left(\frac{n\pi x_0}{L}\right) + (-1)^n f(L) - f(0) + \frac{n\pi}{L}b_n\right]}$$

$$\hookrightarrow f'(x) = \alpha_0 + \sum_{n=1}^{\infty}\alpha_n\cos\left(\frac{n\pi x}{L}\right)$$

By: Herminso Villarraga Gómez

3.2.2 *a) $f(x) = x$



integrando impar!!

$$a_0 = \frac{1}{2L}\int_{-L}^{L} x\,dx \to 0 \qquad \boxed{a_0 = 0}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx \to 0 \quad \to \boxed{a_n = 0}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx$$

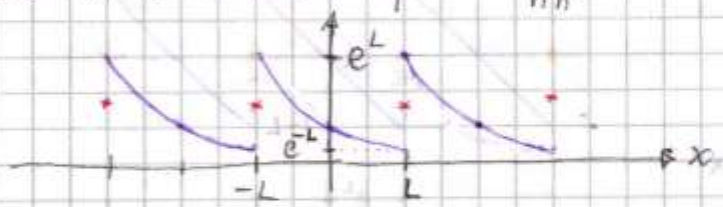
$u = x \qquad dv = \sin(n\pi x/L)\,dx$

$du = dx \qquad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$$= \frac{1}{L}\left\{ -\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx \to 0 \right\}$$

$$b_n = -\frac{1}{n\pi}\left\{L\cos(n\pi) - (-L)\cos(-n\pi)\right\} = -\frac{L}{n\pi}\,2\cos n\pi = \boxed{\frac{2L}{n\pi}(-1)^{n+1}}$$

b) $f(x) = e^{-x}$



$$a_0 = \frac{1}{2L}\int_{-L}^{L} e^{-x}\,dx = \frac{1}{2L}\left(-e^{-x}\right)\Big|_{-L}^{L} = \frac{1}{2L}\left(-e^{-L} + e^{L}\right) = \boxed{\frac{1}{L}\sinh L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\cos\left(\frac{n\pi x}{L}\right)dx$$

$u = \cos\left(\frac{n\pi x}{L}\right) \qquad dv = e^{-x}dx$

$du = -\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)dx \qquad v = -e^{-x}$

$$= \frac{1}{L}\left\{ -e^{-x}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} - \frac{n\pi}{L}\int e^{-x}\sin\left(\frac{n\pi x}{L}\right)dx \right\}$$

$u' = \sin(n\pi x/L)$

$dv' = e^{-x}dx$

$$= \frac{1}{L}\left\{ -e^{-L}(-1)^n + e^{L}(-1)^n - \frac{n\pi}{L}\left[-e^{-x}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} \to 0 + \underbrace{\frac{n\pi}{L}\int e^{-x}\cos\left(\frac{n\pi x}{L}\right)dx}_{n\pi a_n} \right] \right\}$$

$$\left[1 + \left(\frac{n\pi}{L}\right)^2\right] = \frac{1}{L}\left\{(-1)^n\,2\sinh L\right\} \to \boxed{a_n = \frac{2}{L}\,\frac{(-1)^n \sinh L}{1 + (n\pi/L)^2}}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\sin\left(\frac{n\pi x}{L}\right)dx$$

$u = \sin(n\pi x/L) \qquad dv = e^{-x}dx$

$du = (n\pi/L)\cos(n\pi x/L)\,dx \qquad v = -e^{-x}$

$$= \frac{1}{L}\left\{ -e^{-x}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} \to 0 + \int e^{-x}\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx \right\}$$

$u' = \cos(n\pi x/L)$

$dv' = e^{-x}dx$

$$= \frac{n\pi}{L^2}\left[-e^{-x}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} - \underbrace{\frac{n\pi}{L}\int e^{-x}\sin\left(\frac{n\pi x}{L}\right)dx}_{n\pi b_n} \right]$$

$$\left[1 + \left(\frac{n\pi}{L}\right)^2\right] = \frac{n\pi}{L^2}\left[\underbrace{-e^{-L}(-1)^n + e^{L}(-1)^n}_{(-1)^n\,2\sinh L}\right] \longrightarrow \boxed{b_n = \frac{(2n\pi/L^2)(-1)^n \sinh L}{1 + (n\pi/L)^2}}$$

3.4.4 a) Fourier sine series of a continuous function $f(x)$ is

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)$$

If $f'(x)$ is piecewise smoth, then $f'(x)$ has a Fourier cosine series:

$$f'(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{where:}$$

$$\alpha_0 = \frac{1}{L}\int_0^L f'(x)\,dx = \frac{1}{L} f(x)\Big|_0^L = \frac{1}{L}\left[f(L) - f(0)\right]$$

$$\alpha_n = \frac{2}{L}\int_0^L f'(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$u = \cos(n\pi x/L)$ $\quad dv = f'(x)\,dx$

$du = -\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)$ $\quad v = f(x)$

$$= \frac{2}{L}\left\{ f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{n\pi}{L}\underbrace{\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}_{(L/2)B_n} \right\}$$

$$\alpha_n = \frac{2}{L}\left[f(L)(-1)^n - f(0)\right] + \frac{n\pi}{L}B_n \qquad (\star)$$

$$\Rightarrow f'(x) = \frac{1}{L}\left[f(L) - f(0)\right] + \sum_{n=1}^{\infty}\left\{\frac{2}{L}\left[(-1)^n f(L) - f(0)\right] + \frac{n\pi}{L}B_n\right\}\cos\left(\frac{n\pi x}{L}\right)$$

A Fourier sine series may be term-by-term differentiated if we can verify that:

$$f'(x) \sim \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)B_n\cos\left(\frac{n\pi x}{L}\right) \qquad (\star\star)$$

Comparing $(\star)$ and $(\star\star)$ we note that the Fourier sine series of $f(x)$ can only be differentiated term by term if

$$f(L) - f(0) = 0 \quad \text{and} \quad (-1)^n f(L) - f(0) = 0$$

Both of these conditions hold only if $\boxed{f(0) = f(L) = 0}$ ✓

b) Fourier cosine series of a continuous function $f(x)$ is

$$f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)$$

If $f'(x)$ is piecewise smooth, then $f'(x)$ has a Fourier sine series:

$$f'(x) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where:}$$

$u = \sin(n\pi x/L)$ $\quad dv = f'(x)\,dx$

$du = \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)$ $\quad v = f(x)$

$$\beta_n = \frac{2}{L}\int_0^L f'(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left\{ f(x)\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{n\pi}{L}\underbrace{\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx}_{(2/L)A_n} \right\}$$

$$= \frac{2}{L}\left\{ f(L)\cancel{\sin(n\pi)}^{\,0} - f(0)\cancel{\sin(0)}^{\,0} - \frac{n\pi}{2}A_n \right\}$$

$$\beta_n = -\frac{n\pi}{L}A_n \longrightarrow f'(x) = \sum_{n=1}^{\infty} -\frac{n\pi}{L}A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{f'(x) = \frac{d}{dx}\left(A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\right)}$$

Hence the Fourier cosine series for $f(x)$ can be differentiated term by term to obtain the Fourier sine series for $f(x)$.

3.4.5 $f(x) = \cos\left(\frac{\pi x}{L}\right) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$

Where $B_n = \begin{cases} 0 & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)} & n \text{ even} \end{cases}$ See equation (3.3.13) and excercise 3.3.1(a).

By equation (3.4.13):

$$f'(x) \sim \frac{1}{L}\left[f(L)-f(0)\right] + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}B_n + \frac{2}{L}\left((-1)^n f(L) - f(0)\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$\frac{\pi}{L}\sin\left(\frac{\pi x}{L}\right) \sim \frac{1}{L}\left[\cos\pi - \cos 0\right] + \sum_{n \text{ even}}\left[\frac{n\cancel{\pi}}{L}\frac{4n}{\cancel{\pi}(n^2-1)} + \frac{2}{L}\left((-1)^n\cos\pi - \cos 0\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$\frac{\pi}{\cancel{L}}\sin\left(\frac{\pi x}{L}\right) \sim -\frac{2}{\cancel{L}} + \sum_{n \text{ even}}\left[\frac{4n^2}{(n^2-1)\cancel{L}} + \frac{2}{\cancel{L}}\left(\underbrace{(-1)^{n+1}}_{-1 \text{ since } n \text{ even}} - 1\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$\sin\left(\frac{\pi x}{L}\right) \sim \frac{2}{\pi} - \frac{1}{\pi}\sum_{n \text{ even}}\left[\frac{\cancel{4n^2} - 4(\cancel{n^2}-1)}{(n^2-1)}\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$\boxed{\sin\left(\frac{\pi x}{L}\right) \sim \frac{2}{\pi} - \frac{4}{\pi}\sum_{n \text{ even}}\frac{1}{(n^2-1)}\cos\left(\frac{n\pi x}{L}\right)}$$

It's identical to result from exercice (3.3.4)!

3.4.6 $e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$ This equation is not correct.

It's true on $(0,L)$; however $A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$ is an even periodic function, whereas e^x is not.

We can write $e^x \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$ (★)

$$\frac{d}{dx}(e^x) = e^x \sim -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right) A_n \sin\left(\frac{n\pi x}{L}\right)$$

However, to differentiate once more we must use equation **3.4.13** with $f(x) = e^x \sim -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right) A_n \sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow f'(x) = e^x \sim \frac{1}{L}(e^L - e^0) + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}\left(-\frac{n\pi}{L}A_n\right) + \frac{2}{L}\left((-1)^n e^L - e^0\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$e^x \quad \frac{1}{L}(e^L - 1) + \sum_{n=1}^{\infty}\left[-\frac{n^2\pi^2}{L^2}A_n + \frac{2}{L}\left((-1)^n e^L - 1\right)\right]\cos\left(\frac{n\pi x}{L}\right) \quad (★★)$$

Comparing (★) and (★★): $\boxed{A_0 = \frac{1}{L}(e^L - 1)}$ $\left(\text{we can check tat } A_0 = \frac{1}{L}\int_0^L e^x\,dx\right)$

and:

$$A_n = -\frac{n^2\pi^2}{L^2}A_n + \frac{2}{L}\left((-1)^n e^L - 1\right)$$

$$\left(1 + \frac{n^2\pi^2}{L^2}\right)A_n = \frac{2}{L}\left((-1)^n e^L - 1\right) \longrightarrow \boxed{A_n = \frac{2L\left((-1)^n e^L - 1\right)}{(L^2 + n^2\pi^2)}}$$

We can check that $A_n = \frac{2}{L}\int_0^L e^x \cos\left(\frac{n\pi x}{L}\right)dx$.

3.4.7 The Fourier series of a continuous function $u(x,t)$ is

$$u(x,t) \sim g_0(t) + \sum_{n=1}^{\infty} g_n(t)\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} h_n(t)\sin\left(\frac{n\pi x}{2}\right)$$

with $g_0(t) = \frac{1}{2L}\int_{-L}^{L} u(x,t)\,dx$ $\quad g_n = \frac{1}{L}\int_{-L}^{L} u\cos\frac{n\pi x}{L}\,dx$ $\quad h_n = \frac{1}{L}\int_{-L}^{L} u\sin\left(\frac{n\pi x}{L}\right)dx$

If $\partial u/\partial t$ is piecewise, then it can be expressed as a Fourier series:

$$\frac{\partial u}{\partial t} \sim p_0(t) + \sum_{n=1}^{\infty} p_n(t)\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} q_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

with $p_0(t) = \frac{1}{2L}\int_{-L}^{L} \frac{\partial u}{\partial t}\,dx = \frac{\partial}{\partial t}\left(\frac{1}{2L}\int_{-L}^{L} u\,dx\right) = \frac{\partial g_0(t)}{\partial t}$

$$p_n(t) = \frac{1}{L}\int_{-L}^{L} \frac{\partial u}{\partial t}\cos\frac{n\pi x}{L}\,dx = \frac{\partial}{\partial t}\left(\frac{1}{L}\int_{-L}^{L} u\cos\frac{n\pi x}{L}\,dx\right) = \frac{\partial g_n(t)}{\partial t}$$

$$q_n(t) = \frac{1}{L}\int_{-L}^{L} \frac{\partial u}{\partial t}\sin\left(\frac{n\pi x}{L}\right)dx = \frac{\partial}{\partial t}\left(\frac{1}{L}\int_{-L}^{L} u\sin\frac{n\pi x}{L}\,dx\right) = \frac{\partial h_n(t)}{\partial t}$$

$$\Rightarrow \frac{\partial u}{\partial t} \sim \frac{\partial g_0(t)}{\partial t} + \sum_{n=1}^{\infty} \frac{\partial g_n(t)}{\partial t}\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} \frac{\partial h_n(t)}{\partial t}\sin\frac{n\pi x}{L}$$

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(g_0(t) + \sum_{n=1}^{\infty} g_n(t)\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} h_n(t)\sin\frac{n\pi x}{L}\right)}$$

then the Fourier series of a continuous functions $u(x,t)$ can be diferentiated term by term.

3.4.8 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad \frac{\partial u}{\partial x}(0,t) = 0 \quad \frac{\partial u}{\partial x}(L,t) = 0 \quad u(x,0) = f(x)$.

$$u(x,t) \sim A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos\left(\frac{n\pi x}{L}\right) \rightarrow u(x,0) = f(x) \sim A_0(0) + \sum_{n=1}^{\infty} A_n(0)\cos\left(\frac{n\pi x}{L}\right)$$

with $A_0(0) = \frac{1}{L}\int_0^L f(x)\,dx$ and $A_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

If $u(x,t)$ is continuous, then the Fourier cosine series of $u(x,t)$ can be differentiated term by term

$$\frac{\partial u}{\partial x} \sim \sum_{n=1}^{\infty} \frac{-n\pi}{L} A_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

If $\partial u/\partial x$ is also assumed to be continuous, an equality holds here.

Furthermore, the Fourier sines of $\partial u/\partial x$ can be term-by term diferentiated if $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,0) = 0$. Since those are exactly the boundary conditions on $u(x,t)$, it follows that:

$$\frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n(t)\cos\left(\frac{n\pi x}{L}\right) \qquad (*)$$

If we can also term-by-term differentiate with respect to t, then:

$$\frac{\partial u}{\partial t} \sim \frac{\partial A_0}{\partial t} + \sum_{n=1}^{\infty} \frac{\partial A_n}{\partial t}\cos\left(\frac{n\pi x}{L}\right) \qquad (**)$$

Substituing $(*)$ and $(**)$ in the PDE we have:

$$-\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 A_n(t)\cos\left(\frac{n\pi x}{L}\right) = k\frac{\partial A_0}{\partial t} + k\sum_{n=1}^{\infty}\frac{\partial A_n}{\partial t}\cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow k\frac{\partial A_0}{\partial t} = 0 \quad \text{and} \quad \frac{\partial A_n}{\partial t} = -k\left(\frac{n\pi}{L}\right)^2 A_n(t)$$

$$\Rightarrow A_0(t) = C_0 \quad \text{and} \quad A_n(t) = A_n(0)\, e^{-k(n\pi/L)^2 t}$$

Thus $\boxed{u(x,t) = C_0 + \sum_{n=1}^{\infty} A_n(0)\, e^{-k(n\pi/L)^2 t}\cos\left(\frac{n\pi x}{L}\right)}$

where $C_0 = \frac{1}{L}\int_0^L f(x)\,dx$ and $A_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)$

* 3.4.9 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t)$ with $u(0,t)=0$ and $u(L,t)=0$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial u}{\partial x} = +\sum_{n=1}^{\infty} b_n\frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} b_n\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi x}{L}\right)$$

Since $u(0,t)=0$ and $u(L,t)=0$ and assuming u and $\partial u/\partial t$ continuous functions of x and $\partial^2 u/\partial^2 x^2$ and $\partial u/\partial t$ piecewise smooth.

If we can also term by term differentiate with respect to t:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{db_n(t)}{dt}\sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{db_n}{dt}\sin\left(\frac{n\pi x}{L}\right) = -k\sum_{n=1}^{\infty} b_n\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi x}{L}\right) + q(x,t)$$

$$\sum_{n=1}^{\infty}\left[\frac{db_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n\right]\sin\left(\frac{n\pi x}{L}\right) = q(x,t)$$

Multiplying by $\sin\frac{m\pi x}{L}$ and integrating

$$\sum_{n=1}^{\infty}\left[\frac{db_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n\right]\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\delta_{nm}} = \int_0^L q(x,t)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\left[\frac{db_m}{dt} + k\left(\frac{m\pi}{L}\right)^2 b_n\right]\frac{L}{2} = \int_0^L q(x,t)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\boxed{\frac{db_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n = \frac{2}{L}\int_0^L q(x,t)\sin\left(\frac{n\pi x}{L}\right)dx}$$

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3.4.10 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t)$ with $\frac{\partial u}{\partial x}(0,t)=0$ and $\frac{\partial u}{\partial x}(L,t)=0$

$$u(x,t) = \sum_{n=0}^{\infty} b_n(t)\cos\left(\frac{n\pi x}{L}\right)$$

If u is continuous, then $u(x,t)$ can be differentiated

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} -\frac{n\pi}{L} b_n \sin\frac{n\pi x}{L}$$

If $\partial u/\partial x$ is also continuous and $\frac{\partial u}{\partial x}(0,t)=0$ and $\frac{\partial u}{\partial x}(L,t)=0$ then $\partial u/\partial u$ can be term by term diferentiated

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n \cos\frac{n\pi x}{L}$$

If we can also term by tem differentiate with respect t:

$$\frac{\partial u}{\partial t} = \sum_{n=0}^{\infty}\frac{db_n}{dt}\cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{db_n}{\partial t}\cos\left(\frac{n\pi x}{L}\right) = -k\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n\cos\left(\frac{n\pi x}{L}\right) + q(x,t)$$

$$\frac{db_0}{dt} + \sum_{n=1}^{\infty}\left[\frac{db_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n\right]\cos\left(\frac{n\pi x}{L}\right)dx = q(x,t)$$

Multiplying by $\cos\frac{m\pi x}{L}$ and integrating:

$$\cancel{\int_0^L[\ ]}^{\,0} + \sum_{n=1}^{\infty}\left[\frac{db_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n\right]\underbrace{\int_0^L\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\delta_{nm}} = \int_0^L q(x,t)\cos\left(\frac{m\pi x}{L}\right)dx$$

Only integrating obtain:

$$\left[\frac{db_m}{dt} + k\left(\frac{m\pi}{L}\right)^2 b_m\right]\frac{L}{2} = \int_0^L q(x,t)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$\boxed{b_0 = \frac{1}{L}\int_0^L q(x,t)\,dx}$$

$$\boxed{\frac{db_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n = \frac{2}{L}\int_0^L q(x,t)\cos\left(\frac{n\pi x}{L}\right)dx}$$

3.4.11 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x)$ $\quad u(0,t)=0 \quad u(L,t)=0 \quad u(x,0)=f(x)$

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

Assuming that a continuous solution exists with continuous derivatives and with $u(0,t)=u(L,t)=0$ that permit term by term differentiate Fourier sine series.

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{dC_n}{dt}\sin\left(\frac{n\pi x}{L}\right)$$

If we can also term by term differentiate with respect t.

$$\Rightarrow \sum_{n=1}^{\infty}\frac{dC_n}{dt}\sin\frac{n\pi x}{L} = -k\sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right)$$

$$\rightarrow \sum_{n=1}^{\infty}\left[\frac{dC_n}{dt}+k\left(\frac{n\pi}{L}\right)^2 C_n\right]\sin\left(\frac{n\pi x}{L}\right)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\sum_{n=1}^{\infty}\left[\frac{dC_n}{dt}+k\left(\frac{n\pi}{L}\right)^2 C_n\right]\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\delta_{nm}}=\sum_{n=1}^{\infty} B_n\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\delta_{nm}}$$

$$\frac{dC_n}{dt}+k\left(\frac{n\pi}{L}\right)^2 C_n(t)=B_n$$ Multiplying by the integrating factor $e^{k(n\pi/L)^2 t}$

$$\underbrace{e^{k(n\pi/L)^2 t}\frac{dC_n}{dt}+k\left(\frac{n\pi}{L}\right)^2 e^{k(n\pi/L)^2 t}C_n}=B_n e^{k(n\pi/L)^2 t}$$

$$\frac{d}{dt}\left\{e^{k(n\pi/L)^2 t}C_n(t)\right\}=B_n e^{k(n\pi/L)^2 t}$$

$$\Rightarrow e^{k(n\pi/L)^2 t}C_n(t)=B_n\int e^{k(n\pi/L)^2 t}dt=B_n\frac{L^2}{kn^2\pi^2}e^{k(n\pi/L)^2 t}+C_0$$

$$C_n(t)=B_n\frac{L^2}{kn^2\pi^2}+C_0 e^{-k(n\pi/L)^2 t}$$

if $t=0 \rightarrow C_n(0)=B_n\frac{L^2}{kn^2\pi^2}+C_0 \rightarrow C_0=C_n(0)-B_n\frac{L^2}{k(n\pi)^2}$

$$\Rightarrow C_n(t)=B_n\frac{L^2}{k(n\pi)^2}+\left(C_n(0)-B_n\frac{L^2}{kn^2\pi^2}\right)e^{-k(n\pi/L)^2 t}$$

Sustituing this in $u(x,t)$:

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}\left[B_n\frac{L^2}{k(n\pi)^2}+\left(C_n(0)-B_n\frac{L^2}{k(n\pi)^2}\right)e^{-k(n\pi/L)^2 t}\right]\sin\left(\frac{n\pi x}{L}\right)}$$

Where $C_n(0)$ is given by initial condition $u(x,0)=f(x)$

$\rightarrow u(x,0)=\sum_{n=1}^{\infty}C_n(0)\sin\left(\frac{n\pi x}{L}\right)=f(x)$ Multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating:

$$\sum_{n=1}^{\infty}C_n(0)\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\delta_{nm}}=\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$C_m(0)\frac{L}{2}=\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx \rightarrow \boxed{C_m(0)=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

ESTILO

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* 3.4.12 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-t} + e^{-2t}\cos\left(\frac{3\pi x}{L}\right)$ $\left[\text{Assume } 2 \neq k\left(\frac{3\pi}{L}\right)^2\right]$

with $\frac{\partial u}{\partial t}(0,t) = 0$ and $\frac{\partial u}{\partial t}(L,t) = 0$ and $u(x,0) = f(x)$

$$u(x,t) = C_0(t) + \sum_{n=1}^{\infty} C_n(t)\cos\left(\frac{n\pi x}{L}\right)$$

Assuming u continuous then Fourier cosine series can be term by term differentiated.

$$\frac{\partial u}{\partial x} = -\sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L}$$

If $\partial u/\partial x$ is also continuous and $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$ then $\partial u/\partial t$ can be term by term differentiated

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{L}\right)^2\cos\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial u}{\partial t} = \frac{dC_0}{dt} + \sum_{n=1}^{\infty}\frac{dC_n}{dt}\cos\left(\frac{n\pi x}{L}\right)$$

If we can also term by term differentiate with respect t.

$$\Rightarrow \frac{dC_0}{dt} + \sum_{n=1}^{\infty}\frac{dC_n}{dt}\cos\left(\frac{n\pi x}{L}\right) = -k\sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{L}\right)^2\cos\left(\frac{n\pi x}{L}\right) + e^{-t} + e^{-2t}\cos\left(\frac{3\pi x}{L}\right)$$

$$\frac{dC_0}{dt} + \sum_{n=1}^{\infty}\left[\frac{dC_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 C_n(t)\right]\cos\left(\frac{n\pi x}{L}\right) = e^{-t} + e^{-2t}\cos\left(\frac{3\pi x}{L}\right)$$

from which we have thee cases:

$n=0 \rightarrow \frac{dC_0}{dt} = e^{-t} \rightarrow C_0(t) = -e^{-t} + a_0$

$C_0(0) = -1 + a_0 \rightarrow a_0 = 1 + C_0(0)$

$\rightarrow \boxed{C_0(t) = 1 - e^{-t} + C_0(0)}$

$n=3 \rightarrow \frac{dC_3}{dt} + k\frac{9\pi^2}{L^2}C_3(t) = e^{-2t}$ Multiplying by the integrating factor $e^{k(9\pi^2/L^2)t}$ we have:

$$\underbrace{e^{k(9\pi^2/L^2)t}\frac{dC_3}{dt} + k\frac{9\pi^2}{L^2}e^{k(\ )t}C_3(t)} = e^{-2t}e^{k(9\pi^2/L^2)t}$$

$$\frac{d}{dt}\left\{e^{k(9\pi^2/L^2)t}C_3(t)\right\} = e^{-\left(2-k\frac{9\pi^2}{L^2}\right)t}$$

$$e^{k(9\pi^2/L^2)t}C_3(t) = \left(k\frac{9\pi^2}{L^2} - 2\right)^{-1}e^{\left(k\frac{9\pi^2}{L^2}-2\right)t} + a_3$$

if $t=0 \rightarrow$ $C_3(0) = \left(k\frac{9\pi^2}{L^2} - 2\right)^{-1} + a_3$

$a_3 = -\left(k\frac{9\pi^2}{L^2} - 2\right)^{-1} + C_3(0)$

$$\Rightarrow e^{k(9\pi^2/L^2)t}C_3(t) = \left(k\frac{9\pi^2}{L^2} - 2\right)^{-1}\left[e^{\left(k\frac{9\pi^2}{L^2}-2\right)t} - 1\right] + C_3(0)$$

$$\Rightarrow \boxed{C_3(t) = \left(k\frac{9\pi^2}{L^2} - 2\right)^{-1}\left[e^{-2t} - e^{-k(9\pi^2/L^2)t}\right] + C_3(0)\,e^{-k(9\pi^2/L^2)t}}$$

$$n \neq 0,3: \quad \frac{dC_n(t)}{dt} + k\left(\frac{n\pi}{L}\right)^2 C_n(t) = 0 \rightarrow \boxed{C_n(t) = C_n(0)\,e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

$$\Rightarrow \boxed{\begin{aligned} u(t,x) = (1-e^{-t}) + C_0(0) + \left\{\left(k\frac{9\pi^2}{L^2} - 2\right)^{-1}\left[e^{-2t} - e^{-k\frac{9\pi^2}{L^2}t}\right] + C_3(0)\,e^{-k\frac{9\pi^2}{L^2}t}\right\} \\ \times \cos\left(\frac{3\pi x}{L}\right) + \sum_{n\neq 0,3} C_n(0)\,e^{-k\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right) \end{aligned}}$$

Where $C_0(0)$, $C_3(0)$ and $C_n(0)$ are given by initial condition:

$$f(x) = u(0,x) = C_0(0) + C_3(0)\cos\left(\frac{3\pi x}{2}\right) + \sum_{n\neq 0,3} C_n(0)\cos\left(\frac{n\pi x}{L}\right)$$

Integrating $\Rightarrow$ $$\int_0^L f(x)\,dx = C_0(0)\int_0^L dx + C_3(0)\cancelto{0}{\int_0^L \cos\left(\frac{3\pi x}{L}\right)dx} + \sum_{n\neq 0,3} C_n(0)\cancelto{0}{\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx}$$

$$\Rightarrow \boxed{C_0(0) = \frac{1}{L}\int_0^L f(x)\,dx}$$

Multiplying by $\cos\left(\frac{3\pi x}{L}\right)$ and integrating:

$$\int_0^L f(x)\cos\left(\frac{3\pi x}{L}\right)dx = C_0(0)\cancelto{0}{\int_0^L \cos\frac{3\pi x}{L}\,dx} + C_3(0)\overbrace{\int_0^L \cos^2\left(\frac{3\pi x}{L}\right)}^{(L/2)} + \cancelto{0}{\sum_{n\neq 0,3}^{\infty} C_n(0)\underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{3\pi x}{L}\right)dx}_{(L/2)\delta_{n3}}}$$

$$\Rightarrow \boxed{C_3(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{3\pi x}{L}\right)dx}$$

Multiplying by $\cos\left(\frac{m\pi x}{L}\right)$ $m \neq 0,3$ and integrating:

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = C_0(0)\cancelto{0}{\int_0^L \cos\left(\frac{m\pi x}{L}\right)dx} + C_3(0)\cancelto{0}{\int_0^L \cos\left(\frac{3\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx} + \sum_{n\neq 0,3}^{\infty} C_n(0)\underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\delta_{nm}}$$

$m \neq 3$

$$= C_m(0)\,\frac{L}{2} \qquad m \neq 0,3.$$

$$\Rightarrow \boxed{C_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi x}{L}\right)dx} \qquad n \neq 0,3$$

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(3.4.13) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t) = A(t) \quad u(L,t) = 0 \quad u(x,0) = g(x)$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

u cannot be differentiated term by term because $u(0,t) \neq 0$.

$$\frac{\partial u}{\partial x} \sim \frac{1}{L}\left[0 - A(t)\right] + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}b_n(t) + \frac{2}{L}\left((-1)^n \cdot 0 - A(t)\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

where we used (3.4.13) from book. If we suposse $\partial u/\partial x$ continuous then the Fourier cosine series of $\partial u/\partial x$ can be differentiated term by term.

$$\frac{\partial^2 u}{\partial x^2} \sim \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}b_n(t) - \frac{2}{L}A(t)\right]\left(-\frac{n\pi}{L}\right)\sin\left(\frac{n\pi}{L}\right)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{db_n}{dt}\sin\left(\frac{n\pi x}{L}\right)$$

If we can also term by term differentiate with respect t.

$$\Rightarrow \sum_{n=1}^{\infty}\frac{db_n}{dt}\sin\left(\frac{n\pi x}{L}\right) = -k\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)\left[\frac{n\pi}{L}b_n(t) - \frac{2}{L}A(t)\right]\sin\left(\frac{n\pi}{L}\right)$$

$$\boxed{\frac{db}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n(t) = \frac{2kn\pi}{L^2}A(t)}$$

The solution is equal to Homogeneus solution + (particular solution)

$$\Rightarrow b_n(t) = b_n(0)e^{-(n\pi/L)^2 kt} + \underbrace{A_p(t)}_{\text{particular solution}}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[b_n(0)e^{-(n\pi/L)^2 kt} + A_p(t)\right]\sin\left(\frac{n\pi x}{L}\right)}$$

$$g(x) = u(x,0) = \sum_{n=1}^{\infty}\left[b_n(0) + A_p(0)\right]\sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate.

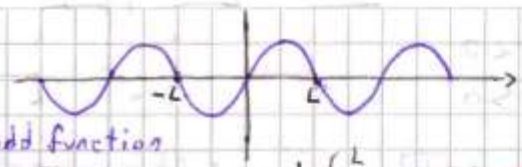
$$\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}\left[b_n(0) + A_p(0)\right]\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\delta_{nm}}$$

$$= \left[b_m(0) + A_p(0)\right]\frac{L}{2}.$$

$$\Rightarrow \boxed{b_n(0) = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx - A_p(0)}$$

where $A_p(0)$ is the particular solution evaluated in $t=0$.

* c) $f(x) = \sin \frac{\pi x}{L}$

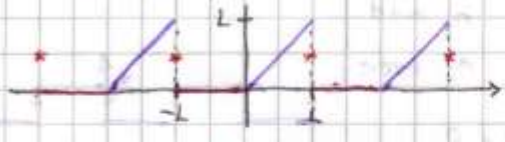


$$a_0 = \frac{1}{2L}\int_{-L}^{L} \cancel{\sin\left(\frac{\pi x}{L}\right)}^{\,0}\, dx \rightarrow \boxed{a_0 = 0}$$ odd function

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\cancel{\sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}}}_{\text{odd function}}^{\,0}\, dx = \boxed{0}$$

$$b_n = \frac{1}{L}\underbrace{\int_{-L}^{L} \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx}_{L\delta_{1m}} = \frac{1}{\cancel{L}}(\cancel{L}\delta_{1m}) \rightarrow \boxed{b_1 = 1} \quad b_n = 0 \text{ if } n \neq 1$$

d) $f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$



$$a_0 = \frac{1}{2L}\left\{\cancel{\int_{-L}^{0} 0\, dx}^{\,0} + \int_0^L x\, dx\right\} = \frac{1}{2\cancel{L}}\frac{L^{\cancel{2}}}{2} = \boxed{\frac{L}{4}}$$

$$a_n = \frac{1}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx \qquad \begin{matrix} u = x & dv = \cos(n\pi x/L)\,dx \\ du = dx & v = \frac{L}{n\pi}\sin(n\pi x/L) \end{matrix}$$

$$= \frac{1}{L}\left\{\frac{L}{n\pi}x\cancel{\sin\frac{n\pi x}{L}}^{\,0}\Big|_0^L - \frac{L}{n\pi}\int \sin\frac{n\pi x}{L}dx\right\} = \frac{L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

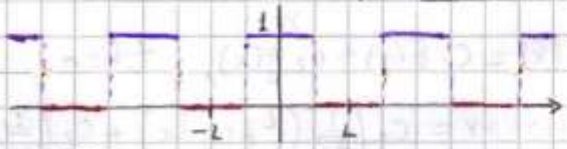
$$= \frac{L}{(n\pi)^2}\left[(-1)^n - 1\right] = \boxed{\begin{cases} -2L/(n\pi)^2 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}}$$

$$b_n = \frac{1}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx \qquad \begin{matrix} u = x & dv = \sin(n\pi x/L)\,dx \\ du = dx & v = -\left(\frac{L}{n\pi}\right)\cos(n\pi x/L) \end{matrix}$$

$$= \frac{1}{L}\left\{-\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$= \frac{1}{L}\left\{-\frac{L}{n\pi}\left[L(-1)^n\right] + \left(\frac{L}{n\pi}\right)^2 \cancel{\sin\left(\frac{n\pi x}{L}\right)}^{\,0}\Big|_0^L\right\} = \boxed{\frac{L}{n\pi}(-1)^{n+1}}$$

e) $f(x) = \begin{cases} 1 & |x| < L/2 \\ 0 & |x| > L/2 \end{cases}$



$$a_0 = \frac{1}{2L}\int_{-L/2}^{L/2} dx = \frac{1}{2L}(L) = \boxed{\frac{1}{2}}$$

$$a_n = \frac{1}{L}\int_{-L/2}^{L/2}\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\cancel{L}}\frac{\cancel{L}}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L/2}^{L/2} = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right)\right]$$

$$\boxed{a_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} (2/n\pi) & \text{if } n = 4K+1 \\ 0 & \text{if } n = 2K \\ -(2/n\pi) & \text{if } n = 4K+3 \end{cases} \quad K \in \mathbb{Z}}$$

$$b_n = \frac{1}{L}\cancel{\int_{-L/2}^{L/2}\sin\left(\frac{n\pi x}{L}\right)dx}^{\,0} \rightarrow \boxed{b_n = 0}$$ odd function

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3.5.1 a) From (3.3.11) and (3.3.12) OR excercise 3.3.3 c)

$$x \simeq \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right) \qquad (3.3.12)$$

and $$\frac{x^2}{2} = \frac{L}{2}x - \frac{4L^2}{\pi^3}\sum_{n\,odd}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi x}{L}\right) \qquad (3.5.6)$$

Substituing (3.3.12) in (3.5.6):

$$\frac{x^2}{2} = \frac{L^2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right) - \frac{4L^2}{\pi^3}\sum_{n\,odd}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi x}{L}\right)$$

$$= -\frac{L^2}{\pi}\left\{\sum_{n\,even}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{L}\right) + \sum_{n\,odd}^{\infty}\left[\frac{(-1)^n}{n} + \frac{4}{\pi^2 n^3}\right]\sin\frac{n\pi x}{L}\right\}$$

$$= -\frac{L^2}{\pi}\left\{\sum_{n\,even}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right) + \sum_{n\,odd}^{\infty}\left(\frac{-\pi^2 n^2 + 4}{\pi^2 n^3}\right)\sin\left(\frac{n\pi x}{L}\right)\right\}$$

then $x^2 \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ with $\boxed{b_n = \begin{cases} -\frac{2L^2}{\pi n} & n \text{ even} \\ \frac{2L^2(\pi^2 n^2 - 4)}{\pi^3 n^3} & n \text{ odd}\end{cases}}$

We can verify this doing $\frac{2}{L}\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right)$

b) The series obtained in item a) is valid for $0 \le x \le L$ because (3.5.6) was derived from sine series (3.5.3) in the book, valid for $0 \le x \le L$.

c) If $x^2 \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) dx$ we can integrate for obtain:

$$\frac{x^3}{3} = -\sum_{n=1}^{\infty} b_n\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) + C_0 \qquad C_0 = \frac{1}{L}\int_0^L \frac{x^3}{3}dx = \frac{L^3}{12}$$

$$\Rightarrow \quad \frac{x^3}{3} = -\sum_{n=1}^{\infty}\left(\frac{L}{n\pi}\right) b_n \cos\left(\frac{n\pi x}{L}\right) + \frac{L^3}{12} \qquad \text{where:}$$

$$-\frac{L}{n\pi} b_n = \begin{cases} +\frac{2L^3}{\pi^2 n^2} & n \text{ even} \\ \frac{2L^3(-\pi^2 n^2 + 4)}{\pi^4 n^4} & n \text{ odd}\end{cases} = 2L^3\frac{(-1)^n}{(n\pi)^2} + 4L^3\frac{[1-(-1)^n]}{(n\pi)^4}$$

$$\Rightarrow \quad \boxed{x^3 = \frac{L^3}{4} + \frac{2L^3}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2} + \frac{2[1-(-1)^n]}{n^4\pi^2}\right]\cos\left(\frac{n\pi x}{L}\right)}$$

Valid for $0 \le x \le L$ since derivation was did from (3.5.12)

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3.5.2 a) $$x \simeq \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right) \qquad (3.3.12)$$

integrating: $$\frac{x^2}{2} = \cancel{-}\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n\cancel{+1}}}{n}\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) + C_0 \qquad C_0 = \frac{1}{L}\int_0^L \frac{x^2}{2}dx = \frac{L^2}{6}$$

$$\boxed{x^2 = \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^2}{3}}$$

b) Integrating from 0 to x yields:

$$\frac{x^3}{3} = \frac{4L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{L}\right) + \frac{L^2}{3}x$$ using here (3.3.12):

$$\frac{x^3}{3} = \frac{4L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{L}\right) + \frac{L^2}{3}\left[\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\boxed{x^3 = \sum_{n=1}^{\infty}\frac{2L^3}{\pi n}(-1)^n\left[\frac{6}{\pi^2 n^2} - 1\right]\sin\left(\frac{n\pi x}{L}\right)}$$

3.5.3 Integrating again the last expression $\qquad C_0 = \frac{1}{L}\int_0^L \frac{x^4}{4} = \frac{L^4}{20}$

$$\frac{x^4}{4} = -\sum_{n=1}^{\infty}\frac{2L^4}{(\pi n)^2}(-1)^n\left[\frac{6}{\pi^2 n^2} - 1\right]\cos\left(\frac{n\pi x}{L}\right) + C_0$$

$$= -\sum_{n=1}^{\infty}\frac{2L^4}{(\pi n)^2}(-1)^n\left[\frac{6}{\pi^2 n^2} - 1\right]\cos\left(\frac{n\pi x}{L}\right) + \frac{L^4}{20}$$

Integrating: $$\frac{x^5}{20} = -\sum_{n=1}^{\infty}\frac{2L^4}{(\pi n)^3}(-1)^n\left[\frac{6}{\pi^2 n^2} - 1\right]\sin\left(\frac{n\pi x}{L}\right) + \frac{L^4}{20}x$$

$$= -\sum_{n=1}^{\infty}\frac{2L^4}{(\pi n)^3}(-1)^n\left[\frac{6}{\pi^2 n^2} - 1\right]\sin\frac{n\pi x}{L} + \frac{L^4}{20}\sum_{n=1}^{\infty}\frac{2L}{\pi}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{x^5 = 2L^5\sum_{n=1}^{\infty}\left[-\frac{120}{(\pi n)^5} + \frac{20}{(\pi n)^3} - \frac{1}{\pi n}\right](-1)^n\sin\left(\frac{n\pi x}{L}\right)}$$

So:

$$x = 2L\sum_{n=1}^{\infty}\left[-\frac{1}{n\pi}\right](-1)^n\sin(n\pi x/L)$$

$$x^3 = 2L^3\sum_{n=1}^{\infty}\left[\frac{}{(n\pi)^3} - \frac{1}{n\pi}\right](-1)^n\sin(n\pi x/L)$$

$$x^5 = 2L^5\sum_{n=1}^{\infty}\left[-\frac{5!}{(n\pi)^5} + \frac{5!/3!}{(n\pi)^3} - \frac{5!/5!}{(n\pi)}\right](-1)^n\sin(n\pi x/L)$$

$\vdots$

$$\boxed{x^m = 2L^m\sum_{n=1}^{\infty}\left[-\frac{1}{n\pi} + \frac{m!/3!}{(n\pi)^3} - \frac{m!/5!}{(n\pi)^5} + \cdots + \frac{m!}{(n\pi)^m}\right]\sin(n\pi x/L)}$$

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* **3.5.4** $\cosh x \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ (*)

Fourier sine series of a continuous function, cannot in general be differentiated term by term. We need use (3.4.13):

a)

$$\sinh x \sim \frac{1}{L}(\cosh L - 1) + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L} b_n + \frac{2}{L}\left((-1)^n \cosh L - 1\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

Since this is a cosine series, it may be differentiated:

$$\cosh x \sim -\sum_{n=1}^{\infty}\left[\left(\frac{n\pi}{L}\right)^2 b_n + \frac{2n\pi}{L^2}\left((-1)^n \cosh L - 1\right)\right]\sin\left(\frac{n\pi x}{L}\right) \quad (**)$$

Comparing (*) with (**):

$$b_n = -\left(\frac{n\pi}{L}\right)^2 b_n - \frac{2n\pi}{L^2}\left((-1)^n \cosh L - 1\right)$$

$$\left[1 + \left(\frac{n\pi}{L}\right)^2\right] b_n = -\frac{2n\pi}{L^2}\left((-1)^n \cosh L - 1\right)$$

$$\boxed{b_n = \frac{2n\pi}{L^2}\,\frac{1-(-1)^n \cosh L}{\left[1+(n\pi/L)^2\right]}}$$

b) Integrating (*) yields:

$$\sinh x = \sum_{n=1}^{\infty}\left(-\frac{L}{n\pi}\right) b_n \cos\left(\frac{n\pi x}{L}\right) + A_0 \qquad A_0 = \frac{1}{L}\int_0^L \sinh x = \frac{1}{L}(\cosh L - 1)$$

Integrating from 0 to x

$$\int_0^x \sinh x\, dx = \sum_{n=1}^{\infty} -\left(\frac{L}{n\pi}\right)^2 b_n \sin\left(\frac{n\pi x}{L}\right) + A_0 x$$

$$\cosh x - 1 = \sum_{n=1}^{\infty} -\left(\frac{L}{n\pi}\right)^2 b_n \sin\left(\frac{n\pi x}{L}\right) + \frac{1}{L}(\cosh L - 1)\, x$$

Using Fourier sine series of x and 1 from (3.3.12) and (3.3.10)

$$\cosh x - \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\left[1-(-1)^n\right]\right) \times \left(\sin\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} -\left(\frac{L}{n\pi}\right)^2 b_n \sin\left(\frac{n\pi x}{L}\right) + \frac{1}{L}(\cosh L - 1)\sum_{n=1}^{\infty}\frac{2L}{\pi}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)$$

$$\cosh x = \sum_{n=1}^{\infty}\left\{-\left(\frac{L}{n\pi}\right)^2 b_n + \frac{2}{n\pi}\left[1-(-1)^n\right] + \frac{1}{L}(\cosh L - 1)\frac{2L}{n\pi}(-1)^{n+1}\right\}\sin\left(\frac{n\pi x}{L}\right) \quad (***)$$

Comparing (*) and (***) we can obtain:

$$\left[1 + \left(\frac{L}{n\pi}\right)^2\right] = \frac{2}{n\pi}\left[1-(-1)^n\right] + \frac{1}{L}(\cosh L - 1)\frac{2L}{n\pi}(-1)^{n+1}$$

OR $\boxed{b_n = \frac{2}{n\pi}\,\frac{1-(-1)^n \cosh L}{1+(L/n\pi)^2}}$ Similar to a)

with $G(-L) = G(L) = 0$ $\qquad \frac{dF}{dx} = f(x)$

3.5.5 $G(x) \equiv F(x) - a_0(x+L)$ with $F(x) = \int_{-L}^{x} f(t)\,dt$

$$G(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

$$B_n = \frac{1}{L}\int_{-L}^{L} G(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{L}\left[F(x) - a_0(x+L)\right]\sin\frac{n\pi x}{L}\,dx$$

$$\int_{-L}^{L} \underbrace{x}_{u}\,\underbrace{\sin\frac{n\pi x}{L}\,dx}_{dv} = -\frac{L}{n\pi}\,x\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{L}{n\pi}\cancelto{0}{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx}$$

$du = dx \qquad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)dx$

$$\int_{-L}^{L} \underbrace{\left[F(x) - a_0 L\right]}_{u}\,\underbrace{\sin\frac{n\pi x}{L}\,dx}_{dv} = -\frac{L}{n\pi}\left(F(x) - a_0 L\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{L}{n\pi}\underbrace{\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx}_{a_n L}$$

$du = \frac{dF}{dx}\,dx = f(x)\,dx \qquad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow B_n = \frac{1}{L}\left\{-\frac{L}{n\pi}\underbrace{\left[F(x) - a_0(x+L)\cos\frac{n\pi x}{L}\right]_{x=-L}^{x=L}}_{G(x)} + \frac{L^2}{n\pi}a_n\right\}$$

$$= \frac{1}{L}\left\{-\frac{L}{n\pi}\cancelto{0}{\left[G(L) - G(-L)\right]} + \frac{L^2}{n\pi}a_n\right\} \rightarrow \boxed{B_n = \frac{a_n}{n\pi/L}}$$

$G(L) = G(-L) = 0$

3.5.6 $x = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos(\pi x/L) + \frac{1}{3^2}\cos\frac{3\pi x}{L} + \frac{1}{5^2}\cos\frac{5\pi x}{L} + \cdots\right) \quad 0 \le x \le L$

if $x = 0 = \frac{L}{2} - \frac{4L}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \rightarrow \boxed{\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi^2}{8}}$ (*)

from exercise (3.5.2) $\quad x^2 = \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^2}{3}$

$\Rightarrow$ in $x = 0 \rightarrow 0 = \frac{4L^2}{\pi^2}\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} + \cdots\right) + \frac{L^2}{3}$

$\rightarrow \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) = -\frac{\pi^2}{12} + \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = -\frac{\pi^2}{12} + \frac{\pi^2}{8}$

$\boxed{\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) = \frac{\pi^2}{24}}$ (**)

from (*) and (**) $\longrightarrow$ $\boxed{\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots\right) = \frac{\pi^2}{6}}$

3.5.7 $\frac{x^2}{2} = \frac{L}{2}x - \frac{4L^2}{\pi^3}\left(\sin\frac{\pi x}{L} + \frac{1}{3^3}\sin\frac{3\pi x}{L} + \frac{1}{5^3}\sin\frac{5\pi x}{L} + \cdots\right)$

at $x = L/2 \Rightarrow \frac{L^2}{8} = \frac{L^2}{4} - \frac{4L^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right)$

$$-\frac{\cancel{L^2}}{8} = -\frac{4\cancel{L^2}}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right)$$

$$\boxed{\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right) = \frac{\pi^3}{32}}$$

* 3.6.1 $f(x) = \begin{cases} 0 & x < x_0 \\ 1/\Delta & x_0 < x < x_0 + \Delta \\ 0 & x > x_0 + \Delta \end{cases}$ $\quad x_0 > -L$, $\quad x_0 + \Delta < L$

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{in\pi x/L}\,dx = \frac{1}{2L}\int_{x_0}^{x_0+\Delta} \frac{1}{\Delta} e^{in\pi x/L}\,dx$$

$$= \frac{1}{2\cancel{L}\Delta}\frac{\cancel{L}}{in\pi} e^{in\pi x/L}\Big|_{x_0}^{x_0+\Delta} = \frac{1}{2in\pi\Delta} e^{in\pi x_0/L}\left(e^{in\pi\Delta/L} - 1\right)$$

$$= \frac{1}{2in\pi\Delta} e^{in\pi(x_0+\Delta/2)/L}\left(e^{in\pi\Delta/2L} - e^{-in\pi\Delta/2L}\right)$$

$$\boxed{C_n = \frac{1}{n\pi\Delta} e^{in\pi(x_0+\Delta/2)/L} \sin(n\pi\Delta/2L)}$$

3.6.2 $C_{-n} = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\,dx$

$$\overline{C_{-n}} = \frac{1}{2L}\int_{-L}^{L} \overline{f(x) e^{-in\pi x/L}}\,dx$$

Since $\overline{f(x)} = f(x)$ for $f(x)$ real.

$$= \underbrace{\frac{1}{2L}\int_{-L}^{L} f(x) e^{in\pi x/L}\,dx}_{C_n}$$

$$\Rightarrow \boxed{\overline{C_{-n}} = C_n}$$

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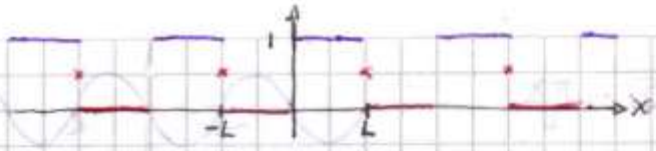
f) $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$



$$a_0 = \frac{1}{2L}\int_0^L dx = \boxed{\frac{1}{2}} \qquad a_n = \frac{1}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx = \frac{-1}{L}\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$b_n = \frac{1}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = \frac{-1}{\cancel{L}}\left(\frac{\cancel{L}}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L = \frac{-1}{n\pi}\left[(-1)^n - 1\right]$$

$$\boxed{b_n = \begin{cases} (2/n\pi) & n \text{ odd} \\ 0 & n \text{ even} \end{cases}}$$

g) $g(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$

$$a_0 = \frac{1}{2L}\left\{\int_{-L}^0 dx + 2\int_0^L dx\right\} = \frac{1}{2\cancel{L}}\left[3\cancel{L}\right] = \boxed{\frac{3}{2}}$$

$$a_n = \frac{1}{L}\left\{\int_{-L}^0 \cos\left(\frac{n\pi x}{L}\right)dx + 2\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx\right\} = \frac{1}{\cancel{L}}\frac{\cancel{L}}{n\pi}\left\{\sin\frac{n\pi x}{L}\Big|_{-L}^0 + 2\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L\right\}$$

$$b_n = \frac{1}{L}\left\{\int_{-L}^0 \sin\left(\frac{n\pi x}{L}\right)dx + 2\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx\right\} = \frac{-1}{\cancel{L}}\frac{\cancel{L}}{n\pi}\left\{\cos\frac{n\pi x}{L}\Big|_{-L}^0 + 2\cos\frac{n\pi x}{L}\Big|_0^L\right\}$$

$$= -\frac{1}{n\pi}\left\{1 - (-1)^n + 2(-1)^n - 2\right\} = -\frac{1}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} (2/n\pi) & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Note that $g(x) = 1 + f(x)$ if $f(x)$ is defined like in **f)** where from **f)** $f(x) = \frac{1}{2} + \sum_{n \text{ odd}}^{\infty} \frac{2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow \boxed{g(x) = \frac{3}{2} + \sum_{n=\text{odd}}^{\infty} \frac{2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)} \rightarrow a_0 = \frac{3}{2} \quad b_n = \frac{2}{n\pi} \quad n \text{ odd}$$

3.2.3 Let $h(x) = c_1 f(x) + c_2 g(x)$, then

$$a_0 = \frac{1}{2L}\int_{-L}^L h(x)\,dx = c_1\left(\frac{1}{2L}\int_{-L}^L f(x)\,dx\right) + c_2\left(\frac{1}{2L}\int_{-L}^L g(x)\,dx\right) = c_1 a_0^f + c_2 a_0^g$$

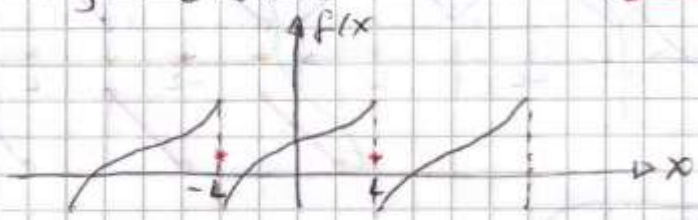
$$a_n = \frac{1}{L}\int_{-L}^L h(x)\cos\left(\frac{n\pi x}{L}\right)dx = c_1\underbrace{\left(\frac{1}{L}\int_{-L}^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right)}_{a_n^f} + c_2\underbrace{\left(\frac{1}{L}\int_{-L}^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx\right)}_{a_n^g}$$

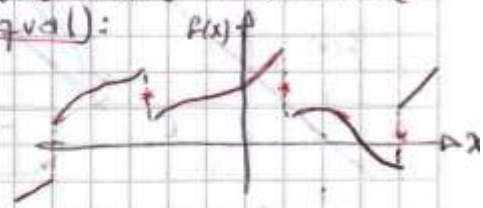
$$b_n = \frac{1}{L}\int_{-L}^L h(x)\sin\left(\frac{n\pi x}{L}\right)dx = c_1\underbrace{\left(\frac{1}{L}\int_{-L}^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)}_{b_n^f} + c_2\underbrace{\left(\frac{1}{L}\int_{-L}^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)}_{b_n^g}$$

$$\Rightarrow h(x) = c_1\left[a_0^f + \sum_{n=1}^{\infty}\left(a_n^f\cos\left(\frac{n\pi x}{L}\right) + b_n^f\sin\left(\frac{n\pi x}{L}\right)\right)\right] + c_2\left[a_0^g + \sum_{n=0}^{\infty} a_n^g\cos(\,) + b_n^g\sin(\,)\right]$$

$$\boxed{h(x) = c_1\,[\text{Fourier Series of } f(x)] + c_2\,[\text{Fourier Series of } g(x)]}$$

3.2.4 A function $f(x)$ is **piecewise smooth** (on some interval) if the interval can be broken up into pices (or sections) such that in each pice the function $f(x)$ is continuous and its derivative df/dx is also continuous. The function $f(x)$ may not be continuous but the only kind of discontinuity allowed is a finite number of jump discontinuities. A function $f(x)$ has a **jump discontinuity** at a point $x = x_0$ if the limit from the left $[f(x_0^-)]$ and the limit from the right $[f(x_0^+)]$ both exist (and are unequal):





Let $\lim_{x \to -L^+} f(x) = f(-L^+)$ and $\lim_{x \to L^-} f(x) = f(L^-)$

then the Fourier series for $f(x)$ converges to $\dfrac{f(-L^+) + f(L^-)}{2}$ at $x = \pm L$. by Convergence theorem for Fourier Series. (See Pg 92 of book).

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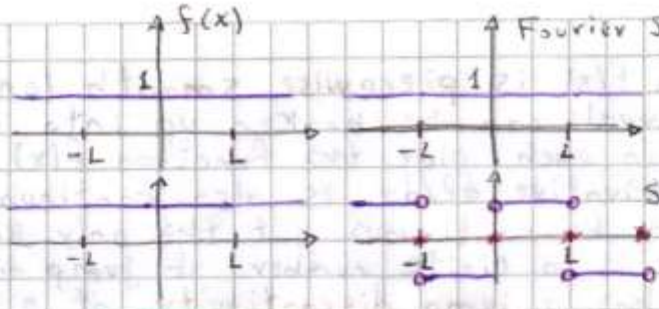
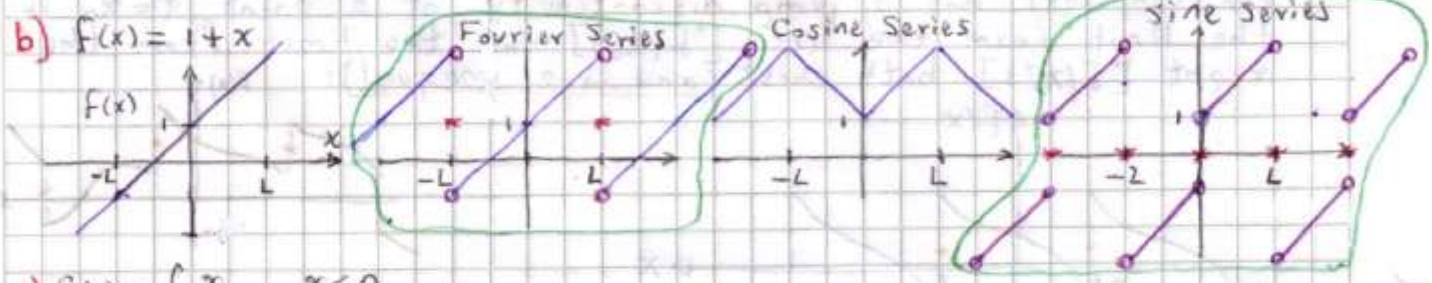
3.3.1 $f(x) = 1$

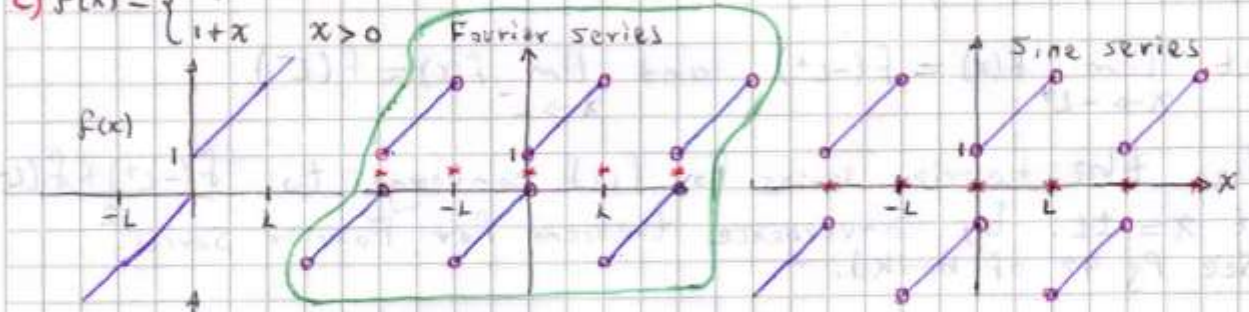
a)

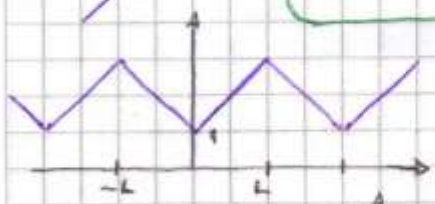


b) $f(x) = 1 + x$

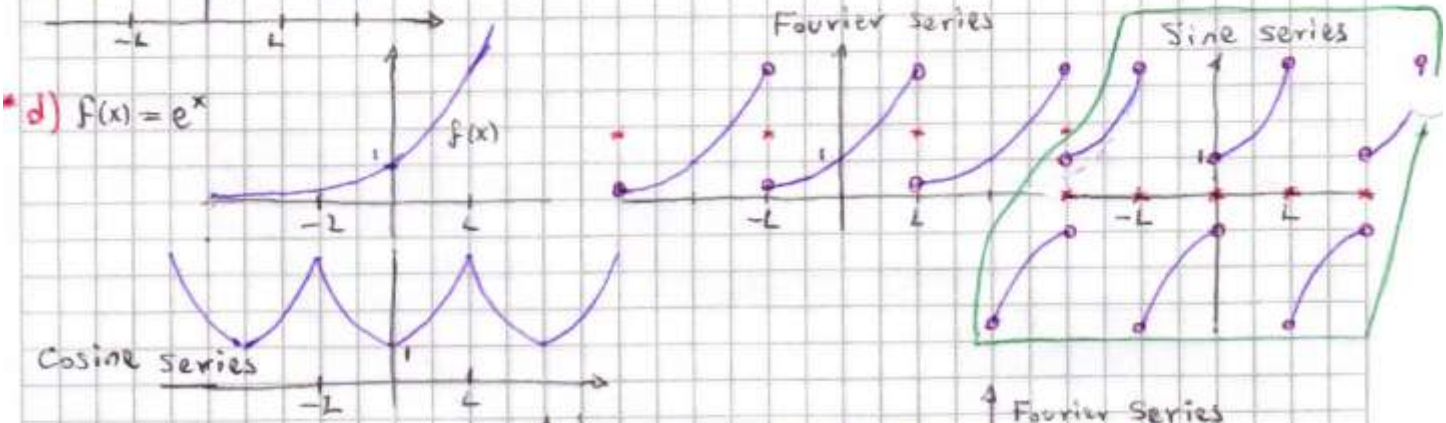


c) $f(x) = \begin{cases} x & x < 0 \\ 1 + x & x > 0 \end{cases}$

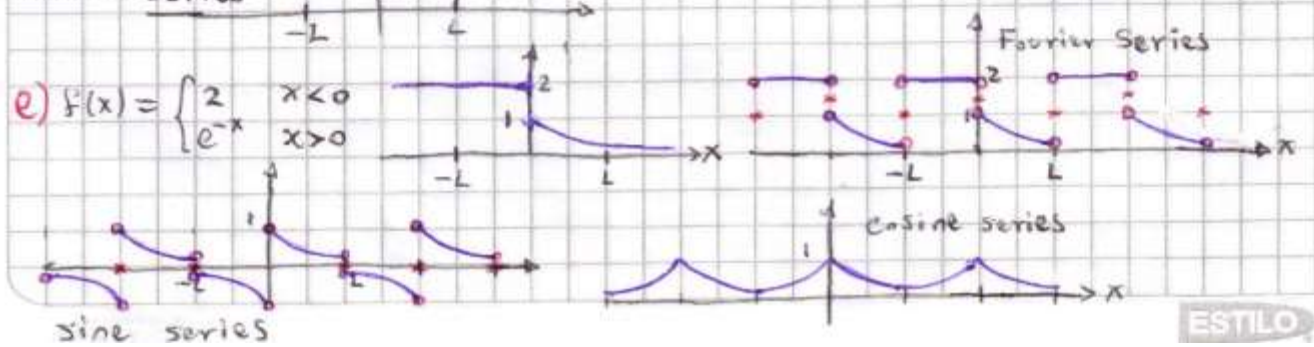




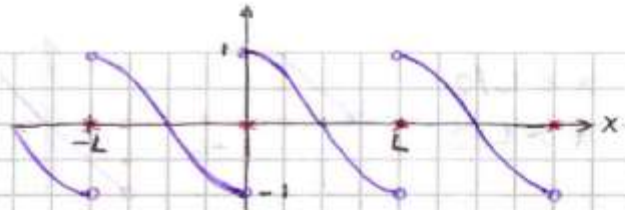
d) $f(x) = e^x$



e) $f(x) = \begin{cases} 2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$



3.3.2 $f(x) = \cos\left(\frac{\pi x}{L}\right)$



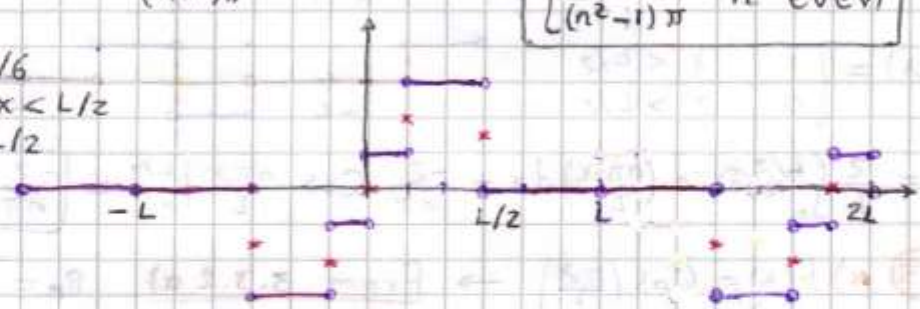
a)

$$B_n = \frac{2}{L}\int_0^L \cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{\cancel{2}}{L}\int_0^L \frac{1}{\cancel{2}}\left[\sin\frac{(n-1)\pi x}{L} + \sin\frac{(n+1)\pi x}{L}\right]dx$$

$$= \frac{-1}{(n-1)\pi}\cos\frac{(n-1)\pi x}{L}\Big|_0^L + \frac{-1}{(n+1)\pi}\cos\frac{(n+1)\pi x}{L}\Big|_0^L$$

$$= \frac{-1}{(n-1)\pi}\left[(-1)^{n-1} - 1\right] + \frac{-1}{(n+1)\pi}\left[(-1)^{n+1} - 1\right] = \boxed{\begin{cases} 0 & n \text{ odd} \\ \dfrac{4n}{(n^2-1)\pi} & n \text{ even} \end{cases}}$$

b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

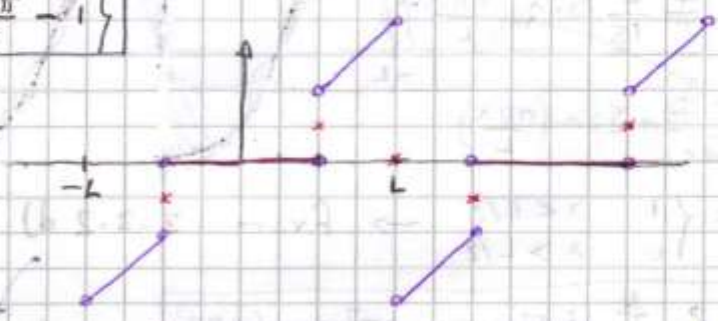


$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left\{\int_0^{L/6}\sin\left(\frac{n\pi x}{L}\right)dx + 3\int_{L/6}^{L/2}\sin\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$= \frac{-2}{n\pi}\left\{\cos\frac{n\pi x}{L}\Big|_0^{L/6} + 3\cos\frac{n\pi x}{L}\Big|_{L/6}^{L/2}\right\} = -\frac{2}{n\pi}\left\{\left[\cos\frac{n\pi}{6} - 1\right] + 3\left[\cos\frac{n\pi}{2} - \cos\frac{n\pi}{6}\right]\right\}$$

$$\boxed{B_n = -\frac{2}{n\pi}\left\{\cos\frac{n\pi}{2} - 2\cos\frac{n\pi}{6} - 1\right\}}$$

c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$



$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_{L/2}^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

$u = x$ $\quad dv = \sin(n\pi x/L)$

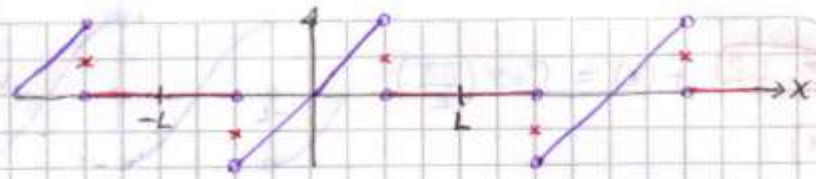
$du = dx$ $\quad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$$= \frac{2}{L}\left\{-\frac{L}{n\pi}x\cos\frac{n\pi x}{L}\Big|_{L/2}^L + \frac{L}{n\pi}\int_{L/2}^L \cos\frac{n\pi x}{L}dx\right\}$$

$$= \frac{2}{n\pi}\left\{-L(-1)^n + \frac{L}{2}\cos\frac{n\pi}{2} + \frac{L}{n\pi}\left[\cancel{\sin n\pi}^{\,0} - \sin\frac{n\pi}{2}\right]\right\}$$

$$\boxed{B_n = \frac{2L}{n\pi}\left\{(-1)^{n+1} + \frac{1}{2}\cos\frac{n\pi}{2} - \frac{1}{n\pi}\sin\frac{n\pi}{2}\right\}}$$

e) $f(x) = \begin{cases} x & x < L/2 \\ 0 & x > L/2 \end{cases}$

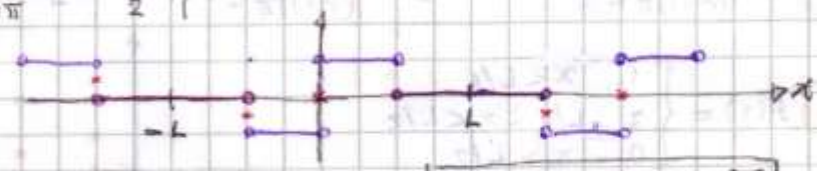


$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx \qquad \begin{matrix} u = x & dv = \sin\frac{n\pi x}{L} \\ du = dx & v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right) \end{matrix}$$

$$= \frac{2}{L}\left\{ -\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2} + \frac{L}{n\pi}\int_0^{L/2}\cos\left(\frac{n\pi x}{L}\right)dx \right\}$$

$$= \frac{2}{n\pi}\left\{ -\frac{L}{2}\cos\frac{n\pi}{2} + \frac{L}{n\pi}\sin\frac{n\pi}{2} \right\}$$

d) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

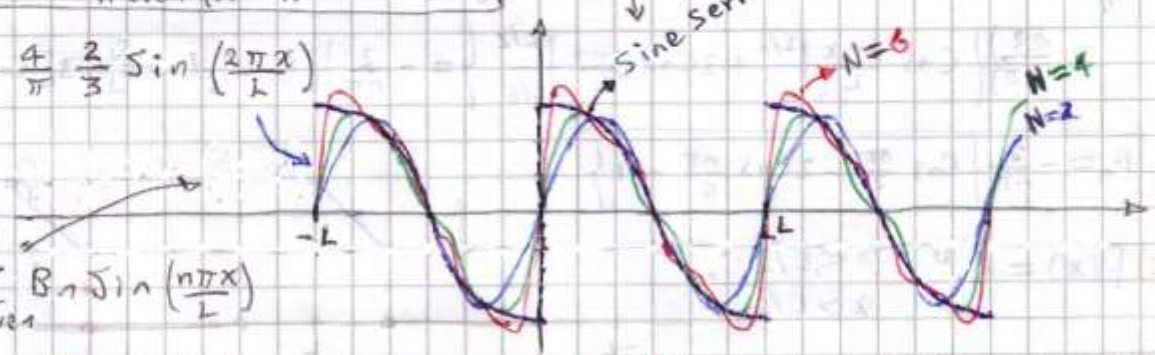


$$B_n = \frac{2}{L}\int_0^{L/2}\sin\left(\frac{n\pi x}{L}\right)dx = \frac{-2}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^{L/2} = \boxed{\frac{-2}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right]}$$

3.3.3 a) $f(x) = \cos\left(\frac{\pi x}{L}\right) \rightarrow$ from 3.3.2 a) $\quad B_n = \begin{cases} 0 & n \text{ odd} \\ \frac{4n}{(n^2-1)\pi} & n \text{ even} \end{cases}$

$$\Rightarrow \boxed{f(x) \simeq \frac{4}{\pi}\sum_{n \text{ even}}^{N} \frac{n}{(n^2-1)}\sin\left(\frac{n\pi x}{L}\right)}$$

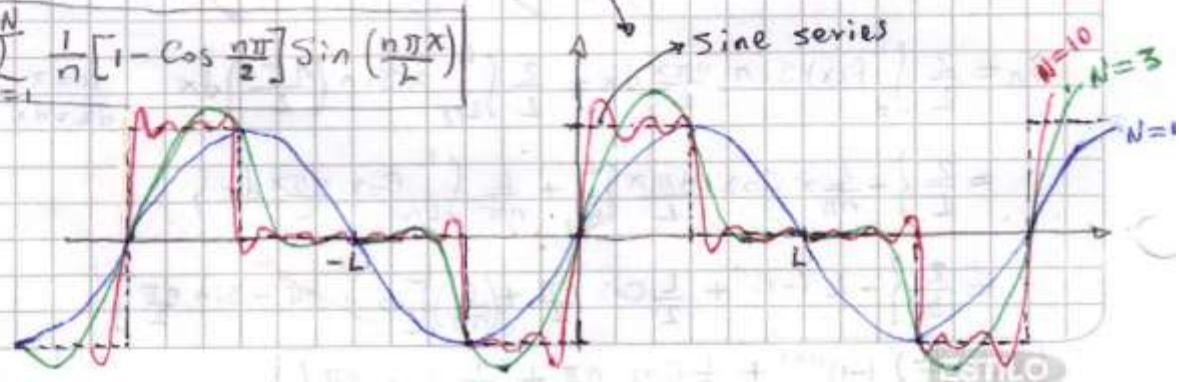
$n = 2 \rightarrow \frac{4}{\pi}\frac{2}{3}\sin\left(\frac{2\pi x}{L}\right)$



$$f(x) \simeq \sum_{n \text{ even}}^{N} B_n \sin\left(\frac{n\pi x}{L}\right)$$

b) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases} \rightarrow$ from 3.3.2 d) $\quad B_n = \frac{2}{n\pi}\left[1 - \cos\frac{n\pi}{2}\right]$

$$\Rightarrow \boxed{f(x) \simeq \frac{2}{\pi}\sum_{n=1}^{N}\frac{1}{n}\left[1 - \cos\frac{n\pi}{2}\right]\sin\left(\frac{n\pi x}{L}\right)}$$



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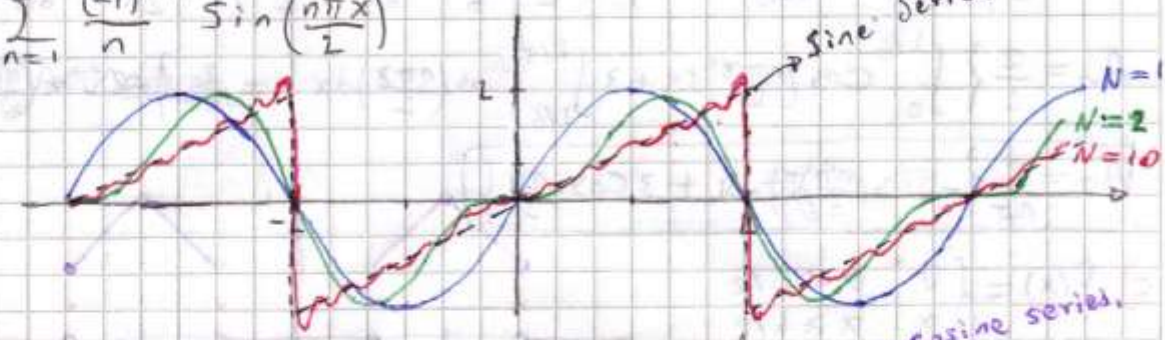
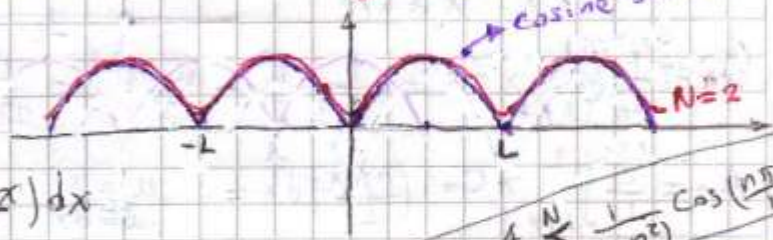
c) $f(x) = x \rightarrow B_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right)dx$ $\quad u = x \quad du = dx \quad dv = \sin(n\pi x/L)\,dx \quad v = \frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$$B_n = \frac{2}{L}\left\{ \frac{-L}{n\pi} x\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{L}{n\pi}\int_0^L \cancel{\cos\left(\frac{n\pi x}{L}\right)}^{\,0} dx \right\} = \frac{2L}{n\pi}(-1)^{n+1}$$

$$f(x) \simeq \frac{2L}{\pi}\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)$$



3.3.4 $f(x) = \sin(\pi x/L)$



$$A_0 = \frac{1}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)dx = \boxed{\frac{2}{\pi}}$$

$$A_n = \frac{2}{L}\int_0^L \sin(\pi x/L)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{\cancel{2}}{L}\int_0^L \frac{1}{\cancel{2}}\left[\sin\frac{(1-n)\pi x}{L} + \sin\frac{(1+n)\pi x}{L}\right]dx$$

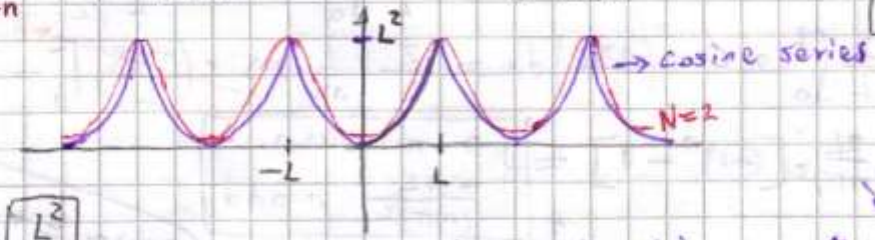
$$\boxed{f \simeq \frac{2}{\pi} + \frac{4}{\pi}\sum_{\text{even}}^{N}\frac{1}{(1-n^2)}\cos\left(\frac{n\pi x}{L}\right)}$$

This aproximation to the function is very well!!

$$= \frac{-1}{(1-n)\pi}\cos\frac{(1-n)\pi x}{L}\Big|_0^L + \frac{-1}{(1+n)\pi}\cos\frac{(1+n)\pi x}{L}\Big|_0^L$$

$$= \frac{-1}{(1-n)\pi}\left[\underset{(-1)^{1+n}}{(-1)^{1-n}} - 1\right] + \frac{-1}{(1+n)\pi}\left[(-1)^{1+n} - 1\right] = -\frac{2}{(1-n^2)\pi}\left[(-1)^{1+n} - 1\right] = \boxed{\begin{cases} 0 & n \text{ odd} \\ \frac{4}{(1-n^2)\pi} & n \text{ even} \end{cases}}$$

3.3.5 $f(x) = x^2$



a)

$$A_0 = \frac{1}{L}\int_0^L x^2\,dx = \boxed{\frac{L^2}{3}}$$

$u = x^2 \quad du = 2x\,dx \qquad dv = \cos(n\pi x/L) \quad v = +\frac{L}{n\pi}\sin(n\pi x/L)$

$u = x \quad du = dx \qquad dv = \sin\left(\frac{n\pi x}{L}\right) \quad v = \frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$$A_n = \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left\{ -\frac{L}{n\pi}x^2\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{2L}{n\pi}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx \right\}$$

$$= \frac{4}{n\pi}\left[+\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \int_0^L \left(\frac{L}{n\pi}\right)\cancel{\cos\frac{n\pi x}{L}}^{\,0}\,dx\right] = \boxed{\frac{4L^2}{(n\pi)^2}(-1)^n}$$

$$\boxed{f(x) \simeq \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{N}\frac{(-1)^n}{n}\cos\left(\frac{n\pi x}{L}\right)}$$

Solutions to problems in Haberman Chapt 4

4.2.1 $\rho_0(x)\dfrac{\partial^2 u}{\partial t^2} = T_0\dfrac{\partial^2 u}{\partial x^2} - g\,\rho_0(x)$ (*) $\quad u(0)=0 \quad u(L)=0$

a)

at the sagged equilibrium position $u(x,t)=u_E(x)$, thus

$$0 = T_0\frac{d^2u_E}{dx^2} - g\,\rho_0(x) \longrightarrow \underbrace{\int_0^x \frac{d^2u_E}{dx'}(x')\,dx'}_{} = \int_0^x \frac{g}{T_0}\rho_0(x')\,dx'$$

$$\frac{du_E}{dx}(x) - \frac{du_E}{dx}(0) = \frac{g}{T_0}\int_0^x \rho(x')\,dx'$$

$$\Rightarrow \underbrace{\int_0^x \frac{du_E}{dx'}\,dx'}_{} = \left(\frac{du_E}{dx}(0)\right)x + \frac{g}{T_0}\int_0^x\int_0^{x'}\rho(x'')\,dx''\,dx' + Cte$$

$$u_E(x) - \cancel{u_E(0)}^{\,0} \Rightarrow u_E(x) = \left(\frac{du_E}{dx}(0)\right)x + \frac{g}{T_0}\int_0^x\int_0^{x'}\rho(x'')\,dx''\,dx' + C$$

$$u_E(0) = 0 = \frac{g}{T_0}\cancel{\int_0^0 (\quad)\,dx'}^{\,0} + C \longrightarrow C=0$$

$$u(L) = 0 = \left(\frac{du_E}{dx}(0)\right)L + \frac{g}{T_0}\int_0^L\int_0^{x'}\rho(x'')\,dx''\,dx'$$

$$\hookrightarrow \frac{du_E}{dx}(0) = -\frac{g}{T_0 L}\int_0^L\int_0^{x'}\rho(x'')\,dx''\,dx'$$

$$\Rightarrow \boxed{u_E(x) = -\left(\frac{g}{T_0 L}\int_0^L\int_0^{x'}\rho(x'')\,dx''\,dx'\right)x + \frac{g}{T_0}\int_0^x\int_0^{x'}\rho(x'')\,dx''\,dx'}$$

If $\rho_0(x) = cte \rightarrow u_E(x) = -\dfrac{g}{T_0 \cancel{L}}\left(\rho_0\dfrac{L^{\cancel{2}}}{2}\right)x + \dfrac{g}{T_0}\left(\rho_0\dfrac{x^2}{2}\right)$

$$\boxed{u_E(x) = \frac{g\rho_0}{2T_0}\left(x^2 - Lx\right)}$$

b) Since $\rho_0(x)\dfrac{\partial^2 u_E}{\partial t^2} = T_0\dfrac{\partial^2 u_E}{\partial x^2} - g\rho_0(x)$ (**) when $u = u_E(x)$ in (*).

If $v(x,t) = u(x,t) - u_E(x) \Rightarrow \dfrac{d^2v}{dt^2} = \dfrac{d^2u}{dt^2} - \dfrac{d^2u_E}{dt^2}$

$$\rho_0(x)\frac{d^2v}{dt^2} = \rho_0(x)\frac{d^2u}{dt^2} - \rho_0(x)\frac{d^2u_E}{dt}$$

Using (*) and (**) here $\Rightarrow$

from here, we can se that $\boxed{\omega_n = b_n = \sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}}$

Method 2

It is simpler to write the general solution of $h(t)$ with Sine and Cosine functions.

$$h(t) = e^{at}(C_1 \cos b_n t + C_2 \sin b_n t)$$

and thus do superposition, etc.....

$$\Rightarrow \boxed{u(x,t) = e^{at} \sum_{n=1}^{\infty} (\alpha_n \cos b_n t + \beta_n \sin b_n t) \sin(n\pi x/L)}$$

$f(x) = u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right) dx$ then, multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating:

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} \alpha_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx}_{(L/2)\delta_{nm}} = \alpha_m L/2.$$

$$\rightarrow \boxed{\alpha_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

$$g(x) = \frac{\partial u}{\partial t}(x,0) = a \underbrace{\sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right)}_{} + \sum_{n=1}^{\infty} \beta_n b_n \sin\left(\frac{n\pi x}{L}\right)$$

Expansion of $f(x)$ in sine series, since

$$g(x) = a f(x) + \sum_{n=1}^{\infty} \beta_n b_n \sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin(m\pi x/L)$ and integrating.

$$\int_0^L (g(x) - a f(x)) \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} \beta_n b_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx}_{(L/2)\delta_{nm}}$$

$$\int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) dx - a\alpha_m = \beta_m b_m \frac{L}{2}$$

$$\rightarrow \boxed{\beta_n = \frac{2}{L b_n}\int_0^L (g(x) - a f(x)) \sin\left(\frac{n\pi x}{L}\right) dx}$$

4.4.4 $\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$ Suposse $u(x,t) \sim \sum_{n=1}^{\infty} B_n(t) \sin \frac{n\pi x}{L}$

Since we hope $u(x,t)$ continuous for physical solutions and $u(0,t) = u(L,t) = 0$ we can derivate:

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 B_n(t) \sin \frac{n\pi x}{L} \quad \text{also} \quad \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} B_n'(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = \sum_{n=1}^{\infty} B_n''(t) \sin\left(\frac{n\pi x}{L}\right) \rightarrow$$

Substituing in partial differential equations:

$$\rho_0\left(\sum_{n=1}^{\infty} B_n''(t)\sin\left(\frac{n\pi x}{L}\right)\right) = T_0\left(-\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 B_n(t)\sin\left(\frac{n\pi x}{L}\right)\right) - \beta\left(\sum_{n=1}^{\infty} B_n'\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\rho_0 B_n''(t) + \beta B'(t) + T_0\left(\frac{n\pi}{L}\right)^2 B_n(t)\right]\sin\left(\frac{n\pi x}{L}\right) = 0$$

$$\boxed{\rho_0 B_n''(t) + \beta B'(t) + T_0\left(\frac{n\pi}{L}\right)^2 B_n(t) = 0}$$

This is the same eq that in problem **4.4.3b** for $h(t)$. The the solution is the same!!

4.4.5 From problem **4.4.3b)** the ~~equati~~ caracteristic equation for $h(t)$ is:

$$\mu^2 + \frac{\beta}{\rho_0}\mu + \frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 = 0$$

$$\mu_\pm = -\frac{\beta}{2\rho_0} \pm \frac{1}{2}\sqrt{\frac{\beta^2}{\rho_0^2} - 4\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2}$$

Since $4\pi^2\rho_0 T_0/L^2 < \beta^2 < 16\pi^2\rho_0 T_0/L^2$ then

for $n=1 \rightarrow$ $\mu_\pm = -\frac{\beta}{2\rho_0} \pm \frac{1}{2}\sqrt{\frac{\beta^2}{\rho_0^2} - 4\frac{T_0}{\rho_0}\left(\frac{\pi}{L}\right)^2} = a \pm b_1$

for $n=2,3,\ldots \rightarrow$ $\mu_\pm = -\frac{\beta}{2\rho_0} \pm \frac{i}{2}\sqrt{4\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{\rho_0}} = a \pm ib_n$

With $a = -\frac{\beta}{2\rho_0}$, $b_1 = \frac{1}{2}\sqrt{\frac{\beta^2}{\rho_0^2} - 4\frac{T_0}{\rho_0}\left(\frac{\pi}{L}\right)^2}$ $\quad b_n = \frac{1}{2}\sqrt{4\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{\rho_0^2}}$

$n = 2, 3, 4, \ldots$

$\Rightarrow h(t) = e^{at}(C_1\cosh b_1 t + C_2\sinh b_1 t)$ if $n=1$

$h(t) = e^{at}(C_n^{(1)}\cos b_n t + C_n^{(2)}\sin b_n t)$ if $n = 2, 3, 4, \ldots$

$u(x,t) = \phi(x)h(t)$ with $\phi_n(x) = B_n\sin(n\pi x/L)$ $n = 1, 2, 3, 4, \ldots$

$$\Rightarrow \boxed{\begin{aligned} u(x,t) &= e^{at}(\alpha_1\cosh b_1 t + \beta_1\sinh b_1 t)\sin(\pi x/L) \\ &\quad + e^{at}\sum_{n=2}^{\infty}(\alpha_n\cos b_n t + \beta_n\sin b_n t)\sin(n\pi x/L) \end{aligned}}$$

the coefficients α_n and β_n are find by initial conditions $f(x) = u(x,0)$ and $g(x) = \frac{\partial u}{\partial t}(x,0)$ multiplying by $\sin(m\pi x/L)$ and integrating as in exercise **4.4.3b)** when $\underline{m \neq 1}$:

$$\boxed{\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx} \quad \boxed{\beta_n = \frac{2}{b_n L}\int_0^L (g(x) - af(x))\sin\left(\frac{n\pi x}{L}\right)dx} \text{ and} \rightarrow$$

To proceed for find α_n, β_n with $n = 2, 3, 4 \ldots$ is as in exercise **4.4.3b**, and for $n=1$ it is as follow when $\underline{m=1}$:

$$\int_0^L f(x) \sin\left(\frac{\pi x}{L}\right) dx = \left[\alpha_1 \overbrace{\cosh(b_1 \cdot 0)}^{1} + \beta_1 \cancel{\sinh(b_1 \cdot 0)}^{0}\right] \frac{L}{2} \delta_{n1}$$

$$\Rightarrow \boxed{\alpha_1 = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi x}{L}\right) dx}$$

$$\int_0^L (g(x) - a f(x)) \sin\left(\frac{\pi x}{L}\right) dx = \left[\alpha_1 b_1 \cancel{\sinh(b_1 \cdot 0)}^{0} + \beta_1 b_1 \overbrace{\cosh b_1 \cdot 0}^{1}\right] \frac{L}{2} \delta_{n1}$$

$$\boxed{\beta_1 = \frac{2}{b_1 L} \int_0^L (g(x) - a f(x)) \sin\left(\frac{\pi x}{L}\right) dx}$$

4.4.6 $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ $\quad u(0,t) = u(L,t) = 0 \quad u(x,0) = f(x) \quad \dfrac{\partial u}{\partial t}(x,0) = g(x)$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi x}{L} \sin \frac{n\pi c t}{L}\right)$$

From trigonometric Identities: $\Rightarrow$ $\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$$\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \left(\frac{A_n}{2}\left[\sin \frac{n\pi}{L}(x-ct) - \sin \frac{n\pi}{L}(x+ct)\right] + \frac{B_n}{2}\left[\cos \frac{n\pi}{L}(x-ct) + \frac{1}{2}\cos \frac{n\pi}{L}(x+ct)\right]\right)$$

$$\Rightarrow u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} \left(A_n \sin \frac{n\pi}{L}(x-ct) + B_n \cos \frac{n\pi}{L}(x-ct)\right) \rightarrow R(x-ct)$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty} \left(-A_n \sin \frac{n\pi}{L}(x+ct) + B_n \cos \frac{n\pi}{L}(x+ct)\right) \rightarrow S(x+ct)$$

$$\boxed{u(x,t) = R(x-ct) + S(x+ct)}$$

4.4.7 Using the result of exercise **4.4.6** we have:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{n=1}^{\infty} \left(A_n\left(-\frac{n\pi c}{L}\right) \cos \frac{n\pi}{L}(x-ct) + B_n\left(\frac{n\pi c}{L}\right) \sin \frac{n\pi}{L}(x-ct)\right)$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty} \left(A_n\left(\frac{n\pi c}{L}\right) \cos \frac{n\pi}{L}(x+ct) + B_n\left(\frac{n\pi c}{L}\right) \sin \frac{n\pi}{L}(x+ct)\right)$$

$$\cancel{g(x)}^{0} = \frac{\partial u}{\partial t}(x,0) = \frac{1}{2}\sum_{n=1}^{\infty} \left(\cancel{A_n\left(-\frac{n\pi c}{L}\right) \cos \frac{n\pi x}{L}} + B_n\left(\frac{n\pi c}{L}\right) \sin \frac{n\pi x}{L}\right)$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty} \left(\cancel{A_n\left(\frac{n\pi c}{L}\right) \cos \frac{n\pi x}{L}} + B_n\left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\right) \Rightarrow$$

$$0 = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \Rightarrow B_n = 0, \quad n \in \mathbb{N}$$

$$\Rightarrow u(x,t) = \frac{1}{2}\Big[\underbrace{\sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}(x-ct)}_{F(x-ct)} + \underbrace{\sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}(x+ct)}_{F(x+ct)}\Big]$$

Since $F(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$ is the odd, $2L$-periodic extension of $f(x) = u(x,0)$.

$$u(x,0) = \frac{1}{2}\left[2\sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}x\right] = F(x)$$

Thus $\boxed{u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right]}$

4.4.8 Using the result of exercise 4.4.6 we have:

$$\cancel{f(x)}^{0} = u(x,0) = \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi x}{L} + \cancel{B_n \cos\frac{n\pi x}{L}}\right) + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi x}{L} - \cancel{B_n \cos\frac{n\pi x}{L}}\right)$$

$$0 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \longrightarrow A_n = 0 \quad \forall n \in \mathbb{N}.$$

$$\Rightarrow u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \cos\frac{n\pi}{L}(x-ct) - \frac{1}{2}\sum_{n=1}^{\infty} B_n \cos(x+ct)\frac{n\pi}{L}$$

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right) B_n \sin\frac{n\pi}{L}(x-ct) + \frac{1}{2}\sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{L}\right)\sin\frac{n\pi}{L}(x+ct)$$

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{L}\right)\sin\frac{n\pi}{L}x = G(x)$$

We can check that $G(x)$ is the odd, $2L$-periodic extension of $g(x)$.

Thus $\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\left[G(x-ct) + G(x+ct)\right]$

Let $\left.\begin{array}{l} \bar{x} = x - ct \rightarrow -\frac{1}{c}d\bar{x} = dt \\ \bar{z} = x + ct \rightarrow \frac{1}{c}d\bar{z} = dt \end{array}\right\}$ and integrating

$$\Rightarrow \int_0^t \frac{\partial u}{\partial t}(x,t')dt' = \frac{-1}{2c}\int_x^{x-ct} G(\bar{x})d\bar{x} + \frac{1}{2c}\int_x^{x+ct} G(\bar{z})d\bar{z} \Rightarrow$$

$\left.\begin{array}{l} \text{If } t' = 0 \rightarrow \bar{x} = x \\ \text{If } t' = t \rightarrow \bar{x} = x - ct \end{array}\right\}$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x} G(\bar{x})\,d\bar{x} + \frac{1}{2c}\int_{x}^{x+ct} G(\bar{z})\,d\bar{z}$$

$$\boxed{u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(u)\,du}$$

4.4.9 $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ Equation (4.4.1)

$$E = \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$\frac{dE}{dt} = \int_0^L \frac{1}{\cancel{2}}\cancel{2}\left(\frac{\partial u}{\partial t}\right)\underbrace{\frac{\partial^2 u}{\partial t^2}}\,dx + \int_0^L \frac{c^2}{2}\,2\left(\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x \partial t}\,dx$$

$$= \int_0^L \left(\frac{\partial u}{\partial t}\right)\left(c^2\frac{\partial^2 u}{\partial x^2}\right)dx + \int_0^L c^2\left(\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x \partial t}\,dx$$

$$= c^2\int_0^L \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right)dx = c^2\int_0^L \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)dx$$

$$\boxed{\frac{dE}{dt} = c^2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)\Bigg|_0^L} = c^2\left(\frac{\partial u}{\partial x}(L,t)\cdot\frac{\partial}{\partial t}u(L,t) - \frac{\partial u}{\partial x}(0,t)\cdot\frac{\partial}{\partial t}u(0,t)\right)$$

4.4.10 a) $u(0,t) = 0 = u(L,t) \longrightarrow \boxed{\frac{dE}{dt} = 0} \rightarrow$ i.e. energy is conserved.

b) $\frac{\partial u}{\partial x}(0,t) = 0 = u(L,t)$ also $\rightarrow \boxed{\frac{dE}{dt} = 0} \rightarrow$

c) $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = -\gamma u(L,t)$ with $\gamma > 0$.

$$\Rightarrow \frac{dE}{dt} = c^2\left(-\gamma u(L,t)\frac{\partial u}{\partial t}(L,t) - \frac{\partial u}{\partial x}(0,t)\cdot 0\right) = -c^2\gamma u(L,t)\frac{\partial u}{\partial t}(L,t)$$

$$= -\frac{c^2\gamma}{2}\frac{d}{dt}\left(u(L,t)\right)^2$$ integrating:

$$\int_0^t \frac{dE(t')}{dt'} = -\frac{c^2\gamma}{2}\left[u^2(L,t) - u^2(L,0)\right] \longrightarrow \boxed{E(t) = \frac{u^2\gamma}{2}\left[u^2(L,0) - u^2(L,t)\right]}$$

d) If $\gamma < 0 \Rightarrow \boxed{E(t) = -\frac{c^2\gamma}{2}\left[u^2(L,0) - u^2(L,t)\right]}$

4.4.11 $u = R(x-ct) = R(v)$ since $v \equiv x - ct$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial R}{\partial v}\frac{\partial v}{\partial t} = -c\frac{\partial R}{\partial v} \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial R}{\partial v} \Rightarrow$$

$$K = \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{1}{2}c^2\int_0^L \left(\frac{\partial R}{\partial v}\right)^2 dx$$
$$U = \int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{1}{2}c^2\int_0^L \left(\frac{\partial R}{\partial v}\right)^2 dx$$
$$\boxed{K = U}$$

4.4.12 $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t)=0$, $u(L,t)=0$ $\quad u(x,0)=f(x)$ $\quad \frac{\partial u}{\partial t}(x,0)=g(x)$

From exercise (4.4.9) we have that:

$$\frac{dE}{dt} = c^2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)\Big|_0^L \quad \text{since } E=\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx \quad (1)$$

$$= c^2\left(\frac{\partial u}{\partial x}(L,t)\cdot\frac{\partial}{\partial t}\cancelto{0}{u(L,t)} - \frac{\partial u}{\partial x}(0,t)\cdot\frac{\partial}{\partial t}\cancelto{0}{u(0,t)}\right) \rightarrow E = \text{constant with time}$$

then (1) is constant for all t, in particular for $t=0$

$$\Rightarrow E = E(t=0) = \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,0)\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\frac{\partial}{\partial x}u(x,0)\right)^2 dx = cte.$$

$$E = \int_0^L \frac{1}{2}\left(g(x)\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\frac{\partial f(x)}{\partial x}\right)^2 dx \quad \forall t\geq 0. \quad (2)$$

We want to show the uniqueness of the solution of the wave equation for $0\leq x\leq L$ subject to Dirichlet boundary conditions $u(0,t)=0=u(L,t)$ and the initial conditions. We suppose, to the contrary, that there exist two solutions u and v satisfying the same boundary conditions and initial conditions. Let:

$$W = u - v \quad \text{then}$$

$$\frac{\partial^2 W}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - c^2\frac{\partial^2 u}{\partial x^2}$$ since u, v are solutions of wave equation

$$\frac{\partial^2 W}{\partial t^2} = c^2\frac{\partial^2}{\partial x^2}(u-v) = c^2\frac{\partial^2 W}{\partial x^2}$$ then W also satisfy the wave equation.

$$\left.\begin{aligned} W(0,t) &= u(0,t) - v(0,t) = 0 \\ W(L,t) &= \quad (L,t) - \quad (L,t) = 0 \end{aligned}\right\}$$ Since u, v satisfy the same boundary conditions

$$\left.\begin{aligned} F(x) = W(x,0) &= u(x,0) - v(x,0) = f(x) - f(x) \\ W(x,0) &= 0 = F(x) \\ G(x) = \frac{\partial W}{\partial t}(x,0) &= \frac{\partial u}{\partial t}(x,0) - \frac{\partial v}{\partial t}(x,0) = g(x) - g(x) \\ \frac{\partial W}{\partial t}(x,0) &= 0 = G(x) \end{aligned}\right\}$$ Since u, v satisfy the same initial conditions

Applying the deffinition (1) for the wave W:

$$E_w = \int_0^L \frac{1}{2}\left(\frac{\partial w}{\partial t}\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx$$

Since w satisfy wave equation, then

in similar way as in exercise (4.4.9) we can show that

$$\frac{dE_w}{dt} = c^2\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\right)\Big|_0^L = c^2\left(\frac{\partial w}{\partial x}(L,t)\cdot \underbrace{\cancel{\frac{\partial}{\partial t}w(L,t)}}_{0} - \frac{\partial w}{\partial x}(0,t)\cdot \underbrace{\cancel{\frac{\partial}{\partial t}w(0,t)}}_{0}\right)$$

Since W also satisfy boundary conditions $w(0,t) = 0 = w(L,t)$.

$$\frac{dE_w}{dt} = 0 \rightarrow \underline{E_w = cte}$$

Energy for wave w is also constant, in particular for $t=0$:

$$E_w = E(t=0) = \int_0^L \frac{1}{2}\left(\underbrace{\cancel{\frac{\partial w}{\partial t}(x,0)}}_{0}\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\underbrace{\cancel{\frac{\partial}{\partial x}w(x,0)}}_{0}\right)^2$$

$$\Rightarrow E_w = 0 \quad \forall t \geq 0$$

Since u and v satisfy the same initial conditions (Remember $\underline{w = u - v}$)

This is true for any time $t \geq 0$ $\Rightarrow$ $\int_0^L \frac{1}{2}\left(\frac{\partial w}{\partial t}(x,t)\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\frac{\partial}{\partial x}w(x,t)\right) dx = 0$ (3)

$$\left.\begin{aligned} &\int_0^L \frac{1}{2}\left(\frac{\partial w}{\partial t}(x,t)\right)^2 dx \geq 0 \\ &\int_0^L \frac{c^2}{2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2 dx \geq 0 \end{aligned}\right\}$$

Because both are integrals of quantities (or functions) positives Then, for satisfy (3) necesarily:

$$\left.\begin{aligned} &\int_0^L \frac{1}{2}\left(\frac{\partial w}{\partial t}(x,t)\right)^2 dx = 0 \rightarrow \frac{\partial w}{\partial t} = 0 \\ &\int_0^L \frac{c^2}{2}\left(\frac{\partial}{\partial x}w(x,t)\right)^2 dx = 0 \rightarrow \frac{\partial w}{\partial x} = 0 \end{aligned}\right\} \boxed{v(x,t) = u(x,t) + C_0}$$

Thus the solution for the wave equation is unique apart from an unimportant arbitrary additive constant.

(4.4.13) a) $u_n(x,t) = \left[A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)\right] \sin \frac{n\pi x}{L}$

$$\frac{\partial u_n}{\partial x} = \left[A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L}\right]\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial u_n}{\partial t} = \left[-\frac{n\pi c}{L}A_n \sin \frac{n\pi c t}{L} + \frac{n\pi c}{L}B_n \cos \frac{n\pi c t}{L}\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$\left(\frac{\partial u_n}{\partial x}\right)^2 = \left[A_n^2 \cos^2\left(\frac{n\pi c t}{L}\right) + B_n^2 \sin^2\left(\frac{n\pi c t}{L}\right) + 2A_nB_n \cos \frac{n\pi c t}{L}\sin \frac{n\pi c t}{L}\right]\left(\frac{n\pi}{L}\right)^2 \cos^2(n\pi x/L)$$

$$\left(\frac{\partial u_n}{\partial t}\right)^2 = \left[A_n^2 \sin^2 \frac{n\pi c t}{L} + B_n^2 \cos^2 \frac{n\pi c t}{L} - 2A_nB_n \cos \frac{n\pi c t}{L}\sin \frac{n\pi c t}{L}\right]\left(\frac{n\pi c}{L}\right)^2 \sin^2\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow E_n = \frac{1}{2}\int_0^L \left[\left(\frac{\partial u_n}{\partial t}\right)^2 + c^2\left(\frac{\partial u_n}{\partial x}\right)^2\right] dx$$

$$E_n = \frac{1}{2}\left(\frac{n\pi c}{L}\right)^2 \int_0^L \left\{\left[\sin^2\frac{n\pi ct}{L}\sin^2\frac{n\pi x}{L} + \cos^2\frac{n\pi ct}{L}\cos^2\frac{n\pi x}{L}\right]A_n^2 + \left[\cos^2\frac{n\pi ct}{L}\sin^2\frac{n\pi x}{L} + \sin^2\frac{n\pi ct}{L}\cos^2\frac{n\pi x}{L}\right]B_n^2 + 2A_nB_n\cos\frac{n\pi ct}{L}\sin\frac{n\pi ct}{L}\left[\cos^2\frac{n\pi x}{L} - \sin^2\frac{n\pi x}{L}\right]\right\} dx$$

see exercise (4.4.9) and (4.4.10)

Since energy E_n conserved when $u_n(x,t)$ satisfy Dirichlet condition OR conditions of fixed extrems $u_n(0,t) = 0 = u_n(L,t)$, then $E = \text{constant}$ for all $t \geq 0$, in particular for $t = 0$, then

$$\boxed{E_n = \frac{1}{2}\left(\frac{n\pi c}{L}\right)^2 \int_0^L \left\{A_n^2\cos^2\frac{n\pi x}{L} + B_n^2\sin^2\frac{n\pi x}{L}\right\} dx} \quad (1)$$

There is an expression more explicit for this in the next page →

b) $$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$E = \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

Since energy is conserved $E = \text{constant}$ $\forall t \geq 0$. in particular for $t = 0$:

$$E = E(t=0) = \frac{1}{2}\int_0^L \left(\frac{\partial u}{\partial t}(x,0)\right)^2 dx + \frac{c^2}{2}\int_0^L \left(\frac{\partial}{\partial x}u(x,0)\right)^2 dx \quad (2)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty}\left[-\frac{n\pi c}{L}A_n\cancelto{0}{\sin\frac{n\pi c(0)}{L}} + \frac{n\pi c}{L}B_n\overbrace{\cos\frac{n\pi c(0)}{L}}^{1}\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right)B_n\sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial}{\partial x}u(x,0) = \frac{\partial}{\partial x}\sum_{n=1}^{\infty}\left[A_n\overbrace{\cos\frac{n\pi c(0)}{L}}^{1} + B_n\cancelto{0}{\sin\frac{n\pi c(0)}{L}}\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)A_n\cos\left(\frac{n\pi x}{L}\right)$$

$$\left(\frac{\partial u}{\partial t}(x,0)\right)^2 = \left(\frac{\pi c}{L}\right)^2\left\{\sum_{n=1}^{\infty} n^2B_n^2\sin^2\frac{n\pi x}{L} + 2\left(\sum_{n=2}^{\infty} B_1B_n\sin\left(\frac{\pi x}{L}\right)\sin\frac{n\pi x}{L} + \sum_{n=3}^{\infty} B_2B_n\sin\left(\frac{2\pi x}{L}\right)\sin\frac{n\pi x}{L} + \sum_{n=4}^{\infty}\cdots\right)\right\}$$

$$\left(\frac{\partial}{\partial x}u(x,0)\right)^2 = \left(\frac{\pi}{L}\right)^2\left\{\sum_{n=1}^{\infty} n^2A_n^2\cos^2\frac{n\pi x}{L} + 2\left(\sum_{n=2}^{\infty} A_1A_n\cos\frac{\pi x}{L}\cos\frac{n\pi x}{L} + \sum_{n=3}^{\infty} A_2A_n\cos\left(\frac{2\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + \sum\cdots\right)\right\}$$

We used $(x+y+z+\cdots)^2 = x^2+y^2+z^2+\cdots + 2(xy+xz+yz+\cdots)$

$$\int_0^L \left(\frac{\partial u}{\partial t}(x,0)\right)^2 dx = \left(\frac{\pi c}{L}\right)^2 \left\{ \sum_{n=1}^{\infty} n^2 \int_0^L B_n^2 \sin^2 \frac{n\pi x}{L}\, dx \right.$$

by orthogonality !!

$$+ 2\left(\sum_{n=2}^{\infty} B_1 B_n \int_0^L \sin\frac{\pi x}{L} \sin\frac{n\pi x}{L}\, dx + \sum_{n=3}^{\infty} B_2 B_n \int_0^L \sin\frac{2\pi x}{L} \sin\frac{n\pi x}{L}\, dx + \sum_{n=4}^{\infty} \cdots \right) \Bigg\}$$

$$= \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right)^2 \int_0^L B_n^2 \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$\int_0^L \left(\frac{\partial}{\partial x} u(x,0)\right)^2 dx = \left(\frac{\pi}{L}\right)^2 \left\{ \sum_{n=1}^{\infty} n^2 \int_0^L A_n^2 \cos^2\left(\frac{n\pi x}{L}\right) dx + 2\left(\sum_{n=2}^{\infty} A_1 A_n \int_0^L \cos\frac{\pi x}{L} \cos\frac{n\pi x}{L}\, dx \right.\right.$$

$$\left.\left. + \sum_{n=3}^{\infty} A_2 A_n \int_0^L \cos\frac{2\pi x}{L} \cos\frac{n\pi x}{L}\, dx + \sum_{n=4}^{\infty} \cdots \right) \right\}$$

$$= \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 \int_0^L A_n^2 \cos^2\left(\frac{n\pi x}{L}\right) dx$$ Substuting in ②

$$\Rightarrow E = E(t=0) = \sum_{n=1}^{\infty} \underbrace{\frac{1}{2}\left(\frac{n\pi c}{L}\right)^2 \int_0^L \left\{ A_n^2 \cos^2\frac{n\pi x}{L} + B_n^2 \sin^2\frac{n\pi x}{L} \right\} dx}_{E_n}$$

$$\rightarrow \boxed{E_T = \sum_{n=1}^{\infty} E_n}$$ Thus total energy is the sum of the energies E_n contained in each mode.

$$E_n = \frac{1}{2}\left(\frac{n\pi c}{L}\right)^2 \left\{ A_n^2 \underbrace{\int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx}_{L/2} + B_n^2 \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx}_{L/2} \right\}$$

$$\boxed{E_n = \frac{1}{L}\left(\frac{\pi c}{2}\right)^2 (A_n^2 + B_n^2)\, n^2}$$ Energy of one normal mode ✓

By: Herminso Villarraga Gómez

4.5.1 For a membrane the Hooke's law says that there is a restoring force proportional to displacement Δu. According to Newton's law:

Analog to section 4.3 from book → $m\dfrac{d^2u}{dt^2} = -K\Delta u + \text{other forces on mass.}$

Where K is a constant of proporcionality for Hooke's law. The other forces on the mass are a tensile force applied and a force $g(t)$ representing any other external forces on the mass in direction of Δu. Recall that we must be restricted to small angles, such that the tension is nearly constant, T_0.

Let $\hat{n}$ a vector in direction in wich displacement diferences Δu are greatest ($\hat{n}$ coincides with normal vector to the surface of vibrating membrane). The tensile force resulting on mass m is proportional (with proporcionality T_0) to the rate of change of displacement in this direction

$$\Delta u = u(\vec{r}+\Delta\vec{r}, t) - u(\vec{r}, t) \approx \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z$$

In the direction $\hat{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$, $\Delta\vec{r} = \Delta s\,\vec{n}$, where Δs is the distance between $\vec{r}$ and $\vec{r}+\Delta\vec{r}$. Thus the rate of change of the displacement in the direction $\hat{n}$ is the directional derivative:

$$\lim_{\Delta s\to 0}\frac{\Delta u}{\Delta s} = n_1\frac{\partial u}{\partial x} + n_2\frac{\partial u}{\partial y} + n_3\frac{\partial u}{\partial z} = \hat{n}\cdot\vec{\nabla}u$$

$\Rightarrow$ force tensile resulting $= T_0(\hat{n}\cdot\vec{\nabla}u)$

With this, $$m\frac{d^2u}{dt^2} = -K\Delta u + T_0(\vec{\nabla}u\cdot\hat{n}) + g(t)$$

Let us consider some special cases in wich there are no external forces on the mass, $g(t)=0$. If in addition, the mass is sufficiently small so that the forces on the mass are in balance, then

$$\boxed{T_0(\vec{\nabla}u\cdot\hat{n}) = K\Delta u}$$

If u is defined along $\hat{n}$, note that $\Delta u = \Delta s$, then in that case $\lim_{\Delta s\to 0}\frac{\Delta u}{\Delta s} = 1 = \hat{n}\cdot\nabla u \rightarrow T_0 = K\Delta u$, this is the case in wich tensile force T_0 acts in same direction of Δu.

By: Herminso Villarraga Gómez

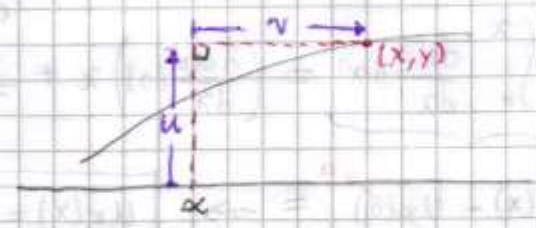
$$\rho_0(x)\frac{\partial^2 v}{\partial t^2} = \left(T_0\frac{\partial^2 u}{\partial x^2} - \cancel{g\rho_0(x)}\right) - \left(T_0\frac{\partial^2 u_E}{\partial x^2} - \cancel{g\rho_0(x)}\right) \qquad c^2 = T_0/\rho_0$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2}{\partial x^2}\underbrace{\left(u(x,t) - u_E(x)\right)}_{v} \Rightarrow \boxed{\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2}}$$

4.2.2 $c^2 = \dfrac{T_0 \longrightarrow \text{force}}{\rho_0(x) \longrightarrow \text{linear density}} \Rightarrow c^2 \to \dfrac{[\cancel{Kg}\cdot m/s^2]}{[\cancel{Kg}/m]}$

wich is velocity squared $\longleftarrow$ $c^2 \to \left[\dfrac{m^2}{s^2}\right] = \left[\dfrac{m}{s}\right]^2$

4.2.3 From figure (4.2.1):



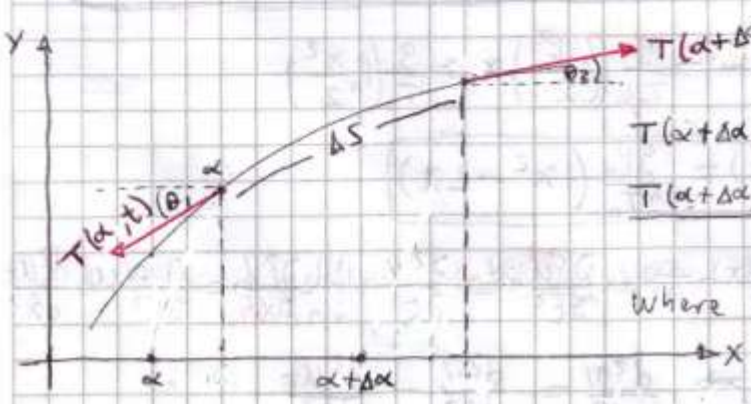
$x = \alpha + v$
$y = u$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\alpha}{dx/d\alpha} = \boxed{\frac{\partial u/\partial\alpha}{1 + \partial v/\partial\alpha}}$$

4.2.4 Let α be the x-coordinate of a particular point (molecule) when the string is in its equilibrium position. Then the motion of the string can be described by giving the coordinates (u, v) of each poit α at each time t. That is, it is sufficient to know the two functions:

$$x = x(\alpha, t) \qquad \text{OR} \qquad x = \alpha + v(\alpha, t)$$
$$y = y(\alpha, t) \qquad\qquad y = u(\alpha, t)$$

Such that in $t = 0 \to (x, y) = (\alpha, 0) \to$ Equilibrium position since $v(t) = u(t) = 0$.



2ª Newton's Law

$$T(\alpha+\Delta\alpha, t)\cos\theta_2 - T(\alpha, t)\cos\theta_1 = (\rho_0\Delta\alpha)\frac{\partial^2 x}{\partial t^2}$$

$$\frac{T(\alpha+\Delta\alpha, t)\cos\theta_2 - T(\alpha, t)\cos\theta_1}{\Delta\alpha} = \rho_0(\alpha)\frac{\partial^2 x}{\partial t^2}$$

where $\theta_1 = \theta(\alpha, t)$
$\theta_2 = \theta(\alpha+\Delta\alpha, t)$

Similarly for y-direction $T(\alpha+\Delta\alpha, t)\sin\theta_2 - T(\alpha, t)\sin\theta_1 = (\rho_0\Delta\alpha)\dfrac{\partial^2 y}{\partial t^2}$

By: Herminso Villarraga Gómez

4.6.1 $u(\vec{x},t) = A e^{i(k_1 x + k_2 y + k_3 z - wt)} = A e^{i(\vec{k}\cdot\vec{x} - wt)}$

If the propagation of wave is in the $\vec{k}$ direction, then $\vec{k}\cdot\vec{x} = kx$ because in that case $\vec{r}$ is parallel to $\vec{k}$.

$u(x,t) = A e^{i(kx - wt)} = A\left[\cos(kx - wt) + i\sin(kx - wt)\right]$

The wave length is defined as the lowest spatial propagation length necessary for that u take the same value:

$u(x,t) = u(x+\lambda, t)$ with λ the wave lenght.

$\cos[kx - wt] + i\sin[kx - wt] = \cos[k(x+\lambda) - wt] + i\sin[k(x+\lambda) - wt]$

$$\left.\begin{array}{l}\cos(kx - wt) = \cos[(kx - wt) + k\lambda] \\ \sin(kx - wt) = \sin[(kx - wt) + k\lambda]\end{array}\right\} \Rightarrow k\lambda = 2\pi \rightarrow \boxed{\lambda = \frac{2\pi}{k}}$$

then if the distance of propagation of wave is $d = 2\pi$ in the $\vec{k}$-direction, the number of waves is $\underline{d/\lambda = k}$

4.6.2 $\rightarrow$ If the wave is moving in the $\vec{k}$-direction:

$\varphi(x,t) = kx - wt \rightarrow$ Phase of plane wave

$$\Rightarrow \frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial x}\underbrace{\frac{dx}{dt}}_{c \rightarrow \text{velocity}} + \frac{\partial\varphi}{\partial t}\cancel{\frac{dt}{dt}}^{1} = kc - w, \quad \text{but } c = \frac{w}{k}$$

$$\Rightarrow \frac{d\varphi}{dt} = 0 \longrightarrow \boxed{\varphi(x,t) = \text{constant}}$$

4.6.3 $\dfrac{\sin\theta_T}{\sin\theta_I} = \dfrac{c_-}{c_+} = \dfrac{c_{light}/n_-}{c_{light}/n_+}$ since $n = \dfrac{c_{light}}{c}$

$\Rightarrow \boxed{\dfrac{\sin\theta_T}{\sin\theta_I} = \dfrac{n_+}{n_-}}$ with n_- the index of refraction of incident medium and n_+ of transmision medium

4.6.4 $1 + R = T$

$$1 - R = \frac{T k_I}{i k_{3I}}\sqrt{\sin^2\theta_i - \frac{c_t^2}{c_-^2}}$$

$$2 = T\left(1 + \frac{k_I}{i k_{3I}}\sqrt{\sin^2\theta_i - \frac{c_t^2}{c_-^2}}\right) \Rightarrow$$

$$T = \frac{2 i K_{3I}}{i K_{3I} + K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}} = \boxed{\frac{2 K_{3I}}{K_{3I} - i K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}}}$$

Substituting this in $1 + R = T$, then:

$$R = T - 1 = \frac{2 K_{3I} - \left(K_{3I} - i K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}\right)}{K_{3I} - i K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}}$$

$$\boxed{R = \frac{K_{3I} + i K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}}{K_{3I} - i K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}}}$$

4.6.5 • If $K_{3I} = \pm i\beta = \pm i K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}$

$$\Rightarrow T = \frac{\pm 2 i K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}}{(\pm 1 - 1) i K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}} \rightarrow T = \frac{\mp 2}{(1 \mp 1)} \qquad (*)$$

and $$R = \frac{(\pm 1 + 1)}{(\pm 1 - 1)} \rightarrow R = -\frac{(1 \pm 1)}{(1 \mp 1)} \qquad (**)$$

For no singularities write $(*)$ and $(**)$ as follows:

$$\left.\begin{aligned} (1 \mp 1) T &= \mp 2 \\ (1 \mp 1) R &= -(1 \pm 1) \end{aligned}\right\} \longrightarrow$$ upper sing lead to $0 = -2$, it is imposible, then we use lower sign:

$\Rightarrow \boxed{T = 1}$ and $\boxed{R = 0}$ Total transmision

If $K_{3I} = \underset{\text{lower sign}}{-} i\beta \rightarrow K_{3R} = +i\beta$ since the z-component of the reflected wave must be minus the z-component of incident wave (See equation (4.6.9) in book)

$$\Rightarrow u_R(x,y,z) = R\, e^{+K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}\, z}\, e^{i(K_{1I} x + K_{2I} y - \omega t)}$$

↳ This solution for reflected wave decay exponentially as $z \rightarrow +\infty$.

~~• We want analyze other case, when~~

~~$K_{3I} = \pm i\beta = \pm i K_I \sqrt{C_+^2/C_-^2 - \sin^2\theta_I} = \mp i K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}$~~

~~as $T = \frac{\mp 2 K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}}{(\mp 1 - 1) K_I \sqrt{\sin^2\theta_I - C_+^2/C_-^2}} = \frac{\mp 2}{\mp 1 - 1}$~~

~~and $R = \frac{\mp 1 + 1}{\mp 1 - 1} = \frac{\pm 1 \mp 1}{\mp 1 - 1}$~~

taking limit $\Delta\alpha \to 0$ en Excercise (4.2.4), then:

$X = \alpha + v(\alpha,t)$

(4.2.5) $$\frac{\partial}{\partial\alpha}\left(T(\alpha,t)\cos\theta(\alpha,t)\right) = \rho_0(\alpha)\frac{\partial^2 X}{\partial t^2} = \rho_0(\alpha)\frac{\partial^2 v}{\partial t^2}$$

For small angles we can suposse $\Delta S \approx \Delta\alpha$, then

$\cos\theta(\alpha,t) \approx \frac{\Delta v}{\Delta\alpha} \to \frac{\partial v}{\partial\alpha}$ in the limit $\Delta\alpha \to 0$

$$\Rightarrow \frac{\partial}{\partial\alpha}\left(T\frac{\partial v}{\partial\alpha}\right) = \rho_0(\alpha)\frac{\partial^2 v}{\partial t^2}$$

If there is external forces in x ~~forces~~ aditionate $\rho_0 Q_x$

$$\Rightarrow \boxed{\frac{\partial}{\partial\alpha}\left(T\frac{\partial v}{\partial\alpha}\right) + \rho_0(\alpha)Q_x(\alpha,t) = \rho_0(\alpha)\frac{\partial^2 v}{\partial t^2}}$$

Where $Q_x(\alpha,t)$ is the component en x-direction of the body force per unit lenght

Simillarly for y-direction:

$$T(\alpha+\Delta\alpha,t)\sin\theta_2 - T(\alpha,t)\sin\theta_1 = (\rho_0\Delta\alpha)\frac{\partial^2 Y}{\partial t^2}$$

Dividing by $\Delta\alpha$ and takin limit $\Delta\alpha \to 0$:

$$\frac{\partial}{\partial\alpha}\left(T\sin\theta(\alpha,t)\right) = \rho_0(\alpha)\frac{\partial^2 Y}{\partial t^2} = \rho_0\frac{\partial^2 u}{\partial t^2} \qquad \text{sin } Y = u(\alpha,t)$$

$\sin\theta(\alpha,t) \approx \frac{\Delta u}{\Delta\alpha} \to \frac{\partial u}{\partial\alpha}$ in limit $\Delta\alpha \to 0$.

$$\Rightarrow \frac{\partial}{\partial\alpha}\left(T\frac{\partial u}{\partial\alpha}\right) = \rho_0(\alpha)\frac{\partial^2 u}{\partial t^2}$$

If there is external forces in y

$$\Rightarrow \boxed{\frac{\partial}{\partial\alpha}\left(T\frac{\partial u}{\partial\alpha}\right) + \rho_0(\alpha)Q_y(\alpha,t) = \rho_0(\alpha)\frac{\partial^2 u}{\partial t^2}}$$

where $Q_y(\alpha,t)$ is the componenten en y-direction of the body force per unit lenght

If $\rho_0(\alpha) = \rho_0$, $T(\alpha,t) = T$ and $Q = 0$.

$$\Rightarrow \frac{\partial^2 v}{\partial\alpha^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \qquad \text{with } c^2 = T_0/\rho_0.$$

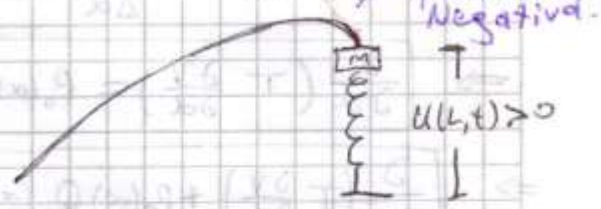
$$\frac{\partial^2 u}{\partial\alpha^2} = \frac{1}{c^2}\frac{\partial u}{\partial t^2}$$

By: Herminso Villarraga Gómez

4.3.1 $T_0 \frac{\partial u}{\partial x}(L,t) = -K_0(u(L,t) - \cancel{u_E(t)})$ → 0 Equilibrium position

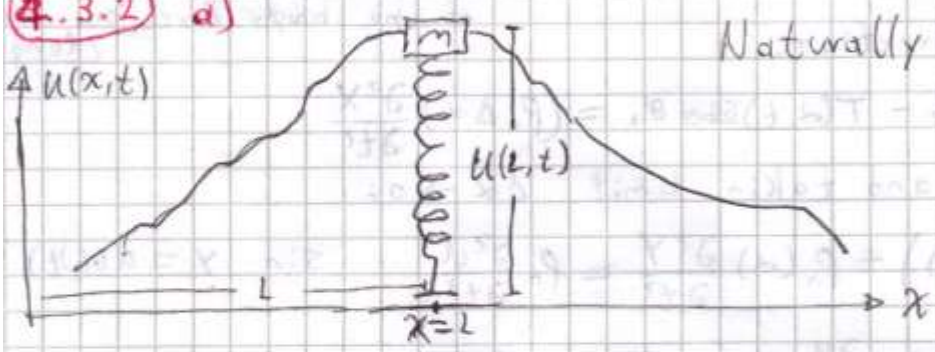
$T_0 > 0$, $K_0 > 0$ ⇒ If $u(L,t) > 0$ → $\frac{\partial u}{\partial x}(L,t) < 0$

thus correct diagram is (b)



4.3.2 a)

Naturally $\boxed{u(L_-,t) = u(L_+,t)}$



$T_0 \frac{\partial u}{\partial x}(L_-,t) = -K\,u(L_-,t)$

But $u(L_+,t) = u(L_-,t)$

$T_0 \frac{\partial u}{\partial x}(L_+,t) = K\,u(L_+,t)$

$\Rightarrow \frac{\partial u}{\partial x}(L_-,t) = -\frac{\partial u}{\partial x}(L_+,t)$ (*)

If $\lim_{x\to L^+} u(x,t) = \lim_{x\to L^-} u(x,t) \rightarrow \frac{\partial u}{\partial x}(L_-,t) = \frac{\partial u}{\partial x}(L_+,t)$ (**)

The function $u(x,t)$ is continuous.

Comparing (*) and (**) → $\boxed{\frac{\partial u}{\partial x}(L,t) = 0}$

b) $u(L_-,t) = u(L_+,t)$ ⇝ Analogous to thermic contact !!

$\frac{\partial u}{\partial x}(L,t) = 0$ ⇝ Analogous to condition of insulated!!

$c = \sqrt{T_0/\rho_0}$

4.4.1 *a) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ $\quad u(0,t)=0 \quad u(L,t)=0$

$u(x,t) = \phi(x)h(t) \rightarrow \phi(x)\dfrac{d^2h}{dt^2} = c^2 h(t)\dfrac{d^2\phi}{dx^2}$ dividing by $\phi(x)h(t)$.

$\Rightarrow \underbrace{\dfrac{1}{c^2}\dfrac{1}{h}\dfrac{d^2h}{dt^2}}_{-\lambda} = \underbrace{\dfrac{1}{\phi}\dfrac{d^2\phi}{dx^2}}_{-\lambda} \Rightarrow \dfrac{d^2h}{dt^2} = -\lambda c^2 h \qquad \dfrac{d^2\phi}{dx^2} = -\lambda\phi \qquad \phi(0)=0,\ \phi(L)=0$

$\lambda = 0: \rightarrow h(t) = \cancel{C_1}^{\,0} t + C_0$ → Solutions growing in time

$\lambda < 0: \rightarrow h(t) \sim e^{\pm c\sqrt{-\lambda}\,t}$ (and decaying if $h \sim e^{-c\sqrt{-\lambda}\,t}$)

$\lambda > 0: \rightarrow h(t) = A\cos c\sqrt{\lambda}\,t + B\sin c\sqrt{\lambda}\,t$

(we reject solutions growing or decaying in time) → Since we are solving a vibration string, it should seem more reasonable that the time-dependent solution oscillate.

$\lambda = 0: \rightarrow \phi(x) = A_0 x + B_0 \rightarrow \phi(0)=0 \rightarrow B_0 = 0;\ \phi(L) = 0 = A_0 L \rightarrow A_0 = 0.$

$\lambda > 0: \rightarrow \phi(x) = A'\cos\sqrt{\lambda}\,x + B'\sin\sqrt{\lambda}\,x \qquad \phi(0)=0 \rightarrow A'=0$

$\phi(L)=0 \rightarrow \sqrt{\lambda}\,L = n\pi$

$\rightarrow \boxed{\lambda = \left(\dfrac{n\pi}{L}\right)^2} \qquad n = 1,2,3,\ldots$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi c}{L}t + B_n\sin\frac{n\pi c}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)}$$

from which we can see that the angular frecquency is: (or Natural frequency)

$\boxed{\omega_n = \dfrac{n\pi c}{L}} \qquad n = 1,2,3,\ldots$

*b) $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2} \qquad u(0,t)=0 \qquad T_0\dfrac{\partial u}{\partial x}(\ ,t) = 0$

↳ the separation of variables is the same in a) and we reject $\lambda < 0$, because it will lead to exponentially growing or decaying solutions.

$\lambda = 0 \rightarrow h(t) = \cancel{C_1}^{\,0} t + C_0$ → solution growing in time.

$\lambda = 0: \rightarrow \phi(x) = A_0 x + B_0 \rightarrow \phi(0) = 0 \rightarrow B_0 = 0;\ \phi'(H) = A_0 = 0.$

$\lambda > 0: \rightarrow \phi(x) = A'\cos\sqrt{\lambda}\,x + B'\sin\sqrt{\lambda}\,x \rightarrow \phi(0) = 0 \rightarrow A' = 0$

$\phi'(x) = B'\sqrt{\lambda}\cos\sqrt{\lambda}\,x \rightarrow \phi'(H) = 0 \rightarrow \sqrt{\lambda}\,H = (2n-1)\dfrac{\pi}{2}$

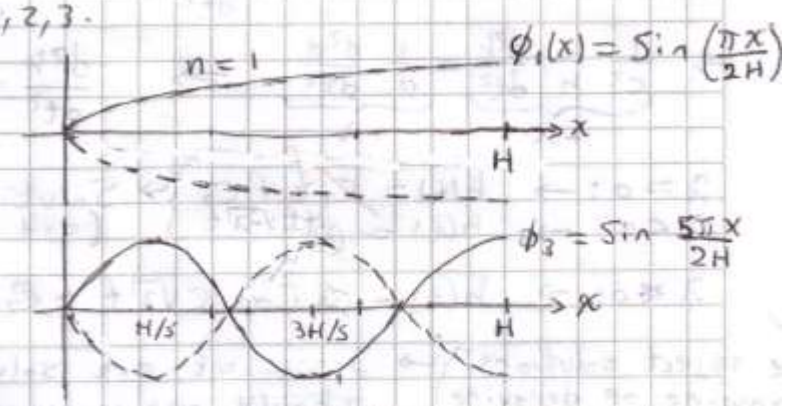
$\rightarrow \boxed{\lambda = \left(\dfrac{(2n-1)\pi}{2H}\right)^2} \qquad n = 1,2,3,\ldots \Rightarrow$

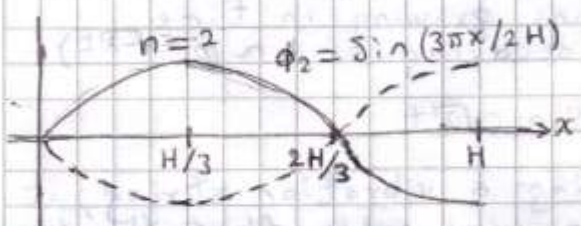
$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{(2n-1)\pi c}{2H} t + B_n \sin \frac{(2n-1)\pi c}{2H} t \right) \sin \frac{(2n-1)\pi x}{2H}}$$

which means that the angular frequency is:

$$\boxed{\omega_n = \frac{(2n-1)\pi c}{2H}} \qquad n = 1, 2, 3.$$

$$\phi_n(x) = \sin \frac{(2n-1)\pi x}{2H} \rightarrow$$





c) If $H = L/2 \rightarrow \frac{(n-1/2)\pi c}{H} = \frac{(2n-1)\pi c}{2H} = \frac{(2n-1)\pi c}{L} \quad \forall n \geq 1$

then the $(2n-1)$th harmonic in part a) is the same as the nth harmonic in part b), and they have the same natural frequencies.

4.4.2 $\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u$

a) If $\alpha < 0$ then $\rho_0 \frac{\partial^2 u}{\partial t^2}$ and αu are of opposite signs and hence the acceleration of the string in the vertical direction is opposite the body force per unit lenght. Thus the body force is restoring.
If $\alpha > 0$ then the aceleration and the body force have the same sign an thus the body force acts to push the string away from equilibrium.

b) Let $\rho_0 \equiv \rho_0(x)$ and $\alpha \equiv \alpha(x)$ but T_0 is a constant

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha(x) u \qquad \text{let } u(x,t) = \phi(x) h(t)$$

$\rightarrow \rho_0(x) \phi(x) h''(t) = T_0 h(t) \phi''(x) + \alpha(x) \phi(x) h(t)$ Divide by $\rho_0 \phi h T_0$

$$\rightarrow \underbrace{\frac{1}{T_0} \frac{h''(t)}{h(t)}}_{-\lambda} = \underbrace{\frac{\phi''(x)}{\rho_0(x)\phi(x)} + \frac{\alpha(x)}{T_0 \rho_0(x)}}_{-\lambda} \rightarrow \boxed{h''(t) + \lambda T_0 h(t) = 0}$$

$$\phi''(x) + \frac{\alpha(x)}{T_0} \phi(x) = -\lambda \rho_0(x) \phi(x)$$

$$\boxed{\phi''(x) + \left[\frac{\alpha(x)}{T_0} + \lambda \rho_0(x) \right] \phi(x) = 0}$$

The time differential equation is: $h''(t) = -\lambda T_0 h(t)$

$\lambda = 0$: $h(t) = C_1 t + C_0 \rightarrow$ growing solution.
$\lambda > 0$: $h(t) = A\cos\sqrt{\lambda T_0}\,t + B\sin\sqrt{\lambda T_0}\,t \rightarrow$ oscillatory solution
$\lambda < 0$: $h(t) \sim e^{\pm\sqrt{\lambda T_0}\,t} \rightarrow$ growing and decaying solutions.

* c) Now consider the constant coefficients case where $\alpha < 0$.

$$\phi''(x) + \underbrace{\left[\lambda\rho_0 + \frac{\alpha}{T_0}\right]}_{\gamma}\phi(x) = 0 \qquad \phi(0) = 0 \quad \phi(L) = 0$$

$\gamma = 0$: $\phi(x) = C_1 x + C_0 \rightarrow \phi(0) = 0 \rightarrow C_0 = 0$, $\phi(L) = 0 \rightarrow C_1 = 0$.

$\gamma > 0$: $\phi(x) = A\cos\sqrt{\gamma}\,x + B\sin\sqrt{\gamma}\,x$ $\quad \phi(0) = 0 \rightarrow A = 0$
$\phi(L) = 0 \rightarrow \sin\sqrt{\gamma}\,L = 0$

$\rightarrow \sqrt{\gamma}\,L = n\pi \rightarrow \boxed{\gamma = \left(\frac{n\pi}{L}\right)^2}$ $\quad \rightarrow \boxed{\phi(x) = B\sin\left(\frac{n\pi x}{L}\right)}$

$n = 1, 2, 3, \ldots$

$\gamma < 0$: $\phi(x) = A\cosh\sqrt{\gamma}\,x + B\sinh\sqrt{\gamma}\,x$ It's no solution because is no oscillatory and don't satisfy Boundary Conditions.

$$\gamma = \left(\frac{n\pi}{L}\right)^2 \rightarrow \lambda\rho_0 + \frac{\alpha}{T_0} = \left(\frac{n\pi}{L}\right)^2 \rightarrow \boxed{\lambda = \frac{1}{\rho_0}\left[\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}\right]}$$

If $\alpha < 0 \rightarrow \lambda > 0$ for all n, then the solution for $h(t)$ is oscilatory

$$h(t) = A'\cos\sqrt{\lambda T_0}\,t + B'\sin\sqrt{\lambda T_0}\,t$$

$$u(x,0) = 0 \rightarrow h(0) = 0 = A \rightarrow h_n(t) = B'\sin\left(\sqrt{\lambda T_0}\,t\right)$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\frac{T_0}{\rho_0}\left[\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}\right]}\,t\right)\sin\left(\frac{n\pi x}{L}\right)}$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} B_n \sqrt{\frac{T_0}{\rho_0}\left[\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}\right]}\cos\left(\sqrt{\frac{T_0}{\rho_0}\left[\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}\right]}\,t\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \sqrt{\frac{T_0}{\rho_0}\left[\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}\right]}\sin\left(\frac{n\pi x}{L}\right)$$

Multiplyin by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} B_n \sqrt{}\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\,\delta_{nm}}$$

$$= B_m \sqrt{\frac{T_0}{\rho_0}\left[\left(\frac{m\pi}{L}\right)^2 - \frac{\alpha}{T_0}\right]}\,\frac{L}{2}$$

$$\boxed{B_n = \frac{2}{L\sqrt{\frac{T_0}{\rho_0}\left[\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}\right]}}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

The frequencies of vibration are

$$\omega_n = \sqrt{\frac{T_0}{\rho_0}\left[\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}\right]} \qquad c = \sqrt{\frac{T_0}{\rho_0}} \rightarrow \boxed{\omega_n = c\left[\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}\right]^{1/2}}$$

4.4.3 $\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$

a) Note that $\frac{\partial^2 u}{\partial t^2}$ represents the acceleration, and $\frac{\partial u}{\partial t}$ is the velocity. For the damped vibration, the acceleration is opposite to the velocity, and hence $\beta > 0$.

*b) Assume $u(x,t) = \phi(x)h(t) \rightarrow \rho_0 \phi \frac{d^2 h}{dt^2} = T_0 \frac{d^2\phi}{dx^2} h - \beta \phi \frac{dh}{dt}$

$$\Rightarrow \underbrace{\frac{1}{h}\left(\rho_0 \frac{d^2 h}{dt^2} + \beta \frac{dh}{dt}\right)}_{-\lambda} = \underbrace{\frac{T_0}{\phi}\frac{d^2\phi}{dx^2}}_{-\lambda} \rightarrow \quad T_0 \phi''(x) = -\lambda \phi(x)$$

$$\rho_0 h''(t) + \beta h'(t) + \lambda h(t) = 0$$

$\lambda = 0: \rightarrow \phi(x) = c_1 x + c_0 \rightarrow \phi(0) = 0 = c_0$; $\phi(L) = 0 = c_1 L \rightarrow c_1 = 0$.

$\lambda > 0: \rightarrow \phi(x) = A\cos\left(\sqrt{\frac{\lambda}{T_0}}x\right) + B\sin\left(\sqrt{\frac{\lambda}{T_0}}x\right)$ $\quad \phi(0) = 0 = A$

$\phi(L) = 0 = B\sin\left(\sqrt{\frac{\lambda}{T_0}}x\right)$

$$\boxed{\frac{\lambda}{T_0} = \left(\frac{n\pi}{L}\right)^2} \quad n = 1, 2, \ldots \rightarrow \boxed{\phi_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)}$$

$\lambda < 0: \rightarrow \phi(x) = A\cosh\left(\sqrt{\frac{\lambda}{T_0}}x\right) + B\sinh\left(\sqrt{\frac{\lambda}{T_0}}x\right) \rightarrow$ It's no solution because no oscillates and don't satisfy Boundary conditions.

$$\Rightarrow \rho_0 h''(t) + \beta h'(t) + T_0\left(\frac{n\pi}{L}\right)^2 h(t) = 0 \rightarrow h''(t) + \frac{\beta}{\rho_0}h'(t) + \frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 h(t) = 0$$

The caractoristic equation is $\mu^2 + \frac{\beta}{\rho_0}\mu + \frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 = 0$

$$\Rightarrow \mu_\pm = -\frac{\beta}{2\rho_0} \pm \frac{1}{2}\sqrt{\frac{\beta^2}{\rho_0^2} - 4\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2}$$

if $\beta^2 < \frac{4\pi^2}{L^2}\rho_0 T_0 \rightarrow \frac{\beta^2}{\rho_0^2} - 4\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 < 0 \qquad n = 1, 2, 3, \ldots$

hence $\mu_\pm = a \pm i b_n$ with $a = -\beta/2\rho_0$, $b_n = \frac{1}{2}\sqrt{4\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{\rho_0^2}}$

The general solution is: $\boxed{h(t) = e^{at}\left(c_1 e^{ib_n t} + c_2 e^{-ib_n t}\right)}$

Then by superposition:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} e^{at}\left(A_n e^{ib_n t} + B_n e^{-ib_n t}\right) \sin\left(\frac{n\pi x}{L}\right)}$$

with $\boxed{a = -\frac{\beta}{2\rho_0}}$ and $\boxed{b_n = \frac{1}{2}\sqrt{4c^2\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{\rho_0^2}}}$ $\quad c = \sqrt{\frac{T_0}{\rho_0}}$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} (A_n + B_n) \sin\left(\frac{n\pi x}{L}\right)$$ Multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating.

$$\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \sum_{n=1}^{\infty} (A_n + B_n) \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx}_{(L/2)\delta_{nm}}$$

$$= (A_m + B_m)\frac{L}{2} \rightarrow A_n + B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad (*)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} \left[a(A_n + B_n) + (ib_n A_n - ib_n B_n)\right] \sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin\left(\frac{n\pi x}{L}\right)$, integrating and using orthogonality of sine functions:

$$a(A_n + B_n) + (ib_n A_n - ib_n B_n) = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad (**)$$

Call $\frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = F_n$, $\frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx = G_n$ then from $(*)$ $A_n = F_n - B_n$ and sustituying in $(**)$:

$$a(F_n - \cancel{B_n} + \cancel{B_n}) + (ib_n F_n - 2ib_n B_n) = G_n$$

$$\Rightarrow B_n = \frac{i}{2b_n}\left(G_n - aF_n - ib_n F_n\right) = \frac{i}{2b_n}\left(G_n - aF_n\right) + \frac{F_n}{2}.$$

$$A_n = F_n - B_n = \frac{F_n}{2} - \frac{i}{2b_n}\left(G_n - aF_n\right)$$

$$\boxed{\begin{Bmatrix} A_n \\ B_n \end{Bmatrix} = \frac{1}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \mp \frac{i}{Lb_n}\int_0^L \left[g(x) - af(x)\right] \sin\left(\frac{n\pi x}{L}\right) dx}$$

Thus the solution looks like follows:

$$u(x,t) = \sum_{n=1}^{\infty} e^{at}\left[\left(\frac{F_n}{2} - \frac{i}{2b_n}(G_n - aF_n)\right)e^{ib_n t} + \left(\frac{F_n}{2} + \frac{i}{2b_n}(G_n - aF_n)\right)e^{-ib_n t}\right] * \sin(n\pi x/L)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} e^{at}\left[F_n \cos(b_n t) + \frac{G_n - aF_n}{b_n}\sin(b_n t)\right] \sin(n\pi x/L)}$$

where $\boxed{F_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$ $\boxed{G_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx}$

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Solution to excersices from HABERMAN - CHap 5

* **5.3.1** $\rho_0(x)\dfrac{\partial^2 u}{\partial t^2} = T_0\dfrac{\partial^2 u}{\partial x^2} + \alpha(x)u$ let $u(x,t) = \phi(x)h(t)$

$\Rightarrow \quad \rho_0(x)\phi(x)\dfrac{d^2u}{dt^2} = T_0\dfrac{d^2\phi}{dx^2}h(t) + \alpha(x)\phi(x)h(t)$ dividing by ϕh

$$\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{\rho_0(x)\phi(x)}\left[T_0\frac{d^2\phi}{dx^2} + \alpha(x)\phi(x)\right]$$

$\Rightarrow \quad \dfrac{d^2h}{dt^2} = -\lambda h(t)$ and $T_0\dfrac{d^2\phi}{dx^2} + \alpha(x)\phi(x) + \lambda\rho_0(x)\phi(x) = 0$

Sturm-Liouville problem $\boxed{\dfrac{d}{dx}\left(T_0\dfrac{d\phi}{dx}\right) + \alpha(x)\phi(x) + \lambda\rho_0(x)\phi = 0}$

5.3.2 $\rho\dfrac{\partial^2 u}{\partial t^2} = T_0\dfrac{\partial^2 u}{\partial x^2} + \alpha u + \beta\dfrac{\partial u}{\partial t}$

a) This is a more general form of the wave equation. The term $\beta\dfrac{\partial u}{\partial t}$ can be thought of as a frictional force. In order for this force to be physical we need $\beta < 0$, so that the frictional force is opposite the direction of motion. The term αu can be thought of as an elastic force. In order for this force to be physical we need $\alpha < 0$, so that for ~~this~~ positive displacements from equilibrium the force is restorative.

b) assume $\rho \equiv \rho(x)$, $\alpha \equiv \alpha(x)$ and $\beta \equiv \beta(x)$

$\rho(x)\dfrac{\partial^2 u}{\partial t^2} = T_0\dfrac{\partial^2 u}{\partial x^2} + \alpha(x)u + \beta(x)\dfrac{\partial u}{\partial t}$ let $u(x,t) = \phi(x)h(t)$

$\rho(x)\phi(x)\dfrac{d^2h}{dt^2} = T_0 h(t)\dfrac{d^2\phi}{dx^2} + \alpha(x)\phi(x)h(t) + \beta(x)\phi(x)\dfrac{dh}{dt}$ dividing by ϕh.

$$\frac{\rho(x)}{h}\frac{d^2h}{dt^2} - \frac{\beta(x)}{h}\frac{dh}{dt} = \frac{T_0}{\phi}\frac{d^2\phi}{dx^2} + \alpha(x)$$

- If $\beta(x) = c\rho(x) \Rightarrow \underbrace{\dfrac{1}{ch}\dfrac{d^2h}{dt^2} - \dfrac{1}{h}\dfrac{dh}{dt}}_{-\lambda} = \underbrace{\dfrac{T_0}{c\rho(x)}\dfrac{d^2\phi}{dx^2} + \dfrac{\alpha(x)}{c\rho(x)}}_{-\lambda}$

  and we can separate variables.
- If $\beta(x) \neq c\rho(x)$ where c is a constant, then we cannot eliminate function of x from the lef-hand side, and thus cannot separate variables.

Note: compare to problem (5.3.3), dividying by x^2 we get

$$\alpha(x)=\frac{1}{x}, \quad \beta(x)=\frac{1}{x^2} \quad \gamma(x)=0 \rightarrow \frac{d^2\phi}{dx^2}+\alpha\frac{d\phi}{dx}+\lambda\beta\phi=0$$

Multiply by $H(x)=e^{\int\alpha(x)dx}=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$

get $p(x)=H(x)=x$, $q=H\gamma=0$, $\sigma=H\beta=x\frac{1}{x^2}=\frac{1}{x}$ ✓

b) $$\lambda=\frac{-p\phi(d\phi/dx)\Big|_1^b+\int_1^b[p(d\phi/dx)^2-q\phi^2]dx}{\int_1^b\phi^2\sigma dx}$$ since $\phi(1)=0=\phi(b)$

$$\lambda=\frac{\int_1^b x(d\phi/dx)^2dx}{\int_1^b\phi^2\frac{1}{x}dx}\geq 0 \quad \text{since} \quad \int_1^b\phi^2\frac{1}{x}dx>0.$$

$\lambda=0 \rightarrow (d\phi/dx)=0 \rightarrow \phi=c_0 \rightarrow \phi(1)=0=c_0 \rightarrow \phi=0 \ \forall x$ trivial soluti

then $\boxed{\lambda>0}$.

*c) $x^2\frac{d^2\phi}{dx^2}+x\frac{d\phi}{dx}+\lambda\phi=0$ $\lambda>0$ try $\phi(x)=x^\rho$ $\frac{d\phi}{dx}=\rho x^{\rho-1}$

$\frac{d^2\phi}{dx^2}=\rho(\rho-1)x^{\rho-2}$

$\Rightarrow \rho(\rho-1)x^{\rho-2+2}+\rho x^{\rho-1+1}+\lambda x^\rho=0$

$\rho(\rho-\cancel{1})+\cancel{\rho}+\lambda=0 \rightarrow \rho^2+\lambda=0 \rightarrow \rho=\pm i\sqrt{\lambda}$

$\Rightarrow \phi\propto x^{\pm i\sqrt{\lambda}}=(e^{\ln x})^{\pm i\sqrt{\lambda}}=e^{\pm i(\sqrt{\lambda}\ln x)}$

$\Rightarrow \phi=A\cos(\sqrt{\lambda}\ln x)+B\sin(\sqrt{\lambda}\ln x) \rightarrow \phi(1)=0=A$

$\phi(b)=B\sin(\sqrt{\lambda}\ln b)=0 \rightarrow \sqrt{\lambda}\ln b=n\pi$ $\boxed{\lambda_n=\left(\frac{n\pi}{\ln b}\right)^2}$

$\lambda=1,2,3,\ldots$

$\boxed{\phi_n(x)=B_n\sin\left(n\pi\frac{\ln x}{\ln b}\right)}$

$\lambda_1=\left(\frac{\pi}{\ln b}\right)^2<\lambda_2<\lambda_3<\cdots<\lambda_n<\cdots$ $\lim_{n\to\infty}\lambda_n=\infty$

↳ smallest ↳ No largest.

• Check $\lambda=0 \rightarrow x^2\frac{d^2\phi}{dx^2}+x\frac{d\phi}{dx}=x\left(x\frac{d^2\phi}{dx^2}+\frac{d\phi}{dx}\right)=x\frac{d}{dx}\left(x\frac{d\phi}{dx}\right)=0$

$\rightarrow x\frac{d\phi}{dx}=C_0 \rightarrow \frac{d\phi}{dx}=\frac{C_0}{x} \rightarrow \phi=C_0\ln x+C_1$

$\phi(1)=0=C_1$

$\phi(b)=0=C_0\ln b \rightarrow C_0=0$

trivial solution.

So $\lambda=0$ is not an eigenvalue (as should by Rayleigh Quotient)

d) $\langle \phi_n, \phi_m \rangle = \int_1^b \sin\left(n\pi \frac{\ln x}{\ln b}\right) \sin\left(m\pi \frac{\ln x}{\ln b}\right) \frac{1}{x}\, dx$ $\quad u = \frac{\ln x}{\ln b}$, $\quad du = \frac{1}{x \ln b}\, dx$

$$= \ln b \underbrace{\int_0^1 \sin(n\pi u) \sin(m\pi u)\, du}_{(1/2)\delta_{nm}}$$

Analogously with basis

$$\Rightarrow \boxed{\langle \phi_n, \phi_m \rangle = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2} \ln b & \text{if } n = m \end{cases}}$$

e) $\phi_n(x) = \sin\left(n\pi \frac{\ln x}{\ln b}\right)$ since $1 < x < b \rightarrow 0 < \frac{\ln x}{\ln b} < 1$

Let $\theta = \frac{\ln x}{\ln b} \rightarrow \sin(n\pi\theta)$ has $n-1$ zeros in the open interval $0 < \theta < 1$ when $n\pi\theta = m\pi \quad m = 1, 2, 3 \ldots$

$$0 < \theta = \frac{m}{n} < 1$$

then $m = 1, 2, 3, \ldots, n-1$.

So $\sin\left(n\pi \frac{\ln x}{\ln b}\right)$ has $n-1$ zeros betwen $x = 1$ and $x = b$.

5.3.10 a) It's similar to (5.3.9 a)

b) It's as in (5.3.9 b) since $\frac{d\phi}{dx}(1) = 0 = \frac{d\phi}{dx}(b) \rightarrow \left.\frac{d\phi}{dx}\right|_1^b = 0$

c) $\phi(x) = A \cos(\sqrt{\lambda} \ln x) + B \sin(\sqrt{\lambda} \ln x)$

$\phi'(x) = -A \frac{\sqrt{\lambda}}{x} \sin(\sqrt{\lambda} \ln x) + B \frac{\sqrt{\lambda}}{x} \cos(\sqrt{\lambda} \ln x)$

$\phi'(1) = 0 = B\sqrt{\lambda} \rightarrow B = 0$

$\phi'(b) = 0 = -A \frac{\sqrt{\lambda}}{b} \sin(\sqrt{\lambda} \ln b) \rightarrow \sqrt{\lambda} \ln b = (2n-1)\frac{\pi}{2}$

$$\boxed{\lambda_n = \frac{(2n-1)\pi}{2 \ln b}}$$

$$\boxed{\phi_n = A_n \cos\left[\frac{(2n-1)\pi \ln x}{2 \ln b}\right]} \quad n = 1, 2, 3, \ldots$$

$\lambda_1 = \frac{\pi}{2 \ln b} < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \quad \lim_{n \to \infty} \lambda_n = \infty$

↳ smallest ↳ No largest

d) $\langle \phi_n, \phi_m \rangle = \int_0^b \cos\left[\frac{(2n-1)\pi \ln x}{2 \ln b}\right] \cos\left[\frac{(2m-1)\pi \ln x}{2 \ln b}\right] dx \quad u = \frac{\ln x}{\ln b}$

$$= \ln b \int_0^1 \cos\left[\frac{(2n-1)\pi u}{2}\right] \cos\left[\frac{(2m-1)\pi u}{2}\right] du = \begin{cases} 0 & \text{if } n \neq m \\ \frac{\ln b}{2} & \text{if } n = m \end{cases}$$

e) Let $\theta = \frac{\ln x}{\ln b} \rightarrow 0 < \theta < 1$

$\cos\left[\frac{(2n-1)}{2}\pi\theta\right]$ has zeros when $\frac{(2n-1)}{2}\pi\theta = \frac{(2m-1)}{2}\pi$

$\rightarrow 0 < \theta = \frac{2m-1}{2n-1} < 1 \rightarrow \boxed{m = 1, 2, \ldots, n-1} \rightarrow n-1$ zeros !!

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5.4.1 $c\rho \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(K_0 \dfrac{\partial u}{\partial x}\right) + \alpha u$ $\quad u(0,t) = 0 = u(L,t)$, $\quad u(x,0) = f(x)$

Set $u(x,t) = \phi(x)h(t)$ so that: $c\rho\phi \dfrac{dh}{dt} = h\dfrac{d}{dx}\left(K_0 \dfrac{d\phi}{dx}\right) + \alpha\phi h$

$$\Rightarrow \underbrace{\frac{1}{h}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{c\rho\phi}\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \frac{\alpha}{c\rho}}_{-\lambda} \Rightarrow \boxed{\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \alpha\phi + \lambda c\rho\phi = 0}$$

Sturm-Liouville equation with $p = K_0$, $q = \alpha$ and $\sigma = c\rho$

$$\frac{dh}{dt} = -\lambda h(t) \rightarrow \boxed{h(t) = c\,e^{-\lambda t}}$$

a) Since p, q and σ are real and continuous everywhere, and $p > 0$ and $\sigma > 0$ everywhere, and the boundary conditions are of the appropiate form, we have a regular Sturm-Liouville problem. Therefore, we can write the Rayleigh quotient:

$$\lambda = \frac{\cancelto{0}{-p\phi\frac{d\phi}{dx}\Big|_a^b} + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]dx}{\int_a^b \phi^2\sigma\,dx}$$

$$\lambda = \frac{\int_0^L \left[K_0\left(\frac{d\phi}{dx}\right)^2 - \alpha\phi^2\right]dx}{\int_0^L \phi^2 c\rho\,dx}$$

Since the first term in the denominator gives zero on substituing BC's.

We have assumed, based on physical principles, that $K_0(x), c(x), \rho(x) > 0$. Thus if $\alpha < 0$, $K_0\left(\frac{d\phi}{dx}\right)^2 - \alpha\phi^2 \geq 0$ and $\phi^2 c\rho \geq 0$. Note that $\phi^2 c\rho = 0 \longrightarrow \phi \equiv 0$ which cannot be an eigenfunction. Hence $\int_0^L \phi^2 c\rho\,dx > 0$.

Likewise $K_0\left(\frac{d\phi}{dx}\right)^2 - \alpha\phi^2 \geq -\alpha\phi^2 > 0 \rightarrow \int_0^L \left[K_0\left(\frac{d\phi}{dx}\right)^2 - \alpha\phi^2\right]dx > 0$

and hence $\boxed{\lambda > 0}$

b) Assume that the known eigenfunctions are $\phi_n(x)$, so that: $\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}}$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

Multiplying by $\phi_m(x)\sigma(x)$ and integrating.

$$\int_0^L f(x)\phi_m(x)\sigma(x)\,dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \phi_n(x)\phi_m(x)\sigma(x)\,dx}_{0 \text{ if } n \neq m}$$

$$= a_m \int_0^L [\phi_m(x)]^2 \cdot \sigma(x)\,dx \qquad \sigma = c\rho$$

$$\Rightarrow \boxed{a_n = \frac{\int_0^L f(x)\phi_n(x)\,c\rho\,dx}{\int_0^L [\phi_n(x)]^2 c\rho\,dx}}$$

with $c \equiv c(x)$, $\rho \equiv \rho(x)$.

c) Since λ_n is positive for all n,

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t} = \sum_{n=1}^{\infty} a_n \phi_n(x) \lim_{t\to\infty} e^{-\lambda_n t} = 0$$

Thus $\lim_{t\to\infty} u(x,t) = 0$ as expected, since both sides of the rod are kept at zero. Note that this only true in general when $\alpha \leq 0$ since then all λ_n's are positive. If $\alpha > 0$, then heat is being generated in the rod, so $u(x,t)$ may not go to 0.

* 5.4.2 $c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right)$ $\quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \quad u(x,0) = f(x)$

Let $u(x,t) = h(t)\phi(x) \rightarrow c\rho\phi \frac{dh}{dt} = h \frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right)$ dividin by $c\rho\phi$

$$\underbrace{\frac{1}{h}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{c\rho\phi}\frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right)}_{-\lambda} \rightarrow \boxed{\frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) + c\rho\lambda\phi(x) = 0}$$

Sturm - Lioville equation with $p = K_0$, $q = 0$ and $\sigma = c\rho$

$$\frac{dh}{dt} = -\lambda h(t) \rightarrow \boxed{h(t) = ce^{-\lambda t}}$$

a) $$\lambda = \frac{-K_0 \phi \cancel{\frac{\partial \phi}{\partial x}\Big|_0^L}^{\to 0} + \int_0^L K_0 \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2 c\rho\, dx} \geq 0$$

λ could be zero if $\frac{d\phi}{dx} = 0 \rightarrow \phi(x) = cte.$

b) $u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}$ with $\lambda_1 = 0$ $\phi_1(x) = 1$

$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x)$ Multiplying by $\phi_m(x)\sigma(x)$ and integrating:

$$\int_0^L f(x)\phi_m(x)\sigma\, dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \phi_n(x)\phi_m(x)\sigma\, dx}_{0 \text{ if } n \neq m} = a_m \int_0^L [\phi_m(x)]^2 \sigma\, dx$$

$$\rightarrow \boxed{a_n = \frac{\int_0^L f(x)\phi_n(x)\, c\rho\, dx}{\int_0^L [\phi_n(x)]^2 c\rho\, dx}}$$ with $c \equiv c(x)$ $\rho \equiv \rho(x)$.

c) For a large t, $u(x,t) \approx a_1 \phi_1 e^{-\lambda_1 t} = a_1$ since $\lambda_1 = 0$ $\phi_1(x) = 1$

Thus $\boxed{\lim_{t\to\infty} u(x,t) = a_1}$ $\quad a_1 = \frac{\int_0^L f(x)\, c\rho\, dx}{\int_0^L c\rho\, dx}$ since $\phi_1(x) = 1$,

wher $c \equiv c(x)$ and $\rho \equiv \rho(x)$.

* **5.4.3** $\frac{\partial u}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$ $\quad u(a,t)=0 \quad |u(0,t)|<\infty$, $\quad u(r,0)=f(r)$

$u(r,t)=h(t)\phi(r) \rightarrow \phi\frac{dh}{dt} = k\frac{1}{r}h\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)$ dividing by $k\phi h$

$$\underbrace{\frac{1}{kh}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)}_{-\lambda} \Rightarrow \boxed{\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)+\lambda r\phi=0}$$

Sturm-Liouville problem with $p=r$, $q=0$ and $\sigma=r$

$\frac{dh}{dt}=-\lambda k h \rightarrow \boxed{h=ce^{-\lambda k t}}$

Assume that the known eigenfunctions are $\phi_n(r)$, so that

$u(r,t)=\sum_{n=1}^{\infty} a_n e^{-k\lambda_n t}\phi_n(r)$ assume that $\phi_n(r)$'s are complete, then multiplying by $\phi_m(r)$ and integrating.

$f(r)=u(r,0)=\sum_{n=1}^{\infty} a_n\phi_n(r)$

$$\int_0^a f(r)\phi_m(r)\sigma\,dr = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^a \phi_n(r)\phi_m(r)\sigma\,dr}_{0 \text{ if } n\neq m} = a_m\int_0^a[\phi_m(r)]^2\sigma\,dr$$

$$\rightarrow \boxed{a_n=\frac{\int_0^a f(r)\phi_m(r)r\,dr}{\int_0^a[\phi_m(r)]^2 r\,dr}} \text{ since } \sigma=r.$$

Rayleigh quotient: $\lambda = \frac{-\cancel{r\phi(d\phi/dr)|_0^a}^{\,0} + \int_0^a\left[r\left(\frac{d\phi}{dr}\right)^2-\cancel{q}^{\,0}\phi^2\right]dr}{\int_a^b \phi^2 r\,dr} \geq 0$

if $\lambda=0 \rightarrow \frac{d\phi}{dr}=0 \rightarrow \phi=cte$, but $\phi(a)=0=cte \rightarrow \lambda=0$ is not eigenvalue

$\Rightarrow \boxed{\lambda>0}$

5.4.4 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ with $\frac{\partial u}{\partial x}(0,t)=0$ and $u(L,t)=0$

$u(x,0)=\sin(\pi x/L)$

Let $u(x,t)=h(t)\phi(x) \rightarrow \phi\frac{dh}{dt}=kh\frac{d^2\phi}{dx^2}$ dividing by kh, then

$$\underbrace{\frac{1}{kh}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{-\lambda} \rightarrow \boxed{\frac{d}{dx}\left(\frac{d\phi}{dx}\right)+\lambda\phi=0}$$

Sturm-Liouville problem with $p=1$, $q=0$, $\sigma=1$.

$\frac{dh}{dt}=-\lambda k h \rightarrow \boxed{h(t)=ce^{-\lambda k t}}$

Rayleigh quotient: $\lambda = \frac{-\cancel{\phi\frac{d\phi}{dx}\Big|_0^L}^{\,0} + \int_0^L\left[\left(\frac{d\phi}{dx}\right)^2-\cancel{q}^{\,0}\phi^2\right]dx}{\int_0^L \phi^2\,dx} \geq 0$

if $\lambda=0 \rightarrow \frac{d\phi}{dx}=cte \rightarrow \frac{d\phi}{dx}(0,t)=0=cte \rightarrow \lambda=0$ is not eigenvalue

$\Rightarrow$ **$\lambda>0$:** $\phi = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$\phi' = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$

$\phi'(0)=0=B\sqrt{\lambda} \rightarrow B=0$

$$\phi(L) = 0 = A\cos\sqrt{\lambda}L \rightarrow \sqrt{\lambda}L = (2n-1)\frac{\pi}{2} \rightarrow \lambda_n = \left[\frac{(2n-1)\pi}{2L}\right]^2$$

$$\boxed{\phi_n(x) = A_n \cos\left[\frac{(2n-1)\pi x}{2L}\right]} \qquad n = 1, 2, 3, \ldots$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n k t} \cos\left[\frac{(2n-1)\pi x}{2L}\right]}$$

$$\sin\left(\frac{\pi x}{L}\right) = u(x,0) = \sum_{n=1}^{\infty} A_n \cos\left[\frac{(2n-1)\pi x}{2L}\right]$$

Multiplying by $\cos\left[\frac{(2m-1)\pi x}{2L}\right]$ and integrating:

$$\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left[\frac{(2m-1)\pi x}{2L}\right]dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos\left[\frac{(2n-1)\pi x}{2L}\right]\cos\left[\frac{(2m-1)\pi x}{2L}\right]dx}_{(L/2)\delta_{nm}}$$

$$= A_m \frac{L}{2}$$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left[\frac{(2n-1)\pi x}{2L}\right]dx$$

$$= \frac{\cancel{2}}{L}\int_0^L \frac{1}{\cancel{2}}\left\{\sin\frac{(2n+1)\pi x}{2L} - \sin\frac{(2n-3)\pi x}{2L}\right\}dx$$

$$= \frac{1}{\cancel{L}}\left\{\frac{-2\cancel{L}}{(2n+1)\pi}\cos\frac{(2n+1)\pi x}{2L} + \frac{2\cancel{L}}{(2n-3)\pi}\cos\frac{(2n-3)\pi x}{2L}\right\}\Bigg|_{x=0}^{x=L}$$

$$A_n = \frac{2}{(2n+1)\pi} - \frac{2}{(2n-3)\pi} = \frac{2}{\pi}\left[\frac{(\cancel{2n}-3)-(\cancel{2n}+1)}{(2n+1)(2n-3)}\right] = \boxed{\frac{-8}{\pi(2n+1)(2n-3)}}$$

5.4.5 $\rho\dfrac{\partial^2 u}{\partial t^2} = T_0\dfrac{\partial^2 u}{\partial x^2} + \alpha u \qquad u(0,t) = 0 = u(L,t) \qquad u(x,0) = f(x)$, $\dfrac{\partial u}{\partial x}(x,0) = g(x)$

Let $u(x,t) = \phi(x)h(t) \rightarrow \rho\phi\dfrac{d^2h}{dt^2} = T_0 h\dfrac{d^2\phi}{dx^2} + \alpha h\phi$ dividing by $\rho h\phi$

$$\underbrace{\frac{1}{h}\frac{d^2h}{dt^2}}_{-\lambda} = \underbrace{\frac{1}{\rho\phi}\left[T_0\frac{d^2\phi}{dx^2} + \alpha\phi\right]}_{-\lambda} \rightarrow \boxed{\frac{d}{dx}\left(T_0\frac{d\phi}{dx}\right) + \alpha\phi + \lambda\rho\phi = 0}$$

$$\boxed{\frac{d^2h}{dt^2} + \lambda h = 0}$$

Sturm-Liouville problem with $p = T_0$, $q = \alpha$, $\sigma = \rho$

Rayleigh quotient: $\lambda = \dfrac{-\cancel{T_0\phi(d\phi/dx)\Big|_0^L}^{\,0} + \int_0^L\left[T_0(d\phi/dx)^2 - \alpha\phi^2\right]dx}{\int_0^L \phi^2\rho\,dx}$

if $\alpha(x) < 0$ then $\lambda \geq 0$ if $T_0 > 0$.
The only way to get $\lambda = 0$ would be to have $\phi'(x) = 0$ and $\phi(x) = cte$ but $\phi(0,t) = 0 \rightarrow cte = 0 \rightarrow \phi(x) \equiv 0 \ \forall x$ it's trivial solution, then $\lambda = 0$ is not eigenvalue. $\boxed{\lambda > 0}$.

$\lambda > 0$: $\rightarrow h(t) = C_1 \sin\sqrt{\lambda}\,t + C_2 \cos\sqrt{\lambda}\,t$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x)\left[C_{1n}\sin\sqrt{\lambda_n}\,t + C_{2n}\cos\sqrt{\lambda_n}\,t\right]$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \phi_n(x)\sin\sqrt{\lambda_n}\,t + \sum_{n=1}^{\infty} B_n \phi_n(x)\cos\sqrt{\lambda_n}\,t}$$

$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \phi_n(x)$ Multiplying by $\phi_m(x)\rho$ and integrating

$$\int_0^L f(x)\phi_m(x)\rho\,dx = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^L \phi_n(x)\phi_m(x)\rho(x)\,dx}_{0 \text{ if } n\neq m} = B_m \int_0^L [\phi_m(x)]^2 \rho\,dx$$

$$\rightarrow \boxed{B_n = \frac{\int_0^L f(x)\phi_n(x)\rho(x)\,dx}{\int_0^L [\phi_n(x)]^2 \rho(x)\,dx}}$$

$g(x) = \frac{\partial u}{\partial x}(x,0) = \sum_{n=1}^{\infty} A_n \sqrt{\lambda_n}\,\phi_n(x)$ Multiplying by $\phi_m(x)\rho(x)$ and integrating.

$$\int_0^L g(x)\phi_m(x)\rho\,dx = \sum_{n=1}^{\infty} A_n \sqrt{\lambda_n} \underbrace{\int_0^L \phi_n(x)\phi_m(x)\rho(x)\,dx}_{0 \text{ if } n\neq m} = A_m\sqrt{\lambda_m}\int_0^L [\phi_m(x)]^2 \rho\,dx$$

$$\rightarrow \boxed{A_m = \frac{\int_0^L g(x)\phi_n(x)\rho(x)\,dx}{\sqrt{\lambda_n}\int_0^L [\phi_n(x)]^2 \rho(x)\,dx}}$$

* 5.4.6 $\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t) = 0$ $\quad u(x,0) = f(x)$

$\frac{\partial u}{\partial x}(L,t) = -\frac{K}{T_0}u(L,t)$ $\quad \frac{\partial u}{\partial t}(x,0) = 0$ initially at rest

$u(x,t) = \phi(x)h(t) \rightarrow \rho_0\phi\frac{d^2h}{dt^2} = T_0 h\frac{d^2\phi}{dx^2}$ dividing by $\rho\phi h$.

$$\underbrace{\frac{1}{h}\frac{d^2h}{dt^2}}_{-\lambda} = \underbrace{\frac{T_0}{\rho_0}\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{-\lambda} \rightarrow \boxed{\frac{d}{dx}\left(\frac{d\phi}{dx}\right) + \lambda\frac{\rho_0}{T_0}\phi = 0}$$

Sturm-Liouville problem $p=1,\ q=0,\ \sigma = \rho_0/T_0$.

$\boxed{\frac{d^2h}{dt^2} + \lambda h = 0}$ Rayleight Quotient $\lambda = \dfrac{-\phi(d\phi/dx)\big|_0^L + \int_0^L (d\phi/dx)^2\,dx}{\int_0^L \phi^2(\rho_0/T_0)\,dx}$

$$\lambda = \frac{-\phi(L,t)\left(-\frac{K}{T_0}\phi(L,t)\right) + \cancelto{0}{\phi(0,t)}\frac{d\phi}{dx}(0,t) + \int_0^L (d\phi/dx)^2\,dx}{\int_0^L \phi^2(\rho_0/T_0)\,dx}$$

$$\lambda = \frac{(K/T_0)\phi(L,t) + \int_0^L (d\phi/dx)^2\,dx}{\int_0^L \phi^2(\rho_0/T_0)\,dx} \geq 0 \quad \text{since } (K/T_0)\phi(L,t) > 0$$

$\lambda = 0$ if $\phi(L,t) = 0$ and $d\phi/dx = 0 \rightarrow \phi = cte$, but $\phi(0,t) = 0 = cte$

$\rightarrow \lambda = 0$ is not eigenvalue. $\Rightarrow \boxed{\lambda > 0} \rightarrow$

$h(t) = C\cos\sqrt{\lambda}\,t + D\sin\sqrt{\lambda}\,t$ with $\lambda > 0$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \phi_n(x)\left[C_n\cos\sqrt{\lambda_n}\,t + D_n\sin\sqrt{\lambda}\,t\right]}$$

We will assume the eigenvalues of the boundary value problem are $\{\lambda_n\}_{n\in\mathbb{N}}$ with corresponding eigenfunctions $\{\phi_n(x)\}_{n\in\mathbb{N}}$

$f(x) = u(x,0) = \sum_{n=1}^{\infty} C_n\phi_n(x)$ Multiplying by $\phi_m(x)\rho_0/T_0$ and integrating

$$\int_0^L f(x)\phi_m(x)\frac{\rho_0}{T_0}\,dx = \sum_{n=1}^{\infty} C_n \underbrace{\int_0^L \phi_n(x)\phi_m(x)\frac{\rho_0}{T_0}\,dx}_{0 \text{ if } n\neq m} = C_m\int_0^L[\phi_m(x)]^2\frac{\rho_0}{T_0}\,dx$$

$$\rightarrow \boxed{C_n = \frac{\int_0^L f(x)(\rho_0/T_0)\,dx}{\int_0^L[\phi_n(x)]^2(\rho_0/T_0)\,dx}}$$

$$0 = \frac{\partial u}{\partial x}(x,0) = \sum_{n=1}^{\infty} D_n\sqrt{\lambda_n}\,\phi_n(x) \longrightarrow \boxed{D_n = 0}$$

5.5.1 a) $$p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_0^L = p(L)\left(\cancelto{0}{u(L)}\frac{dv}{dx}(L)-\cancelto{0}{v(L)}\frac{du}{dx}(L)\right) - p(0)\left(\cancelto{0}{u(0)}\frac{dv}{dx}(0)-\cancelto{0}{v(0)}\frac{du}{dx}(0)\right)=0$$

b) $$p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_0^L = p(L)\left(\cancelto{0}{u(L)}\frac{dv}{dx}(L)-\cancelto{0}{v(L)}\frac{du}{dx}(L)\right) - p(0)\left(u(0)\cancelto{0}{\frac{dv}{dx}(0)}-v(0)\cancelto{0}{\frac{du}{dx}(0)}\right)=0$$

c) $$p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_0^L = p(L)\left(u(L)\cancelto{0}{\frac{dv}{dx}(L)}-v(L)\cancelto{0}{\frac{du}{dx}(L)}\right) - p(0)\left(u(0)\frac{dv}{dx}(0)-v(0)\frac{du}{dx}(0)\right)$$

$$= \cancel{-p(0)\left(u(0)\,h\,v(0)-v(0)\,h\,u(0)\right)} = 0$$

d) $$p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_a^b = p(b)\left(u(b)\frac{dv}{dx}(b)-v(b)\frac{du}{dx}(b)\right) - p(a)\left(u(a)\frac{dv}{dx}(a)-v(a)\frac{du}{dx}(a)\right)$$

$$= \cancel{p(a)\left(u(a)\frac{dv}{dx}(a)-v(a)\frac{du}{dx}(a)\right)} - \cancel{p(a)\left(u(a)\frac{dv}{dx}(a)-v(a)\frac{du}{dx}(a)\right)}=0$$

e) $$p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_a^b = p(b)\left(u(b)\frac{dv}{dx}(b)-v(b)\frac{du}{dx}(b)\right) - p(a)\left(u(a)\frac{dv}{dx}(a)-v(a)\frac{du}{dx}(a)\right)$$

$$= p(b)\left(u(a)\frac{dv}{dx}(a)-v(a)\frac{du}{dx}(a)\right) - p(a)\left(u(a)\frac{dv}{dx}(a)-v(a)\frac{du}{dx}(a)\right)$$

$$= \left[p(b)-p(a)\right]\left(u(a)\frac{dv}{dx}(a)-v(a)\frac{du}{dx}(a)\right)$$

$$= 0 \quad \text{only if } p(b)=p(a)$$

f) $$p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_0^L = p(L)\left(\cancelto{0}{u(L)}\frac{dv}{dx}(L)-\cancelto{0}{v(L)}\frac{du}{dx}(L)\right) - p(0)\left(u(0)\frac{dv}{dx}(0)-v(0)\frac{du}{dx}(0)\right)$$

$$= -p(0)\left[\cancelto{0}{\lim_{x\to 0} u(x)\frac{dv}{dx}(x)} + \cancelto{0}{\lim_{x\to 0} v(x)\frac{du}{dx}(x)}\right] = 0$$

~~$h(t) = C\cos\sqrt{\lambda}t + D\sin\sqrt{\lambda}t$ with $\lambda > 0$~~

$$\Rightarrow \cancel{\boxed{u(x,t) = \sum_{n=1}^{\infty} \phi_n(x)\left[C_n \cos\sqrt{\lambda_n}t + D_n \sin\sqrt{\lambda_n}t\right]}}$$

~~We will assume the eigenvalues of the boundary value problem are $\{\lambda_n\}_{n\in\mathbb{N}}$ with ~~eigen~~ corresponding eigenfunctions $\{\phi_n(x)\}_{n\in\mathbb{N}}$~~

~~$f(x) = u(x,0) = \sum_{n=1}^{\infty} C_n \phi_n(x)$ Multiplying by $\phi_m(x)\,\rho_0/T_0$ and integrating.~~

$$\cancel{\int_0^L f(x)\phi_m(x)\frac{\rho_0}{T_0}dx = \sum_{n=1}^{\infty} C_n \underbrace{\int_0^L \phi_n(x)\phi_m(x)\frac{\rho_0}{T_0}dx}_{\text{so if } n\neq m} = C_m \int_0^L [\phi_m(x)]^2 \frac{\rho_0}{T_0}dx}$$

$$\cancel{\rightarrow \boxed{C_m = \frac{\int_0^L f(x)\phi_m(x)\frac{\rho_0}{T_0}dx}{\int_0^L [\phi_m(x)]^2\frac{\rho_0}{T_0}dx}}}$$

~~$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} D_n \sqrt{\lambda_n}\cos\sqrt{\lambda_n}t\,\phi_n(x) \rightarrow \boxed{D_n = 0}$~~

*9) If $\rho = \text{constant} \rightarrow \rho\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_0^L = 0 \Longleftrightarrow \left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_0^L = 0$

$$\Longleftrightarrow u(L)v'(L) - v(L)u'(L) - u(0)v'(0) + v(0)u'(0) = 0.$$

$$[\alpha u(0) + \beta u'(0)][\gamma v(0) + \delta v'(0)] - [\alpha v(0) + \beta v'(0)][\gamma u(0) + \delta u'(0)] - u(0)v'(0) + v(0)u'(0) = 0.$$

$$\Longleftrightarrow \cancel{\alpha\gamma u(0)v(0)} + \beta\gamma u'(0)v(0) + \alpha\delta u(0)v'(0) + \cancel{\beta\delta u'(0)v'(0)} - [\cancel{\alpha\gamma u(0)v(0)} + \beta\gamma v'(0)u(0) + \alpha\delta v(0)u'(0) + \cancel{\beta\delta u'(0)v'(0)}] - u(0)v'(0) + v(0)u'(0) = 0$$

$$\Longleftrightarrow (\beta\gamma + 1)u'(0)v(0) + (\alpha\delta - 1)u(0)v'(0) - \beta\gamma u(0)v'(0) - \alpha\delta u'(0)v(0) = 0$$

$$(\beta\gamma - \alpha\delta + 1)u'(0)v(0) + (\alpha\delta - \beta\gamma - 1)u(0)v'(0) = 0$$

$$(\alpha\delta - \beta\gamma - 1)u'(0)v(0) - (\alpha\delta - \beta\gamma - 1)u(0)v'(0) = 0$$

$$(\alpha\delta - \beta\gamma - 1)\left(u'(0)v(0) - u(0)v'(0)\right) = 0$$

$$\boxed{\alpha\delta - \beta\gamma = 1}$$

Condition for that the boundary value problem can be <u>self-adjoint</u>

c) Assume $\beta(x) = c\rho(x)$, then the spatial equation obtained by separation of variables has the form:

$$\frac{1}{\phi(x)}\frac{T_0}{\beta(x)}\frac{d^2\phi}{dx^2} + \frac{\alpha(x)}{\beta(x)} = -\lambda \rightarrow T_0\frac{d^2\phi}{dx^2} + \alpha(x)\phi(x) + \lambda\beta(x)\phi(x) = 0$$

Sturm - Lioville form $\rightarrow$ $\boxed{\frac{d}{dx}\left(T_0\frac{d\phi}{dx}\right) + (\alpha(x) + \lambda\beta(x))\phi(x) = 0}$

The time dependent equation has the form:

$$\frac{1}{ch(t)}\frac{d^2h}{dt^2} - \frac{1}{h(t)}\frac{dh}{dt} = -\lambda \rightarrow \frac{d^2h}{dt^2} - c\frac{dh}{dt} + \lambda c h(t) = 0$$

wich is a 2nd order linear constant coefficients homogeneus ODE. Its characteristic equation has the form:

$$r^2 - cr + \lambda c = 0 \rightarrow r = \frac{c \pm (c^2 - 4\lambda c)^{1/2}}{2}$$

let $r_1 = \frac{c + (c^2 - 4\lambda c)^{1/2}}{2}$ and $r_2 = \frac{c - (c^2 - 4\lambda c)^{1/2}}{2}$

then $\boxed{h(t) = Ae^{r_1 t} + Be^{r_2 t}}$

* 5.3.3 $\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + (\lambda\beta(x) + \gamma(x))\phi = 0$ multiply by $H(x)$.

$$H(x)\frac{d^2\phi}{dx^2} + \alpha(x)H(x)\frac{d\phi}{dx} + (\lambda\beta(x) + \gamma(x))H(x)\phi = 0$$

we would like to write $H(x)\frac{d^2\phi}{dx^2} + \alpha(x)H(x)\frac{d\phi}{dx}$ in the form $\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right]$ for some $p(x)$. Note that

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] = p(x)\frac{d^2\phi}{dx^2} + p'(x)\frac{d\phi}{dx} = H(x)\frac{d^2\phi}{dx^2} + \alpha(x)H(x)\frac{d\phi}{dx}$$

$\Rightarrow p'(x) = \alpha(x)H(x)$ and $p(x) = H(x)$

$$\frac{p'(x)}{p(x)} = \alpha(x) \rightarrow \int\frac{p'(x)}{p(x)}dx = \int\alpha(x)dx$$

$$\ln|p(x)| + c = \int\alpha(x)dx \rightarrow |p(x)| = ce^{\int\alpha(x)dx}$$

Assuming $p(x) > 0 \ \forall x \Rightarrow p(x) = ce^{\int\alpha(x)dx} = H(x)$

for simplicity choose $c = 1 \longrightarrow \boxed{H(x) = e^{\int\alpha(x)dx}}$

Integrating factor!!

5.5.2 Let λ_n and λ_m be eigenvalues with corresponding eigenfunctions ϕ_n and ϕ_m respectively. Let the operator L be:

$$L(\phi) = \frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi$$

then $L(\phi_n) + \lambda_n \sigma(x)\phi_n = 0$ and $L(\phi_m) + \lambda_m \sigma(x)\phi_m = 0$

using the Green's formula ec (5.5.8) with $u = \phi_m$ $v = \phi_n$:

$$\int_1^2 [\phi_m L(\phi_n) - \phi_n L(\phi_m)]\,dx = p(x)\left(\phi_m \frac{d\phi_n}{dx} - \phi_n \frac{d\phi_m}{dx}\right)\Big|_1^2$$

$$\int_1^2 [-\phi_m \lambda_n \phi_n \sigma + \phi_n \lambda_m \phi_m \sigma]\,dx = p(2)\left(\phi_m(2)\frac{d\phi_n}{dx}(2) - \phi_n(2)\frac{d\phi_m}{dx}(2)\right) - p(1)\left(\cancelto{0}{\phi_m(1)}\frac{d\phi_n}{dx}(1) - \cancelto{0}{\phi_n(1)}\frac{d\phi_m}{dx}(1)\right)$$

$$(\lambda_m - \lambda_n)\int_1^2 \phi_n \phi_m \sigma\,dx = p(2)\left(\phi_m(2)\frac{\phi_n(2)}{2} - \phi_n(2)\frac{\phi_m(2)}{2}\right) = 0$$

If $\lambda_m \neq \lambda_n \rightarrow \boxed{\int_1^2 \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0}$ $m \neq n$

↳ The weighting function.

5.5.3 If $L(\phi) = -\lambda\sigma(x)\phi$, then:

$$\underbrace{\int_a^b [uL(v) - vL(u)]\,dx}_{0} = \int_a^b [u(-\lambda_v \sigma v) - v(-\lambda_u \sigma u)]\,dx$$

$$= (\lambda_u - \lambda_v)\int_a^b uv\sigma(x)\,dx$$

If $\lambda_u \neq \lambda_v \rightarrow \boxed{\int_a^b uv\sigma(x)\,dx = 0}$ $u \neq v$

↳ weight.

5.5.4 Let $L = \frac{d^2}{dx^2}$ such that $L(\phi) = -\lambda\phi$ with periodic boundary conditions $\phi(-L) = \phi(L)$, $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$

$\Rightarrow \frac{d^2\phi}{dx^2} = -\lambda\phi$ OR $\boxed{\frac{d^2\phi}{dx^2} + \lambda\phi = 0}$

$\lambda > 0$: $\phi = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$ and by BC's

$\phi(-L) = \cancel{A\cos\sqrt{\lambda}(-L)} + B\sin\sqrt{\lambda}(-L) = \cancel{A\cos\sqrt{\lambda}L} + B\sin\sqrt{\lambda}L = \phi(L)$

$-B\sin\sqrt{\lambda}L = B\sin\sqrt{\lambda}L \rightarrow 2B\sin\sqrt{\lambda}L = 0$

$\frac{d\phi}{dx}(-L) = -A\sqrt{\lambda}\sin\sqrt{\lambda}(-L) + \cancel{B\sqrt{\lambda}\cos\sqrt{\lambda}(-L)}$ $\rightarrow \boxed{\lambda = (n\pi/L)^2}$

$= -A\sqrt{\lambda}\sin\sqrt{\lambda}L + \cancel{B\sqrt{\lambda}\cos\sqrt{\lambda}L} = \frac{d\phi}{dx}(L)$

$2A\sqrt{\lambda}\sin\sqrt{\lambda}L = 0 \rightarrow \boxed{\lambda = (n\pi/L)^2}$ $n = 1, 2, 3, \ldots$

Then, for $\lambda = \left(\frac{n\pi}{L}\right)^2$ there are two eigenfunctions $\rightarrow$ $\boxed{\begin{array}{l}\phi_1 = \cos\left(\frac{n\pi x}{L}\right) \\ \phi_2 = \sin(n\pi x/L)\end{array}}$ It's are orthogonals!

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5.5.5 $L = \frac{d^2}{dx^2} + 6\frac{d}{dx} + 9$

a) $L(e^{rx}) = r^2 e^x + 6re^x + 9e^x = (r^2+6r+9)e^x = (r+3)^2 e^x$

b) $L(y) = 0 \rightarrow \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$ from part a) let $y = e^{rx}$

$L(e^{rx}) = 0 \rightarrow (r+3)^2 e^x = 0 \rightarrow (r+3)^2 = 0 \rightarrow \boxed{r=-3}$

$\Rightarrow \boxed{y = e^{-3x}}$ with c a constant.

c) $z \equiv z(x,r) \rightarrow \frac{\partial}{\partial r} L(z) = \frac{\partial}{\partial r}\left[\frac{d^2z}{dx^2} + 6\frac{dz}{dx} + 9z\right]$

$= \frac{d^2}{dx^2}\left(\frac{\partial z}{\partial r}\right) + 6\frac{d}{dx}\left(\frac{\partial z}{\partial r}\right) + 9\left(\frac{\partial z}{\partial r}\right) = L\left(\frac{\partial z}{\partial r}\right)$

d) If $z = e^{rx} \rightarrow L\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r} L(z) = \frac{\partial}{\partial r} L(e^{rx}) = \frac{\partial}{\partial r}\left[(r+3)^2 e^{rx}\right]$

$= 2(r+3)e^{rx} + x(r+3)^2 e^{rx}$

$= [2 + x(r+3)](r+3)e^{rx}$

e) Let $y = \frac{\partial z}{\partial r} \Rightarrow L(y) = 0 \rightarrow L\left(\frac{\partial z}{\partial r}\right) = 0 \rightarrow [2+x(r+3)](r+3)e^{rx} = 0$

$\boxed{r=-3}$

if $z = e^{rx} \rightarrow y = \frac{\partial z}{\partial r} = xe^{rx} \rightarrow \boxed{y = xe^{-3r}}$

5.5.6 $f(x) = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = \sum_{n=0}^{6} a_n x^n$

Note if there is some z such that $f(z) = 0$
(with $z = \alpha + i\beta$, an complex number) $\overline{f(z)} = \overline{0} = 0$

$\overline{f(z)} = \overline{\sum_{n=0}^{\infty} a_n z^n} = \sum_{n=0}^{\infty} \overline{a_n}\,\overline{z}^n = \underbrace{\sum_{n=1}^{\infty} a_n \overline{z}^n}_{f(\overline{z})}$ since the a_n's are real numbers.

$\boxed{\overline{f(z)} = f(\overline{z})}$

then, if x is a root of $f(x) \rightarrow$ $f(x) = 0$
$\overline{f(x)} = \overline{0} = 0$
$\boxed{f(\overline{x}) = 0}$

thus $\overline{x}$ is also a root.

5.5.7 $L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$ Let $\phi = u + iv$ with u, v real.

$L(\phi) = \frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi = \frac{d}{dx}\left(p\frac{d(u+iv)}{dx}\right) + q(u+iv)$

$= \left[\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right] + i\left[\frac{d}{dx}\left(p\frac{dv}{dx}\right) + qv\right] = L(u) + iL(v)$

$$\Rightarrow \quad L(u+iv) = L(u) + i\,L(v) \quad \text{and}$$
$$L(u-iv) = L(u) + i\,L(-v) = L(u) - i\,L(v) \quad \underline{\text{since } L(-v) = -L(v)}$$

Note that $\overline{L(u+iv)} = L(u) - iL(v) = L(u-iv) = L(\overline{u+iv})$

$$\Rightarrow \quad \boxed{\overline{L(\phi)} = L(\overline{\phi})}$$

5.5.8 $L = (d^4/dx^4)$

a) $$uL(v) = u\,\frac{d^4v}{dx^4} = \frac{d}{dx}\left(u\,\frac{d^3v}{dx^3}\right) - \frac{du}{dx}\,\frac{d^3v}{dx}$$
$$= \frac{d}{dx}\left(u\,\frac{d^3v}{dx^3}\right) - \frac{d}{dx}\left(\frac{du}{dx}\,\frac{d^2v}{dx^2}\right) + \frac{d^2u}{dx^2}\,\frac{d^2v}{dx^2}$$

and $$vL(u) = \frac{d}{dx}\left(v\,\frac{d^3u}{dx^3}\right) - \frac{d}{dx}\left(\frac{dv}{dx}\,\frac{d^2u}{dx^2}\right) + \frac{d^2v}{dx^2}\,\frac{d^2u}{dx^2}.$$

$$\Rightarrow \quad uL(v) - vL(u) = \frac{d}{dx}\left[u\,\frac{d^3v}{dx^3} - v\,\frac{d^3u}{dx^3} - \frac{du}{dx}\,\frac{d^2v}{dx^2} + \frac{dv}{dx}\,\frac{d^2u}{dx^2}\right]$$

b) $$\int_0^1 [uL(v) - vL(u)]\,dx = \left[u\,\frac{d^3v}{dx^3} - v\,\frac{d^3u}{dx^3} - \frac{du}{dx}\,\frac{d^2v}{dx^2} + \frac{dv}{dx}\,\frac{d^2u}{dx^2}\right]_{x=0}^{x=1}$$

c) $$= \cancelto{0}{u(1)}\,\frac{d^3v}{dx^3}(1) - \cancelto{0}{v(1)}\,\frac{d^3v}{dx^3}(1) - \frac{du}{dx}(1)\,\cancelto{0}{\frac{d^2v}{dx^2}(1)} + \frac{dv}{dx}(1)\,\cancelto{0}{\frac{d^2u}{dx^2}(1)}$$
$$- \cancelto{0}{u(0)}\,\frac{d^3v}{dx^3}(0) + \cancelto{0}{v(0)}\,\frac{d^3v}{dx^3}(0) + \cancelto{0}{\frac{du}{dx}(0)}\,\frac{d^2v}{dx^2}(0) - \cancelto{0}{\frac{dv}{dx}(0)}\,\frac{d^2u}{dx^2}(0)$$

$$\int_0^1 [uL(v) - vL(u)]\,dx = 0$$

d) For example: $\frac{d\phi}{dx}(0) = 0 \quad \frac{d^3\phi}{dx^3}(0) = 0, \quad \phi(1) = 0 \quad \frac{d\phi}{dx}(1) = 0$

OR $\frac{d^3\phi}{dx^3}(0) = 0 = \frac{d^3\phi}{dx^3}(1) \qquad \frac{d\phi}{dx}(0) = 0 = \frac{d\phi}{dx}(1)$

OR $\frac{d^2\phi}{dx^2}(0) = 0 = \frac{d^2\phi}{dx^2}(1) \qquad \phi(0) = 0 = \phi(1)$

OR $\phi(0) = \phi(1), \quad \phi'(0) = \phi'(1) \quad \frac{d^2\phi}{dx^2}(0) = \frac{d^2\phi}{dx^2}(1)$

and $\frac{d^3\phi}{dx^3}(0) = \frac{d^3\phi}{dx^3}(1) \longrightarrow$ Periodic Boundary conditions.

⋮

Mixed...

e) $\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0 \rightarrow L(\phi) + \lambda e^x \phi = 0$ OR $L(\phi) = -\lambda e^x \phi$

Suppose ϕ_m and ϕ_n ar eigenfunctions associated with the eigenvalues λ_m and λ_n, and $\lambda_m \neq \lambda_n$. From part c) we know:

$$0 = \int_0^L [\phi_n L(\phi_m) - \phi_m L(\phi_n)]\, dx$$

$$0 = \int_0^L [\phi_m(-\lambda_n e^x \phi_n) - \phi_n(-\lambda_m e^x \phi_m)]\, dx$$

$$0 = (\lambda_m - \lambda_n) \int_0^L \phi_m \phi_n e^x\, dx$$

weighting function

Since $\lambda_m \neq \lambda_n$, we must have $\boxed{\int_0^L \phi_m \phi_n e^x\, dx = 0}$

* 5.5.9 $\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0$ Multiplying by ϕ and integrating.

$$\int_0^1 \phi \frac{d^4\phi}{dx^4}\, dx + \lambda \int_0^1 e^x \phi^2\, dx = 0 \rightarrow \lambda = \frac{-\int_0^1 \phi \frac{d^4\phi}{dx^4}\, dx}{\int_0^1 \phi^2 e^x\, dx}$$

$$\int_0^1 \phi \frac{d^4\phi}{dx^4}\, dx = \cancel{\phi \frac{d^3\phi}{dx^3}\Big|_0^1}^{\,0} - \int_0^1 \frac{d\phi}{dx}\frac{d^3\phi}{dx^3}\, dx \quad \text{(integration by parts)}$$

$$= \cancel{-\frac{d\phi}{dx}\frac{d^2\phi}{dx^2}\Big|_0^1}^{\,0} + \int_0^1 \left(\frac{d^2\phi}{dx^2}\right)^2 dx$$

$$\Rightarrow \lambda = \frac{-\int_0^1 \left(\frac{d^2\phi}{dx^2}\right)^2 dx}{\int_0^1 \phi^2 e^x\, dx} \leq 0$$

if $\lambda = 0 \rightarrow \frac{d^2\phi}{dx^2} = 0 \rightarrow \phi = C_0 + C_1 x$

$$\left.\begin{aligned} \phi(0) &= 0 = C_0 \\ \frac{d\phi}{dx}(0) &= 0 = C_1 \end{aligned}\right\} \quad \begin{aligned} \phi &= C_1 x \\ \phi(1) &= 0 = C_1 \end{aligned}$$

$\Rightarrow \phi \equiv 0 \rightarrow \lambda = 0$ is not eigenvalue $\rightarrow \boxed{\lambda < 0}$

5.5.10 a) $p\left(\phi_1 \frac{d\phi_2}{dx} - \phi_2 \frac{d\phi_1}{dx}\right) =$ constant, where ϕ_1 and ϕ_2 satisfy at least $\beta_1 \phi\Big|_{\text{Boundary}} + \beta_2 \frac{d\phi}{dx}\Big|_{\text{Boundary}} = 0$ at one boundary

$\Rightarrow \frac{d\phi}{dx}\Big|_{\text{Boundary}} = -\frac{\beta_1}{\beta_2}\phi\Big|_{\text{Boundary}} \Rightarrow$ On that boundary:

$$p\left[\phi_1\left(-\frac{\beta_1}{\beta_2}\phi_2\right)-\phi_2\left(-\frac{\beta_1}{\beta_2}\phi_1\right)\right]_{\text{Boundary}} = \text{Constant}$$

$$\frac{\beta_1}{\beta_2}p(x)\left[\cancel{\phi_1(-\phi_2)-\phi_2(-\phi_1)}\right]_{\text{Boundary}} = \text{constant}$$

$$\Rightarrow \text{Constant} = 0 \rightarrow \boxed{p\left[\phi_1\frac{d\phi_2}{dx}-\phi_2\frac{d\phi_1}{dx}\right]=0}$$

b) The singular case occurs if the coeficient of the second derivative of the diferential operator is zero at ~~the~~ an endpoint; for example if $p(x)=0$ at $x=0$ [i.e $p(0)=0$]. So evaluate **(5.5.22)** at singular edpoint.

$$\cancel{p\Big|_{x\to\text{Singular point}}}^{\,0}\left(\phi_1\frac{d\phi_2}{dx}-\phi_2\frac{d\phi_1}{dx}\right)\Big|_{\text{singular point}} = \text{constant}$$

$$\rightarrow \text{constant} = 0 \rightarrow \boxed{p\left[\phi_1\frac{d\phi_2}{dx}-\phi_2\frac{d\phi_1}{dx}\right]=0}$$

5.5.11 $L = p(x)\dfrac{d^2}{dx^2}+r(x)\dfrac{d}{dx}+q(x)$

* a) $\displaystyle\int_a^b vL(u)\,dx = \int_a^b v\left[p\frac{d^2u}{dx^2}+r\frac{du}{dx}+qu\right]dx$

$$\int_a^b pv\frac{d^2u}{dx^2}dx = pv\frac{du}{dx}\Big|_a^b-\int_a^b\frac{d(pv)}{dx}\frac{du}{dx}dx \quad \text{(integrating by parts).}$$

$$= pv\frac{du}{dx}\Big|_a^b-\frac{d(pv)}{dx}u\Big|_a^b+\int_a^b\frac{d^2(pv)}{dx^2}u\,dx$$

$$\int_a^b rv\frac{du}{dx}dx = rvu\Big|_a^b-\int_a^b\frac{d(rv)}{dx}u\,dx$$

$$\Rightarrow \int_a^b vL(u)\,dx = \left(pv\frac{du}{dx}-\frac{d(pv)}{dx}u+rvu\right)\Big|_a^b + \int_a^b\left(u\frac{d^2(pv)}{dx^2}-u\frac{d(rv)}{dx}+qvu\right)dx$$

$$\frac{d^2(pv)}{dx^2} = \frac{d}{dx}\left(p\frac{dv}{dx}+\frac{dp}{dx}v\right) = p\frac{d^2v}{dx^2}+\frac{d^2p}{dx^2}v+2\frac{dp}{dx}\frac{dv}{dx}$$

$$\rightarrow \int_a^b vL(u)\,dx = (\qquad)\Big|_a^b+\int_a^b u\left(p\frac{d^2v}{dx^2}+\frac{d^2p}{dx^2}v+2\frac{dp}{dx}\frac{dv}{dx}-\frac{dr}{dx}v-r\frac{dv}{dx}-qv\right)dx$$

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$$\int_a^b v L(u)\,dx = \left(\quad \right)\Big|_a^b + \int_a^b u \underbrace{\left[p\frac{d^2v}{dx^2} + \left(2\frac{dp}{dx} - r\right)\frac{dv}{dx} + \left(\frac{d^2p}{dx^2} - \frac{dr}{dx} + q\right)v \right]}_{L^*(v)} dx$$

$$\Rightarrow \int_a^b \left[v L(u) - u L^*(v) \right] dx = \left(p v \frac{du}{dx} - \frac{d(pv)}{dx} u + r v u \right)\Big|_a^b$$

$$\int_a^b \left[u L^*(v) - v L(u) \right] dx = \underbrace{-\left[p v \frac{du}{dx} - p\frac{dv}{dx}u - \frac{dp}{dx} v u + r v u \right]_a^b}_{H(x)}$$

$$\Rightarrow \boxed{H(x) = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right) + uv\left(\frac{dp}{dx} - r\right)}$$

$$\underline{L = L^*} \Rightarrow \cancel{p\frac{d^2}{dx^2}} + r\frac{d}{dx} + q = \cancel{p\frac{d^2}{dx^2}} + \left(2\frac{dp}{dx} - r\right)\frac{d}{dx} + \left(\frac{d^2p}{dx^2} - \frac{dr}{dx} + q\right)$$

$$\left.\begin{aligned} &\Rightarrow 2\frac{dp}{dx} - r = r \rightarrow \boxed{\frac{dp}{dx} = r} \\ &\frac{d^2p}{dx^2} - \frac{dr}{dx} + \cancel{q} = \cancel{q} \rightarrow \frac{d^2p}{dx^2} = \frac{dr}{dx} \end{aligned}\right\}$$

with this, the operator L take form of Sturm-Liouvil' problem:

$$L(\phi) = p\frac{d^2\phi}{dx^2} + r\frac{d\phi}{dx} + q = p\frac{d^2\phi}{dx^2} + \frac{dp}{dr}\frac{d\phi}{dx} + q$$

$$\boxed{L(\phi) = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)} \rightarrow \text{Sturm-Liouville operator!!}$$

b) $$H(x)\Big|_0^L = p(L)\left(u(L)\frac{dv}{dx}(L) - v(L)\frac{du}{dx}(L)\right) + u(L)v(L)\left(\frac{dp}{dx}(L) - r\right)$$
$$- p(0)\left(\cancelto{0}{u(0)}\frac{dv}{dx}(0) - v(0)\frac{du}{dx}(0)\right) - \cancelto{0}{u(0)}v(0)\left(\frac{dp}{dx}(0) - r\right)$$

$$= p(L)\left(u(L)\frac{dv}{dx}(L) + v(L)u(L)\right) + u(L)v(L)\left(\frac{dp}{dx}(L) - r\right) + p(0)v(0)\frac{du}{dx}(0)$$

$$H(x)\Big|_0^L = u(L)p(L)\left(\frac{dv}{dx}(L) + v(L)\right) + u(L)v(L)\left(\frac{dp}{dx}(L) - r\right) + p(0)v(0)\frac{du}{dx}(0)$$

Adjoint boundary Conditions!!

For $H(x)\Big|_0^L = 0$ we need $\boxed{\frac{dv}{dx}(L) + v(L) = 0}$ $\boxed{v(L) = 0}$ and $\boxed{v(0) = 0}$.

If $L = L^* \Rightarrow \frac{dp}{dx} = r \rightarrow$ only need $\frac{dv}{dx}(L) + v(L) = 0$ and $v(0) = 0$.

5.5.12 $L(\phi) = p(x)\dfrac{d^2\phi}{dx^2} + r(x)\dfrac{d\phi}{dx} + q(x)\phi$

$$\overline{L(\phi)} = \overline{p(x)}\frac{d^2\overline{\phi}}{dx^2} + \overline{r(x)}\frac{d\overline{\phi}}{dx} + \overline{q}(x)\overline{\phi}$$

$$= p(x)\frac{d^2\overline{\phi}}{dx^2} + r(x)\frac{d\overline{\phi}}{dx} + q(x)\overline{\phi}$$ since p, q an r are real functions.

$$\boxed{\overline{L(\phi)} = L(\overline{\phi})}$$

a) $L(\phi) + \lambda\sigma\phi = 0 \rightarrow$ Eigenvalue problem.

$$\overline{L(\phi) + \lambda\sigma\phi} = \overline{0} \rightarrow \overline{L(\phi)} + \overline{\lambda}\,\overline{\sigma}\,\overline{\phi} = 0$$ $\overline{\sigma} = \sigma$ since σ is a real function of x.

Since $\overline{L(\phi)} = L(\overline{\phi}) \Rightarrow \boxed{L(\overline{\phi}) + \overline{\lambda}\sigma\overline{\phi} = 0}$ $\hookrightarrow$ $\overline{\lambda}$ is an eigenvalue with eigenfunction $\overline{\phi}$

b) From part a) in exercise 5.5.11:

$$\int_a^b [\phi L^*(\psi) - \psi L(\phi)]\,dx = \cancel{H(x)\Big|_a^b} \rightarrow 0$$ for adjoint boundary conditions in ψ an ϕ (see exercise 5.5.11 b)

$$\int_a^b [\phi(-\nu\sigma\psi) - \psi(-\lambda\sigma\phi)]\,dx = 0$$

$$(\lambda - \nu)\int_a^b \phi\psi\sigma\,dx = 0 \rightarrow \text{If } \lambda \neq \nu \Rightarrow \boxed{\int_a^b \phi\psi\sigma\,dx = 0}$$

5.5.13 $L(u) = f(x)$

$L(\phi_n) = 0 \rightarrow \phi_n$'s are all homogeneus solutions.

$$\int_a^b [u\underbrace{L^*(\phi_n)}_{0} - \phi_n L(u)]\,dx = \cancel{H(x)\Big|_a^b} \rightarrow 0$$ for homogeneus boundary conditions!!

$$\int_a^b [u\cdot 0 - \phi_n f(x)]\,dx = 0 \rightarrow \boxed{\int_a^b f(x)\phi_n\,dx = 0}$$

5.5.14 $L = p(x)\dfrac{d}{dx}$

$$\int_a^b vL(u)\,dx = \int_a^b v\,p(x)\frac{du}{dx}\,dx = vpu\Big|_a^b - \int_a^b \frac{d(vp)}{dx}u\,dx$$

$$= vpu\Big|_a^b - \int_a^b u\underbrace{\left[p\frac{dv}{dx} + \frac{dp}{dx}v\right]}_{-L^*(v)}dx$$

$$\Rightarrow \boxed{\int_a^b [uL^*(v) - vL(u)]\,dx = \underbrace{-vpu\Big|_a^b}_{B(x)}}$$ wit $\boxed{L^*(v) = -p\dfrac{dv}{dx} - \dfrac{dp}{dx}v}$

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5.5A.1 Suppose that u is an eigenvector of A corresponding to eigenvalue λ. $\rightarrow$ $Au = \lambda u \rightarrow \overline{Au} = \overline{\lambda u} = \bar{\lambda}\bar{u}$

$\bar{A} = A$ since A is a ~~symmetric~~ real matrix $\rightarrow A\bar{u} = \bar{\lambda} u$

For symetric matrices we can use (5.5.30) with $v = \bar{u}$

$$\rightarrow \quad u \cdot A\bar{u} - \bar{u} \cdot Au = 0$$
$$u \cdot (\bar{\lambda}\bar{u}) - \bar{u} \cdot (\lambda u) = 0 \rightarrow (\bar{\lambda} - \lambda)\, u \cdot \bar{u} = 0$$

$u \cdot \bar{u} = \sum_i u_i \bar{u}_i = \sum_i |u_i|^2 \geq 0$ It's zero only if $u = 0$, but $u \neq 0$ in general, then $(\bar{\lambda} - \lambda)\, u \cdot \bar{u} = 0 \rightarrow \boxed{\bar{\lambda} = \lambda}$

5.5A.2 a) $\begin{vmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \rightarrow \boxed{\lambda = 1}$

$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 2a = 0 \rightarrow \left.\begin{matrix} a = 0 \\ b \text{ independent} \end{matrix}\right\} \boxed{v = \begin{pmatrix} 0 \\ b \end{pmatrix}}$

b) $\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \rightarrow \boxed{\lambda = 1}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow a, b$ independent $\rightarrow \boxed{v_1 = \begin{pmatrix} a \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ b \end{pmatrix}}$

5.5A3 $\begin{vmatrix} 6-\lambda & 4 \\ 1 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda) - 4 = 18 - 9\lambda + \lambda^2 - 4 = 0$

$$\lambda^2 - 9\lambda + 14 = 0 \rightarrow \lambda = \frac{9}{2} \pm \frac{1}{2}\sqrt{81 - 56}$$

$\lambda = \frac{9}{2} \pm \frac{5}{2} \rightarrow \boxed{\lambda_+ = 7}$ and $\boxed{\lambda_- = 2}$

$\underline{\lambda_+ = 7}$: $\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow a = 4b \rightarrow \boxed{v_+ = \begin{pmatrix} 4b \\ b \end{pmatrix}}$

$\underline{\lambda_- = 2}$: $\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow b' = -a' \rightarrow \boxed{v_- = \begin{pmatrix} a' \\ -a' \end{pmatrix}}$

a) $\vec{v}_+ \cdot \vec{v}_- = 4ba' - ba' = 3ba' \neq 0$ if $a', b \neq 0$ for don't have trivial solution.

b) $\vec{v}_+ \cdot \vec{v}_- = \frac{1}{\cancel{4}}(\cancel{4}ba') - ba' = 0 \rightarrow \vec{v}_+$ an $\vec{v}_-$ are orthogonals with these definition of dot product.

5.5A.4 a) $\begin{vmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda) - 4 = 18 - 9\lambda + \lambda^2 - 4 = 0$

$$\lambda^2 - 9\lambda + 14 = 0 \rightarrow \lambda = \frac{9}{2} \pm \frac{1}{2}\sqrt{81-56}$$

$$\lambda = \frac{9}{2} \pm \frac{5}{2} \rightarrow \boxed{\lambda_+ = 7} \text{ and } \boxed{\lambda_- = 2}$$

$\lambda_+ = 7$: $\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{matrix} a = 2b \\ 2a - 4b = 0 \end{matrix}\right\} \rightarrow \boxed{v_+ = \begin{pmatrix} 2b \\ b \end{pmatrix}}$

$\lambda_- = 2$: $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{matrix} 4a' + 2b' = 0 \\ b' = -2a' \end{matrix}\right\} \rightarrow \boxed{v_- = \begin{pmatrix} a' \\ -2a' \end{pmatrix}}$

According to (5.5.37) the general solution is:

$$\boxed{v(t) = c_1 e^{7t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

$$v(0) = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow \left.\begin{matrix} 2c_1 + c_2 = 1 \\ c_1 - 2c_2 = 2 \quad \cdot(-2) \end{matrix}\right\}$$

$$5c_2 = -3$$

$$\Rightarrow \boxed{c_2 = -\frac{3}{5}} \text{ and } 2c_1 = 1 + \frac{3}{5} = \frac{8}{5} \rightarrow \boxed{c_1 = \frac{4}{5}}$$

b) $\begin{vmatrix} -1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = -(1+\lambda)(4-\lambda) - 4 = -(4 + 3\lambda - \lambda^2) - 4 = 0$

$$\lambda^2 - 3\lambda - 8 = 0 \rightarrow \lambda = \frac{3}{2} \pm \frac{1}{2}\sqrt{9+32}$$

$$\lambda = \frac{3}{2} \pm \frac{1}{2}\sqrt{41} \rightarrow \boxed{\lambda_+ = \frac{3+\sqrt{41}}{2}} \text{ and } \boxed{\lambda_- = \frac{3-\sqrt{41}}{2}}$$

$\boxed{\text{let } \begin{matrix} \alpha = 3/2 \\ \beta = \sqrt{41}/2 \end{matrix}} \Rightarrow \lambda_+ = \alpha + \beta$ and $\lambda_- = \alpha - \beta$

For λ_+: $\begin{pmatrix} -1-(\alpha+\beta) & 2 \\ 2 & 4-(\alpha+\beta) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} [+1+(\alpha+\beta)]a/2 = b \\ 2a + [4-(\alpha+\beta)]b = 0 \end{matrix}$

$$2a + [4-(\alpha+\beta)][1+(\alpha+\beta)]\frac{a}{2} = 0$$

$$\underbrace{\left[2 + \{4 + 3(\alpha+\beta) - (\alpha+\beta)^2\}\right]}_{0} a = 0$$

$$\Rightarrow \boxed{v_+ = \begin{pmatrix} a \\ \left(\frac{1+\alpha+\beta}{2}\right)a \end{pmatrix}}$$

For λ_-: $\begin{pmatrix} -1-(\alpha-\beta) & 2 \\ 2 & 4-(\alpha-\beta) \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} [1+(\alpha-\beta)]a/2 = b' \\ 2a' + [4-(\alpha-\beta)]b' = 0 \end{matrix}$

$$\underbrace{\left[2 + \{4 + 3(\alpha-\beta) - (\alpha-\beta)^2\}\right]}_{0} a' = 0$$

$$\Rightarrow \boxed{v_- = \begin{pmatrix} a' \\ \left(\frac{1+\alpha-\beta}{2}\right)a' \end{pmatrix}}$$

According to (5.5.37) the general solution is:

$$\boxed{v(t) = c_1 e^{(\alpha+\beta)t}\begin{pmatrix} 4 \\ \alpha+\beta+2 \end{pmatrix} + c_2 e^{(\alpha-\beta)t}\begin{pmatrix} 4 \\ \alpha-\beta+2 \end{pmatrix}}$$

$$v(0) = c_1\begin{pmatrix} 4 \\ \alpha+\beta+2 \end{pmatrix} + c_2\begin{pmatrix} 4 \\ \alpha-\beta+2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 &= 1/2 \\ c_1(\alpha+\beta+2) + c_2(\alpha-\beta+2) &= 3 \end{aligned}\right\}$$

c_1 and c_2 are given by solution of this system !!

5.5A.5 a) $\begin{vmatrix} 2-\lambda & 1 \\ 1 & -4-\lambda \end{vmatrix} = -(2-\lambda)(4+\lambda) - 1 = -(8-2\lambda-\lambda^2) - 1 = 0$

$$\lambda^2 + 2\lambda - 9 = 0 \rightarrow \lambda = \frac{-2}{2} \pm \frac{1}{2}\sqrt{4+36}$$

$$\rightarrow \lambda = -1 \pm \sqrt{10} \rightarrow \boxed{\lambda_+ = -1+\sqrt{10}} \text{ and } \boxed{\lambda_- = -1-\sqrt{10}}$$

For λ_+: $\begin{pmatrix} 3-\sqrt{10} & 1 \\ 1 & -3-\sqrt{10} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow$

$$b = (-3+\sqrt{10})a$$
$$a - (3+\sqrt{10})b = 0$$
$$a - (3+\sqrt{10})(-3+\sqrt{10})a = 0$$
$$\cancel{a - (-3^2+10)a = 0} \checkmark$$

$$\Rightarrow \boxed{v_+ = \begin{pmatrix} a \\ (-3+\sqrt{10})a \end{pmatrix}}$$

For λ_-: $\begin{pmatrix} 3+\sqrt{10} & 1 \\ 1 & -3+\sqrt{10} \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \boxed{v_- = \begin{pmatrix} a' \\ (-3-\sqrt{10})a' \end{pmatrix}}$

$$v_+ \cdot v_- = aa' - (-3+\sqrt{10})(3+\sqrt{10})aa' = aa' - (-9+10)aa' = 0 \checkmark$$

b) $\begin{vmatrix} 3-\lambda & 1-i \\ 1+i & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) - (1+i)(1-i) = (3-4\lambda+\lambda^2) - (1-i^2) = 0$

$$\lambda^2 - 4\lambda + 1 = 0 \rightarrow \lambda = 2 \pm \frac{1}{2}\sqrt{16-4} = 2 \pm \frac{1}{2}\sqrt{12}$$

$$\lambda = 2 \pm \sqrt{3} \rightarrow \boxed{\lambda_+ = 2+\sqrt{3}} \text{ and } \boxed{\lambda_- = 2-\sqrt{3}}$$

For λ_+: $\begin{pmatrix} 1-\sqrt{3} & 1-i \\ 1+i & -1-\sqrt{3} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow v_+ = \begin{pmatrix} -\frac{(1-i)}{1-\sqrt{3}}b \\ b \end{pmatrix}$

For λ_-: $\begin{pmatrix} 1+\sqrt{3} & 1-i \\ 1+i & -1+\sqrt{3} \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow v_- = \begin{pmatrix} a' \\ +\frac{1+i}{1-\sqrt{3}}a' \end{pmatrix} \Rightarrow \overline{v}_- = \begin{pmatrix} \overline{a'} \\ \frac{1-i}{1-\sqrt{3}}\overline{a'} \end{pmatrix}$

conjugate.

$$v_+ \cdot \overline{v}_- = -\frac{(1-i)}{1-\sqrt{3}}b\overline{a'} + \frac{1-i}{1-\sqrt{3}}b\overline{a'} = 0 \checkmark$$

$$\rightarrow \frac{d}{dx}\left[H(x)\frac{d\phi}{dx}\right] + \left[\lambda\beta(x)H(x) + \gamma(x)H(x)\right]\phi(x) = 0 \quad \text{OR}$$

$$\boxed{\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + \left[\lambda\sigma(x) + q(x)\right]\phi = 0}$$ Sturm - Liouville problem.

With: $p(x) = e^{\int \alpha(x)\,dx}$, $\sigma(x) = \beta(x)e^{\int \alpha(x)dx}$ $\quad q(x) = \gamma(x)e^{\int \alpha(x)dx}$

5.3.4 $\dfrac{\partial u}{\partial t} = K\dfrac{\partial^2 u}{\partial x^2} - V_0\dfrac{\partial u}{\partial x}$

a) Suppose $u(x,t) = \phi(x)h(t) \rightarrow \phi(x)\dfrac{dh}{dt} = K h(t)\dfrac{d^2\phi}{dx^2} - V_0 h(t)\dfrac{d\phi}{dx}$

$$\Rightarrow \frac{1}{h}\frac{dh}{dt} = \frac{K}{\phi}\frac{d^2\phi}{dx^2} - \frac{V_0}{\phi}\frac{d\phi}{dx} \rightarrow \underbrace{\frac{1}{Kh}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{\phi}\left[\frac{d^2\phi}{dx^2} - \frac{V_0}{K}\frac{d\phi}{dx}\right]}_{-\lambda}$$

the spatial equation is $\boxed{\dfrac{d^2\phi}{dx^2} - \dfrac{V_0}{K}\dfrac{d\phi}{dx} + \lambda\phi(x) = 0}$

This is not in Sturm-Liouville form. Usig the result of exercise 5.3.3 we can put this O.D.E in Sturm-Liouville form by multiplying both sides by

$$H(x) = e^{-\int \alpha(x)\,dx} = e^{-\int (V_0/K)dx} = e^{-V_0 x/K + C_0}$$

choose $C_0 = 0$ for simplicity:

$$e^{-V_0 x/K}\frac{d^2\phi}{dx^2} - \frac{V_0}{K}e^{-V_0 x/K}\frac{d\phi}{dx} + \lambda e^{-V_0 x/K}\phi(x) = 0$$

$$\boxed{\frac{d}{dx}\left[e^{-V_0 x/K}\frac{d\phi}{dx}\right] + \lambda e^{-V_0 x/K}\phi(x) = 0}$$ Now the equation is in Sturm-Liouville form.

b) $\dfrac{d^2\phi}{dx^2} - \dfrac{V_0}{K}\dfrac{d\phi}{dx} + \lambda\phi(x) = 0 \qquad \phi(0) = 0,\ \phi(L) = 0$

$$r^2 - \frac{V_0}{K}r + \lambda = 0 \longrightarrow r = \frac{V_0}{2K} \pm \frac{1}{2}\sqrt{\frac{V_0^2}{K^2} - 4\lambda}$$

$\dfrac{V_0^2}{K^2} - 4\lambda \gtreqless 0 \longrightarrow$ three cases, but we need oscilatory solution for satisfy boundary conditions $\phi(0) = \phi(L) = 0$.

$$\rightarrow \frac{V_0^2}{K^2} - 4\lambda < 0 \rightarrow r = \frac{V_0}{2K} \pm i\underbrace{\frac{1}{2}\sqrt{4\lambda - \frac{V_0^2}{K^2}}}_{D} \longrightarrow$$

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Suppose that u is an eigenvector of A:

5.5A-6 a) $Au = \lambda u$ (*) the transpose of this equation is:

$u^t A^t = u^t \lambda$ where the subscript t denote transpose.

$\Rightarrow \overline{u}^t \overline{A^t} = \overline{u}^t \overline{\lambda}$ but $\overline{A^t} = A^H$ by notation

$\overline{u}^t A^H = \overline{u}^t \overline{\lambda}$ (**)

Multiplying (*) at left by $\overline{u}^t$ and (**) at right by u and substracting:

$$\overline{u}^t A u - \overline{u}^t A^H u = \overline{u}^t \lambda u - \overline{u}^t \overline{\lambda} u$$

$$\overline{u}^t (\underbrace{A - A^H}_{0}) u = (\lambda - \overline{\lambda}) \overline{u}^t \cdot u$$

Since $A^H = A \rightarrow 0 = (\lambda - \overline{\lambda}) \overline{u}^t \cdot u$.

$\overline{u}^t \cdot u = \sum_i u_i \overline{u}_i = \sum_i |u_i|^2 \geq 0$ It's zero only if $u = 0$

but $u \neq 0$ in general $\Rightarrow (\overline{\lambda} - \lambda) \overline{u}^t u = 0 \rightarrow \boxed{\overline{\lambda} = \lambda}$

b) Suppose that u and v are eigenvector of A with eigenvalues λ_1 and λ_2 respectively:

$Au = \lambda_1 u$ (*) and $Av = \lambda_2 v$

$v^t A^t = v^t \lambda_2$ transpose eq.

$\overline{v}^t \overline{A^t} = \overline{v}^t \lambda_2$

OR $\overline{v}^t A^H = \overline{v}^t \lambda_2$ (**)

Multiplying (*) at left by $\overline{v}^t$ and (**) at right by u and substracting:

$$\overline{v}^t A u - \overline{v}^t A^H u = \overline{v}^t \lambda_1 u - \overline{v}^t \lambda_2 u$$

$$v^t (\underbrace{A - A^H}_{0}) u = (\lambda_1 - \lambda_2) \overline{v}^t \cdot u$$

Since $A^H = A \rightarrow 0 = (\lambda_1 - \lambda_2) \overline{v}^t \cdot u$

if $\lambda_1 \neq \lambda_2 \rightarrow \boxed{\overline{v}^t \cdot u = 0}$

5.6.1 a) $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \qquad \frac{d\phi}{dx}(0) = 0 \qquad \phi(1) = 0$

$$\frac{d}{dx}\left(\frac{d\phi}{dx}\right) - x^2\phi + \lambda\phi = 0 \longrightarrow \text{Sturm-Liouville problem with } p=1,\ q=-x^2,\ \sigma=1.$$

$$\Rightarrow \lambda_1 = \min \frac{\cancel{-u(du/dx)\Big|_0^1}^{\,0} + \int_0^1 \left[\left(\frac{du}{dx}\right)^2 + x^2u^2\right]}{\int_0^1 u^2 dx^2}$$

$$\lambda_1 \le \frac{\int_0^1 \left[\left(\frac{du_T}{dx}\right)^2 + x^2u_T^2\right]dx}{\int_0^1 u_T^2\, dx}$$ for any trial function u_T satisfying the boundary conditions.

- Let $u_T = \cos(\pi x/2)$ be a trial function, then:

$$\lambda_1 \le \frac{\int_0^1 \left[\left(-\frac{\pi}{2}\sin\frac{\pi x}{2}\right)^2 + x^2\cos^2\frac{\pi x}{2}\right]dx}{\int_0^1 \cos^2\left(\frac{\pi x}{2}\right)dx} = \frac{\frac{\pi^2}{4}\int_0^1 \sin^2\frac{\pi x}{2} + \int_0^1 x^2\cos^2\frac{\pi x}{2}dx}{\int_0^1 \cos^2\frac{\pi x}{2}dx}$$

$$\lambda_1 \le \frac{(\pi^2/4)(1/2) + \left(\frac{1}{6} - \frac{1}{\pi^2}\right)}{1/2} \approx 2.50007$$

- Try $u_T = 1 - x^2 \rightarrow du/dx = -2x$ and $u_T^2 = 1 - 2x^2 + x^4$

$$\Rightarrow \int_0^1 \left[\left(\frac{du}{dx}\right)^2 + x^2u^2\right]dx = \int_0^1 (4x^2 + x^2 - 2x^4 + x^6)\,dx = \underbrace{\frac{4}{3} + \frac{1}{3} - \frac{2}{5} + \frac{1}{7}}_{148/105}$$

$$\int_0^1 u^2 dx = \int_0^1 (1 - 2x^2 + x^4)\,dx = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$$

$$\Rightarrow \lambda_1 \le \frac{148/105}{8/15} = \frac{32}{14} \approx 2.643. \longrightarrow \boxed{\lambda_1 \le 2.50007}$$

b) $\frac{d^2\phi}{dx^2} + (\lambda - x)\phi = 0 \qquad \frac{d\phi}{dx}(0) = 0 \qquad \frac{d\phi}{dx}(1) + 2\phi(1) = 0.$

$$\frac{d}{dx}\left(\frac{d\phi}{dx}\right) - x\phi + \lambda\phi = 0 \longrightarrow \text{Sturm-Liouville problem with } p=1 \quad q=-x \text{ and } \sigma=1.$$

$$\lambda_1 \le \frac{-u(du/dx)\Big|_0^1 + \int_0^1 \left[\left(\frac{du}{dx}\right)^2 + xu^2\right]dx}{\int_0^1 u^2 dx} = \frac{-u(1)\frac{du}{dx}(1) + \int_0^1 [\quad]dx}{\int_0^1 u^2 dx}$$

$$\lambda_1 \le \frac{+2u^2(1) + \int_0^1 \left[(du/dx)^2 + xu^2\right]dx}{\int_0^1 u^2 dx}$$

We want to find a simple trial function. Assume

$u_T(x) = ax^2 + bx + c$, then $\frac{du_T}{dx} = 2ax + b$

$\frac{du_T}{dx}(0) = 0 = b$

hence $u_T(x) = ax^2 + c \rightarrow \frac{du_T}{dx}(1) + 2u_T(1) = 2a + 2(a+c) = 0$

$c = -2a$

Thus we can use $u_T(x) = ax^2 - 2a = a(x^2-2)$ as a trial function

$$\Rightarrow \lambda_1 \leq \frac{2a^2(-1)^2 + \int_0^1 [(2ax)^2 + x a^2 (x^2-2)^2]\,dx}{\int_0^1 a^2(x^2-2)^2\,dx}$$

$$\lambda_1 \leq \frac{2 + \int_0^1 [4x^2 + x(x^2-2)^2]\,dx}{\int_0^1 (x^2-2)^2\,dx} = \frac{2 + 5/2}{43/15} = 135/86$$

$$\boxed{\lambda_1 \leq 1.56977}$$

*c) $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ with $\phi(0) = 0$ and $\frac{d\phi}{dx}(1) + \phi(1) = 0$.

$\frac{d}{dx}\left(\frac{d\phi}{dx}\right) + \lambda\phi = 0 \rightarrow$ Sturm-Liouville problem with: $p = 1$ $q = 0$ and $\sigma = 1$

$$\Rightarrow \lambda_1 \leq \frac{-u(du/dx)\big|_0^1 + \int_0^1 (du/dx)^2\,dx}{\int_0^1 u^2\,dx} = \frac{-u(1)\frac{du}{dx}(1) + \int_0^1 \left(\frac{du}{dx}(1)\right)^2 dx}{\int_0^1 u^2\,dx}$$

$$\lambda_1 \leq \frac{u_T^2(1) + \int_0^1 (du_T/dx)^2\,dx}{\int_0^1 u_T^2\,dx}$$ where u_T is a trial function satisfying the boundary conditions.

We will try to find a simple trial function.

Let $u(x) = ax^2 + bx + c$. Since $u(0) = 0 \rightarrow c = 0$.

So $u(x) = (ax+b)x \rightarrow u(1) = a + b$ while $u'(1) = 2a + b$

$u'(1) + u(1) = 0 = (2a+b) + (a+b) = 3a + 2b \rightarrow b = -\frac{3}{2}a$.

Thus we can let $u(x) = ax^2 - \frac{3}{2}ax = a\left(x^2 - \frac{3}{2}x\right)$.

$$\Rightarrow \lambda_1 \leq \frac{a^2(1 - 3/2)^2 + \int_0^1 a^2(2x - 3/2)^2\,dx}{a^2\int_0^1 (2x-3/2)^2\,dx} = \frac{(-1/2)^2 + \int_0^1 (2x-3/2)^2\,dx}{\int_0^1 (x^2 - 3/2\,x)^2\,dx}$$

$$= \frac{1/4 + 7/12}{1/5} = \frac{25}{6}$$

$$\boxed{\lambda_1 \leq 4.1\overline{6}}$$

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5.6.2 $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \qquad \frac{d\phi}{dx}(0) = 0 = \frac{d\phi}{dx}(1)$

$\frac{d}{dx}\left(\frac{d\phi}{dx}\right) - x^2\phi + \lambda\phi = 0 \rightarrow$ Sturm-Liouville problem with: $p=1,\ q=-x^2\ \ \sigma=1$

The Rayleigh quotient then have the form:

$$\lambda = \frac{-\cancel{\phi(x)(d\phi/dx)\Big|_0^1}^{\,0} + \int_0^1 \left[(d\phi/dx)^2 + x^2\phi^2\right]dx}{\int_0^1 \phi^2 dx} \geq 0$$

Since $\phi(x)$ is assumed to be an eigenfunction $\phi(x) \not\equiv 0$ on $[0,1]$ hence $\phi^2 > 0$ on some interval contained within $[0,1]$. Therefore the denominator is positive and $\left(\frac{d\phi}{dx}\right)^2 + x^2\phi^2 > 0$ on some interval contained with $[0,1]$.

Hence the numerator is positive. Thus $\lambda \geq 0$.

λ cannot be zero, otherwise, by continuity we will have: $\left(\frac{d\phi}{dx}\right)^2 + x^2\phi^2 \equiv 0 \rightarrow \phi = 0$ of course, this gives a contradiction, because ϕ cannot be null (trivial solution), being and eigenfunction

$\Rightarrow \boxed{\lambda > 0}$

5-6-3 Assume $L(u)/\sigma$ is piecewise smooth so that

$$\frac{L(u)}{\sigma} = \sum_{n=1}^{\infty} b_n \phi_n(x) \quad \text{where } u = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \text{Equat (5.6.9)}$$

since u and du/dx are continuous and if u satisfy the same homogeneous ~~equation~~ BC's as the eigenfunctions $\phi_n(x)$. We can use the Sturm-Liouville differential equation

$L(\phi_n) + \lambda_n \sigma(x)\phi = 0$ to apply for u:

$$L(u) = \sum_{n=1}^{\infty} a_n L(\phi_n) = \sum_{n=1}^{\infty} a_n(-\lambda_n \sigma \phi_n)$$

$$\Rightarrow \frac{L(u)}{\sigma} = \sum_{n=1}^{\infty} a_n(-\lambda_n \phi_n) = \sum_{n=1}^{\infty} b_n \phi_n \rightarrow b_n = -a_n\lambda_n$$

So $\boxed{L(u) = -\sum_{n=1}^{\infty} a_n \lambda_n \sigma \phi_n}$ Equation (5.6.11)

Wave equation:

* **5.7.1** $\rho(x)\dfrac{\partial^2 u}{\partial t^2} = T_0 \dfrac{\partial^2 u}{\partial x^2}$ $\qquad u(0,t)=0$ and $u(1,t)=0$.

let $u(x,t) = \phi(x)h(t) \rightarrow \rho(x)\phi\dfrac{d^2h}{dt^2} = T_0 h \dfrac{d^2\phi}{dx^2}$ dividing by $\rho\phi h$.

$$\underbrace{\frac{1}{h}\frac{d^2h}{dt^2}}_{-\lambda} = \underbrace{\frac{T_0}{\rho}\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{-\lambda} \rightarrow \boxed{\frac{d}{dx}\left(T_0\frac{d\phi}{dx}\right) + \lambda\rho(x)\phi = 0} \qquad \begin{matrix}\phi(0)=0\\ \phi(1)=0.\end{matrix}$$

Sturm-Liouville problem with $p = T_0$, $q=0$ and $\sigma = \rho(x)$.

$$\boxed{\frac{d^2h}{dt^2} = -\lambda h(t)}$$

The Rayleigh quotient can be used to obtain additional information about the lowest eigenvalue λ_1. We know ~~that~~ that:

$$\lambda_1 = \min \frac{\cancel{-T_0\phi(d\phi/dx)|_0^1} + \int_0^1 [T_0(d\phi/dx)^2]\,dx}{\int_0^1 u^2\rho\,dx}$$

Let us suppose, as is usual, that the variable mass density has upper and lower bounds,

$$0 < \rho_{min} \leq \rho(x) \leq \rho_{máx}$$

For any $u(x)$ it follows that:

$$\rho_{min}\int_0^1 u^2\,dx \leq \int_0^1 u^2\rho(x)\,dx \leq \rho_{máx}\int_0^1 u^2\,dx$$

$$\Rightarrow \frac{T_0}{\rho_{máx}}\min\frac{\int_0^1 (du/dx)^2\,dx}{\int_0^1 u^2\,dx} \leq \lambda_1 \leq \frac{T_0}{\rho_{min}}\min\frac{\int_0^1 (du/dx)^2\,dx}{\int_0^1 u^2\,dx}$$

we can evaluate the expressions abouve, since we recognize the minimum of $\dfrac{\int_0^1 (du/dx)^2\,dx}{\int_0^1 u^2\,dx}$ subject to $\begin{matrix}u(0)=0\\ u(1)=0\end{matrix}$

as the lowest eigenvalue of a different problem: namely, one with constant coefficients:

$$\frac{d^2\phi}{dx^2} + \bar{\lambda}\phi = 0 \qquad \text{with } \phi(0)=0 \text{ and } \phi(1)=0$$

we already know that $\bar{\lambda} = (n\pi)^2$ and hence the lowest eigenvalue for this problem is $\bar{\lambda}_1 = \pi^2$. But the minimization property of Rayleigh quotient implies that

$$\underbrace{\bar{\lambda}_1}_{\pi^2} = \min\frac{\int_0^1 (du/dx)^2\,dx}{\int_0^1 u^2\,dx}$$

$$\Rightarrow \frac{T_0}{\rho_{max}}\bar{\lambda}_1 \le \lambda_1 \le \frac{T_0}{\rho_{min}}\bar{\lambda}_1 \quad \text{OR} \quad \frac{T_0}{\rho_{max}}\pi^2 \le \lambda_1 \le \frac{T}{\rho_{min}}\pi^2$$

since $\bar{\lambda}_1 = \pi^2$. By taking square roots:

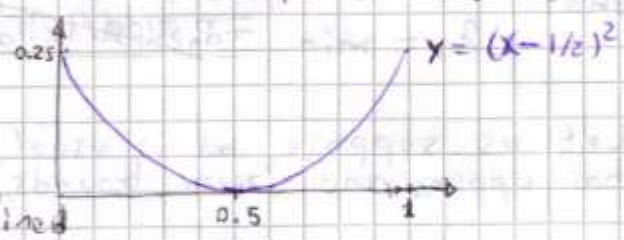
$$\pi\sqrt{\frac{T_0}{\rho_{max}}} \le \sqrt{\lambda_1} \le \pi\sqrt{\frac{T_0}{\rho_{min}}} \quad \text{OR} \quad \pi c_{min} \le \sqrt{\lambda_1} \le \pi c_{máx}$$

since $c = \sqrt{T_0/\rho}$, where $c^2 = 1 + 4\alpha^2(x - 1/2)^2$

For find $c_{máx}$ and c_{min} we need derivate:

$$\cancel{2}c\frac{dc}{dx} = 4\alpha^2 \cdot \cancel{2}(x - 1/2) \rightarrow \frac{dc}{dx} = \frac{4\alpha^2(x-1/2)}{1+4\alpha^2(x-1/2)^2} = 0$$

$\rightarrow \boxed{x = 1/2}$ minim of c. Note that



The minimum of c over $[0,1]$ is reached at $x = 1/2$, while the maximum is obtained at the ends of the interval $x = 0, 1$.
By substituting thes in the previous inequality

$$\pi\sqrt{1 + 4\alpha^2\underbrace{\cancel{(1/2 - 1/2)^2}}_{0}} \le \sqrt{\lambda_1} \le \pi\sqrt{1 + \cancel{4}\alpha^2(\cancel{0.25})}$$

$\boxed{\pi \le \sqrt{\lambda_1} \le \pi\sqrt{1+\alpha^2}}$ This represent the requerid bounds for the circular frequency, while for the frequency per 1 unit of time, $\sqrt{\lambda_1}/2\pi$ we have:

$$\frac{1}{2} \le \frac{\sqrt{\lambda_1}}{2\pi} \le \frac{1}{2}\sqrt{1+\alpha^2}.$$

5.7.2 $c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)$ $\rightarrow$ heat equation with $u(0,t) = 0 = u(L,t)$

with nonconstant thermal propierties, wher c, ρ and K_0 depend on x.

Let $u(x,t) = \phi(x)h(t) \rightarrow c\rho\phi\frac{dh}{dt} = h\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right)$ dividing by $c\rho\phi h$.

$$\underbrace{\frac{1}{h}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{c\rho\phi}\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right)}_{-\lambda} \rightarrow \boxed{\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \lambda c(x)\rho(x)\phi = 0} \quad \begin{matrix}\phi(0)=0\\ \phi(L)=0\end{matrix}$$

$\boxed{h(t) = ce^{-\lambda t}}$

Sturm-Liouville problem with $p = K_0$, $q = 0$ and $r = c\rho$. $\Rightarrow$

$$\lambda_{min} = \min \frac{\cancelto{0}{-K_0\,\phi\left(\frac{d\phi}{dx}\right)\Big|_0^1} + \int_0^1 K_0\left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^1 u^2 c\rho\, dx}$$

we are told $K_{min} \le K_0 \le K_{max}$ and $c\rho_{min} \le c\rho \le c\rho_{máx}$

$$\rightarrow \lambda_1 = \min \frac{\int_0^1 K_0 (d\phi/dx)^2 dx}{\int_0^1 u^2 c\rho\, dx} \le \min \frac{K_{max}\int_0^1 (du/dx)^2 dx}{c\rho_{min}\int_0^1 u^2 dx}$$

$$= \frac{K_{max}}{c\rho_{min}}\left[\min \frac{\int_0^1 (du/dx)^2 dx}{\int_0^1 u^2 dx}\right]$$

and

$$\lambda_1 = \min \frac{\int_0^1 K_0 (d\phi/dx)^2 dx}{\int_0^1 u^2 c\rho\, dx} \ge \min \frac{K_{min}}{c\rho_{max}} \frac{\int_0^1 (du/dx)^2 dx}{\int_0^1 u^2 dx}$$

$$= \frac{K_{min}}{c\rho_{max}}\left[\min \frac{\int_0^1 (du/dx)^2 dx}{\int_0^1 u^2 dx}\right]$$

Next we observe that $\min \frac{\int_0^1 (du/dx)^2 dx}{\int_0^1 u^2 dx}$ is the smallest eigenvalue of te Sturm-Liouvil problem for

$$\frac{d^2\phi}{dx^2} + \bar{\lambda}\phi = 0 \quad \text{with} \quad \phi(0) = 0 \text{ and } \phi(L) = 0.$$

wich is $\bar{\lambda}_1 = (\pi/L)^2$. Combining relation abovre, we have:

$$\frac{K_{min}}{c\rho_{max}}\bar{\lambda}_1 \le \lambda_1 \le \frac{K_{max}}{c\rho_{min}}\bar{\lambda}_1 \quad \text{ó R} \quad \boxed{\frac{K_{min}}{c\rho_{max}}\left(\frac{\pi}{L}\right)^2 \le \lambda_1 \le \frac{K_{max}}{c\rho_{min}}\left(\frac{\pi}{L}\right)^2}$$

Since the time-dependent part of the solution is $e^{-\lambda_1 t}$, these bounds on λ_1 give upper and lower bounds on the slowest exponential rate of decay.

By: Herminso Villarraga Gómez

5.8.1 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t)=0 \quad \frac{\partial u}{\partial x}(L,t) = -h\,u(L,t)$

$u(x,0) = f(x)$.

Let $u(x,t) = G(t)\phi(x) \longrightarrow \phi\frac{\partial G}{\partial t} = kh\frac{\partial^2\phi}{\partial x^2}$ dividing by $k\phi G$

$$\Rightarrow \underbrace{\frac{1}{kG}\frac{dG}{dt}}_{-\lambda} = \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{-\lambda} \longrightarrow \boxed{\frac{d^2\phi}{dx^2} + \lambda\phi = 0} \quad \phi(0)=0,\ \frac{d\phi}{dx}(L) + h\phi(L) = 0$$

$$\frac{dG}{dt} = -\lambda k G(t) \rightarrow \boxed{G(t) = Ce^{-\lambda k t}}$$

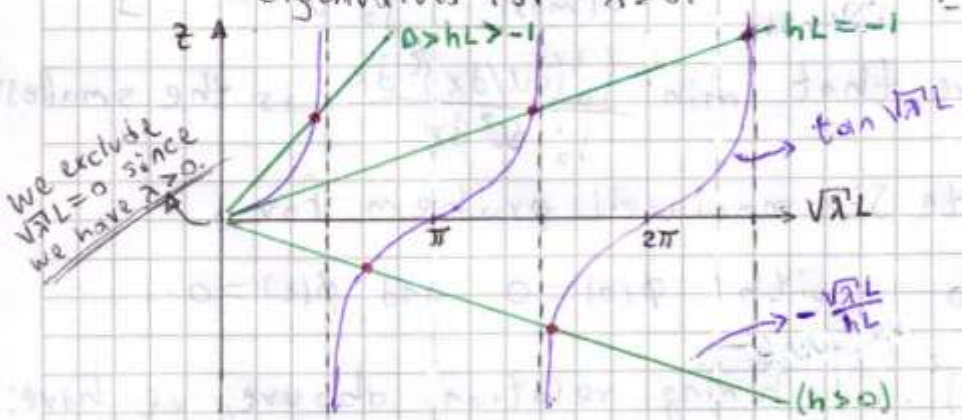
$\underline{\lambda > 0}$: $\phi(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x \qquad \phi(0) = 0 = C_1$

$$\rightarrow \frac{d\phi}{dx} = 0 + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$\frac{d\phi}{dx}(L) + h\phi(L) = C_2\left(\sqrt{\lambda}\cos\sqrt{\lambda}L + h\sin\sqrt{\lambda}L\right) = 0$$

$C_2 \neq 0$ for non-trivial sol. $\Rightarrow \sqrt{\lambda}\cos\sqrt{\lambda}L + h\sin\sqrt{\lambda}L = 0$

this equation give the eigenvalues for $\lambda > 0$. $\longleftarrow \boxed{\tan\sqrt{\lambda}L = -\frac{\sqrt{\lambda}L}{hL}}$



There is an infinite number of intersection for each case. Each corresponds to a positive eigenvalue.

$$\boxed{\phi_n(x) = C_{2n}\sin\sqrt{\lambda_n}x}$$

Where the allowable λ_n are determined graphically.

$\underline{\lambda = 0}$: $\phi = C_0 x + C_1 \qquad \phi(0) = 0 = C_1 \rightarrow$

$$\frac{d\phi}{dx} = C_0 \rightarrow \frac{d\phi}{dx}(L) + h\phi(L) = C_0(1 + hL) = 0 \rightarrow hL = -1 \text{ OR } C_0 = 0$$

$$\boxed{\begin{array}{l}\text{If } hL = -1 \rightarrow \phi_0(x) = C_0 x \\ \text{If } hL \neq -1 \rightarrow \lambda = 0 \text{ is not eigenvalue}\end{array}}$$

$\underline{\lambda < 0}$: let $\underline{s = -\lambda} \rightarrow \phi = C_1\cosh\sqrt{s}x + C_2\sinh\sqrt{s}x \qquad \phi(0) = 0 = C_1$

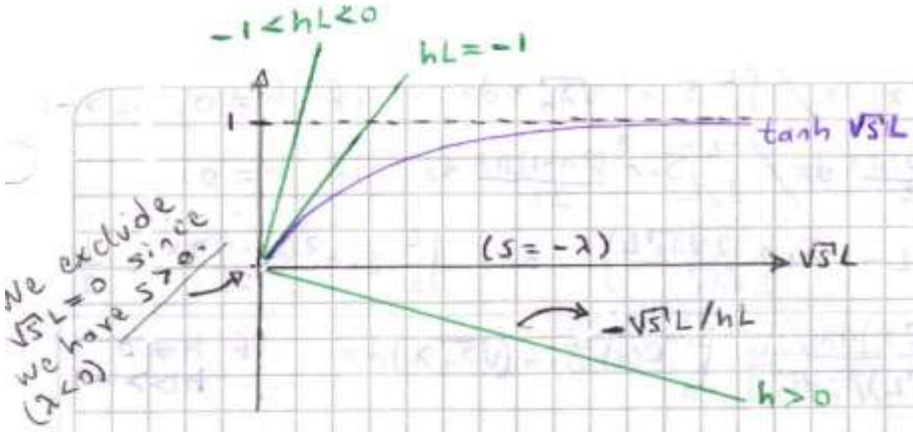
$$\phi'(x) = C_2\sqrt{s}\cosh\sqrt{s}x$$

$$\frac{d\phi}{dx}(L) + h\phi(L) = C_2\left(\sqrt{s}\cosh\sqrt{s}L + h\sinh\sqrt{s}L\right) = 0$$

$C_2 \neq 0$ for non-trivial solutions

$$\rightarrow \sqrt{s}\cosh\sqrt{s}L + h\sinh\sqrt{s}L = 0$$

$$\rightarrow \boxed{\tanh\sqrt{s}L = -\frac{\sqrt{s}L}{hL}} \rightarrow \text{graphicating}$$



If $hL>-1$ there are no intersections with $\sqrt{s}L>0$; there are no negative eigenvalues

If $hL=-1$ the slope of the straight line is equals 1, but the slope of $\tanh\sqrt{s}L$ equal 1 at $\sqrt{s}L=0$ and decreases to $\sqrt{s}L>0$.

There are no intersections for $hL=-1$ when $\sqrt{s}L>0$, then there are no negatives eigenvalues for $hL=-1$.

$\Rightarrow$ $\boxed{\text{There are nonnegative eigenvalues if } hL\geq -1}$

Special case $h=0$: $\dfrac{d^2\phi}{dx^2}+\lambda\phi=0$ $\quad\phi(0)=0$ and $\phi'(L)=0$.

- $\lambda>0$: $\phi=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$ $\quad\phi(0)=0=C_1$, $\phi'(L)=0=C_2\sqrt{\lambda}\cos\sqrt{\lambda}L$

$\rightarrow \sqrt{\lambda}L=(2n-1)\dfrac{\pi}{2} \rightarrow \boxed{\lambda_n=\left[\dfrac{(2n-1)\pi}{2L}\right]^2} \rightarrow \boxed{\phi_n(x)=C_{2n}\sin\left[\dfrac{(2n-1)\pi x}{2L}\right]}$

- $\lambda=0$: $\phi=C_1x+C_0 \rightarrow \phi(0)=0=C_0$ $\quad\phi'(L)=0=C_1 \rightarrow$ No eigenvalues for $\lambda=0$.
- $\lambda<0$: $s=-\lambda$ $\quad\phi=C_1\cosh\sqrt{s}x+C_2\sinh\sqrt{s}x$ $\quad\phi(0)=0=C_1$, $\phi'(L)=0=C_2\sqrt{\lambda}\cosh\sqrt{\lambda}L$

$\Rightarrow C_2=0 \rightarrow \phi(x)=0 \rightarrow$ No eigenvalues for $\lambda<0$

a) **If $hL>-1$:** According to the principle of superposition we attemp to satisfy the initial value problem with a linear combination of all posible production solutions:

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}a_n\phi_n(x)e^{-\lambda_n kt}} \quad \text{with} \quad \boxed{\phi_n=\begin{cases}\sin\sqrt{\lambda_n}x & \text{if } h\neq 0\\ \sin\dfrac{(2n-1)\pi x}{2L} & \text{if } h=0\end{cases}}\ hL>-1$$

where λ_n is given by intesections of $z=\tan\sqrt{\lambda}L$ and $z=-\dfrac{\sqrt{\lambda}L}{hL}$ when $\sqrt{\lambda}L>0$ and $hL>-1$ $(h\neq 0)$.

$f(x)=u(0,t)=\sum_{n=1}^{\infty}a_n\phi_n(x)$ $\quad$ Multiplying by $\phi_m(x)$ and integrating

$$\int_0^L f(x)\phi_m(x)dx=\sum_{n=1}^{\infty}a_n\underbrace{\int_0^L\phi_n(x)\phi_m(x)dx}_{0 \text{ if } n\neq m}=a_m\int_0^L[\phi_m(x)]^2dx$$

$$\rightarrow a_m=\frac{\int_0^L f(x)\phi_m(x)dx}{\int_0^L[\phi_m(x)]^2dx}$$

$$a_n = \begin{cases} \int_0^L f(x)\sin(\sqrt{\lambda_n}\,x)\,dx \Big/ \int_0^L \sin^2\sqrt{\lambda_n}\,x\,dx & \text{if } h\neq 0,\ hL>-1 \\ \int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx \Big/ \int_0^L \sin^2\frac{(2n-1)\pi x}{2L}\,dx & \text{if } h=0 \end{cases}$$

$$\int_0^L \sin^2(\sqrt{\lambda_n}\,x)\,dx = \frac{1}{2}\left(L - \frac{\sin 2\sqrt{\lambda_n}L}{2\sqrt{\lambda_n}}\right) \quad \text{and} \quad \int_0^L \sin^2\frac{(2n-1)\pi x}{2L} = \frac{L}{2}$$

$$\boxed{a_n = \begin{cases} \dfrac{2}{[L-(\sin 2\sqrt{\lambda_n}L)/2\sqrt{\lambda_n}]}\displaystyle\int_0^L f(x)\sin(\sqrt{\lambda_n}\,x)\,dx & \text{if } h\neq 0,\ hL>-1 \\ \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx & \text{if } h=0 \end{cases}}$$

b) If $hL=-1$ $\quad \boxed{u(x,t) = c_0 x + \sum_{n=1}^{\infty} c_n \sin(\sqrt{\lambda_n}\,x)\,e^{-\lambda_n k t}}$

where λ_n is given by intesections of $z=\tan\sqrt{\lambda}L$ and $z=\sqrt{\lambda}L$ when $\sqrt{\lambda}L>0$ $(hL=-1)$.

OR $u(x,t) = \sum_{n=0}^{\infty} c_n \phi_n(x) e^{-\lambda_n k t}$ with $\phi_n = \begin{cases} \sin\sqrt{\lambda_n}\,x & \text{if } n\neq 0 \\ x & \text{for } n=0 \end{cases}$

$f(x) = u(0,t) = \sum_{n=0}^{\infty} c_n \phi_n(x)$ Multiplying by $\phi_m(x)$ and integrating

$$\int_0^L f(x)\phi_m(x)\,dx = \sum_{n=0}^{\infty} c_n \underbrace{\int_0^L \phi_n(x)\phi_m(x)\,dx}_{0 \text{ if } n\neq m} = c_m \int_0^L [\phi_m(x)]^2\,dx$$

0 if $n\neq m$ $\rightarrow$ according to orthogonality of solutions for Sturm-Liouville problem!!

$$\Rightarrow c_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L [\phi_n(x)]^2\,dx}$$

If $n=0 \rightarrow \int_0^L [\phi_n(x)]^2\,dx = \int_0^L x^2\,dx = \frac{L^3}{3}$

If $n\neq 0 \rightarrow \int_0^L [\phi_n(x)]^2\,dx = \int_0^L \sin^2(\sqrt{\lambda_n}\,x)\,dx = \frac{1}{2}\left(L - \frac{\sin 2\sqrt{\lambda_n}L}{2\sqrt{\lambda_n}}\right)$

$$\Rightarrow \boxed{c_n = \frac{2}{[L-(\sin 2\sqrt{\lambda_n}L)/2\sqrt{\lambda_n}]}\int_0^L f(x)\sin(\sqrt{\lambda_n}\,x)\,dx} \qquad n\neq 0$$

$$\boxed{c_0 = \frac{3}{L^3}\int_0^L x f(x)\,dx}$$

$$\phi(x) = e^{\frac{V_0}{2K}x}\left(A\cos Dx + B\sin Dx\right)$$

$$\phi(0) = 0 = A$$

$$\phi(L) = 0 = e^{\frac{V_0}{2K}L} B\sin(DL) \Rightarrow DL = (n\pi) \rightarrow \boxed{D = \frac{n\pi}{L}}$$

$$\Rightarrow \boxed{\phi_n(x) = B_n e^{(V_0/2K)x}\sin\left(\frac{n\pi x}{L}\right)} \qquad n = 1, 2, 3, \ldots$$

By other hand $\frac{dh}{dt} + K\lambda h(t) = 0$ is the temporal equation

$$\Rightarrow h(t) = e^{-K\lambda t} \Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} B_n e^{-K\lambda_n t} e^{(V_0/2K)x}\sin\left(\frac{n\pi x}{L}\right)}$$

with

$$D = \frac{1}{2}\sqrt{4\lambda - \frac{V_0^2}{K^2}} = \frac{n\pi}{L} \rightarrow \boxed{\lambda_n = -\left(\frac{n\pi}{L}\right)^2 + \left(\frac{V_0}{2K}\right)^2}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n e^{(V_0/2K)x}\sin\left(\frac{n\pi x}{L}\right)$$

multiplying by $e^{-(V_0/2K)x}\sin\left(\frac{m\pi x}{L}\right)$ and integrating:

$$\int_0^L f(x)\, e^{-(V_0/2K)x}\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{(L/2)\delta_{nm}}$$

$$= B_m \frac{L}{2} \rightarrow \boxed{B_n = \frac{2}{L}\int_0^L f(x)\, e^{-\frac{V_0}{2K}x}\sin\left(\frac{n\pi x}{L}\right)dx}$$

c) $\frac{d^2\phi}{dx^2} - \frac{V_0}{K}\phi(x) + \lambda\phi(x) = 0 \qquad \phi'(0) = 0, \quad \phi'(L) = 0$

$$\phi(x) = e^{\frac{V_0}{2K}x}\left(A\cos Dx + B\sin Dx\right) \rightarrow \text{oscilatory solution}$$

$$\phi'(x) = \frac{V_0}{2K}e^{\frac{V_0}{2K}x}\left(A\cos Dx + B\sin Dx\right) + e^{\frac{V_0}{2K}x}\left(-AD\sin Dx + BD\cos Dx\right)$$

$$\phi'(0) = \frac{V_0}{2K}A + BD = 0 \rightarrow B = -\frac{V_0}{2KD}A$$

$$\phi'(L) = 0 = \frac{V_0}{2K}e^{V_0L/2K}\left(A\cos DL + B\sin DL\right) + e^{\frac{V_0}{2K}L}\left(-AD\sin DL + BD\cos DL\right)$$

$$\Rightarrow \cancel{\left(\frac{V_0}{2K}A + BD\right)}^{\,0}\cos(DL) = \underbrace{\left(AD - \frac{V_0}{2K}B\right)}_{\neq 0}\sin DL$$

since $B \neq -\frac{V_0}{2KD}A$

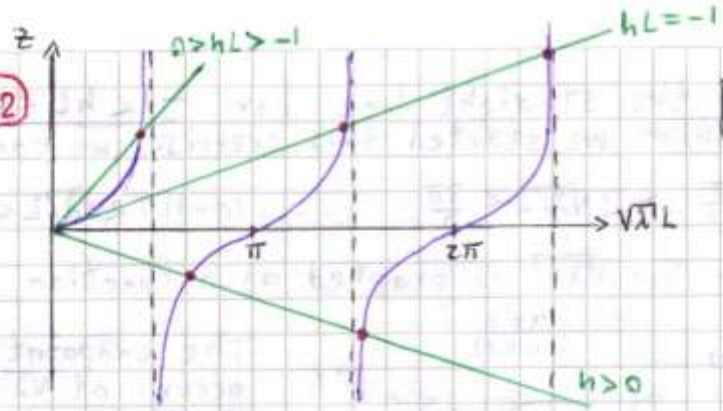
$A \neq 0$ for no trivial solution.

$$\Rightarrow \sin(DL) = 0 \rightarrow DL = n\pi$$

$$\boxed{D = \frac{n\pi}{L}} \qquad n = 1, 2, 3, \ldots$$

By: Herminso Villarraga Gómez

5.8.2



For $\lambda > 0$:

$$\boxed{\phi_n(x) = \sin \sqrt{\lambda_n}\, x}$$

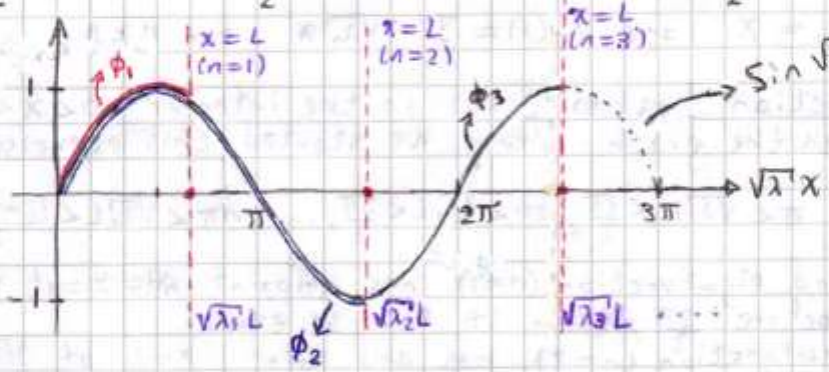
Where the allowable λ_n are determined by the graphycal intersections.

For $\lambda = 0$: $\boxed{\begin{array}{l} \text{If } hL = -1 \rightarrow \phi_0(0) = c_0 x \\ \text{If } hL \neq -1 \rightarrow \lambda = 0 \text{ is not eigenvalue.} \end{array}}$

For $\lambda < 0$: There are nonnegative eigenvalues if $hL \geq 1$

a) $h > 0$: From the graph we can note that:

$$\frac{\pi}{2} < \sqrt{\lambda_1}L < \pi, \quad \frac{3\pi}{2} < \sqrt{\lambda_2}L < 2\pi, \ \ldots\ldots, \quad \frac{(2n-1)}{2} < \sqrt{\lambda_n}L < n\pi$$



The endpoints of the eigenfunctions ϕ_n depends on λ and occurrs at $\sqrt{\lambda_n}L$ when $x = L$.

For the first eigenfunction $(n=1)$, the endpoint occurs at $\sqrt{\lambda_1}L$. This occurs before $\pi \Rightarrow$ the eigenfunction has no zeros for $x < 0 < L$.
For the second eigenfunction $(n=2)$, the endpoint $(x=L)$ occurs at $\sqrt{\lambda_2}L$. It occurs after $3\pi/2$ and before $2\pi \Rightarrow$ has 1 zero.
For $\phi_3(x)$ the endpoint occur after $5\pi/2$ and befor $3\pi \Rightarrow$ 2 zeros.
⋮
In general, the nth eigenfunction has $n-1$ zeroes in the interior $(0 < x < L)$.

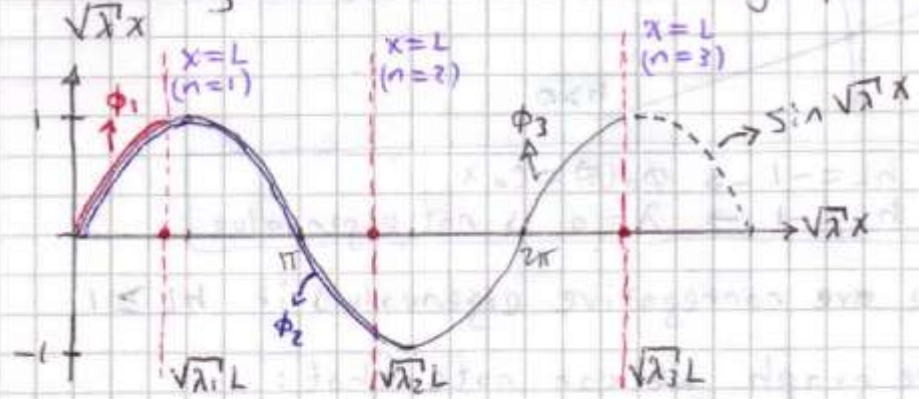
b) $h = 0$: Special case $\rightarrow \phi_n(x) = \sin\left[\frac{(2n-1)\pi x}{L}\right]$, this function has zeros when $\frac{(2n-1)\pi x}{L} = (2m-1)\pi$ with $m = 1, 2, 3 \ldots$

$\Rightarrow \quad x = \left(\frac{2m-1}{2n-1}\right)L$, but $0 < x < L \rightarrow 0 < \frac{2m-1}{2n-1} < 1$, this inequality is true when $m = 1, 2, \ldots, n-1 \rightarrow$ $\boxed{\phi_n \text{ has } (n-1) \text{ zeros for } 0 < x < L}$

c) By extension of the straight line for $-1 < hL < 0$ on the graph which we started this exercise, we can see that:

$$0 < \sqrt{\lambda_1}L < \frac{\pi}{2},\ \pi < \sqrt{\lambda_2}L < \frac{3\pi}{2},\ \ldots\ldots,\ (n-1)\pi < \sqrt{\lambda_n}L < \frac{(2n-1)\pi}{2}$$

The eigenfunction $\sin\sqrt{\lambda}x$ is graphed as a function of $\sqrt{\lambda}x$



The endpoint $x=L$ occurs at $\sqrt{\lambda}L$.
For ϕ_1 occurs before $\pi/2 \rightarrow$ No zero
For ϕ_2 occurs before $3\pi/2$ and after π then ϕ_2 has 1 zero.
For $\phi_3 \rightarrow$ 2 zeroes
⋮
For $\phi_n \rightarrow (n-1)$ zeros in the interior.

d) $hL = -1$: $\phi_0 = x$ and $\phi_n(x) = \sin\sqrt{\lambda_n}x$ for $n \geq 1$

First eigenfunction has no zeros in the interior $0 < x < 1$.
For $n \geq 1$, see the graph which we started this exercise, we can see that:

$$\pi < \sqrt{\lambda_1}L < \frac{3\pi}{2},\ 2\pi < \sqrt{\lambda_2}L < \frac{5\pi}{2},\ \ldots,\ n\pi < \sqrt{\lambda_n}L < \frac{(2n+1)\pi}{2}.$$

then, the second eigenfunction $(n=1)$ has endpoint $x=L$ at $\sqrt{\lambda_1}L$ after π and before $\frac{3\pi}{2}$, then it has 1 zero.
The third eigenfunction $(n=2)$ has end point $x=L$ at $\sqrt{\lambda_2}L$ after 2π and befor $5\pi/2 \Rightarrow$ It has 2 zeroes.
⋮
the nth eigenfunction has $(n-1)$ zeroes.

5.8.3 $\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad \phi'(0) = 0 \qquad \phi'(L) + h\phi(L) = 0 \qquad \boxed{h > 1}$

↳ Sturm-Liouville problem with $p = 1$ $q = 0$ and $\sigma = 1$

a) $$\lambda = \frac{-p\phi\phi'\big|_0^L + \int_0^L [p(\phi')^2 - \cancel{q}^{\,0}\phi^2]\,dx}{\int_0^L \phi^2\sigma\,dx} = \frac{-\phi(L)(-h\phi(L)) + \int_0^L [\phi'(x)]^2\,dx}{\int_0^L [\phi(x)]^2\,dx}$$

$$= \frac{h[\phi(L)]^2 + \int_0^L [\phi'(x)]^2\,dx}{\int_0^L [\phi(x)]^2\,dx} \geq 0 \quad \text{since } h > 0.$$

$\lambda = 0 \rightarrow \phi'(x) = 0,\ \phi(L) = 0 \rightarrow \phi(x) = C \rightarrow \phi(0) = 0 = C \rightarrow \phi(x) \equiv 0$ trivial Solution

$\Rightarrow \lambda = 0$ is not eigenvalue $\Rightarrow \boxed{\lambda > 0}$

b) If $\lambda > 0 \Rightarrow \phi = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$

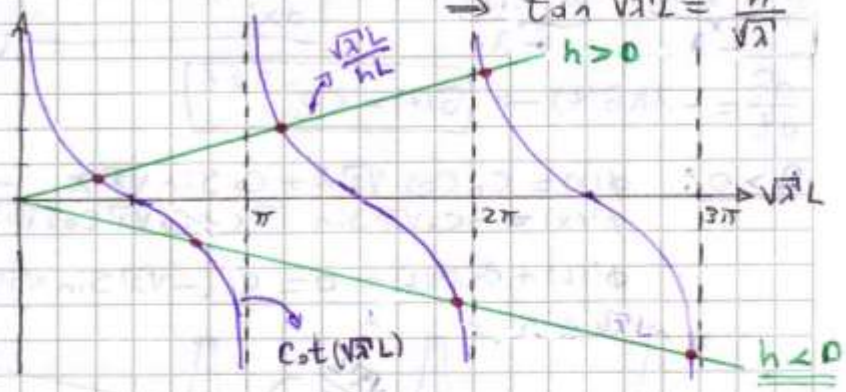
$\phi' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x \qquad \phi'(0) = 0 = c_2\sqrt{\lambda} \quad c_2 = 0$

$\phi'(L) + h\phi(L) = c_1(-\sqrt{\lambda}\sin\sqrt{\lambda}L + h\cos\sqrt{\lambda}L) = 0$

$c_1 = 0 \rightarrow$ trivial solution, $\Rightarrow -\sqrt{\lambda}\sin\sqrt{\lambda}L + h\cos\sqrt{\lambda}L = 0$

$\rightarrow \tan\sqrt{\lambda}L = \dfrac{h}{\sqrt{\lambda}}$

OR $\boxed{\cot(\sqrt{\lambda}L) = \dfrac{\sqrt{\lambda}L}{hL}}$



The allowable λ_n are determined by the intersections in the graph.

$\Rightarrow \boxed{\phi_n(x) = \cos(\sqrt{\lambda_n}\,x)}$

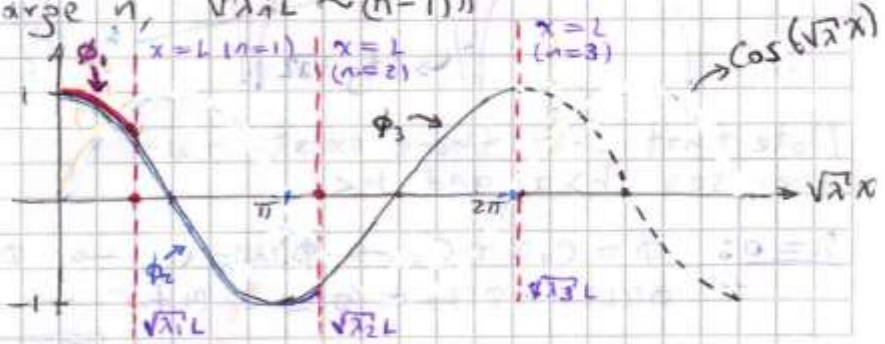
we exclude $\sqrt{\lambda}L = 0$ because $\lambda > 0$.

Note from graph that: $0 < \sqrt{\lambda_1}L < \frac{\pi}{2}$, $\pi < \sqrt{\lambda_2}L < \frac{3\pi}{2}, \ldots, (n-1)\pi < \sqrt{\lambda_n}L < \frac{(2n-1)\pi}{2}$

OR $\boxed{(n-1)^2\left(\frac{\pi}{L}\right)^2 < \lambda_n < (2n-1)^2\left(\frac{\pi}{2L}\right)^2}$

Asymptotically for large n, $\sqrt{\lambda_n}L \sim (n-1)\pi$

c) The endpoints of the $\phi_n(x)$ depends on λ and occurs at $\sqrt{\lambda_n}L$ when $x = L$.



the endpoit

For ϕ_1, $x = L$ occurs at $\sqrt{\lambda_1}L$ before $\pi/2 \rightarrow$ No zeros.

For ϕ_2, $x = L$ occurs at $\sqrt{\lambda_2}L$ before $3\pi/2$ and after $\pi \rightarrow$ 1 zero

For ϕ_3, $x = L$ " at $\sqrt{\lambda_3}L$ before $5\pi/2$ and after $2\pi \rightarrow$ 2 zeroes

⋮

For ϕ_n, $x = L$ " at $\sqrt{\lambda_n}L$ before $(2n-1)\pi/2$ and after $(n-1)\pi$, then $\boxed{\phi_n \text{ has } (n-1) \text{ zeroes}}$

8.5.4 It's similar to part b) of excersice (5.8.3) only change the straight line because it has negative slope. See the graph above on the top of this page.

In that case, $\frac{\pi}{2} < \sqrt{\lambda_1}L < \pi$, $\frac{3\pi}{2} < \sqrt{\lambda_2}L < 2\pi, \ldots, \boxed{\frac{(2n-1)\pi}{2} < \sqrt{\lambda_n}L < n\pi}$

Thus $\boxed{\phi_n \text{ has } (n-1) \text{ zeros}}$ since the endpoint $x = L$ occurs at $\sqrt{\lambda_n}L$ before $n\pi$ and after $(2n-1)\pi/2$.

By: Herminso Villarraga Gómez

5.8.5 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad \frac{\partial u}{\partial x}(0,t)=0 \quad \frac{\partial u}{\partial x}(L,t) = -h\,u(L,t)$

$u(x,0) = f(x)$

$u(x,t) = G(t)\phi(x) \rightarrow \phi\frac{\partial G}{\partial t} = kh\frac{\partial^2\phi}{\partial x^2}$ dividing by $k\phi G$

$\underbrace{\frac{1}{kG}\frac{dG}{dx}}_{-\lambda} = \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{-\lambda} \rightarrow \boxed{\frac{d^2\phi}{dx^2} + \lambda\phi = 0}$ $\quad \phi'(0)=0 \quad \phi'(L)+h\phi(L)=0$

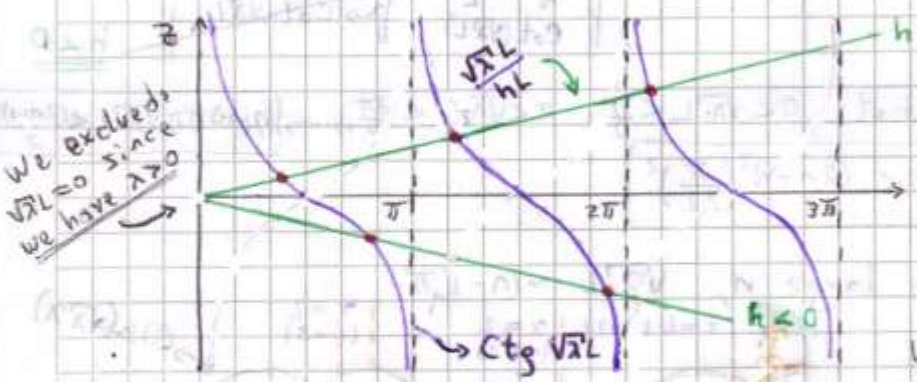
↳ Sturm-Liouville problem.

$\frac{dG}{dt} = -\lambda k G(t) \rightarrow \boxed{G(t) = Ce^{-\lambda k t}}$

$\lambda > 0$: $\phi(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$

$\phi'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x \rightarrow \phi'(0) = 0 = C_2$

$\phi'(L) + h\phi(L) = 0 = C_1(-\sqrt{\lambda}\sin\sqrt{\lambda}L + h\cos\sqrt{\lambda}L) \rightarrow \boxed{\operatorname{Ctg}\sqrt{\lambda}L = \frac{\sqrt{\lambda}L}{hL}}$



There are an infinite number of intesection in both cases, each correspond to a positive eigenvalue.

$\boxed{\phi_n(x) = \cos(\sqrt{\lambda_n}x)}$

Where de allowable λ_n's are determined by the intersections of graphs.

Note that only there exist two subcases $h>0$ and $h<L$.

$\lambda = 0$: $\phi = C_1x + C_0 \rightarrow \phi'(x) = C_1 \rightarrow \phi'(0) = 0 = C_1$

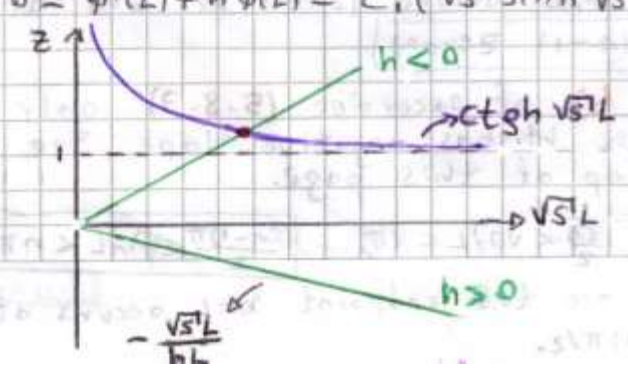
$\phi'(L) + h\phi(L) = 0 = \cancel{C_1}^{\,0} + hC_0 \rightarrow C_0 = 0$ since $h \neq 0$.

$\rightarrow$ $\boxed{\text{There are not eigenvalues for } \lambda = 0 \text{ since } h \neq 0.}$

$\lambda < 0$: let $s = -\lambda \rightarrow \phi(x) = C_1\cosh\sqrt{s}x + C_2\sinh\sqrt{s}x$

$\phi'(x) = C_1\sqrt{s}\sinh\sqrt{s}x + C_2\sqrt{s}\cosh\sqrt{s}x \quad \phi'(0) = 0 = C_2$

$0 = \phi'(L) + h\phi(L) = C_1(\sqrt{s}\sinh\sqrt{s}x + h\cosh\sqrt{s}L) \rightarrow \boxed{\operatorname{Ctgh}\sqrt{s}L = -\frac{\sqrt{s}L}{hL}}$



$\boxed{\text{For } h>0 \rightarrow \text{There are no negative eigenvalues}}$

$\boxed{\text{For } h<0 \rightarrow \text{There is only one negative eigenvalue in } s = s_1 \rightarrow \lambda = -s_1 \rightarrow \boxed{\phi = \cosh\sqrt{s_1}x}}$

a) $h>0$: There are eingenvalues only for $\lambda>0$, then

$$u(x,t)=\sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n k t} = \boxed{\sum_{n=1}^{\infty} a_n \cos(\sqrt{\lambda_n}\, x) e^{-\lambda_n k t}}$$

$f(x)=u(x,0)=\sum_{n=1}^{\infty} a_n \phi_n(x)$ Multiplying by $\phi_m(x)$ and integrating.

$$\int_0^L f(x)\phi_m(x)\,dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \phi_n(x)\phi_m(x)\,dx}_{0 \text{ if } n\neq m} = a_m \int_0^L [\phi_m(x)]^2 dx$$

$$\Rightarrow a_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L [\phi_n(x)]^2 dx} \longrightarrow \int_0^L \cos^2(\sqrt{\lambda_n}\, x)\,dx = \frac{1}{2}\left(L + \frac{\sin 2\sqrt{\lambda_n} L}{2\sqrt{\lambda_n}}\right)$$

$$\Rightarrow \boxed{a_n = \frac{2}{[L + (\sin 2\sqrt{\lambda_n} L)/2\sqrt{\lambda_n}]} \int_0^L f(x)\cos(\sqrt{\lambda_n}\, x)\,dx}$$

where λ_n is given by intersections of $z=\cot\sqrt{\lambda}L$ and $z=\frac{\sqrt{\lambda}L}{hL}$ when $\sqrt{\lambda}L>0$ and $h>0$.

b) $h<0$: There are an infinite number of eigenvalues positives λ_n with corresponding eigenfunction $\phi_n(x)=\cos\sqrt{\lambda_n}\,x$ where λ_n is solution of $\cot\sqrt{\lambda}\,x=\sqrt{\lambda}/h$; and there is only one negative eigenvalue $\lambda=-s_1$ with corresponding eigenfunction $\cosh(\sqrt{s_1}\,x)$, where s_1 is determines as the unique solution of $\coth\sqrt{s}L=-\sqrt{s}/h$.

$$\Rightarrow \boxed{u(x,t)=a_0\cosh(\sqrt{s_1}\,x)e^{s_1 k t}+\sum_{n=1}^{\infty} a_n \cos(\sqrt{\lambda_n}\,x)e^{-\lambda_n k t}}$$

OR $u(x,t)=\sum_{n=0}^{\infty} a_n\phi_n(x)e^{-\lambda_n k t}$ with $\phi_n=\begin{cases}\cosh\sqrt{s_1}\,x & n=0\\ \cos\sqrt{\lambda_n}\,x & n\geq 1\end{cases}$ and $\lambda_0=-s_1$

$f(x)=u(x,0)=\sum_{n=0}^{\infty} a_n\phi_n(x)$ Multiplying by $\phi_m(x)$ and integrating

$$\int_0^L f(x)\phi_m(x)\,dx = \sum_{n=0}^{\infty} a_n \underbrace{\int_0^L \phi_n(x)\phi_m(x)\,dx}_{0 \text{ if } n\neq m} = a_m\int_0^L [\phi_m(x)]^2 dx$$

$$\Rightarrow a_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L [\phi_n(x)]^2 dx}$$

If $n=0 \rightarrow \int_0^L [\phi_0(x)]^2 dx = \int_0^L \cosh^2(\sqrt{s_1}\,x)\,dx = \frac{1}{2}\left(L+\frac{\sinh 2\sqrt{s_1}L}{2\sqrt{s_1}}\right)$

$$\Rightarrow \boxed{a_0 = \frac{2}{\left[L + (\sinh 2\sqrt{s_1}L)/2\sqrt{s_1}\right]} \int_0^L f(x)\cosh(\sqrt{s_1}x)\,dx}$$

If $n \geq 1 \rightarrow \int_0^L [\phi_n(x)]^2 dx = \int_0^L \cos^2(\sqrt{\lambda_n}x)\,dx = \frac{1}{2}\left(L + \frac{\sin \sqrt{\lambda_n}x}{2\sqrt{\lambda_n}}\right)$

$$\Rightarrow \boxed{a_n = \frac{2}{\left[L + (\sin 2\sqrt{\lambda_n}L)/2\sqrt{\lambda_n}\right]} \int_0^L f(x)\cos(\sqrt{\lambda_n}x)\,dx}$$

5.8.6 $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\quad \frac{\partial u}{\partial x}(L,t) = 0$ $\quad \frac{\partial u}{\partial x}(0,t) - \overset{(h>0)}{h}u(0,t) = 0$ $\quad u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(x,0) = g(x)$

Let $u(x,t) = \phi(x)G(t) \rightarrow \phi(x)\frac{d^2G}{dt^2} = c^2 G \frac{d^2\phi}{dx^2}$ dividing by $c^2 G\phi$

$$\underbrace{\frac{1}{c^2 G}\frac{d^2G}{dt^2}}_{-\lambda} = \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{-\lambda} \rightarrow \boxed{\frac{d^2\phi}{dx^2} + \lambda\phi = 0}$$

Sturm-Liouville problem with $p=1$ $q=0$ and $\sigma=1$

$\phi'(L) = 0$

$\phi'(0) - h\phi(0) = 0$

$$\boxed{\frac{d^2G}{dt^2} + \lambda c^2 G(t) = 0}$$

a) by Rayleight quotient: $\lambda = \dfrac{-p\phi\frac{d\phi}{dx}\Big|_0^L + \int_0^L \left[p\left(\frac{d\phi}{dx}\right)^2 - \cancel{q}\phi^2\right]dx}{\int_0^L \phi^2\sigma\,dx}$

$$\lambda = \frac{-\phi(L)(-h\phi(L)) + \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2 dx} = \frac{h[\phi(L)]^2 + \int_0^L \left(\frac{d\phi}{dx}\right)^2}{\int_0^L \phi^2 dx} \geq 0.$$

since $h>0$

If $\lambda = 0 \rightarrow \phi(L) = 0$

$d\phi/dx = 0 \rightarrow \phi(x) = C \rightarrow \cancel{\phi'(0)}^{0} - h\phi(0) = 0 \rightarrow C = 0$

$\Rightarrow \phi(x) \equiv 0 \rightarrow$ hence $\lambda = 0$ is not eigenvalue $\rightarrow \boxed{\lambda > 0}$

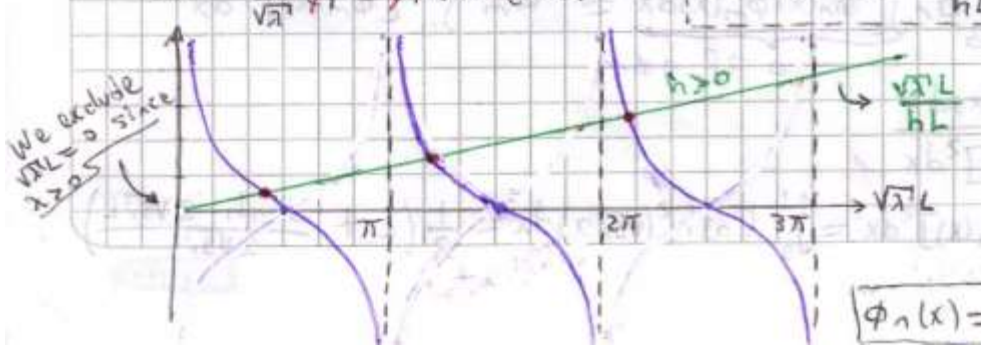
$\underline{\lambda > 0}$: $\rightarrow \phi(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$

$\phi'(x) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$\phi'(0) - h\phi(0) = 0 \rightarrow C_2\sqrt{\lambda} - hC_1 = 0 \rightarrow \boxed{C_2 = \frac{h}{\sqrt{\lambda}}C_1}$

$\phi'(L) = 0 = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}L + C_2\sqrt{\lambda}\cos\sqrt{\lambda}L \rightarrow C_2 = C_1\tan(\sqrt{\lambda}L)$

OR $\frac{h}{\sqrt{\lambda}}\cancel{C_1} = \cancel{C_1}\tan(\sqrt{\lambda}L) \rightarrow \boxed{\cot(\sqrt{\lambda}L) = \frac{\sqrt{\lambda}L}{hL}}$



There are an infinite number of eigenvalues corresponding to the intersections in the graph, with eigenfunctions:

$$\boxed{\phi_n(x) = \cos(\sqrt{\lambda_n}x) + \frac{h}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}x}$$

b) Note that: $0 < \sqrt{\lambda_1}L < \frac{\pi}{2}$, $\pi < \sqrt{\lambda_2}L < \frac{3\pi}{2}, \ldots, (n-1)\pi < \sqrt{\lambda_n}L < \frac{(2n-1)\pi}{2}$

and see in the graph that $\sqrt{\lambda_n}L \to \frac{(2n-1)\pi}{2}$ for n large

$\Rightarrow \boxed{\sqrt{\lambda_n} \sim \left(\frac{2n-1}{2L}\right)\pi}$ Spatial frequency for n large!!

Since $\lambda > 0$ and $\frac{d^2G}{dt^2} + \lambda c^2 G(t) = 0 \to G_n(t) = a_n \cos(c\sqrt{\lambda_n}t) + b_n \sin(c\sqrt{\lambda_n}t)$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos(c\sqrt{\lambda_n}t) + b_n \sin(c\sqrt{\lambda_n}t)\right]\left[\cos\sqrt{\lambda_n}x + \frac{h}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}x\right]}$$

The angular frequency is $\boxed{\omega_n = c\sqrt{\lambda_n} \sim \left(\frac{2n-1}{L}\right)\pi c}$ for large n!!

c) $u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos(c\sqrt{\lambda_n}t) + b_n \sin(c\sqrt{\lambda_n}t)\right]\phi_n(x)$

$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x)$ Multiplying by $\phi_m(x)$ and integrating.

$$\int_0^L f(x)\phi_m(x)\,dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \phi_n(x)\phi_m(x)\,dx}_{0 \quad n\neq m} = a_m \int_0^L [\phi_m(x)]^2\,dx$$

$$\to a_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L [\phi_n(x)]^2\,dx} \Rightarrow \boxed{a_n = \frac{\int_0^L f(x)\left[\cos\sqrt{\lambda_n}x + \frac{h}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}x\right]dx}{\int_0^L \left[\cos\sqrt{\lambda_n}x + \frac{h}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}x\right]^2 dx}}$$

$g(x) = \frac{\partial u}{\partial x}(x,0) = \sum_{n=1}^{\infty} b_n\sqrt{\lambda_n}c\,\phi_n(x)$ Multiplyin by $\phi_n(x)$ and integrating

$$\int_0^L g(x)\phi_m(x)\,dx = \sum_{n=1}^{\infty} b_n\sqrt{\lambda_n}c \underbrace{\int_0^L \phi_n(x)\phi_m(x)\,dx}_{0 \quad n\neq m} = b_m\sqrt{\lambda_m}c\int_0^L [\phi_m(x)]^2\,dx$$

$$\to b_n = \frac{\int_0^L g(x)\phi_n(x)\,dx}{\sqrt{\lambda_n}c\int_0^L [\phi_n(x)]^2\,dx} \to \boxed{b_n = \frac{\int_0^L g(x)\left[\cos\sqrt{\lambda_n}x + \frac{h}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}x\right]dx}{\sqrt{\lambda_n}c\int_0^L \left[\cos\sqrt{\lambda_n}x + \frac{h}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}x\right]^2 dx}}$$

* 5.8.7 $\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \phi(0) = 0 \quad \phi(\pi) - 2\frac{d\phi}{dx}(0) = 0$

a) $$\int_0^{\pi}\left(u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2}\right)dx = u\frac{dv}{dx}\Big|_0^{\pi} - \cancel{\int_0^{\pi}\frac{dv}{dx}\frac{du}{dx}\,dx} - v\frac{du}{dx}\Big|_0^{\pi} + \cancel{\int_0^{\pi}\frac{du}{dx}\frac{dv}{dx}\,dx}$$

Integrating by parts

$$= \left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_0^{\pi}$$

by Boundary conditions!!

$$= u(\pi)\frac{dv}{dx}(\pi) - v(\pi)\frac{du}{dx}(\pi) - \cancel{u(0)}^{0}\frac{dv}{dx}(0) + \cancel{v(0)}^{0}\frac{du}{dx}(0)$$

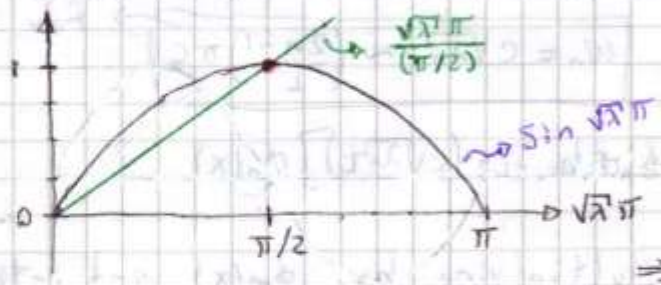
$$= 2\frac{du}{dx}(0)\frac{dv}{dx}(\pi) - 2\frac{dv}{dx}(0)\frac{du}{dx}(\pi)$$

$$\boxed{\int_0^\pi \left(u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2}\right)dx = 2\left[\frac{du}{dx}(0)\frac{dv}{dx}(\pi) - \frac{dv}{dx}(0)\frac{du}{dx}(\pi)\right]} \neq 0 \text{ in general!}$$

b) $\lambda > 0$: $\phi(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$ $\quad \phi(0) = 0 \rightarrow C_1 = 0$

$\phi'(x) = C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$$\phi(\pi) - 2\phi'(0) = 0 = C_2 \sin\sqrt{\lambda}\pi - 2C_2\sqrt{\lambda} \rightarrow \boxed{\sin\sqrt{\lambda}\pi = \frac{2\sqrt{\lambda}\pi}{\pi}}$$



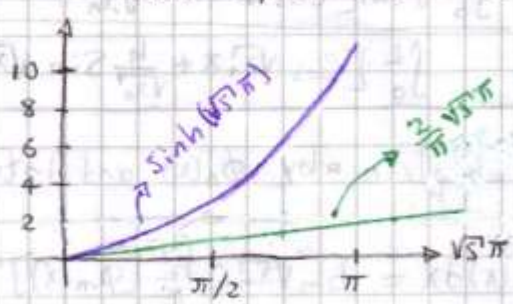
Note there only one intesection at $\sqrt{\lambda}\pi = \pi/2$, therefore the unique eigenvalue is: $\boxed{\lambda = 1/4}$

$$\Rightarrow \boxed{\phi(x) = \sin\left(\tfrac{1}{2}x\right)}$$

c) $\lambda < 0$: let $s = -\lambda \Rightarrow \phi(x) = C_1\cosh\sqrt{s}x + C_2\sinh\sqrt{s}x$ $\quad \phi(0) = 0 = C_1$

$\phi'(x) = C_2\sqrt{s}\cosh(\sqrt{s}x)$

$$\phi(\pi) - 2\phi'(0) = 0 = C_2\sinh\sqrt{s}\pi - 2C_2\sqrt{s} \rightarrow \boxed{\sinh\sqrt{s}\pi = \frac{2\sqrt{s}\pi}{\pi}}$$



No intersections in the graph

$\Rightarrow$ $\boxed{\text{No eigenvalues for } \lambda < 0}$

d) $\lambda = 0$: $\phi(x) = C_1 x + C_0$ $\quad \phi(0) = 0 = C_0 \rightarrow \phi(x) = C_1 x$ and $\phi'(x) = C_1$

$\phi(\pi) - 2\phi'(0) = 0 = C_1(\pi - 2) \rightarrow C_1 = 0 \rightarrow \phi(x) \equiv 0$

$\rightarrow$ $\boxed{\lambda = 0 \text{ is not eigenvalue}}$

e) Yes. Since the boundary condition for this eigenvalue problem are not of the form

$$\beta_1\phi(\pi) + \beta_2\phi'(\pi) = 0$$

this is not a regular Sturm-Liouville problem. Thus, there may be complex eigenvalues (with a non-zero imaginary part), no obtained in parts (b) - (d).

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5.8.8 $\dfrac{d^2\phi}{dx^2} + \lambda\phi = 0$ with $\left.\begin{array}{l}\phi(0) - \phi'(0) = 0 \\ \phi(1) + \phi'(1) = 0\end{array}\right\}$ Sturm-Liouville problem with $p=1$, $q=0$, $\sigma=1$

a) $$\lambda = \frac{-\phi\phi'\big|_0^1 + \int_0^1 \phi'^2\,dx}{\int_0^1 \phi^2\,dx} = \frac{-\phi(1)\phi'(1) + \phi(0)\phi'(0) + \int_0^1 [\phi'(x)]^2 dx}{\int_0^1 [\phi(x)]^2 dx}$$

$$= \frac{-\phi(1)(-\phi(1)) + \phi(0)(\phi(0)) + \int_0^1 [\phi'(x)]^2 dx}{\int_0^1 [\phi(x)]^2 dx}$$

$$\lambda = \frac{[\phi(1)]^2 + [\phi(0)]^2 + \int_0^1 [\phi'(x)]^2 dx}{\int_0^1 [\phi(x)]^2 dx} \geq 0$$

$\lambda = 0 \rightarrow \phi(1) = \phi(0) = 0$ and $\phi'(x) = 0 \rightarrow \phi(x) = C_1$, $\phi'(1) + \phi(1) = 0 \rightarrow C_1 = 0$ $\Big\}$ $\phi(x) \equiv 0$

No eigenvalues for $\lambda = 0$ $\longrightarrow$ $\boxed{\lambda > 0}$

b) Let λ_n and λ_m be eigenvalues with corresponding eigenfunctions $\phi_n(x)$ and $\phi_m(x)$.

$$L(\phi_n) + \lambda_n\phi_n = 0 \quad \text{and} \quad L(\phi_m) + \lambda_m\phi_m = 0$$

where $L = d^2/dx^2$.

$$\int_0^1 [\phi_m L(\phi_n) - \phi_n L(\phi_m)]\,dx = \cancel{p}^{\,1}\left(\phi_m \frac{d\phi_n}{dx} - \phi_n \frac{d\phi_m}{dx}\right)\Big|_0^1 \rightarrow \text{by Green's formula.}$$

$$\int_0^1 [\phi_m(-\lambda_n\phi_n) - \phi_n(-\lambda_m\phi_m)]\,dx = \phi_m(1)\phi_n'(1) - \phi_n(1)\phi_m'(1) - \phi_m(0)\phi_n'(0) + \phi_n(0)\phi_m'(0)$$

$$(\lambda_m - \lambda_n)\int_0^1 \phi_m\phi_n\,dx = \cancel{\phi_m(1)(-\phi_n'(1))} - \cancel{\phi_n(1)(-\phi_m'(1))} - \cancel{\phi_m(0)\phi_n'(0)} + \cancel{\phi_n(0)\phi_m'(0)}$$

$$= 0 \rightarrow \text{if } \lambda_m \neq \lambda_m \Rightarrow \boxed{\int_0^1 \phi_m\phi_n\,dx = 0} \text{ if } n \neq m$$

*c) $\lambda > 0$: $\phi(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$

$\phi'(x) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$\phi(0) - \phi'(0) = 0 = C_1 - C_2\sqrt{\lambda} \longrightarrow C_1 = C_2\sqrt{\lambda}$

$\phi(1) + \phi'(1) = 0 = C_2(\sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda}) - C_2\sqrt{\lambda}(\sqrt{\lambda}\sin\sqrt{\lambda} + \cos\sqrt{\lambda})$

dividing by $C_2\cos\sqrt{\lambda}$: $(\sqrt{\lambda} + \tan\sqrt{\lambda}) - \sqrt{\lambda}(\sqrt{\lambda}\tan\sqrt{\lambda} + 1) = 0$

$$2\sqrt{\lambda} + (1-\lambda)\tan\sqrt{\lambda} = 0$$

$$\boxed{\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}}$$

Discuss the function on the right hand side $\longrightarrow$

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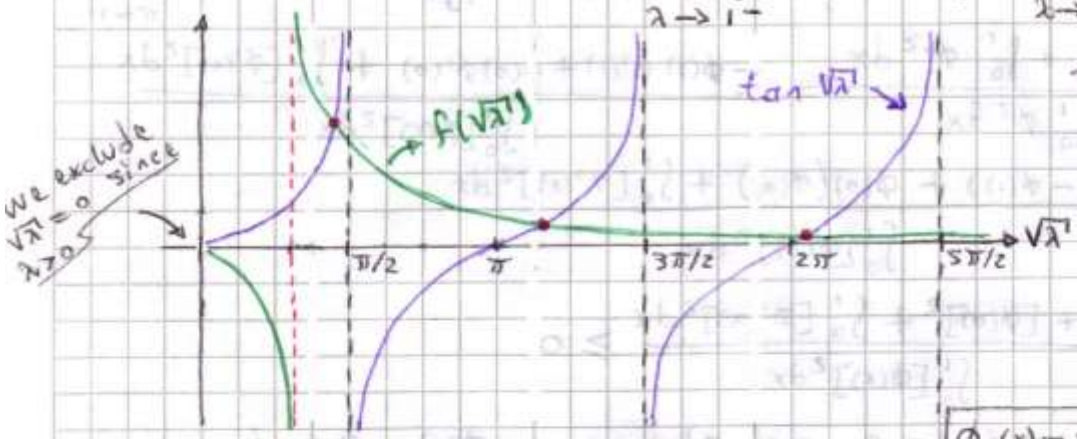
Let $f(\sqrt{\lambda}) = \dfrac{2\sqrt{\lambda}}{\lambda - 1} \rightarrow f(0) = 0$ and $f(+\infty) \rightarrow 0$.

$\lim_{\lambda \to 1^-} f(\sqrt{\lambda}) = -\infty \quad \lim_{\lambda \to 1^+} f(\sqrt{\lambda}) = +\infty$



There are an infinite intersections that correspond to the solution for λ. Thus, there are infinite number of eigenvalues λ_n corresponding to the eigenfunctions:

$$\boxed{\phi_n(x) = \sqrt{\lambda_n}\cos\sqrt{\lambda_n}\,x + \sin\sqrt{\lambda_n}\,x}$$

Note from graph that for large n $\sqrt{\lambda_n} \sim (n-1)\pi$

These eigenfunctions are orthogonals from part b)

d) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ $\quad u(0,t) - \dfrac{\partial u}{\partial x}(0,t) = 0$, $\quad u(1,t) + \dfrac{\partial u}{\partial x}(1,t) = 0$, $\quad u(x,0) = f(x)$.

Let $u(x,t) = h(t)\phi(x) \rightarrow \phi\dfrac{dh}{dt} = kh\dfrac{d^2\phi}{dt^2}$ dividying by $kh\phi$:

$$\underbrace{\frac{1}{kh}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{-\lambda} \rightarrow \frac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad \left.\begin{array}{l}\phi(0) - \phi'(0) = 0\\ \phi(1) + \phi'(1) = 0\end{array}\right\}$$

Problem solved above

$$\frac{dh}{dt} = -\lambda k h(t) \rightarrow \boxed{h(t) = c\,e^{-\lambda k t}}$$

$$\boxed{\phi_n(x) = \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\,x) + \sin(\sqrt{\lambda_n}\,x)}$$

where λ_n is given by graphical intersection that are solutions of $\tan\sqrt{\lambda} = 2\sqrt{\lambda}/(\lambda - 1)$.

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n k t}\phi_n(x) = \boxed{\sum_{n=1}^{\infty} a_n e^{-\lambda_n k t}\left[\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\,x + \sin\sqrt{\lambda_n}\,x\right]}$$

$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n\phi_n(x)$ Multiplying by $\phi_m(x)$ and integrating

$$\int_0^1 f(x)\phi_m(x)\,dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^1 \phi_n(x)\phi_m(x)\,dx}_{0 \text{ if } n \neq m} = a_m\int_0^1 [\phi_m(x)]^2\,dx$$

$$\rightarrow a_n = \frac{\int_0^1 f(x)\phi_n(x)\,dx}{\int_0^1 [\phi_n(x)]^2\,dx} \qquad a_n = \frac{\int_0^1 f(x)\left[\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\,x + \sin\sqrt{\lambda_n}\,x\right]dx}{\int_0^1 \left[\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\,x + \sin\sqrt{\lambda_n}\,x\right]^2 dx}$$

$$D = \frac{1}{2}\sqrt{4\lambda - \frac{V_0^2}{K^2}} = \frac{n\pi}{L} \rightarrow \boxed{\lambda_n = -\left(\frac{n\pi}{L}\right)^2 + \left(\frac{V_0}{2K}\right)^2}$$

$$\Rightarrow \boxed{\phi_n(x) = e^{(V_0/2K)x} A_n\left(\cos\frac{n\pi x}{L} - \frac{V_0 L}{2Kn\pi}\sin\frac{n\pi x}{L}\right)}$$

Since $h(t) = e^{-K\lambda t}$ from temporal equation, then:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t} e^{(V_0/2K)x}\left(\cos\frac{n\pi x}{L} - \frac{V_0 L}{2Kn\pi}\sin\frac{n\pi x}{L}\right)}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} \underbrace{A_n e^{(V_0/2K)x}\left(\cos\frac{n\pi x}{L} - \frac{V_0 L}{2Kn\pi}\sin\left(\frac{n\pi x}{L}\right)\right)}_{\phi_n(x)}$$

$$f(x) = \sum_{n=1}^{\infty} A_n \phi_n(x)$$

Multiplying by $\phi_m^*(x)\sigma(x)$ and using the inner product. for integrate:

$$\int_0^L f(x)\phi_m^*(x)\sigma\,dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \phi_n(x)\phi_m^*(x)\sigma(x)\,dx}_{0 \text{ if } n\neq m} = A_m\int_0^L \phi_m(x)\phi_m^*(x)\sigma\,dx$$

$$\Rightarrow \boxed{A_n = \frac{\int_0^L f(x)\phi_n^*(x)\sigma(x)\,dx}{\int_0^L |\phi_n(x)|^2\sigma(x)\,dx}}$$

where $\sigma(x) = e^{-V_0 x/K}$ if you compar the last eq in a) with Sturm-Liouville problem.

5.3.5 $\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad \frac{d\phi}{dx}(0) = 0 \qquad \frac{d\phi}{dx}(L) = 0$

$\lambda > 0 \rightarrow \phi = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$\phi' = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x \rightarrow \phi'(0) = 0 = B\sqrt{\lambda} \rightarrow B = 0$

$\phi'(L) = 0 = -A\sqrt{\lambda}\sin\sqrt{\lambda}L = 0$

$\rightarrow \sqrt{\lambda}L = n\pi \rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \ldots \quad \boxed{\phi_n(x) = A_n\cos\left(\frac{n\pi x}{L}\right)}$

$\lambda = 0 \rightarrow \phi = C_1 x + A_0 \rightarrow \phi'(x) = C_1$

$\phi'(0) = 0 = C_1 \quad \phi'(L) = 0 = C_1 \rightarrow \phi_0 = A_0$

$\lambda < 0 \rightarrow$ No eigenvalues because no oscilotes.

$\Rightarrow \boxed{\phi_n(x) = A_n\cos\left(\frac{n\pi x}{L}\right)} \qquad n = 0, 1, 2, 3, \ldots$

a) $\lambda_n = \frac{n\pi x}{L}$, $n = 0, 1, 2, \ldots$ infty eigenvalues with the smallest: $\underline{\lambda_0 = 0}$ but not largest $n \to \infty \rightarrow \lambda_n \to \infty$

with:

$$\left[\sqrt{\lambda_n}\cos\sqrt{\lambda_n}x+\sin\sqrt{\lambda_n}x\right]^2=\lambda_n\cos^2\sqrt{\lambda_n}x+\sin^2\sqrt{\lambda_n}x+2\sqrt{\lambda_n}\cos\sqrt{\lambda_n}x\sin\sqrt{\lambda_n}x$$

$$=\frac{\lambda_n}{2}\left(1+\cos(2\sqrt{\lambda}x)\right)+\frac{1}{2}\left(1-\cos(2\sqrt{\lambda}x)\right)+2\sqrt{\lambda}\sin(2\sqrt{\lambda}x)$$

$$=\frac{1}{2}(1+\lambda_n)+\frac{1}{2}(\lambda_n-1)\cos(2\sqrt{\lambda_n}x)+\sqrt{\lambda_n}\sin(2\sqrt{\lambda_n}x)$$

$$\Rightarrow\int_0^1\left[\sqrt{\lambda_n}\cos\sqrt{\lambda_n}x+\sin\sqrt{\lambda_n}x\right]^2dx=\int_0^1\left[\frac{1+\lambda_n}{2}+\frac{\lambda_n-1}{2}\cos 2\sqrt{\lambda_n}x+\sqrt{\lambda_n}\sin 2\sqrt{\lambda_n}x\right]dx$$

$$=\left[\frac{1+\lambda_n}{2}x+\frac{\lambda_n-1}{2}\frac{\sin 2\sqrt{\lambda_n}x}{2\sqrt{\lambda_n}}-\frac{\cos 2\sqrt{\lambda_n}x}{2}\right]_0^1$$

$$=\frac{1+\lambda_n}{2}+\frac{\lambda_n-1}{2}\frac{\sin 2\sqrt{\lambda_n}}{2\sqrt{\lambda_n}}-\frac{\cos 2\sqrt{\lambda_n}}{2}+\frac{1}{2}$$

$$=\frac{2+\lambda_n}{2}-\frac{\cos 2\sqrt{\lambda_n}}{2}+\frac{(\lambda_n-1)}{2}\frac{\sin 2\sqrt{\lambda_n}}{2\sqrt{\lambda_n}}$$

$$\Rightarrow a_n=2\left[(2+\lambda_n)-\cos 2\sqrt{\lambda_n}+(\lambda_n-1)\frac{\sin 2\sqrt{\lambda_n}}{2\sqrt{\lambda_n}}\right]^{-1}\cdot\int_0^1 f(x)\phi_n(x)\,dx$$

$$\boxed{a_n=\frac{2\int_0^1 f(x)\left[\sqrt{\lambda_n}\cos\sqrt{\lambda_n}x+\sin\sqrt{\lambda_n}x\right]dx}{(2+\lambda_n)-\cos 2\sqrt{\lambda_n}+(\lambda_n-1)\frac{\sin 2\sqrt{\lambda_n}}{2\sqrt{\lambda_n}}}}$$

5.8.9 $\frac{d^2\phi}{dx^2}+\lambda\phi=0 \qquad \phi(0)=\phi'(0)$ and $\phi(1)=\beta\phi'(1)$

$\underline{\lambda=0}\rightarrow\frac{d^2\phi}{dx^2}=0\rightarrow\phi'=C_1\rightarrow\phi(x)=C_1x+C_0$

$\phi(0)=\phi'(0)\rightarrow C_0=C_1\rightarrow\phi=C_0(x+1)$

$\phi(1)=\beta\phi'(1)\rightarrow 2\cancel{C_0}=\beta\cancel{C_1}\rightarrow\boxed{\beta=2}$

5.8.10 $\frac{d^2\phi}{dx^2}+\lambda\phi=0 \qquad \phi(0)=0 \qquad \phi'(1)+\phi(1)=0$ → Sturm-Liouville problem $p=1$, $q=0$, $\sigma=1$

* a) By Rayleigh quotient $\lambda=\dfrac{-\phi\phi'(x)\big|_0^1+\int_0^1[\phi'(x)]^2dx}{\int_0^1\phi^2dx}$ $\quad\phi(0)=0$ and $\phi'(1)=-\phi(1)$

$$\Rightarrow\lambda=\frac{[\phi(1)]^2+\int_0^1[\phi'(x)]^2dx}{\int_0^1[\phi(x)]^2dx}\geq 0$$

If $\lambda=0\rightarrow\phi'(x)=0\rightarrow\phi(x)=C_0$, but $\phi(0)=0\rightarrow C_0=0\rightarrow\phi(x)\equiv 0$

$\phi(1)=0$

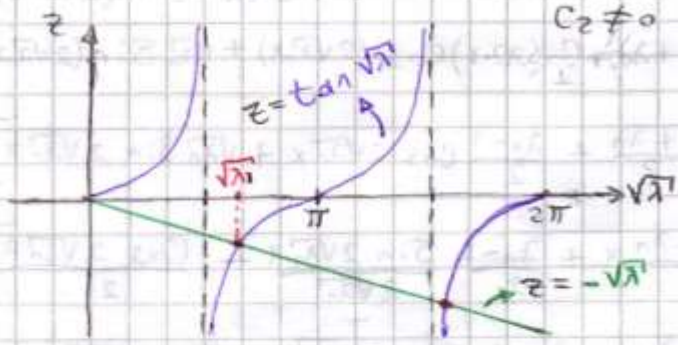
$\rightarrow\lambda=0$ is not eigenvalue $\Rightarrow\boxed{\lambda>0}$

$\underline{\lambda > 0}$: $\phi(x) = C_1 \cos\sqrt{\lambda}\,x + C_2 \sin\sqrt{\lambda}\,x$ $\quad \phi(0) = 0 \rightarrow C_1 = 0$

$\phi'(x) = C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x$

$\phi'(1) + \phi(1) = 0 \rightarrow C_2(\sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda}) = 0$

$C_2 \neq 0$ for no trivial solutions. $\Rightarrow \boxed{\tan\sqrt{\lambda} = -\sqrt{\lambda}}$



The eigenvalues correspond to the solution of equation, that is given by the intersections of the graphs.

$\boxed{\phi_n(x) = \sin\sqrt{\lambda_n}\,x}$

Note that $\frac{\pi}{2} < \sqrt{\lambda_1} < \pi$. Using Mathematica in my PC:

$\tan(2.03) = -2.022419$ and $\tan(2.02) = -2.074373 \rightarrow \sqrt{\lambda_1} \approx 2.03$

$\boxed{\lambda_1 \approx 4.12}$

*b) Using Newton's method (letting $x = \sqrt{\lambda_1}$) with $f(x) = x + \tan x$, yields:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n + \tan x_n}{1 + \sec^2 x_n}$$

In this manner, we obtain $\sqrt{\lambda_1} = \lim_{n\to\infty} x_n = 2.028758$

OR $\boxed{\lambda_1 = 4.11586}$

c) By Rayleight quotient $\lambda = \dfrac{[\phi(1)]^2 + \int_0^1 [\phi'(x)]^2 dx}{\int_0^1 [\phi(x)]^2 dx}$

and by Minimization principle $\lambda_1 \leq \dfrac{[u(1)]^2 + \int_0^1 [u'(x)]^2 dx}{\int_0^1 [u(x)]^2 dx}$

where $u(x)$ is a trial function, that we choose with no zeros in the interior for find the bound of the lowest eigenvalue. In addition $u(x)$ need satisfy the boundary conditions of our problem!!

We will try to find a simple trial function, Let

$u(x) = ax^2 + bx + c \rightarrow u(0) = 0 = c$

$u(1) + u'(1) = 0 \rightarrow 3a + 2b = 0 \rightarrow b = -\frac{3}{2}a$

$\Rightarrow u(x) = a\left(x^2 - \frac{3}{2}x\right) \rightarrow \lambda_1 \leq \dfrac{\cancel{a^2}(1 - 3/2)^2 + \int_0^1 \cancel{a^2}(2x - 3/2)^2 dx}{\int_0^1 \cancel{a^2}(x^2 - 3x/2)^2 dx}$

$$\lambda_1 \leq \frac{(1/4) + 7/12}{1/5} = \frac{25}{6} \approx 4.1\overline{6}$$

It is a very good bound closely to results in (a)-(b) $\Rightarrow$ $\boxed{\lambda_1 \leq 4.1\overline{6}}$ Note that Results in (a)-(b) are less that this bound.

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5.8.11 $\frac{d^2\phi}{dx^2} + (5+\lambda)\phi = 0$ with $\phi(0)=0$ and $\phi(\pi)=0$

Let $\gamma = 5+\lambda \rightarrow \underbrace{\frac{d^2\phi}{dx^2} + \gamma\phi = 0 \quad \phi(0)=0 \quad \phi(\pi)=0}_{\text{Sturm-Liouville problem with } p=1,\ q=0,\ \sigma=1}$

By Rayleight quotient: $\gamma = \dfrac{\cancel{p\phi(x)\phi'(x)\Big|_0^\pi}^{\,0} + \int_0^\pi [p[\phi'(x)]^2 - \cancel{q}^{\,0}\phi^2]dx}{\int_0^\pi [\phi(x)]^2 dx} \geq 0$

If $\gamma = 0 \rightarrow \phi''(x) = 0 \rightarrow \phi(x) = C_0$, but $\phi(0)=0 \rightarrow C_0 = 0$.
$\Rightarrow \phi(x) \equiv 0$ trivial solution $\rightarrow$ $\gamma = 0$ is not eigenvalue

Thus $\gamma > 0$. Hence $5+\lambda > 0 \rightarrow \boxed{\lambda > -5}$

For $\gamma > 0$: $\phi(x) = C_1 \cos\sqrt{\gamma}\,x + C_2 \sin\sqrt{\gamma}\,x$ $\quad \phi(0) = 0 = C_1$
$\phi(\pi) = 0 = C_2 \sin\sqrt{\gamma}\,\pi$

$C_2 \neq 0$ for no trivial solutions, then: $\sqrt{\gamma}\,\pi = n\pi$

$\Rightarrow \boxed{\gamma = n^2}$ and $\boxed{\phi_n(x) = C_n \sin(nx)}$

Since $\lambda = \gamma - 5 \Rightarrow \boxed{\lambda = n^2 - 5}$ $\quad n = 1, 2, 3, 4, \ldots$

$\Rightarrow \boxed{\lambda_1 = -4,\ \lambda_2 = -1} \rightarrow$ only these two λ_n are negatives!!

5.8.12 $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t) = 0$ $\quad m\frac{\partial^2 u}{\partial t^2}(L,t) = -T_0 \frac{\partial u}{\partial x}(L,t) - K u(L,t)$

↳ Wave equation of a vibrating string

a) $u(0,t) = 0 \rightarrow$ fixed end at $x=0$.

$m\frac{\partial^2 u}{\partial t^2}(L,t) = \underbrace{-T_0 \frac{\partial u}{\partial x}(L,t)}_{\text{damping force}} \underbrace{- K u(L,t)}_{\text{Hooke's Law for spring-mass system}}$

Fixed end at $x=0$

String attached to spring-mass system at $x=L$ in damping media.

$\begin{pmatrix}\text{Second Law's}\\ \text{Newton}\end{pmatrix} = \begin{pmatrix}\text{damping}\\ \text{Force}\end{pmatrix} + \begin{pmatrix}\text{Hooke's Law}\\ \text{for spring-mass}\\ \text{system}\end{pmatrix}$

Let $u(x,t) = h(t)\phi(x) \rightarrow \phi\frac{d^2h}{dt^2} = c^2 h \frac{d^2\phi}{dx^2}$ dividing by $c^2 h\phi$

$\underbrace{\frac{1}{c^2 h}\frac{d^2h}{dt^2}}_{-\lambda} = \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{-\lambda} \rightarrow \frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \phi(0) = 0$

$\boxed{\frac{d^2h}{dt^2} = -\lambda c^2 h(t)} \longrightarrow m\phi(L)\frac{d^2h}{dt^2} = -T_0 h(t)\phi'(L) - K h(t)\phi(L)$

$m\phi(L)(-\lambda c^2 \cancel{h(t)}) = -\cancel{h(t)}(T_0\phi'(L) - K\phi(L))$

Thus the last boundary condition is reduced to:

$$T_0\phi'(L)+(K-m\lambda c^2)\phi(L)=0 \Rightarrow \boxed{\frac{d^2\phi}{dx^2}+\lambda\phi=0 \qquad \phi(0)=0 \quad \phi'(L)=\frac{(m\lambda c^2-K)}{T_0}\phi(L)}$$

Sturm-Liouville problem with $p=1$, $q=0$ and $\sigma=1$

by Rayleight quotient $\lambda=\dfrac{-\phi\phi'|_0^L+\int_0^L[\phi'(x)]^2dx}{\int_0^L[\phi(x)]^2dx}$

$$\lambda=\frac{(K-m\lambda c^2)[\phi(L)]^2+\int_0^L[\phi'(x)]^2dx}{\int_0^L[\phi(x)]^2dx}$$

$$\left(1+\frac{mc^2[\phi(L)]^2}{\int_0^L[\phi(x)]^2dx}\right)\lambda=\frac{K[\phi(L)]^2+\int_0^L[\phi'(x)]^2dx}{\int_0^L[\phi(x)]^2dx}$$

OR $$\lambda=\left(1+\frac{mc^2[\phi(L)]^2}{\int_0^L[\phi(x)]^2dx}\right)^{-1}\frac{K[\phi(L)]^2+\int_0^L[\phi'(x)]^2dx}{\int_0^L[\phi(x)]^2dx}\geq 0$$

$\lambda\geq 0$ since $K>0$ and $m>0$ as usual by physical requirements (the mass is always positive and also K, the constant of restoring force in Hooke's Law).

$\lambda=0 \rightarrow \phi'(x)=0 \rightarrow \phi(x)=C_0 \rightarrow \phi(0)=0=C_0$
$\phi(L)=0 \rightarrow$ No sense since the end of spring is no fixed at $x=L$ $\Big\}$ $\phi(x)\equiv 0$ trivial solution

$\Rightarrow \lambda=0$ is not eigenvalue.

$\Rightarrow \boxed{\lambda>0} \Rightarrow$ Only there exists oscillatory solutions!!

b) $\lambda>0$: $\phi(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x \qquad \phi(0)=0=C_1$
$\phi'(x)=C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$$\phi'(L)=\left(\frac{m\lambda c^2-K}{T_0}\right)\phi(L)\rightarrow C_2\sqrt{\lambda}\cos\sqrt{\lambda}L=\left(\quad\right)C_2\sin\sqrt{\lambda}L$$

OR $$\boxed{\tan(\sqrt{\lambda}L)=\left(\frac{mc^2}{T_0L}\right)\sqrt{\lambda}L-\left(\frac{KL}{T_0}\right)\frac{1}{\sqrt{\lambda}L}}$$ and the eigenfunctions are $\boxed{\phi_n(x)=\sin(\sqrt{\lambda_n}x)}$

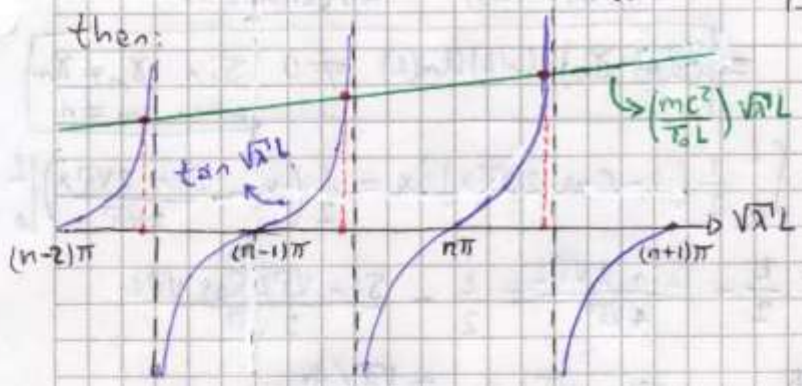
The engenvalues are determined by the intersections of the graph of $\tan(\sqrt{\lambda}L)$ and $f(\sqrt{\lambda}L)=\left(\frac{mc^2}{T_0L}\right)\sqrt{\lambda}L-\left(\frac{KL}{T_0}\right)\frac{1}{\sqrt{\lambda}L}$

That give the spatial oscilatory frequencies $\sqrt{\lambda_n}$ and the temporal oscillatory frequencies $\omega=\sqrt{\lambda_n}c$.

Since $\lambda>0 \Rightarrow \boxed{h_n(t)=A_n\cos(\sqrt{\lambda_n}ct)+B_n\sin(\sqrt{\lambda_n}ct)}$

For a large n, $\sqrt{\lambda}L \gg \frac{1}{\sqrt{\lambda}L} \Rightarrow \boxed{\tan(\sqrt{\lambda}L) \sim \left(\frac{mc^2}{T_0 L}\right)\sqrt{\lambda}L}$

then:



From graph we can see that the intesections there are at

$$\sqrt{\lambda}L \approx \left(n+\tfrac{1}{2}\right)\pi$$

$$\rightarrow \boxed{\lambda_n \approx \left[\left(n+\tfrac{1}{2}\right)\frac{\pi}{L}\right]^2}$$

for large n $(n \in \mathbb{N})$

c) $\underline{\lambda = 0}$: $\phi(x) = C_1 x + C_0 \rightarrow \phi(0) = 0 = C_0$

$$\phi'(L) = \frac{m\cancelto{0}{\lambda} c^2 - K}{T_0}\phi(L) \rightarrow C_1 = -\frac{K}{T_0}C_1 L \rightarrow T_0 = -KL$$

It's no sense physically sin $T_0 > 0$ $(K>0,\ L>0) \Rightarrow \phi \equiv 0$ trivial sol.

$\Rightarrow$ $\boxed{\lambda = 0 \text{ is not eigenvalue}}$

$\underline{\lambda < 0}$: Let $s = -\lambda \rightarrow \phi(x) = C_1 \cosh\sqrt{s}x + C_2 \sinh\sqrt{s}x$ $\quad \phi(0) = 0 = C_1$

$\phi'(x) = C_2\sqrt{s}\cosh\sqrt{s}x$

$$\phi'(L) = \frac{m\lambda c^2 - K}{T_0}\phi(L) \rightarrow C_2\sqrt{s}\cosh\sqrt{s}L = (\quad)\, C_2 \sinh\sqrt{s}L$$

$$\Rightarrow \tanh(\sqrt{s}L) = \frac{-msc^2 - K}{\sqrt{s}\,T_0} \rightarrow \boxed{\tanh(\sqrt{s}L) = -\left(\frac{mc^2}{T_0 L}\right)\sqrt{s}L - \left(\frac{KL}{T_0}\right)\frac{1}{\sqrt{s}L}}$$

OR $\tanh(\sqrt{s}L) = f(\sqrt{s}L)$ with $f(\sqrt{s}L) = -\left(\frac{mc^2}{T_0 L}\right)\sqrt{s}L - \left(\frac{KL}{T_0}\right)\frac{1}{\sqrt{s}L}$

Note that $\tanh(\sqrt{s}L)$ is a positive function for all $\sqrt{s}L > 0$ $(\lambda < 0)$ while $f(\sqrt{s}L)$ is a negative function in the same range, then these two functions never intersects

$\hookrightarrow$ Since physical parameters m, T_0, L, K are all positives!!

Thus, $\boxed{\lambda < 0 \text{ else is not eigenvalue}}$

d) $$\int_0^L \left(u_n \frac{d^2u_m}{dx^2} - u_m \frac{d^2u_n}{dx^2}\right)dx = u_n \frac{du_m}{dx}\Big|_0^L - \cancel{\int_0^L \frac{du_m}{dx}\frac{du_n}{dx}} - u_m \frac{du_n}{dx}\Big|_0^L + \cancel{\int_0^L \frac{du_n}{dx}\frac{du_m}{dx}}$$

Integrating by parts

$$= u_n(L)u_m'(L) - u_m(L)u_n'(L) - \cancelto{0}{u_n(0)}u_m'(0) + \cancelto{0}{u_m(0)}u_n'(0)$$

by BC's

Using te BC: $\phi'(L) = \underbrace{\left(\frac{m\lambda_n c^2 - K}{T_0}\right)}_{\gamma_n}\phi(L) = \gamma_n \phi(L)$

$\hookrightarrow$ then $\Rightarrow$

$n = 0, 1, 2, 3 \ldots$ $\qquad$ $m = 0, 1, 2, 3 \ldots$

b) $\phi_n(x) = \cos \frac{n\pi x}{L} = 0$ on $[0, L]$ $\Rightarrow$ $\frac{n\cancel{\pi} x}{L} = \frac{(2m+1)\cancel{\pi}}{2}$

$\rightarrow x = \frac{(2m+1)L}{2n}$ and $0 < \frac{(2m+1)L}{2n} < L \longleftrightarrow 0 < \frac{2m+1}{2n} < 1$

This is true for m from $m=0$ to $m = n-1$. Hence $\phi_n(x)$ has $(n-1)$ zeros on $[0, L]$.

c) $\int_0^L \phi_m(x)\phi_n(x)\sigma(x)\,dx = \int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} \frac{L}{2}\delta_{nm} & \text{if } n \geq 1 \\ L & \text{if } n = 0. \end{cases}$

then, the eigenfunctions are orthogonal.

Let $f(x)$ be a piecewise smooth function, then

$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$ multiplying by $\phi_m^*(x)\sigma(x)$, integrating

$$\int_0^L f(x)\phi_m^*(x)\sigma(x)\,dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \phi_n(x)\phi_m^*(x)\sigma(x)\,dx}_{0 \text{ if } n \neq m} = a_m \int_0^L |\phi_m(x)|^2 \sigma(x)\,dx$$

$$\rightarrow \boxed{a_n = \frac{\int_0^L f(x)\phi_n^*(x)\sigma(x)\,dx}{\int_0^L \|\phi_n(x)\|^2 \sigma(x)\,dx}} = \frac{\int_0^L f(x)\cos(n\pi x/L)\,dx}{(L/2)} = \boxed{\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

$\Rightarrow$ Thus the eigenfunctions are complete.

d) $\lambda = \dfrac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2 \sigma\,dx}$ $\qquad$ where $a=0$, $b=L$, $\frac{d\phi}{dx}(0) = 0 = \frac{d\phi}{dx}(L)$

Comparing with (5.3.2) $\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \rightarrow p = 1$, $q = 0$ and $\sigma = 1$. Thus this quotient reduces to: $\lambda = \dfrac{\int_0^L (d\phi/dx)^2\,dx}{\int_0^L \phi^2\,dx} \geq 0$

Then $\lambda = 0$ can be an eigenvalue if $\phi(x) = A_0$ is its corresponding function. There can be no negative eigenvalues.

5.3.6 $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$.

$\lambda = 0 \rightarrow \phi = C_1 x + C_0 \rightarrow \phi'(0) = 0 = C_1$ $\qquad$ $\phi(L) = C_0 = 0$

$\lambda > 0 \rightarrow \phi = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$\phi' = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$ $\qquad$ $\phi'(0) = B\sqrt{\lambda} = 0 \rightarrow B = 0$

$\phi(L) = A\cos\sqrt{\lambda}L = 0$

$\Rightarrow \sqrt{\lambda}L = (2n-1)\frac{\pi}{2}$

$$\boxed{\lambda_n = \left[\frac{(2n-1)\pi}{2}\right]^2} \longrightarrow \boxed{\phi_n(x) = A_n\cos\left[\frac{(2n-1)\pi x}{L}\right]}$$

$n = 1, 2, 3, \ldots$

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a) $\lambda_1 = \frac{\pi x}{2L} < \lambda_2 = \frac{3\pi x}{2} < \lambda_3 = \frac{5\pi x}{2} < \cdots < \lambda_n = \frac{(2n-1)\pi}{L} < \cdots < \infty$

↳ smallest, infinite number, no largest – $\lim_{n\to\infty} \lambda_n = \infty$

b) $\phi_n = \cos\left[\frac{(2n-1)\pi x}{2L}\right] = 0$ for $0 < x < L$ fixed n.

$\Rightarrow \frac{(2n-1)\cancel{\pi} x}{\cancel{2}L} = \frac{(2m-1)\cancel{\pi}}{\cancel{2}}$ $m = 1, 2, 3 \ldots$ $\rightarrow x = \frac{2m-1}{2n-1} \cdot L \rightarrow 0 < \frac{2m-1}{2n-1} \cdot L < L$

OR $0 < \frac{2m-1}{2n-1} < 1 \rightarrow 2m-1 < 2n-1$ OR $m < n$

then $\boxed{m = 1, 2, 3, \ldots, n-1} \rightarrow n-1$ internal zeros at $x = \frac{2m-1}{2n-1} L$.

c) $\int_0^L \phi_n \phi_m^* \sigma \, dx = \underbrace{\int_0^L \cos\left[\frac{(2n-1)\pi x}{2L}\right] \cos\left[\frac{(2m-1)\pi x}{2L}\right] dx}_{(L/2)\delta_{nm}}$ $\sigma = 1$

then eigenfunctions are orthogonals.

$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x) = \sum_{n=1}^{\infty} c_n \cos\left[\frac{(2n-1)\pi x}{2L}\right]$ multiplying by $\cos\frac{(2m-1)\pi x}{L}$ and integrating

$\int_0^L f(x) \cos \frac{(2m-1)\pi x}{L} dx = \sum_{n=1}^{\infty} c_n \underbrace{\int_0^L \cos \frac{(2n-1)\pi x}{2L} \cos \frac{(2m-1)\pi x}{2L} dx}_{(L/2)\delta_{nm}}$

$= c_n \frac{L}{2}$ $\rightarrow$ $\boxed{c_n = \frac{2}{L} \int_0^L f(x) \cos\left[\frac{(2m-1)\pi x}{2L}\right] dx}$

$\boxed{c_n = \frac{\int_0^L f(x) \phi_n^*(x) \sigma(x) dx}{\int_0^L |\phi_n(x)|^2 \sigma \, dx}}$ since $\int_0^L |\phi_n(x)|^2 \sigma \, dx = L/2$ ✓

d) $\lambda = \frac{-p\phi \, d\phi/dx \Big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2] dx}{\int_a^b \phi^2 \sigma \, dx}$ where $a = 0$, $b = L$

$\frac{d\phi}{dx}(0) = 0$, $\phi(L) = 0$

comparing $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ with (5.3.2) $\rightarrow p = 1, q = 0$ an $\sigma = 1$

Thus this quotient reduces to: $\lambda = \frac{2}{L} \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \geq 0$

since $\int_0^L \phi^2 dx = L/2$. because $\left(\frac{d\phi}{dx}\right)^2 \geq 0$.

Then $\lambda = 0 \rightarrow \frac{d\phi}{dx} = 0 \rightarrow \phi = c$ but $\phi(L) = 0 \rightarrow c = 0$

thus $\lambda = 0$ is not eigenvalue. $\Rightarrow$ $\boxed{\lambda > 0}$ all eigenvalues are positive.

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5.3.7 $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ with $\phi(-L) = \phi(L)$ $\quad \frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$

we start with the Rayleigh quotient:

$$\lambda = \frac{-p\phi(d\phi/dx)\Big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2 \sigma\, dx}$$

here we have $a = -L$, $b = L$, $p = 1$, $q = 0$, $\sigma = 1$ thus:

$$\lambda = \frac{\cancel{-p\phi(d\phi/dx)\Big|_{-L}^{L}}^{\,0} + \int_{-L}^{L} (d\phi/dx)^2\,dx}{\int_{-L}^{L} \phi^2\,dx} \geq 0 \quad \text{thus } \lambda \geq 0 \text{ always.}$$

$\underline{\lambda = 0}$: $\rightarrow \phi(x) = C_1 x + C_0 \rightarrow \phi(-L) = \phi(L) \rightarrow C_1 = 0$

$$\boxed{\phi_0 = C_0}$$

$\underline{\lambda > 0}$: $\rightarrow \phi(x) = C_1 \cos\sqrt{\lambda}\,x + C_2 \sin\sqrt{\lambda}\,x$

$\phi(-L) = \phi(L) \rightarrow \cancel{C_1 \cos\sqrt{\lambda}(-L)}\ C_2 \sin\sqrt{\lambda}(-L) = \cancel{C_1 \cos\sqrt{\lambda}L} + C_2 \sin\sqrt{\lambda}L$

$-C_2 \sin\sqrt{\lambda}L = C_2 \sin\sqrt{\lambda}L$

$\rightarrow 2C_2 \sin\sqrt{\lambda}L = 0 \rightarrow \sqrt{\lambda}L = n\pi \rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3$

$$\boxed{\phi = C_1 \cos\left(\frac{n\pi x}{L}\right) + C_2 \sin\left(\frac{n\pi x}{L}\right)}$$

$$\phi' = -C_1\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + C_2\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

$\phi'(-L) = \phi'(L) \rightarrow -C_1 \sin\sqrt{\lambda}(-L) + \cancel{C_2 \cos\sqrt{\lambda}(-L)} = -C_1 \sin\sqrt{\lambda}L + \cancel{C_2 \cos(\sqrt{\lambda}L)}$

$\underbrace{C_1 \sin\frac{n\pi}{L}(-L)}_{0} = \underbrace{-C_1 \sin\frac{n\pi}{L}L}_{0}$

Hence the eigenfunctions are of the form $\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$.

Including $\left.\begin{array}{l}\lambda_0 = 0\\ \phi_0 = C_0\end{array}\right\}$ $\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$ $\boxed{\phi_n = C_1 \cos\left(\frac{n\pi x}{L}\right) + C_2 \sin\left(\frac{n\pi x}{L}\right)}$

hence theorems 1 and 2 are true. Theorem 3 is not true since there are two eigenfunctions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ which are not constant multiplers of one another that correspond to the same eigenvalue $\lambda_n = \left(\frac{n\pi}{L}\right)^2$.

Theorem 4 is true since

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = L\delta_{nm}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = L\delta_{nm}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0.$$

Theorem 5 is true if

$$\lambda = \frac{\int_a^b (d\phi/dx)^2\,dx}{\int_a^b \phi^2\,dx} = \frac{(n\pi/L)\int_{-L}^{L}\left\{\begin{array}{l}\cos n\pi x/L\\ -\sin n\pi x/L\end{array}\right\}^2 dx}{\int_{-L}^{L}\left\{\begin{array}{l}\sin n\pi x/L\\ \cos n\pi x/L\end{array}\right\}^2 dx} = \boxed{\frac{(n\pi/L)\cancel{L}}{\cancel{L}} = \lambda_n}$$

5.3.8 $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ $\frac{d\phi}{dx}(1) = 0$

Multiplying the equation by ϕ yields:

$$\phi \frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi^2 = \frac{d}{dx}\left(\phi \frac{d\phi}{dx}\right) - \left(\frac{d\phi}{dx}\right)^2 + \lambda\phi^2 - x^2\phi^2 = 0$$

integrating this identity over the interval $0 \le x \le 1$ allow us to conclude:

$$\phi \frac{d\phi}{dx}\Big|_0^1 - \int_0^1 \left[\left(\frac{d\phi}{dx}\right)^2 + x^2\phi^2\right] dx = -\lambda \int_0^1 \phi^2 dx$$

If ϕ is an eigenfunction and hence not identically zero, we can now divide by $-\int_0^1 \phi^2 dx$ to obtain the standard Rayleigh quotient:

$$\lambda = \frac{-\phi(d\phi/dx)\Big|_0^1 + \int_0^1 \left[(d\phi/dx)^2 + x^2\phi^2\right] dx}{\int_0^1 \phi^2 dx}$$

The first derivatives of ϕ vanish at the boundary wich means:

$$\boxed{\lambda = \frac{\int_0^1 \left[(d\phi/dx)^2 + x^2\phi^2\right] dx}{\int_0^1 \phi^2 dx} \ge 0}$$ because the integrand is ≥ 0.

If $\lambda = 0$, we would then have $0 = \int_0^1 \left(\frac{d\phi}{dx}\right)^2 dx + \int_0^1 x^2\phi^2 dx \ge 0$

since each term in the integrand is nonnegative. This implies that each integral vanishes, and that each integrand vanishes. In particular, $x^2\phi^2(x) = 0$ for all $0 \le x \le 1$, wich implies that $\phi(x) = 0$ for all $0 < x \le 1$ and hence $\phi(0) = 0$ by continuity. Thus ϕ is the trivial function and hence $\underline{\lambda = 0}$ cannot be an eigenvalue !!

5.3.9 $x^2 \frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0$ with $\phi(1) = 0$ and $\phi(b) = 0$

a) Multiplying by $\frac{1}{x} \Rightarrow$ $\underbrace{x\frac{d^2\phi}{dx^2} + \frac{d\phi}{dx}}_{} + \frac{\lambda}{x}\phi = 0$

$$\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) + \lambda\left(\frac{1}{x}\right)\phi = 0$$

OR $\boxed{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0}$ with $p(x) = x$ $q = 0$ $\sigma(x) = \frac{1}{x}$

↳ Sturm - Liouville form $1 \le x \le b$

this is a regular Sturm - Liouville problem.

$$\int_0^L \left(u_n \frac{d^2 u_m}{dx^2} - u_m \frac{d^2 u_n}{dx^2} \right) dx = u_n(L)\left(\gamma_m u_m(L)\right) - u_m(L)\left(\gamma_n u_n(L)\right)$$

$$= \boxed{(\gamma_m - \gamma_n)\, u_n(L)\, u_m(L) \neq 0 \quad \text{sin } \gamma_m \neq \gamma_n \text{ when } m \neq n}$$

* 5.8.13 $$\int_0^L \sin^2(\sqrt{\lambda}\, x)\, dx = \int_0^L \frac{1}{2}\left(1 - \cos 2\sqrt{\lambda}\, x\right) dx = \frac{1}{2}\left(x - \frac{\sin 2\sqrt{\lambda}\, x}{2\sqrt{\lambda}}\right)\Big|_0^L$$

$$= \frac{L}{2} - \frac{\sin 2\sqrt{\lambda}\, L}{4\sqrt{\lambda}} = \frac{L}{2} - \frac{\sin \sqrt{\lambda}\, L \cos \sqrt{\lambda}\, L}{2\sqrt{\lambda}}$$

$$\tan \sqrt{\lambda}\, L = -\frac{\sqrt{\lambda}}{h} = \frac{-\sqrt{\lambda}/h}{1} \rightarrow \sin \sqrt{\lambda}\, L = \frac{-\sqrt{\lambda}/h}{\sqrt{1 + \lambda/h^2}}$$

$$\cos \sqrt{\lambda}\, L = \frac{1}{\sqrt{1 + \lambda/h^2}}$$

$$\Rightarrow \int_0^L \sin^2(\sqrt{\lambda}\, x)\, dx = \frac{L}{2} + \frac{\cancel{\sqrt{\lambda}}/h}{2\cancel{\sqrt{\lambda}}\,(1 + \lambda/h^2)}$$

$$\boxed{\int_0^L \sin^2(\sqrt{\lambda}\, x)\, dx = \frac{L}{2} + \frac{1}{2h(1 + \lambda/h^2)}}$$

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Solutions to problems of Haberman Chap 7

7.2.1 $\boxed{\dfrac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u}$ with $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

Let $u(x,y,t) = h(t)\,\phi(x,y) \rightarrow \phi(x,y)\dfrac{d^2h}{dt^2} = c^2 h(t)\nabla^2\phi$, divide by $c^2 h\phi$:

$$\Rightarrow \underbrace{\frac{1}{c^2 h}\frac{d^2h}{dt^2}}_{-\lambda} = \underbrace{\frac{1}{\phi}\nabla^2\phi}_{-\lambda}$$

separating variables: $\boxed{\begin{aligned}\frac{d^2h}{dt^2} &= -\lambda c^2 h(t)\\ \nabla^2\phi &= -\lambda\phi(x,y)\end{aligned}}$

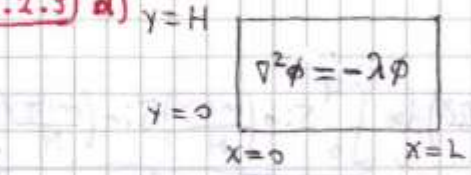
7.2.2 $\boxed{\dfrac{\partial u}{\partial t} = K\nabla^2 u}$ with $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

Let $u(x,y,t) = \phi(x,y)\,h(t) \rightarrow \phi(x,y)\dfrac{dh}{dt} = Kh(t)\nabla^2\phi$ dividing by $Kh\phi$:

$$\Rightarrow \underbrace{\frac{1}{Kh}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{\phi}\nabla^2\phi}_{-\lambda}$$

separing variables: $\boxed{\begin{aligned}\frac{dh}{dt} &= -\lambda K h(t)\\ \nabla^2\phi &= -\lambda\phi(x,y)\end{aligned}}$

7.2.3 **a)**



$\phi(0,y)=0 \quad \phi(x,0)=0$
$\phi(L,y)=0 \quad \phi(x,H)=0$

Let $\phi(x,y) = f(x)g(y) \rightarrow \nabla^2\phi = g(y)\dfrac{d^2f}{dx^2} + f(x)\dfrac{d^2g}{dy^2} = -\lambda[f(x)g(y)]$

dividing by fg: $\underbrace{\dfrac{1}{f}\dfrac{d^2f}{dx^2}}_{-\gamma} + \underbrace{\dfrac{1}{g}\dfrac{d^2g}{dy^2}}_{-\delta} = -\lambda \Rightarrow \boxed{\begin{aligned}\frac{d^2f}{dx^2} &= -\gamma f(x)\\ \frac{d^2g}{dy^2} &= -\delta g(y)\end{aligned}}$

such that $\gamma + \delta = \lambda$. and $f(0) = f(L) = 0$, $g(0) = g(H) = 0$.
Since the boundary conditions are homogeneus, we expect oscillatory eigenfunctions that exist only when $\gamma, \delta > 0$:

$f(x) = A_1 \cos\sqrt{\gamma}\,x + A_2 \sin\sqrt{\gamma}\,x$, $\quad f(0) = 0 = A_1$, $f(L) = 0 \rightarrow \sqrt{\gamma}\,L = n\pi$
$g(y) = B_1 \cos\sqrt{\delta}\,y + B_2 \sin\sqrt{\delta}\,y$, $\quad g(0) = 0 = B_1$, $g(H) = 0 \rightarrow \sqrt{\delta}\,H = m\pi$

$$\Rightarrow \phi(x,y) = f(x)g(y) = A_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right) \qquad n, m = 1, 2, 3, \ldots$$

There are not solution for $\gamma, \delta \leq 0$ that satisfy the homogeneous boundary conditions.

b) The time-depended differential equation from separation of temporal variable in the wave equation is:

$\dfrac{d^2h}{dt^2} = -\lambda_{nm} c^2 h(t)$ where $\boxed{\lambda_{nm} = \gamma_n + \delta_m = \left(\dfrac{n\pi}{L}\right)^2 + \left(\dfrac{m\pi}{H}\right)^2}$

from part **a)**.

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$$\lim_{t\to\infty} u = 0$$

$$\Rightarrow \boxed{u(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{q=1}^{\infty} D_{nmq} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left[\frac{(2q-1)\pi z}{2W}\right] e^{-\lambda_{nmq} k t}}$$

$$f(x,y,z) = u(x,y,z,0) = \sum_{n=1}^{\infty}\sum_{q=1}^{\infty} D_{n0q} \sin\left(\frac{n\pi x}{L}\right)\cos\left[\frac{(2q-1)\pi z}{2W}\right]$$
$$+ \sum_{n=1}\sum_{m=1}\sum_{q=1} D_{nmq} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left[\frac{(2q-1)\pi z}{W}\right]$$

multiplying by $\sin(n'\pi x/L)\cos[(2q'-1)\pi z/2W]$ and integrating:

$$\int_0^W\int_0^H\int_0^L f(x,y,z)\sin\left(\frac{n'\pi x}{L}\right)\cos\left[\frac{(2q'-1)\pi z}{2W}\right]dx\,dy\,dz$$
$$= \sum_{n=1}^{\infty}\sum_{q=1}^{\infty} D_{n0q}\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}}\overbrace{\int_0^H dy}^{H}\underbrace{\int_0^W \cos\frac{(2q-1)\pi z}{2W}\cos\frac{(2q'-1)\pi z}{2W}dz}_{(W/2)\delta_{qq'}}$$
$$+ \sum_{\substack{n=1\\m=1\\q=1}}^{\infty} D_{nmq}\int_0^L[\ ]dx\cancel{\int_0^H \cos\left(\frac{m\pi y}{H}\right)dy}_{0}\int_0^W[\ ]dz$$
$$= \frac{LHW}{4}D_{n'0q'} \rightarrow \boxed{D_{n0q} = \frac{4}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\sin\frac{n\pi x}{L}\cos\frac{(2q-1)\pi z}{2W}dx\,dy\,dz}$$

Similarly, multiplying by $\sin\left(\frac{n'\pi x}{L}\right)\cos\left(\frac{m'\pi y}{H}\right)\cos\left[\frac{(2q'-1)\pi z}{2W}\right]$ and integrating:

$$\boxed{D_{nmq} = \frac{8}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left[\frac{(2q-1)\pi z}{2W}\right]dx\,dy\,dz}$$

* b) $f'(0) = f'(L) = 0$, $g'(0) = g'(H) = 0$ and $p'(0) = p'(W) = 0$.

By extension of problem 7.3.1(b) to 3-Dimensions, we have:

$$\boxed{u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{\infty} C_{nmq} e^{-\lambda_{nmq} k t}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{q\pi z}{W}\right)} \qquad \lim_{t\to\infty} u = C_{000}$$

$n, m, q = 0, 1, 2, \ldots$

$$C_{000} = \frac{1}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\,dx\,dy\,dz \rightarrow \text{The average of initial temperature distribution.}$$

$$C_{n00} = \frac{2}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\cos\left(\frac{n\pi x}{L}\right)dx\,dy\,dz \qquad n>0$$

$$C_{0m0} = \frac{2}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\cos\left(\frac{m\pi y}{H}\right)dx\,dy\,dz \qquad m>0$$

$$C_{00q} = \frac{2}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\cos\left(\frac{q\pi y}{W}\right)dx\,dy\,dz \qquad q>0$$

$$C_{nm0} = \frac{4}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)dx\,dy\,dz \qquad n,m>0$$

$$C_{0mq} = \frac{4}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{q\pi z}{W}\right)dx\,dy\,dz \qquad m,q>0$$

$$C_{n0q} = \frac{4}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{q\pi z}{W}\right)dx\,dy\,dz \qquad n,q>0$$

$$C_{nmq} = \frac{8}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{q\pi z}{W}\right)dx\,dy\,dz \qquad n,m,q>0.$$

add for $(1+\delta_{0m})(1+\delta_{0q})$ for summation all $n,m,q \geq 0$

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7.3.3 $\frac{\partial u}{\partial t} = K_1 \frac{\partial^2 u}{\partial x^2} + K_2 \frac{\partial^2 u}{\partial y^2}$ let $u(x,y,t) = \phi(x,y)h(t)$

$\Rightarrow \phi(x,y)\frac{dh}{dt} = K_1 h \frac{\partial^2 \phi}{\partial x^2} + K_2 h \frac{\partial^2 \phi}{\partial y^2}$ dividying by $h(t)\phi(x,y)$:

$\underbrace{\frac{1}{h}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{K_1}{\phi}\frac{\partial^2 \phi}{\partial x^2} + \frac{K_2}{\phi}\frac{\partial^2 \phi}{\partial y^2}}_{-\lambda}$ Separing $\Rightarrow$ $\boxed{\frac{dh}{dt} = -\lambda h(t)}$

$K_1 \frac{\partial^2 \phi}{\partial x^2} + K_2 \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi$

Let $\phi(x,y) = f(x)g(y) \rightarrow K_1 g \frac{d^2 f}{dx^2} + K_2 f \frac{d^2 g}{dy^2} = -\lambda f g$ dividing by fg

$\underbrace{\frac{K_1}{f}\frac{d^2 f}{dx^2}}_{-\gamma} + \underbrace{\frac{K_2}{g}\frac{d^2 g}{dy^2}}_{-\delta} = -\lambda \rightarrow \boxed{\begin{array}{l} f''(x) = -(\gamma/K_1) f \\ g''(y) = -(\delta/K_2) g \end{array}}$

$f(x) = A_1 \cos\sqrt{\gamma/K_1}\, x + A_2 \sin\sqrt{\gamma/K_1}\, x$ $\quad f(0) = 0 = A_1$

$f'(L) = 0 = A_2\sqrt{\gamma/K_1}\cos\sqrt{\gamma/K_1}\, L \rightarrow \sqrt{\gamma/K_1}\, L = \left(\frac{2n-1}{2}\right)\pi \rightarrow \gamma_n = \left[\frac{(2n-1)\pi}{2L}\right]^2 K_1$

$\Rightarrow f(x) = A_n \sin\left[\frac{(2n-1)\pi}{2L}\right]$ $\quad n = 1,2,3,\ldots$

Similarly: $g(y) = B_n \sin\left[\frac{(2m-1)\pi}{2H}\right]$ $\quad m = 1,2,3\ldots$ and $\delta_m = \left[\frac{(2m-1)\pi}{2H}\right]^2 K_2$

There are for $\gamma \leq 0$, $\delta \leq 0$ that satisfy the boundary conditions.

$\lambda = \lambda_{nm} = \gamma_n + \delta_m > 0 \rightarrow h(t) = C_{nm} e^{-\lambda_{nm} t}$

$$\Rightarrow \boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} e^{-\lambda_{nm} t} \sin\left[\frac{(2n-1)\pi}{2L}\right] \sin\left[\frac{(2m-1)\pi}{2H}\right]}$$

$$f(x,y) = u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin\left[\frac{(2n-1)\pi}{2L}\right] \sin\left[\frac{(2m-1)\pi}{2H}\right]$$

multiplying by $\sin\frac{(2n'-1)\pi}{2L} \sin\frac{(2m'-1)\pi}{2H}$ and integrating we obtain:

$$\boxed{C_{nm} = \frac{4}{LH}\int_0^H\int_0^L f(x,y) \sin\left[\frac{(2n-1)\pi x}{2L}\right] \sin\left[\frac{(2m-1)\pi y}{2H}\right] dx\, dy}$$

7.3.4 $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$, by problem 7.2.1 $\rightarrow$ $\frac{d^2 h}{dt^2} = -\lambda c^2 h(t)$

$\nabla^2 \phi = -\lambda \phi(x,y)$.

Let $\phi(x,y) = f(x)g(y) \rightarrow$ by problem 7.2.3(a) $\rightarrow$ $f''(x) = -\gamma f(x)$

such that $\lambda = \gamma + \delta$. $\quad g''(y) = -\delta g(x)$

a) $f(0) = f(L) = 0 \rightarrow f(x) = A_n \sin(n\pi x/L)$ with $\gamma_n = (n\pi/L)^2$ $\quad n = 1,2,3,\ldots$

There are not solutions for $\gamma \leq 0$.

$g'(0) = g'(H) = 0 \rightarrow g(y) = B_n \cos(m\pi y/H)$ with $\delta_m = (m\pi/H)^2$ $\quad m = 1,2,3\ldots$

$g_0(y) = B_0$ when $\delta = 0$ or $\delta_0 = 0$ if $m = 0$.

Not solutions for $\delta < 0$

$\lambda = \lambda_{nm} = \gamma_n + \delta_m > 0 \rightarrow h(t) = C_{nm}\cos c\sqrt{\lambda_{nm}}\, t + D_{nm}\sin c\sqrt{\lambda_{nm}}\, t \Rightarrow$

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Summarize: $$\boxed{D_{nm} = \frac{4}{(1+\delta_{0m})LHc\sqrt{\lambda_{nm}}}\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)dx\,dy}$$

for all $n>0,\ m\geq 0$

$$\boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\left[C_{nm}\cos c\sqrt{\lambda_{nm}}\,t + D_{nm}\sin c\sqrt{\lambda_{nm}}\,t\right]}$$

$$u(x,y,0) = 0 = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right) \rightarrow \boxed{C_{nm}=0}$$

$$\frac{\partial u}{\partial t}(x,y,0) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} D_{nm}\,c\sqrt{\lambda_{nm}}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$$

$$= \sum_{n=1}^{\infty} D_{n0}\,c\sqrt{\lambda_{n0}}\sin\left(\frac{n\pi x}{L}\right) + \sum_{\substack{n=1\\m=1}} D_{nm}\,c\sqrt{\lambda_{nm}}\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}$$

Multiplying by $\sin(n'\pi x/L)$ and integrating between $0<x<L$, $0<y<H$

$$\boxed{D_{n0} = \frac{2}{LHc\sqrt{\lambda_{n0}}}\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)dx\,dy} \quad n>1$$

Multiplying by $\sin(n'\pi x/L)\cos(m'\pi y/H)$ and integrating:

$$\boxed{D_{nm} = \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)dx\,dy}$$

* b) $f'(0)=f'(L)=0 \rightarrow f(x)=A_n\cos(n\pi x/L)$ with $\gamma_n=(n\pi/L)^2$ $n=1,2,3,\ldots$

$f(x)=A_0$ when $\gamma=\gamma_0=0$ $n=0$.

$g'(0)=g'(H)=0 \rightarrow g(y)=B_n\cos(m\pi y/H)$ with $\delta_m=(m\pi/H)^2$ $m=1,2,3\ldots$

$g(y)=B_0$ when $\delta=\delta_0=0$ $m=0$.

There are not solutions when $\gamma<0$ or $\delta<0$.

$$\left.\begin{array}{l}\lambda=\lambda_{nm}=\gamma_n+\delta_m>0\\ \text{if } n>0 \text{ or } m>0\end{array}\right\} \Rightarrow h(t) = C_{nm}\cos c\sqrt{\lambda_{nm}}\,t + D_{nm}\sin c\sqrt{\lambda_{nm}}\,t$$

$$\left.\begin{array}{l}\lambda=\lambda_{00}=\gamma_0+\delta_0=0\\ \text{if } n=0 \text{ and } m=0\end{array}\right\} \Rightarrow \frac{d^2h(t)}{dt^2}=0 \rightarrow h(t)=C_{00}+D_{00}t$$

$$u(x,y,0)=0=\phi(x,y)h(0) \rightarrow h(0)=0 \Rightarrow C_{nm}=0 \text{ and } C_{00}=0$$

$$\Rightarrow \boxed{u(x,y,t) = D_{00}t + \sum_{\substack{n,m\\ \text{No both } 0}} D_{nm}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\sin c\sqrt{\lambda_{nm}}\,t}$$

$$\frac{\partial u(x,y,0)}{\partial t} = f(x,y) = D_{00} + \sum_{n=1}^{\infty} D_{n0}\,c\sqrt{\lambda_{n0}}\cos\left(\frac{n\pi x}{L}\right) + \sum_{m=1}^{\infty} D_{0m}\,c\sqrt{\lambda_{0m}}\cos\left(\frac{m\pi x}{H}\right)$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm}\,c\sqrt{\lambda_{nm}}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{H}\right)$$

Manipulating:

$$D_{00} = \frac{1}{LH}\int_0^H\int_0^L f(x,y)\,dx\,dy$$

Add $\frac{1}{(1+\delta_{0m})(1+\delta_{0m})}$ for summarize $n\geq 0,\ m\geq 0$

$$D_{n0} = \frac{2}{LHc\sqrt{\lambda_{n0}}}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)dx\,dy \qquad n>0$$

$$D_{0m} = \frac{2}{LHc\sqrt{\lambda_{0m}}}\int_0^H\int_0^L f(x,y)\cos\left(\frac{m\pi x}{H}\right)dx\,dy \qquad m>0$$

$$D_{nm} = \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{H}\right)dx\,dy \qquad n,m>0.$$

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7.3.5 a) That is a combination of wave equation plus diffusion OR atenuation equation (similar to heat equation).

b) $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - K\frac{\partial u}{\partial t}$ $\quad K>0$ let $u(x,y,t) = f(x)g(y)h(t)$

$fgh''(t) + Kfgh'(t) = c^2(gh f''(x) + fh g''(y))$ dividing by $fghc^2$:

$$\underbrace{\frac{1}{c^2 h}\left[h''(t) + Kh'(t)\right]}_{-\lambda} = \underbrace{\frac{1}{f}f''(x)}_{-\gamma} + \underbrace{\frac{1}{g}g''(y)}_{-\delta} \rightarrow \boxed{\begin{aligned} f''(x) &= -\gamma f(x) \\ g''(y) &= -\delta g(y) \\ h''(t) + Kh'(t) &= -\lambda c^2 h(t) \end{aligned}}$$

If $\gamma, \delta > 0 \rightarrow \lambda > 0$, these equations above describe waves oscilatory in space without attenuation but wave oscilatory attenuated in time.

7.3.6 a)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Let $u(x,y,z) = f(x)g(y)h(z)$

$\rightarrow gh f''(x) + fh g''(y) + fg h''(z) = 0$ dividing by fgh

$$\underbrace{\frac{1}{f}f''(x)}_{-\gamma} + \underbrace{\frac{1}{g}g''(y)}_{-\delta} + \underbrace{\frac{1}{h}h''(z)}_{\lambda} = 0 \qquad \lambda = +\gamma + \delta$$

$$\boxed{\begin{aligned} f''(x) &= -\gamma f(x) \\ g''(y) &= -\delta g(y) \\ h''(z) &= \lambda h(z) \end{aligned}} \Leftarrow$$

* b)

$0 < x < L$ $\quad u(0,y,z) = u(L,y,z) = 0$

$0 < y < W$ $\quad u(x,0,z) = u(x,W,z) = 0$

$0 < z < H$ $\quad \frac{\partial u}{\partial z}(x,y,0) = 0 \quad u(x,y,H) = f(x,y)$

Since $u(x,y,z) = f(x)g(y)h(z)$ by separation of variables of part a)

$f(0) = f(L) = 0 \rightarrow f(x) = A_n \sin(n\pi x/L) \qquad \gamma_n = (n\pi/L)^2 \qquad n = 1,2,3,\ldots$

$g(0) = g(L) = 0 \rightarrow g(x) = B_n \sin(m\pi y/W) \qquad \delta_n = (m\pi/W)^2 \qquad m = 1,2,3,\ldots$

$\lambda = \lambda_{nm} = \gamma_n + \delta_m > 0 \Rightarrow h(z) = C_{nm}\cosh\sqrt{\lambda_{nm}}\,z + D_{nm}\sinh\sqrt{\lambda_{nm}}\,z$

$h'(0) = 0 = D_{nm}\sqrt{\lambda_{nm}} \rightarrow D_{nm} = 0$

$$\Rightarrow \boxed{u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\cosh\left(\sqrt{\lambda_{nm}}\,z\right)\sin(n\pi x/L)\sin(m\pi y/W)}$$

$$f(x,y) = u(x,y,H) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\cosh\left(\sqrt{\lambda_{nm}}H\right)\sin(n\pi x/L)\sin(m\pi y/W)$$

Multiplying by $\sin(n'\pi x/L)\sin(m'\pi y/W)$ and integrating for $0 < x < L$, $0 < y < W$

$$\Rightarrow \boxed{C_{nm} = \frac{4}{LW\cosh\left(\sqrt{\lambda_{nm}}H\right)}\int_0^W\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{W}\right)dx\,dy}$$

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Using separation of variables. From exercise 7.3.5:

7.3.7 a) $f'(0)=f'(L)=0 \rightarrow f(x)=A_n\cos(n\pi x/L) \qquad \gamma_n=(n\pi/L)^2 \qquad n=1,2,3\ldots$

$f(x)=A_0$ when $\gamma=\gamma_0=0 \qquad n=0.$

$g(0)=g(W)=0 \rightarrow g(x)=B_m\sin(m\pi y/W) \qquad \delta_m=(m\pi/W)^2 \qquad m=1,2,\ldots$

$\lambda_{nm}=\gamma_n+\delta_m>0 \rightarrow h(z)=C_{nm}\cosh\sqrt{\lambda_{nm}}\,z+D_{nm}\sinh\sqrt{\lambda_{nm}}\,z$

$h(H)=0=C_{nm}\cosh\sqrt{\lambda_{nm}}H+D_{nm}\sinh\sqrt{\lambda_{nm}}H$

$\Rightarrow D_{nm}=-\dfrac{C_{nm}}{\tanh\sqrt{\lambda_{nm}}t}$

$$\Rightarrow \boxed{u(x,y,z)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}C_{nm}\left[1-\frac{\tanh\sqrt{\lambda_{nm}}z}{\tanh\sqrt{\lambda_{nm}}H}\right]\cos(\sqrt{\lambda_{nm}}z)\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{W}\right)}$$

$$f(x,y)=u(x,y,0)=\sum_{m=1}^{\infty}C_{0m}\sin\left(\frac{m\pi y}{W}\right)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{W}\right)$$

multiplying by $\sin(m'\pi y/W)$ and integrating for $0<x<L$, $0<y<W$:

$$\Rightarrow C_{0m}=\frac{2}{LW}\int_0^W\int_0^H f(x,y)\sin\left(\frac{m\pi y}{W}\right)dx\,dy \qquad m>0.$$

multiplying by $\cos(n'\pi x/L)\sin(m'\pi y/W)$ and integrating:

$$C_{nm}=\frac{4}{LW}\int_0^W\int_0^H f(x,y)\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{W}\right)dx\,dy \qquad n,m>0.$$

b) $f(0)=f(L)=0 \rightarrow f(x)=A_n\sin(n\pi x/L) \qquad \gamma_n=(n\pi/L)^2 \qquad n=1,2,3,\ldots$

$h(0)=h(H)=0 \rightarrow h(z)=B_m\sin(m\pi z/H) \qquad -\lambda_m=(m\pi/H)^2 \qquad m=1,2,3,\ldots$

$\delta=-\delta_{nm}=\gamma_n-\lambda_m>0 \rightarrow g(y)=C_{nm}\cosh\sqrt{-\delta_{nm}}\,y+D_{nm}\sinh\sqrt{-\delta_{nm}}\,y$

$g(0)=0=C_{nm}$

$$\Rightarrow \boxed{u(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}D_{nm}\sinh\sqrt{-\delta_{nm}}\,y\sin(n\pi x/L)\sin(m\pi z/H)}$$

$$f(x,z)=u(x,W,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}D_{nm}\sinh\sqrt{-\delta_{nm}}\,W\sin(n\pi x/L)\sin(m\pi z/H)$$

Multiplying by $\sin(n'\pi x/L)\sin(m'\pi z/H)$, integrating $\rightarrow$ $D_{nm}=\dfrac{4}{LH\sinh\sqrt{-\delta_{nm}}}\displaystyle\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi z}{H}\right)dx\,dz$

* c) $g'(0)=g'(W)=0 \rightarrow g(y)=A_n\cos(n\pi y/W) \qquad \delta_n=(n\pi/W)^2 \qquad n=1,2,\ldots$

$g(y)=A_0$ when $\delta=\delta_0=0 \qquad n=0$

$h'(0)=h'(H)=0 \rightarrow h(z)=B_n\cos(m\pi z/H) \qquad -\lambda_m=(m\pi/H)^2 \qquad m=1,2,\ldots$

$h(z)=B_0$ when $-\lambda=-\lambda_0=0 \qquad m=0.$

$-\gamma=-\gamma_{nm}=\delta_n-\lambda_m>0$ when $n=0$ OR $m\neq 0$ $\rightarrow f(x)=C_{nm}\cosh\sqrt{-\gamma_{nm}}\,x+D_{nm}\sinh\sqrt{-\gamma_{nm}}\,x$

$f'(0)=0=D_{nm}\sqrt{-\gamma_{nm}} \rightarrow D_{nm}=0$

$-\gamma=-\gamma_{00}=\delta_0-\lambda_0=0 \rightarrow f''(x)=0 \rightarrow f(x)=D_{00}x+C_{00}$

$f'(0)=0=D_{00}$

$$\Rightarrow \boxed{u(x,y,z)=C_{00}+\sum_{\substack{n,m\\ \text{No both } 0}}C_{nm}\cosh(\sqrt{-\gamma_{nm}}\,x)\cos(n\pi y/W)\cos(m\pi z/H)}$$

▷ Summarize: $$\boxed{C_{nm} = \frac{4/\left[(1+\delta_{0n})(1+\delta_{0m})\right]}{WH\sqrt{-\gamma_{n0}}\,\sinh\left(\sqrt{-\gamma_{n0}}\,L\right)}\int_0^H\int_0^W f(x,y)\cos\left(\frac{n\pi y}{W}\right)\cos\left(\frac{m\pi z}{H}\right)dy\,dz}$$

no both m and n zero!!

$$f(y,z) = \frac{\partial u}{\partial x}(L,y,z) = \sum_{n=1}^{\infty} C_{n0}\sqrt{-\gamma_{n0}}\,\sinh\left(\sqrt{-\gamma_{n0}}\,L\right)\cos(n\pi y/W)$$

$$+\sum_{m=1}^{\infty} C_{0m}\sqrt{-\gamma_{0m}}\,\sinh\left(\sqrt{-\gamma_{n0}}\,L\right)\cos\left(\frac{m\pi z}{H}\right) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sqrt{-\gamma_{nm}}\,\sinh\sqrt{-\gamma_{nm}}\,L\cos\left(\frac{n\pi y}{W}\right)\cos\left(\frac{m\pi z}{H}\right)$$

Doing Algebra

$$\Rightarrow C_{n0} = \frac{2}{WH\sqrt{-\gamma_{n0}}\,\sinh\left(\sqrt{-\gamma_{n0}}\,L\right)}\int_0^H\int_0^W f(y,z)\cos(n\pi y/W)\,dy\,dz \qquad n>0.$$

$$C_{0m} = \frac{2}{WH\sqrt{-\gamma_{0m}}\,\sinh\left(\sqrt{-\gamma_{0m}}\,L\right)}\int_0^H\int_0^W f(y,z)\cos(m\pi z/H)\,dy\,dz \qquad m>0.$$

$$C_{nm} = \frac{4}{WH\sqrt{-\gamma_{nm}}\,\sinh\left(\sqrt{-\gamma_{nm}}\,L\right)}\int_0^H\int_0^W f(y,z)\cos\left(\frac{n\pi y}{W}\right)\cos\left(\frac{m\pi z}{H}\right)dy\,dz \qquad m,n>0.$$

C_{00} is arbitrary.

* d) See the solution to 7.3.7(c) above:

$$\boxed{u(x,y,z) = C_{00} + \sum_{\substack{n,m\\ \text{No both } 0}} C_{nm}\cosh\left(\sqrt{-\gamma_{nm}}\,x\right)\cos\left(\frac{n\pi y}{W}\right)\cos\left(\frac{m\pi z}{H}\right)}$$

$$-\gamma_{nm} = \delta_n - \lambda_m = \left(\frac{n\pi}{W}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \qquad n,m = 0,1,2\ldots$$

But now $u(L,y,z) = g(y,z)$:

$$\Rightarrow g(y,z) = C_{00} + \sum_{n=1}^{\infty} C_{n0}\cosh\left(\sqrt{-\gamma_{n0}}\,L\right)\cos\left(\frac{n\pi y}{W}\right) + \sum_{m=1}^{\infty} C_{0m}\cosh\left(\sqrt{-\gamma_{0m}}\,L\right)\cos\left(\frac{m\pi z}{H}\right)$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\cosh\left(\sqrt{-\gamma_{nm}}\,L\right)\cos\left(\frac{n\pi y}{W}\right)\cos\left(\frac{m\pi z}{H}\right)$$

Doing algebraic manipulations:

$$C_{n0} = \frac{2}{WH\cosh\left(\sqrt{-\gamma_{n0}}\,L\right)}\int_0^H\int_0^W g(y,z)\cos\left(\frac{n\pi y}{W}\right)dy\,dz \qquad n>0$$

$$C_{0m} = \frac{2}{WH\cosh\left(\sqrt{-\gamma_{0m}}\,L\right)}\int_0^H\int_0^W g(y,z)\cos\left(\frac{m\pi z}{H}\right)dy\,dz \qquad m>0$$

$$C_{nm} = \frac{4}{WH\cosh\left(\sqrt{-\gamma_{nm}}\,L\right)}\int_0^H\int_0^W g(y,z)\cos\left(\frac{n\pi y}{W}\right)\cos\left(\frac{m\pi z}{H}\right)dy\,dz \qquad n,m>0.$$

$$C_{00} = \frac{1}{WH}\int_0^H\int_0^W g(x,y)\,dy\,dz \longrightarrow \text{The mean value of } g(x,y).$$

▷ Summarize: $$\boxed{C_{nm} = \frac{4}{(1+\delta_{0n})(1+\delta_{0m})WH\cosh\left(\sqrt{-\gamma_{nm}}\,L\right)}\int_0^H\int_0^W g(x,y)\cos\left(\frac{n\pi y}{W}\right)\cos\left(\frac{m\pi z}{H}\right)dy\,dz}$$

We would find this using: $\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}(1+\delta_{0n})\delta_{nm}$.

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7.4.1 $\nabla^2\phi + \lambda\phi = 0 \rightarrow \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,y)$

let $\phi = fg \rightarrow g\frac{\partial^2\phi}{\partial x^2} + f\frac{\partial^2\phi}{\partial y^2} = -\lambda fg$ dividying by fg

$$\underbrace{\frac{1}{f}\frac{\partial^2 f}{\partial x^2}}_{-\gamma} + \underbrace{\frac{1}{g}\frac{\partial^2 g}{\partial y^2}}_{-\delta} = -\lambda \longrightarrow \boxed{\begin{array}{l} f''(x) = -\gamma f(x) \\ g''(x) = -\delta g(x) \end{array}} \qquad \gamma + \delta = \lambda$$

* a) $f'(0) = f'(L) = 0 \rightarrow f(x) = A_n \cos(n\pi x/L) \qquad \gamma = \gamma_n = (n\pi/L)^2 \qquad n = 1, 2, \ldots$

$f(x) = A_0$ when $\gamma = \gamma_0 = 0 \qquad n = 0.$

$g(0) = g(H) = 0 \rightarrow g(x) = B_m \sin(m\pi y/H) \qquad \delta = \delta_m = (m\pi/H)^2 \qquad m = 1, 2, 3 \ldots$

$$\Rightarrow \boxed{\phi(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{nm}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)} \qquad \boxed{\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$

↳ eigenvalues.

b) If $L = H \Rightarrow \lambda_{nm} = (n^2 + m^2)\left(\frac{\pi}{L}\right)^2$. Then if $n \neq m$, the eigenvalue $\lambda_{nm} = \lambda_{mn}$ has both eigenfunctions $\phi_{nm}(x,y)$ and $\phi_{mn}(x,y)$ (which are distinct since $n \neq m$). Thus for $n \neq m$ ~~max~~ for most eigenvalues, most eigenvalues have more than one eigenfunction.

c) $$\int_0^H\int_0^L \phi_{nm}(x,y)\,\phi_{n'm'}(x,y)\,dx\,dy = \int_0^H\int_0^L \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\cos\left(\frac{n'\pi x}{L}\right)\sin\left(\frac{m'\pi y}{H}\right)dx\,dy$$

$$= \underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}}\ \underbrace{\int_0^H \sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{m'\pi y}{H}\right)dy}_{(H/2)\delta_{mm'}}$$

$$= \frac{LH}{4}\delta_{nn'}\delta_{mm'} \qquad n, n' \neq 0.$$

If $n = 0 \rightarrow \phi_{0m} = \sin\left(\frac{m\pi y}{H}\right)$

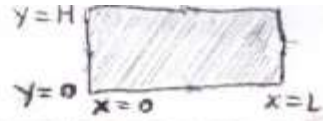
$$\hookrightarrow \int_0^H\int_0^L \phi_{0m}\,\phi_{n'm'}\,dx\,dy = \underbrace{\int_0^L \cos\left(\frac{n'\pi x}{L}\right)dx}_{\substack{L \text{ if } n'=0 \\ 0 \text{ if } n'\neq 0.}}\ \underbrace{\int_0^H \sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{m'\pi y}{H}\right)dy}_{(H/2)\delta_{mm'}}$$

$$= \frac{LH}{2}\delta_{0n'}\delta_{mm'}$$

$$\Rightarrow \boxed{\int_0^H\int_0^L \phi_{nm}(x,y)\,\phi_{n'm'}(x,y)\,dx\,dy = \begin{cases} \frac{LH}{4}\delta_{nn'}\delta_{mm'} & \text{if } n, n' \neq 0 \\ \frac{LH}{2}\delta_{0n'}\delta_{mm'} & \text{if } n = 0 \\ \frac{LH}{2}\delta_{n0}\delta_{mm'} & \text{if } n' = 0 \end{cases}}$$

summarize $\rightarrow = \frac{LH}{4}(1 + \delta_{0n})\delta_{nn'}\delta_{mm'}$

By: Herminso Villarraga Gómez



7.4.2 $\nabla^2\phi + \lambda\phi = 0 \rightarrow \nabla^2\phi = -\lambda\phi$ $\qquad \phi(0,y) = \phi(L,y) = 0$, $\phi(x,0) = \phi(x,H) = 0$

$$\lambda = \frac{-\oint \phi\vec{\nabla}\phi\cdot d\vec{l} + \iint_R |\vec{\nabla}\phi|^2 dx\,dy}{\iint_R \phi^2 dx\,dy} \qquad \vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$$

$$\oint \phi\vec{\nabla}\phi\cdot d\vec{l} = \int_{x=0}^{x=L} \cancelto{0}{\phi(x,0)}\vec{\nabla}\phi\cdot\hat{x}\,dx + \int_{y=0}^{y=H} \cancelto{0}{\phi(L,y)}\vec{\nabla}\phi\cdot\hat{y}\,dy$$
$$+ \int_{x=L}^{x=0} \cancelto{0}{\phi(0,H)}\vec{\nabla}\phi\cdot(-\hat{x}\,dx) + \int_{y=H}^{y=0} \cancelto{0}{\phi(0,y)}\vec{\nabla}\phi\cdot(-\hat{y}\,dy) = 0$$

$$\Rightarrow \lambda = \frac{\iint_R |\vec{\nabla}\phi|^2 dx\,dy}{\iint_R \phi^2 dx\,dy} \geq 0$$

That's is zero if $\vec{\nabla}\phi = 0$

$\Rightarrow \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} = 0 \rightarrow \frac{\partial\phi}{\partial x} = 0,\ \frac{\partial\phi}{\partial y} = 0$

$\rightarrow \phi(x,y) = cte$, but $\phi(0,y) = 0 \rightarrow Cte = 0$

trivial solution

trivial solution
No count $\Rightarrow \boxed{\lambda > 0}$

7.4.3 Using $\vec{\nabla}\cdot(\alpha\vec{\beta}) = \alpha\vec{\nabla}\cdot\vec{\beta} + \vec{\nabla}\alpha\cdot\vec{\beta}$ we have:

$$\left.\begin{aligned}\vec{\nabla}\cdot(u\vec{\nabla}v) &= u\nabla^2 v + \vec{\nabla}u\cdot\vec{\nabla}v\\ \vec{\nabla}\cdot(v\vec{\nabla}u) &= v\nabla^2 u + \vec{\nabla}v\cdot\vec{\nabla}u\end{aligned}\right\}\text{ subtracting:}$$

$\vec{\nabla}\cdot(u\vec{\nabla}v - v\vec{\nabla}u) = u\nabla^2 v - v\nabla^2 u$ integrating over R:

$$\iint_R \vec{\nabla}\cdot(u\vec{\nabla}v - v\vec{\nabla}u)\,dx\,dy = \iint_R (u\nabla^2 v - v\nabla^2 u)\,dx\,dy$$

$$\boxed{\oint (u\vec{\nabla}v - v\vec{\nabla}u)\cdot d\vec{l} = \iint_R (u\nabla^2 v - v\nabla^2 u)\,dx\,dy} \quad \text{in 2-D}$$

By divergence theorem.

OR $$\boxed{\oint (u\vec{\nabla}v - v\vec{\nabla}u)\cdot\hat{n}\,ds = \int_V (u\nabla^2 v - v\nabla^2 u)\,dV} \quad \text{in 3-D}$$

b) Consider two eigenfunctions ϕ_1 and ϕ_2 corresponding to eigenvalues λ_1 and λ_2:

$$\nabla^2\phi_1 = -\lambda_1\phi_1 \qquad \lambda_1 \neq \lambda_2$$
$$\nabla^2\phi_2 = -\lambda_2\phi_2$$

Let $u = \phi_1$ and $v = \phi_2$ above, then:

$$\iint (\phi_1\nabla^2\phi_2 - \phi_2\nabla^2\phi_1)\,dx\,dy = \underbrace{\oint (\phi_1\vec{\nabla}\phi_2 - \phi_2\vec{\nabla}\phi_1)\cdot d\vec{l}}_{0}$$

since ϕ_1 and ϕ_2 satisfy the same homogeneous boundary conditions.

$$\iint [\phi_1(-\lambda_2\phi_2) - \phi_2(-\lambda_1\phi_1)]\,dx\,dy = 0$$

$$(\lambda_1 - \lambda_2)\iint \phi_1\phi_2\,dx\,dy = 0$$

Since $\lambda_1 \neq \lambda_2 \Rightarrow \boxed{\iint_R \phi_1\phi_2\,dx\,dy = 0}$

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7.4.4 $\nabla^2\phi + \lambda\phi = 0$, multiplying by ϕ and integrating:

$$\iint_R \phi\nabla^2\phi\, dx\,dy + \lambda\iint_R \phi^2\, dx\,dy = 0 \rightarrow \lambda = \frac{-\iint_R \phi\nabla^2\phi\, dx\,dy}{\iint_R \phi^2\, dx\,dy}$$

$$\vec{\nabla}\cdot(\phi\vec{\nabla}\phi) = \phi\nabla^2\phi + \vec{\nabla}\phi\cdot\vec{\nabla}\phi \rightarrow \phi\nabla^2\phi = \vec{\nabla}\cdot(\phi\vec{\nabla}\phi) - |\vec{\nabla}\phi|^2$$

$$\Rightarrow \lambda = \frac{-\iint_R [\vec{\nabla}\cdot(\phi\vec{\nabla}\phi) - |\vec{\nabla}\phi|^2]\, dx\,dy}{\iint_R \phi^2\, dx\,dy} = \frac{-\iint_R \vec{\nabla}\cdot(\phi\vec{\nabla}\phi)\, dx\,dy + \iint_R |\vec{\nabla}\phi|^2\, dx\,dy}{\iint_R \phi^2\, dx\,dy}$$

by divergence theorem $\iint_R \nabla\cdot\vec{A}\, dx\,dy = \oint \vec{A}\cdot\hat{n}\, ds$

$$\Rightarrow \boxed{\lambda = \frac{-\oint(\phi\vec{\nabla}\phi)\cdot\hat{n}\, ds + \iint_R |\vec{\nabla}\phi|^2\, dx\,dy}{\iint_R \phi^2\, dx\,dy}} \rightarrow \text{Rayleigh Quotient.}$$

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7.5.1 $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c^2 \nabla^2 \phi$ $\quad u=0$ on the boundary

a) Let $u(x,y,t) = \phi(x,y)h(t) \Rightarrow \phi \frac{d^2h}{dt^2} = h c^2 \nabla^2 \phi$ dividing by ϕh

$$\Rightarrow \underbrace{\frac{1}{h}\frac{d^2h}{dt^2}}_{-\lambda} = \underbrace{\frac{c^2}{\phi}\nabla^2\phi}_{-\lambda} \Rightarrow \begin{array}{l} h''(t) = -\lambda h(t) \\ c^2\nabla^2\phi = -\lambda\phi \rightarrow \boxed{\nabla^2\phi + \lambda\left(\frac{1}{c^2}\right)\phi = 0} \end{array}$$

comparing with $\nabla^2\phi + \lambda\sigma(x,y)\phi = 0 \rightarrow \boxed{\sigma(x,y) = 1/c^2}$

Since $u=0$ on the boundary $\Rightarrow u = \phi(x,y)h(t) = 0$ on the boundary $\Rightarrow \phi(x,y) = 0$ on the boundary.

b) If $\lambda_n > 0 \Rightarrow h''(t) = -\lambda_n h(t) \rightarrow h(t) = A_n \cos\sqrt{\lambda_n}\, t + B_n \sin\sqrt{\lambda_n}\, t$

Thus the frequencies of vibration are $\sqrt{\lambda_n}$.

7.5.2 Use the Green's second identity:

$$\boxed{\iint_R (u\nabla^2 v - v\nabla^2 u)\,dx\,dy = \oint (u\vec{\nabla} v - v\vec{\nabla} u)\cdot\hat{n}\,ds}$$

a) Let $u = \phi_1$ and $v = \phi_2$ with ϕ_1 and ϕ_2 eigenfunctions corresponding to eigenvalues λ_1 and λ_2 such that

$$\nabla^2\phi_1 + \lambda_1 \frac{1}{c^2}\phi_1 = 0 \quad \text{and} \quad \nabla^2\phi_2 + \lambda_2\frac{1}{c^2}\phi_2 = 0$$

also satisfying the same boundary conditions: $\underbrace{\phi_1 = 0,\ \phi_2 = 0}_{\phi_1\phi_2 = 0 \text{ on the boundary.}}$ on the boundary:

$$\Rightarrow \underbrace{\iint_R (\phi_1\nabla^2\phi_2 - \phi_2\nabla^2\phi_1)\,dx\,dy}_{} = \cancel{\oint (\phi_1\vec{\nabla}\phi_2 - \phi_2\vec{\nabla}\phi_1)\cdot\hat{n}\,ds} \rightarrow 0$$

$$\iint_R \left[\phi_1\left(-\frac{\lambda_2}{c^2}\phi_2\right) - \phi_2\left(-\frac{\lambda_1}{c^2}\phi_1\right)\right]dx\,dy = 0$$

$\sigma = \frac{1}{c^2} \rightarrow$ weight!!

$$(\lambda_1 - \lambda_2)\iint_R \phi_1\phi_2\frac{1}{c^2}\,dx\,dy = 0 \rightarrow \boxed{\iint_R \phi_1\phi_2\,\sigma(x,y)\,dx\,dy = 0}$$

b) Let λ be an eigenvalue of $\phi \Rightarrow \nabla^2\phi + \lambda\sigma\phi = 0$

$$\overline{\nabla^2\phi + \lambda\sigma\phi = 0} \qquad \bar{\sigma} = \sigma = \frac{1}{c^2}$$

$\Rightarrow \nabla^2\bar{\phi} + \bar{\lambda}\sigma\bar{\phi} = 0 \rightarrow$ Therefore $\bar{\lambda}$ is also an eigenvalue with corresponding eigenfunction $\bar{\phi}$. From part a) this implies

$$(\lambda - \bar{\lambda})\iint_R \phi\bar{\phi}\sigma\,dx\,dy = 0$$

$$(\lambda - \bar{\lambda})\iint_R |\phi|^2\sigma\,dx\,dy = 0 \rightarrow \lambda = \bar{\lambda} \quad \text{because} \rightarrow$$

Clearly $\lambda_{nm} > 0$, then the solution of differential equation time-dependet is:

$$h(t) = C_{nm} \cos c\sqrt{\lambda_{nm}}\, t + D_{nm} \sin c\sqrt{\lambda_{nm}}\, t$$

then by superposition principle the general solution of the wave equation given by product of $f_{nm}\, g_{nm}\, h_{nm}$ solutions is:

$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\left[C_{nm}\cos c\sqrt{\lambda_{nm}}\, t + D_{nm}\sin c\sqrt{\lambda_{nm}}\, t\right]}$$

The initial value problem is determined specificanding the initial position and velocity forms:

$$u(x,y,0) = \alpha(x,y)$$
$$\frac{\partial u}{\partial t}(x,y,0) = \beta(x,y)$$

$$\Rightarrow \alpha(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

multiplying by $\sin(n'\pi x/L)\sin(m'\pi y/H)$ and integrating:

$$\int_0^L\int_0^H \alpha(x,y)\sin\left(\frac{n'\pi x}{L}\right)\sin\left(\frac{m'\pi y}{H}\right)dx\,dy$$

$$= \sum_{n,m} C_{nm} \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}} \underbrace{\int_0^H \sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{m'\pi y}{H}\right)dy}_{(H/2)\delta_{mm'}}$$

$$= C_{n'm'}\left(\frac{L}{2}\right)\left(\frac{H}{2}\right) \Rightarrow \boxed{C_{nm} = \frac{4}{LH}\int_0^H\int_0^L \alpha(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dx\,dy}$$

The other coefficient D_{nm} can be determined with the initial condition for the velocity:

$$\beta(x,y) = \frac{\partial u}{\partial t}(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}\, D_{nm} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

multiplying by $\sin(n'\pi x/L)\sin(m'\pi y/H)$ and integrating:

Since $\sigma = \frac{1}{c^2}$ is positive and $|\phi|^2$ is nonegative

$\Rightarrow \iint_R |\phi|^2 \sigma \, dx\,dy = 0$ if and only if $|\phi|^2 = 0 \rightarrow \phi = 0$ everywhere

but that would imply that ϕ is not an eigenfunction (It is a trivial solution).

$\Rightarrow \iint_R |\phi|^2 \sigma \, dx\,dy > 0.$

$\Rightarrow (\lambda - \bar{\lambda}) = 0 \rightarrow \bar{\lambda} = \lambda$ and hence λ is real. Thus the eigenvalues are real.

c) See Exercise 7.6.1 after (go ahead!!)

7.5.3 a). If $\vec{\nabla}\phi_1 \cdot \hat{n} = 0$ and $\vec{\nabla}\phi_2 \cdot \hat{n} = 0$ also we have

$$\oint (\phi_1 \vec{\nabla}\phi_2 - \phi_2 \vec{\nabla}\phi_1) \cdot \hat{n} \, ds = 0 \Rightarrow \iint_R (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1) \, dx\,dy = 0$$

$\Rightarrow \iint_R \phi_1 \phi_2 \sigma \, dx\,dy = 0$ according to exercise 7.5.2 (a).

b) If $\vec{\nabla}\phi_1 \cdot \hat{n} + h(x,y)\phi_1 = 0$ and $\vec{\nabla}\phi_2 \cdot \hat{n} + h(x,y)\phi_2 = 0$

$\Rightarrow \oint (\phi_1 \vec{\nabla}\phi_2 - \phi_2 \vec{\nabla}\phi_1) \cdot \hat{n} \, ds = \oint [\phi_1(-h\phi_2) - \phi_2(-h\phi_1)] \, ds = 0$

$\Rightarrow \iint_R \phi_1 \phi_2 \sigma \, dx\,dy = 0$ again according to problem 7.5.2 (a).

c) If $\phi_1 = 0$ and $\phi_2 = 0$ on part S_1 of the boundary S and $\vec{\nabla}\phi_1 \cdot \hat{n} = 0$ and $\vec{\nabla}\phi_2 \cdot n = 0$ on part S_2 of the boundary S.

$$\Rightarrow \oint_S (\phi_1 \vec{\nabla}\phi_2 - \phi_2 \vec{\nabla}\phi_1) \cdot \hat{n} \, ds = \cancel{\int_{S_1} (\phi_1 \vec{\nabla}\phi_2 - \phi_2 \vec{\nabla}\phi_1) \cdot \hat{n} \, ds} \rightarrow 0 \text{ since } \phi_1 = 0,\ \phi_2 = 0 \text{ on the boundary}$$

$$+ \cancel{\int_{S_2} (\phi_1 \vec{\nabla}\phi_2 - \phi_2 \vec{\nabla}\phi_1) \cdot \hat{n} \, ds} \rightarrow 0$$

Since $\vec{\nabla}\phi_1 \cdot \hat{n} = 0$, $\vec{\nabla}\phi_2 \cdot \hat{n} = 0$ on the boundary.

7.5.4 $c\rho \frac{\partial u}{\partial t} = \vec{\nabla} \cdot (k_0 \vec{\nabla} u)$

Let $u(x,y,z,t) = \phi(x,y,z) h(t) \rightarrow c\rho\phi \frac{dh}{dt} = h \vec{\nabla} \cdot (k_0 \vec{\nabla}\phi)$

dividing by $c\rho\phi h \Rightarrow \underbrace{\frac{1}{h}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{c\rho\phi} \vec{\nabla} \cdot (k_0 \vec{\nabla}\phi)}_{-\lambda}$

$\Rightarrow h'(t) = -\lambda h(t)$ and $\boxed{\vec{\nabla} \cdot (k_0 \vec{\nabla}\phi) + \lambda c\rho\phi = 0}$

Thus ϕ satisfy $\vec{\nabla} \cdot (p \vec{\nabla}\phi) + \lambda \sigma(x,y,z)\phi = 0$ with $p = k_0$, $\sigma = c\rho$.

Since $u = 0$ on the boundary and $u = h(t)\phi(x,y,z)$

$\Rightarrow \boxed{\phi = 0 \text{ on the boundary.}}$

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7.5.5 Let $L(\phi) = \vec{\nabla}\cdot(p\vec{\nabla}\phi) \rightarrow \boxed{L(\phi) + \lambda\sigma\phi = 0} \rightarrow$ Eq (7.5.13)

$$\phi_1 L(\phi_2) - \phi_2 L(\phi_1) = \phi_1 \vec{\nabla}(p\vec{\nabla}\phi_2) - \phi_2 \vec{\nabla}\cdot(p\vec{\nabla}\phi_1)$$

$$= \left\{\vec{\nabla}[\phi_1(p\vec{\nabla}\phi_2)] - \cancel{\vec{\nabla}\phi_1(p\vec{\nabla}\phi_2)}\right\} - \left\{\vec{\nabla}[\phi_2(p\vec{\nabla}\phi_1)] - \cancel{\vec{\nabla}\phi_2(p\vec{\nabla}\phi_1)}\right\}$$

$$\int[\phi_1 L(\phi_2) - \phi_2 L(\phi_1)]dV = \int_V \vec{\nabla}\cdot[\phi_1(p\vec{\nabla}\phi_2) - \phi_2(p\vec{\nabla}\phi_1)]dV$$

$$\boxed{\int[\phi_1 L(\phi_2) - \phi_2 L(\phi_1)]dV = \oint[\phi_1(p\vec{\nabla}\phi_2) - \phi_2(p\vec{\nabla}\phi_1)]\cdot\hat{n}\,dS}$$

Integral form of Lagrange's identity OR Green's formula.

a) Since $\phi = 0$ on the boundary in (7.5.13), the rightside of Green's formula vanishes.

$\Rightarrow \int[\phi_1 L(\phi_2) - \phi_2 L(\phi_1)]dV = 0$, but $L(\phi_2) = -\lambda_2\sigma\phi_2$, $L(\phi_1) = -\lambda_1\sigma\phi_1$

$\Rightarrow \int[\phi_1(-\lambda_2\sigma\phi_2) - \phi_2(-\lambda\sigma\phi_1)]dV = 0$

$(\lambda_1 - \lambda_2)\int \phi_1\phi_2\sigma dV = 0$ if $\lambda_1 \neq \lambda_2 \rightarrow \boxed{\int\phi_1\phi_2\sigma dV = 0}$

where $\sigma = c\rho$ according to problem (7.5.4).

b) Let λ be an eigenvalue of $\phi \rightarrow$ $L(\phi) + \lambda\sigma\phi = 0$

$\overline{L(\phi)} = L(\bar{\phi})$ since p is real $\rightarrow$ $\overline{L(\phi)} + \bar{\lambda}\sigma\bar{\phi} = 0$, $L(\bar{\phi}) + \bar{\lambda}\sigma\bar{\phi} = 0$ $\quad \bar{\sigma} = \sigma$ since it's real

Therefore $\bar{\lambda}$ is also an eigenvalue with corresponding eigenfunction $\bar{\phi}$. From part a) this implies:

$(\lambda - \bar{\lambda})\int \phi\bar{\phi}\sigma dV = 0 \rightarrow (\lambda - \bar{\lambda})\int_V |\phi|^2\sigma dV = 0 \rightarrow \boxed{\lambda = \bar{\lambda}}$

because since $\sigma > 0$ and $|\phi|^2$ is nonnegative

$\Rightarrow \int_V |\phi|^2\sigma dV = 0$ if and only if $|\phi|^2 = 0 \rightarrow \phi = 0$.

but that would imply that ϕ is not an eigenfunction (It is a trivial solution) $\Rightarrow \int_V |\phi|^2\sigma dV > 0 \Rightarrow$ if $(\lambda - \bar{\lambda})\underbrace{\int|\phi|^2\sigma dV}_{=0}$

$\Rightarrow (\lambda - \bar{\lambda}) = 0 \rightarrow \boxed{\bar{\lambda} = \lambda}$ and hence λ is real.

c) See Exercise 7.6.3.

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7.5.6 $L(\phi) = \nabla^2\phi + q\phi$ with $q = q(x,y)$

$\Rightarrow uL(v) - vL(u) = u(\nabla^2 v + \cancel{qv}) - v(\nabla^2 u + \cancel{qu}) = u\nabla^2 v - v\nabla^2 u$

$\Rightarrow \iint [uL(v) - vL(u)]\,dx\,dy = \int [u\nabla^2 v - v\nabla^2 u]\,dx\,dy$ by Green's Formula

$$\boxed{\iint [uL(v) - vL(u)]\,dx\,dy = \oint [u\vec{\nabla} v - v\vec{\nabla} u]\cdot\hat{n}\,ds}$$

7.5.7 a) $\nabla^2 u = 0 \rightarrow$ There are not sources of heat inside of S.

$\oint \vec{\nabla} u\cdot\hat{n}\,ds = 0 \rightarrow$ There are not heat flux outside of S. The heat flux is tangential over S only, no in normal direction to S.

b) $\oint \vec{\nabla} u\cdot\hat{n}\,ds = \int_V \vec{\nabla}(\vec{\nabla} u)\,dV$ by divergence theorem

$= \int_V \nabla^2 u\,dV = 0$ becase $\nabla^2 u = 0$.

7.5.8 $\nabla^2\phi = f(x,y,z)$ and $\vec{\nabla}\phi\cdot\hat{n} = 0$ on the boundary

a) $\int_V f\,dV = \int_V \nabla^2\phi\,dV = \underbrace{\int_V \vec{\nabla}\cdot(\vec{\nabla}\phi)\,dV}_{\text{divergence theo}} = \int_S \underbrace{\vec{\nabla}\phi\cdot\hat{n}}_{0}\,dS = 0$

b) $\vec{\nabla}\phi\cdot\hat{n} = 0$ on the boundary implies that the heat flow through S in normal direction to the surface is vanishing $\Rightarrow \int_S \vec{\nabla}\phi\cdot\hat{n}\,dS = 0 \rightarrow$ flux over S vanishing

$= \int_V f\,dV \Rightarrow \int_V f\,dV = 0.$

If $\int_V f\,dV > 0 \rightarrow$ the flux over S is no null.

7.5.9 $\oint_S (u\nabla v - v\nabla u)\cdot\hat{n}\,ds \rightarrow$ boundary term (7.5.8)

$\beta_1\phi + \beta_2\vec{\nabla}\phi\cdot\hat{n} = 0 \rightarrow \vec{\nabla}\phi\cdot\hat{n} = -\frac{\beta_1}{\beta_2}\phi \rightarrow$ Boundary Condition (7.5.2).

a) If $\beta_2 \neq 0 \rightarrow \oint_S (u\vec{\nabla} v - v\vec{\nabla} u)\cdot\hat{n}\,ds = \oint_S [u(-\frac{\beta_1}{\beta_2}v) - v(-\frac{\beta_1}{\beta_2}u)]\,ds$

$= 0$

b) If $\beta_2 = 0 \rightarrow \beta_1\phi = 0$ on the boundary, then $\phi = 0$ on the boundary $\Rightarrow \oint_S (u\vec{\nabla} v - v\vec{\nabla} u)\cdot\hat{n}\,ds = 0$ because $u = 0$, $v = 0$ on the boundary.

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7.6.1 $\boxed{\nabla^2\phi + \lambda\sigma\phi = 0}$ with $\phi = 0$ on the boundary.

a) By Rayleigh quotient $\lambda = \dfrac{-\oint \cancelto{0}{\phi}\vec{\nabla}\phi\cdot\hat{n}\,ds + \int_V |\nabla\phi|^2 dV}{\int_V \phi^2 dV} \geq 0$

b) $\lambda = 0 \rightarrow \nabla\phi = 0 \rightarrow \dfrac{\partial\phi}{\partial x}\hat{i} + \dfrac{\partial\phi}{\partial y}\hat{j} = 0 \rightarrow \dfrac{\partial\phi}{\partial x} = 0,\ \dfrac{\partial\phi}{\partial y} = 0$ everywhere

Thus, ϕ is a constant everywhere, but since $\phi = 0$ on the boundary, $\phi = 0$ everywhere. $\phi = 0$ everywhere is not an eigenfunction and thus we have shown that $\lambda = 0$ is not an eigenvalue. In conclusion, $\boxed{\lambda > 0}$

7.6.2 a) $\nabla\phi\cdot\hat{n} = 0 \longrightarrow \lambda = \dfrac{-\oint \phi\cancelto{0}{\vec{\nabla}\phi\cdot\hat{n}}\,ds + \int_V |\vec{\nabla}\phi|^2 dV}{\int_V \phi^2 dV} \geq 0.$

if $\lambda = 0 \rightarrow \vec{\nabla}\phi = 0 \rightarrow \dfrac{\partial\phi}{\partial x} = 0,\ \dfrac{\partial\phi}{\partial y} = 0$ everywhere $\Rightarrow \phi = cte$ everywhere

b) $\vec{\nabla}\phi\cdot\hat{n} + h(x,y)\phi = 0$ on the boundary $\rightarrow \vec{\nabla}\phi\cdot\hat{n}\big|_S = -h\phi$

$$\Rightarrow \lambda = \frac{-\oint \phi\vec{\nabla}\phi\cdot\hat{n}\,ds + \int_V |\nabla\phi|^2 dV}{\int_V \phi^2 dV} = \frac{-\oint \phi(-h\phi)\,ds + \int_V |\nabla\phi|^2 dV}{\int_V \phi^2 dV}$$

$$= \frac{h\oint \phi^2 ds + \int_V |\nabla\phi|^2 dV}{\int_V \phi^2 dV} \geq 0 \quad \text{since } h > 0.$$

$\lambda = 0 \rightarrow \nabla\phi = 0$ everywhere and $\phi = 0$ over S.

$\hookrightarrow \phi = cte$, but $\phi|_S = 0 \rightarrow \phi = 0$ everywhere.

$\hookrightarrow$ Trivial solution.

we don't want trivial solution $\Rightarrow \boxed{\lambda > 0}$

c) $\phi|_{S_1} = 0$ and $\vec{\nabla}\phi\cdot\hat{n}\big|_{S_2} = 0$ such tat $S = S_1 + S_2$.

$$\lambda = \frac{-\oint_{S_1} \cancelto{0}{\phi}\vec{\nabla}\phi\cdot\hat{n}\,ds - \oint_{S_2} \phi\cancelto{0}{\vec{\nabla}\phi\cdot\hat{n}}\,ds + \int_V |\nabla\phi|^2 dV}{\int_V \phi^2 dV} \geq 0$$

$\lambda = 0 \rightarrow \vec{\nabla}\phi = 0$ everywhere $\rightarrow \phi = cte$,

but $\phi|_{S_1} = 0 \Rightarrow \phi = 0$ everywhere.

7.6.3 $\nabla\cdot(p\nabla\phi) + \lambda\sigma(x,y,z)\phi = 0$ multiplying by ϕ and integrating:

$$\int \phi\nabla\cdot(p\nabla\phi)\,dV + \lambda\int \sigma\phi^2 dV = 0 \rightarrow \lambda = \frac{-\int \phi\nabla\cdot(p\nabla\phi)\,dV}{\int \phi^2\sigma\,dV}$$

$$\lambda = \frac{-\int\left\{\nabla\cdot[\phi(p\nabla\phi)] - \nabla\phi(p\nabla\phi)\right\}dV}{\int \phi^2\sigma\,dV} = \frac{-\oint \phi(p\vec{\nabla}\phi)\cdot\hat{n}\,ds + \int_V p|\nabla\phi|^2 dV}{\int \phi^2\sigma\,dV}$$

$$\Rightarrow \boxed{\lambda = \frac{-\oint \phi(\rho\vec{\nabla}\phi)\cdot\hat{n}\,ds + \int_V \rho|\nabla\phi|^2 dv}{\int_V \phi^2\sigma\, dv}}$$

Rayleigh quotient for $\vec{\nabla}\cdot(\rho\vec{\nabla}\phi)+\lambda\sigma\phi=0$.

a) $\phi|_S = 0 \rightarrow \oint \cancel{\phi}^{\,0}(\rho\vec{\nabla}\phi)\cdot\hat{n}\,ds = 0 \rightarrow \lambda = \dfrac{\int_V \rho|\nabla\phi|^2 dv}{\int_V \phi^2\sigma\, dv} \geq 0$

if $\lambda = 0 \rightarrow \vec{\nabla}\phi = 0$ everywhere $\rightarrow \phi = cte$ but $\phi|_S = 0$, then $\phi = 0$ everywhere $\rightarrow$ trivial solution $\Rightarrow \boxed{\lambda > 0}$

b) $\vec{\nabla}\phi\cdot\hat{n}|_S = 0 \rightarrow \oint \phi\,\cancel{(\rho\vec{\nabla}\phi)\cdot\hat{n}}^{\,0}\,ds = 0 \rightarrow \lambda = \dfrac{\int_V \rho|\nabla\phi|^2 dv}{\int_V \phi^2\sigma\, dv} \geq 0$

If $\lambda = 0 \rightarrow \vec{\nabla}\phi = 0 \rightarrow \phi = cte$.

7.6.4 a) If $\nabla^2\phi = 0$ with $\phi|_S = 0 \Rightarrow \phi = 0$ everywhere

Let use $\nabla^2\phi + \lambda\phi = 0$ with $\lambda = 0 \rightarrow$ by Rayleigh quotient:

$$0 = \lambda = \frac{-\oint \cancel{\phi}^{\,0}(\rho\vec{\nabla}\phi)\cdot\hat{n}\,ds + \int_V \rho|\nabla\phi|^2 dv}{\int_V \phi^2\sigma\, dv} \rightarrow \nabla\phi = 0 \rightarrow \phi = cte \text{ everywhere}$$

but $\phi|_S = 0 \rightarrow \underline{\phi = 0 \text{ everywhere}}$

b) Let u_1 and u_2 two diffent solutions, then:

$$\left.\begin{array}{l}\nabla^2 u_1 = f \\ \nabla^2 u_2 = f\end{array}\right\} \Rightarrow \nabla^2(u_1 - u_2) = 0,$$

If u_1 and u_2 Satisfy the boundary condition, then

$$\left.\begin{array}{l}u_1|_S = g(x,y,z) \\ u_2|_S = g(x,y,z)\end{array}\right\} \Rightarrow (u_1 - u_2)|_S = 0$$

but from part a) if $\nabla^2\phi = 0$ with $\phi|_S = 0 \rightarrow \phi = 0$ everywhere.

Let $\phi = u_1 - u_2$, above we have $\left.\begin{array}{l}\nabla^2(u_1-u_2) = 0 \\ (u_1-u_2)|_S = 0\end{array}\right\} \Rightarrow u_1 - u_2 = 0$ everywhere.

$\Rightarrow u_1 = u_2$ and the solution is unique.

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* **7.7.1**

$u(a,\theta,t)=0 \rightarrow$ Boundary Condition

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \qquad \underbrace{u(r,\theta,0)=0, \quad \frac{\partial u}{\partial t}(r,\theta,0)=\alpha(r)\sin 3\theta}_{\text{Initial conditions}}$$

From separation of variables (Section 7.7.2 in the text):

Let $u(r,\theta,t) = f(r)g(\theta)h(t) \Longrightarrow \dfrac{d^2h}{dt^2} = -\lambda c^2 h(t)$

$$g(-\pi)=g(\pi), \quad \frac{dg}{d\theta}(-\pi)=\frac{dg}{d\theta}(\pi) \qquad \frac{d^2g}{d\theta^2} = -m^2 g(\theta) \longrightarrow g(\theta) = \left\{ \begin{matrix} \sin m\theta \\ \cos m\theta \end{matrix} \right\}$$

$m = 0, 1, 2, 3, \ldots$

Bessel's diff eq $\longrightarrow$ $$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right) f(r) = 0$$

$\boxed{\lambda = 0 \rightarrow f \sim r^{\pm m} \text{ but } f(a)=0 \rightarrow \text{There are Not solution}}$

$$f(r) = C_1 J_m(\sqrt{\lambda}\, r) + C_2 Y_m(\sqrt{\lambda}\, r)$$

zeros of J_m

$|f(0)| < \infty \longrightarrow C_2 = 0$ and $f(a) = 0 = C_1 J_m(\sqrt{\lambda}\, a) \rightarrow \sqrt{\lambda}\, a = z_{mn}$

$\Rightarrow f(a) = C\, J_m(\sqrt{\lambda_{mn}}\, r)$ $\qquad \lambda_{nm} = (z_{mn}/a)^2$

$m \rightarrow$ order of Bessel's function

$n \rightarrow$ nth zero of $J_m(z)$. $\quad n = 1, 2, 3, \ldots$

The zeros of Bessel's functions are $z_{mn} > 0 \rightarrow \lambda_{nm} > 0$

$\Rightarrow h(t) = \left\{ \begin{matrix} \cos c\sqrt{\lambda_{mn}}\, t \\ \sin c\sqrt{\lambda_{mn}}\, t \end{matrix} \right\}$ thus:

$$\boxed{u(r,\theta,t) \sim J_m(\sqrt{\lambda_{mn}}\, r) \left\{ \begin{matrix} \cos m\theta \\ \sin m\theta \end{matrix} \right\} \left\{ \begin{matrix} \cos c\sqrt{\lambda_{mn}}\, t \\ \sin c\sqrt{\lambda_{mn}}\, t \end{matrix} \right\}}$$

$\rightarrow$ Solution for a circular membrane in 2-D with u being fixed on the boundary.

$u(r,\theta,0) = 0 \rightarrow$ implies that all terms with $\cos c\sqrt{\lambda_{mn}}\, t$ vanish

$$\Rightarrow u(r,\theta,t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\, r)\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right] \sin c\sqrt{\lambda_{mn}}\, t$$

$$\alpha(r)\sin 3\theta = \frac{\partial u}{\partial t}(r,\theta,0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\, r)\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right] c\sqrt{\lambda_{mn}}$$

$$\rightarrow \alpha(r)\cancel{\sin 3\theta} = \sum_{n=1}^{\infty} J_3(\sqrt{\lambda_{3n}})\, B_{3n}\, \cancel{\sin(3\theta)} \cdot c\sqrt{\lambda_{3n}} \qquad \begin{matrix} A_{mn} = 0 \\ B_{mn} = 0 \text{ if } m \neq 3 \end{matrix}$$

Multiplying by $J_3(\sqrt{\lambda_{3n'}}\, r)\, r$ and integrating:

$$\int_0^a \alpha(r) J_3(\sqrt{\lambda_{3n'}}\, r)\, r\, dr = \sum_{n=1}^{\infty} B_{3n}\, c\sqrt{\lambda_{3n}} \underbrace{\int_0^a J_3(\sqrt{\lambda_{3n}}\, r)\, J_3(\sqrt{\lambda_{3n'}}\, r)\, r\, dr}_{0 \text{ if } n = n'}$$

$$\int_0^a \alpha(r) J_3(\sqrt{\lambda_{3n'}}\, r)\, r\, dr = B_{3n'}\, c\sqrt{\lambda_{3n'}} \int_0^a J_3^2(\sqrt{\lambda_{3n'}}\, r)\, r\, dr$$

$$\rightarrow \boxed{B_{3n} = \frac{1}{c\sqrt{\lambda_{3n}}} \frac{\int_0^a \alpha(r) J_3(\sqrt{\lambda_{3n}}\, r)\, r\, dr}{\int_0^a J_3^2(\sqrt{\lambda_{3n}}\, r)\, r\, dr}}$$

$\frac{a^2}{2} J_4^2(\sqrt{\lambda_{3n}}\, a)$

$$\Rightarrow \boxed{u(r,\theta,t) = \sum_{n=1}^{\infty} B_{3n} J_3(\sqrt{\lambda_{3n}}\, r) \sin(3\theta) \sin c\sqrt{\lambda_{3n}}\, t}$$

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There are not solutions for $\lambda<0$ or $\mu<0$ that satisfy boundary conditions.

7.7.2

$\frac{\partial u}{\partial r}(a,\theta,t)=0$

$$\frac{\partial^2 u}{\partial t^2}=c^2\nabla^2 u \qquad \text{let } u=f(r)g(\theta)h(t)$$

by separation of variables $\Rightarrow$ $h''(t)=-\lambda h(t)$, $g''(\theta)=-\mu g(\theta)$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right)+(\lambda r^2-\mu)f=0.$$

See section 7.7.2 from text

$\mu>0 \rightarrow g(\theta)=\left\{\begin{matrix}\sin\sqrt{\mu}\,\theta\\ \cos\sqrt{\mu}\,\theta\end{matrix}\right\}$

$g(-\pi)=g(\pi)$, $g'(-\pi)=g'(\pi) \rightarrow \sqrt{\mu}=m=1,2,3,\ldots$

$\mu=0 \rightarrow g(\theta)=d\theta+b \rightarrow d=0 \Rightarrow g(\theta)=\left\{\begin{matrix}\sin m\theta\\ \cos m\theta\end{matrix}\right\}$ $\quad \mu=m^2$, $m=0,1,2,\ldots$

$\lambda>0 \rightarrow f(r)=C_1J_m(\sqrt{\lambda}\,r)+C_2Y_m(\sqrt{\lambda}\,r)$, $|f(0)|<\infty \rightarrow C_2=0$.

$f'(a)=0=C_1J_m'(\sqrt{\lambda}\,a) \longrightarrow \sqrt{\lambda}\,a=z_{mn} \rightarrow$ Zeros of J_m'

$\rightarrow f(r)=CJ_m(\sqrt{\lambda_{mn}}\,r)$ $\quad \lambda_{mn}=(z_{mn}/a)^2$ $\quad n=1,2,3,\ldots$

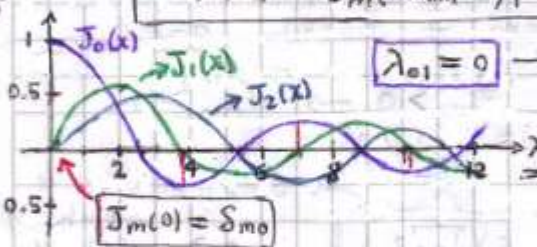
$\lambda_{mn}>0 \Rightarrow h''(t)=-\lambda h(t) \rightarrow h(t)=\left\{\begin{matrix}\sin c\sqrt{\lambda_{mn}}\,t\\ \cos c\sqrt{\lambda_{mn}}\,t\end{matrix}\right\}$

$\lambda=0 \rightarrow f\sim r^{\pm m}$, but $f'(a)=0 \rightarrow m=0 \rightarrow f(r)=C_0$

$\rightarrow g(\theta)=\cos 0=1$

$\lambda=0 \Rightarrow h''(t)=0 \rightarrow h(t)=at+b \Rightarrow \boxed{u(r,\theta,t)=C_0(at+b)}$

and $\boxed{u(r,\theta,t)\sim J_m(\sqrt{\lambda_{mn}}\,r)\left\{\begin{matrix}\cos m\theta\\ \sin m\theta\end{matrix}\right\}\left\{\begin{matrix}\cos c\sqrt{\lambda_{mn}}\,t\\ \sin c\sqrt{\lambda_{mn}}\,t\end{matrix}\right\}}$ for $\lambda>0$.



$\boxed{J_m(0)=\delta_{m0}}$

$\boxed{\lambda_{01}=0} \rightarrow$ Note that $z_{01}=0 \rightarrow$ first zero of J_0'

$J_0(z_{01})=J_0(0)=1$. when $m=0$, $n=1$

$\Rightarrow$ In addition $\boxed{J_0'(x)=-J_1(x)}$

and $\boxed{\int_0^a J_0(\alpha r)\,r\,dr=\frac{a}{\alpha}J_1(\alpha a)=-\frac{a}{\alpha}J_0'(x)}$

a) $u(r,\theta,0)=0 \rightarrow b=0$ for $\lambda=0$ and that all terms with $\cos c\sqrt{\lambda_{mn}}\,t$ vanish for $\lambda>0$. Also $A_{01}=0$.

$$\Rightarrow u(r,\theta,t)=C_0t+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos m\theta+B_{mn}\sin m\theta\right]\sin c\sqrt{\lambda_{mn}}\,t$$

$$\beta(r)\cos 5\theta=\frac{\partial u}{\partial t}(r,\theta,0)=C_0+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos m\theta+B_{mn}\sin m\theta\right]c\sqrt{\lambda_{mn}}$$

$$\Rightarrow \beta(r)\cancel{\cos 5\theta}=\sum_{n=1}^{\infty}J_5(\sqrt{\lambda_{5n}}\,r)\,A_{5n}\,\cancel{\cos 5\theta}\cdot c\sqrt{\lambda_{5n}}$$

$C_0=0$, $A_{mn}=0$ if $m\neq 5$, $B_{mn}=0$.

multiplying by $J_5(\sqrt{\lambda_{5n'}}\,r)\,r$ and integrating:

$$\int_0^a\beta(r)J_5(\sqrt{\lambda_{5n'}}\,r)\,r\,dr=\sum_{n=1}^{\infty}A_{5n}\,c\sqrt{\lambda_{5n}}\underbrace{\int_0^a J_5(\sqrt{\lambda_{5n}}\,r)J_5(\sqrt{\lambda_{5n'}}\,r)\,r\,dr}_{0 \text{ if } n\neq n'}$$

$$\Rightarrow A_{5n'}\,c\sqrt{\lambda_{5n'}}\int_0^a J_5^2(\sqrt{\lambda_{5n'}}\,r)\,r\,dr$$

$$\boxed{\frac{\int_0^a\beta(r)J_5(\sqrt{\lambda_{5n'}}\,r)\,r\,dr}{c\sqrt{\lambda_{5n}}\int_0^a J_5^2(\sqrt{\lambda_{5n}}\,r)\,r\,dr}=A_{5n}} \longrightarrow \frac{a^2}{2}\left[1-(5/\sqrt{\lambda_{5n}}\,a)^2\right]J_5^2(\sqrt{\lambda_{5n}}\,a)$$

$$\Rightarrow \boxed{u(r,\theta,t)=\sum_{n=1}^{\infty}A_{5n}J_5(\sqrt{\lambda_{5n}}\,r)\cos(5\theta)\sin c\sqrt{\lambda_{5n}}\,t}$$

$\lambda_{5n}=\left(\frac{z_{5n}}{a}\right)^2$ with z_{5n} the zeros of J_5'.

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Since $\beta(r)$ has no dependence of $\theta \rightarrow B_{mn}=0$ and $A_{mn}=0$ except for $m=0$.!

b) $u(r,\theta,0)=0 \rightarrow b=0$ and all terms with $\cos c\sqrt{\lambda_{mn}}\,t$ vanish.

$$\Rightarrow u(r,\theta,t)=C_0 t+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos m\theta+B_{mn}\sin m\theta\right]\sin c\sqrt{\lambda_{mn}}\,t$$

$$\beta(r)=\frac{\partial u}{\partial t}(r,\theta,0)=C_0+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos m\theta+B_{mn}\sin m\theta\right]c\sqrt{\lambda_{mn}}$$

$$\Rightarrow \beta(r)=\sum_{n=1}^{\infty} J_0(\sqrt{\lambda_{0n}}\,r)\,A_{0n}\,c\sqrt{\lambda_{0n}} \quad \text{and} \quad C_0=0,\ B_{mn}=0,\ A_{mn}=0 \text{ when } m\neq 0.$$

multiplying by $J_0(\sqrt{\lambda_{0n'}}\,r)\,r$ and integrating, by orthogonality

$$\Rightarrow \boxed{A_{0n}=\frac{\int_0^a \beta(r)\,J_0(\sqrt{\lambda_{0n}}\,r)\,r\,dr}{c\sqrt{\lambda_{0n}}\int_0^a J_0^2(\sqrt{\lambda_{0n}}\,r)\,r\,dr}} \qquad \boxed{u=\sum_{n=1}^{\infty} A_{0n}\,J_0(\sqrt{\lambda_{0n}}\,r)\sin c\sqrt{\lambda_{0n}}\,t}$$

$\frac{a^2}{2}\left[1-(0/\sqrt{\lambda_{0n}}\,a)^2\right]J_0^2(\sqrt{\lambda_{0n}}\,a)$ $\qquad \lambda_{0n}=(z_{0n}/a)^2$, $z_{0n}\rightarrow$ zeros of J_0'.

c) $\frac{\partial u}{\partial t}(r,\theta,0)=0 \rightarrow \alpha=0$ and all terms with $\sin c\sqrt{\lambda_{mn}}\,t$ vanish

$$\Rightarrow \boxed{u(r,\theta,t)=C_0+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos m\theta+B_{mn}\sin m\theta\right]\cos c\sqrt{\lambda_{mn}}\,t}$$

$$\alpha(r,\theta)=u(r,\theta,0)=C_0+A_{01}\underbrace{J_0(\sqrt{\lambda_{01}}\,r)}_{1}+\sum_{n=2}^{\infty}J_0(\sqrt{\lambda_{0n}}\,r)A_{0n}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}J_m(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos m\theta+B_{mn}\sin m\theta\right]$$

since $\lambda_{01}=0$ when $z_{01}=0 \rightarrow$ first zero of J_0'.

$$\Rightarrow \int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\,r\,dr\,d\theta=(C_0+A_{01})(\pi a^2)+\sum_{n=2}^{\infty}A_{0n}\int_{-\pi}^{\pi}d\theta\underbrace{\int_0^a J_0(\sqrt{\lambda_{0n}}\,r)\,r\,dr}_{\frac{a}{\sqrt{\lambda_{0n}}}J_1(\sqrt{\lambda_{0n}}\,a)=\frac{-a}{\sqrt{\lambda_{0n}}}J_0'(z_{0n})\ \to\ 0}$$

$$+\sum_{m=1}\sum_{n=1}\int_0^a J_m(\ )\,r\,dr\left[A_{mn}\underbrace{\int_{-\pi}^{\pi}\cos m\theta\,d\theta}_{0}+B_{mn}\underbrace{\int_{-\pi}^{\pi}\sin m\theta\,d\theta}_{0}\right]$$

$$\Rightarrow \boxed{C_0+A_{01}=\frac{1}{\pi a^2}\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\,r\,dr\,d\theta} \longrightarrow \text{average of the initial distribution.}$$

We can choose $C_0=0$ rescaling that constant.

Now, multiplying by $J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\cos m'\theta$ and integrating

$$\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\cos(m'\theta)\,r\,dr\,d\theta=C_0\underbrace{\int_{-\pi}^{\pi}\cos m'\theta\,d\theta}_{0}\int_0^a J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr$$

$$+\sum_{m,n}\int_0^a J_m J_{m'}\,r\,dr\left[A_{mn}\underbrace{\int_{-\pi}^{\pi}\cos m\theta\cos m'\theta\,d\theta}_{\pi\delta_{mm'}\ (m\neq 0)\ \text{and zero if } m=0}+B_{mn}\underbrace{\int_{-\pi}^{\pi}\sin m\theta\cos m\theta\,d\theta}_{0}\right]$$

multiplying by $J_{0n'}$ and integrating as above

$$=\sum_{n=1}^{\infty}\pi A_{m'n}\underbrace{\int_0^a J_{m'}(\sqrt{\lambda_{m'n}}\,r)\,J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr}_{0\ \text{if}\ n\neq n'}=\pi A_{m'n'}\int_0^a J_{m'}^2(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr$$

$m'\neq 0$ $\qquad (a^2/2)\left[1-(m/\sqrt{\lambda_{mn}}\,a)^2\right]J_m^2(z_{mn})$

$\boxed{A_{0n}=\frac{1}{2\pi}[\ ]}$ $n\neq 1$

$$\Rightarrow \boxed{A_{mn}=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\,J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)\,r\,dr\,d\theta\Big/\int_0^a J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr\right]}$$

$m\neq 0$

Similarly: $$\boxed{B_{mn}=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\,J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\,r\,dr\,d\theta\Big/\int_0^a J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr\right]}$$

$m\neq 0$

$\boxed{B_{0n}=0}$

$\lambda_{mn}=(z_{mn}/a)^2$, $z_{mn}\rightarrow$ zeros of J_m'.

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$J_0(\sqrt{\lambda_{01}}\,r) = J_0(0) = 1$ at the first zero of J_0'

Since $J_0(\sqrt{\lambda_{01}}\,r) = J_0(0) = 1$ $\Rightarrow u(r,\theta,0) = 0 \rightarrow \boxed{A_{01} = 0}$

* d) $u(r,\theta,0) = 0 \rightarrow b = 0$ and all terms with $\cos c\sqrt{\lambda_{mn}}\,t$ vanish

$$\Rightarrow \boxed{u(r,\theta,t) = c_0 t + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos m\theta + B_{mn}\sin(m\theta)\right]\sin c\sqrt{\lambda_{mn}}\,t}$$

$$\beta(r,\theta) = \frac{\partial u}{\partial t}(r,\theta,0) = c_0 + \cancel{A_{01}}\,J_0(\sqrt{\lambda_{01}}\,r)\,c\cancel{\sqrt{\lambda_{01}}} + \sum_{n=2} J_0(\sqrt{\lambda_{0n}}\,r)\,A_{0n}\,c\sqrt{\lambda_{0n}}$$

$$+ \sum_{m=1}\sum_{n=1} J_{mn}(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right] c\sqrt{\lambda_{mn}}$$

$$\Rightarrow \int_{-\pi}^{\pi}\int_0^{\alpha}\beta(r,\theta)\,r\,dr\,d\theta = c_0(\pi\alpha^2) + \sum_{n=2} A_{0n}\,c\sqrt{\lambda_{0n}}\int_{-\pi}^{\pi}d\theta\underbrace{\int_0^{\alpha} J_0(\sqrt{\lambda_{0n}}\,r)\,r\,dr}_{\frac{\alpha}{\sqrt{\lambda_{0n}}}J_1(\sqrt{\lambda_{0n}}\,\alpha) = -\frac{\alpha}{\sqrt{\lambda_{0n}}}\cancel{J_0'(z_{0n})}}$$

$$+ \sum_{m=1}\sum_{n=1}\int_0^{\alpha} J_{mn}(\,)\,r\,dr\left[A_{mn}\cancel{\int_{-\pi}^{\pi}\cos m\theta\,d\theta} + B_{mn}\cancel{\int_{-\pi}^{\pi}\sin m\theta\,d\theta}\right]$$

$$\Rightarrow \boxed{c_0 = \frac{1}{\pi\alpha^2}\int_{-\pi}^{\pi}\int_0^{\alpha}\beta(r,\theta)\,r\,dr\,d\theta} \rightarrow \text{average of the initial distribution.}$$

Now, multiplying by $J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\cos m'\theta$ and integrating in $dS = r\,dr\,d\theta$

$$\int_{-\pi}^{\pi}\int_0^{\alpha}\beta(r,\theta)\,J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\cos m'\theta\,(r\,dr\,d\theta) = c_0\int_0^{\alpha} J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr\cancel{\int_{-\pi}^{\pi}\cos m'\theta\,d\theta}$$

$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}}\int_0^{\alpha} J_m(\sqrt{\lambda_{mn}}\,r)\,J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr\left[A_{mn}\cancel{\int_{-\pi}^{\pi}\cos m\theta\cos m'\theta\,d\theta} + B_{mn}\cancel{\int_{-\pi}^{\pi}\sin m\theta\cos m'\theta\,d\theta}\right]$$

$\pi\delta_{mm'}$ $\begin{matrix} m\neq 0 \\ m=0 \end{matrix}\Big\}$

multiplying by $J_{m'n'}$ as above and integrating

$$= \sum_{n=1}^{\infty}\pi c\sqrt{\lambda_{m'n}}\,A_{m'n}\underbrace{\int_0^{\alpha} J_{m'}(\sqrt{\lambda_{m'n}}\,r)\,J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr}_{0 \text{ if } n\neq n'} \qquad m'\neq 0$$

$$= \pi c\sqrt{\lambda_{m'n'}}\,A_{m'n'}\int_0^{\alpha} J_{m'}^2(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr \qquad m'\neq 0$$

$\frac{\alpha^2}{2}\left[1-\left(\frac{m}{\sqrt{\lambda_{mn}}\,\alpha}\right)^2\right]J^2(\sqrt{\lambda_{mn}}\,\alpha)$

$A_{0n} = \frac{1}{2\pi c}\left[\ \right]$ $n\neq 1$

$$\Rightarrow \boxed{A_{mn} = \frac{1}{\pi c\sqrt{\lambda_{mn}}}\left[\int_{-\pi}^{\pi}\int_0^{\alpha}\beta(r,\theta)\,J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)\,r\,dr\,d\theta \Big/ \int_0^{\alpha} J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr\right]} \quad m\neq 0$$

similar

$$\boxed{B_{mn} = \frac{1}{\pi c\sqrt{\lambda_{mn}}}\left[\int_{-\pi}^{\pi}\int_0^{\alpha}\beta(r,\theta)\,J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\,r\,dr\,d\theta \Big/ \int_0^{\alpha} J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr\right]} \quad m\neq 0$$

$B_{0n} = 0$

$\lambda_{mn} = (z_{mn}/\alpha)^2$ with z_{mn} the zeros of J_{mn}.

7.7.3

$u = 0$ on entire boundary

let $u = f(r)\,g(\theta)\,h(t)$, by separation of variables $\Rightarrow$ $h''(t) = -\lambda h(t)$

$g''(\theta) = -\mu g(\theta)$

$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu)f = 0$

see section 7.7.2

* a)

$\mu > 0 \rightarrow g(\theta) = A\cos\sqrt{\mu}\,\theta + B\sin\sqrt{\mu}\,\theta$ $\quad g(0) = 0 = A$

$g(\pi/2) = 0 = B\sin\sqrt{\mu}\,\pi/2 \rightarrow \sqrt{\mu}\,\frac{\pi}{2} = m\pi$

$g(\theta) = B\sin 2m\theta$ $\quad \mu = (2m)^2 \quad m = 1,2,3,\ldots$

$\mu = 0 \rightarrow g(\theta) = A_0\theta + B_0$ $\quad g(0) = 0 = B_0$

$g(\pi/2) = 0 = A_0\pi/2 \rightarrow A_0 = 0$ $\quad$ There are no solutions for $\mu = 0$.

For $\mu < 0$ there are no solutions such that $g(0) = g(\pi/2) = 0$

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Since $\mu = (2m)^2$ then

$\underline{\lambda > 0} \rightarrow f(r) = C_1 J_{2m}(\sqrt{\lambda}\, r) + C_2 Y_{2m}(\sqrt{\lambda}\, r) \qquad |f(0)| < \infty \rightarrow C_2 = 0$

$f(a) = 0 = C_1 J_{2m}(\sqrt{\lambda}\, a) \rightarrow \sqrt{\lambda}\, a = z_{mn} \longrightarrow$ zeros of J_{2m}

$\lambda_{mn} = (z_{mn}/a)^2 \qquad n = 1, 2, 3, \ldots$

$\Rightarrow f(r) = C J_{2m}(\sqrt{\lambda_{mn}}\, r)$

$\lambda > 0 \Rightarrow h''(t) = -\lambda h(t) \rightarrow h(t) = \begin{Bmatrix} \sin c\sqrt{\lambda_{mn}}\, t \\ \cos c\sqrt{\lambda_{mn}}\, t \end{Bmatrix}$

$$\Rightarrow \boxed{u(r,\theta,t) \sim J_{2m}(\sqrt{\lambda_{mn}}\, r) \sin 2m\theta \begin{Bmatrix} \cos c\sqrt{\lambda_{mn}}\, t \\ \sin c\sqrt{\lambda_{mn}}\, t \end{Bmatrix}}$$

$\underline{\lambda = 0} \rightarrow f \sim r^{\pm 2m}$, but $f(a) = 0 \rightarrow f = 0 \rightarrow$ There are no solution for $\lambda = 0$.

Similarly, there are no solutions for $\lambda < 0$.

thus, the frequencies of vibration are $c\sqrt{\lambda_{mn}}$ where $J_{2m}(\sqrt{\lambda_{mn}}\, a) = 0$

b) $\frac{\partial u}{\partial t}(r,\theta,0) = 0 \rightarrow$ implies that all terms with $\sin c\sqrt{\lambda_{mn}}\, t$ vanish.

$$\Rightarrow \boxed{u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1} A_{mn} J_{2m}(\sqrt{\lambda_{mn}}\, r) \sin(2m\theta) \cos c\sqrt{\lambda_{mn}}\, t}$$

$$g(r,\theta,0) = u(r,\theta,0) = \sum_{m=1}^{\infty}\sum_{n=1} A_{mn} J_{2m}(\sqrt{\lambda_{mn}}) \sin(2m\theta)$$

multiplying by $J_{2m'}(\sqrt{\lambda_{m'n'}}) \sin 2m'\theta$ and integrating:

$$\int_0^{\frac{\pi}{2}}\int_0^a g(r,\theta) J_{2m'}(\sqrt{\lambda_{m'n'}}) \sin(2m'\theta)\, r\,dr\,d\theta$$

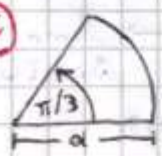
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \int_0^a J_{2m}(\sqrt{\lambda_{mn}}\, r) J_{2m'}(\sqrt{\lambda_{m'n'}}\, r)\, r\,dr \underbrace{\int_0^{\frac{\pi}{2}} \sin 2m\theta \sin 2m'\theta\, d\theta}_{\frac{\pi}{4}\delta_{mm'}}$$

$$= \sum_{n=1}^{\infty} A_{m'n} \pi \underbrace{\int_0^a J_{2m'}(\sqrt{\lambda_{m'n}}\, r) J_{2m'}(\sqrt{\lambda_{m'n'}}\, r)\, r\,dr}_{0 \text{ if } n \neq n'}$$

$$= A_{m'n} \frac{\pi}{4} \int_0^a J_{2m'}^2(\sqrt{\lambda_{m'n'}}\, r)\, r\,dr$$

$$\Rightarrow \boxed{A_{mn} = \frac{4}{\pi}\left[\int_0^{\frac{\pi}{2}}\int_0^a g(r,\theta) J_{2m'}(\sqrt{\lambda_{m'n'}}\, r) \sin(2m'\theta)\, r\,dr\,d\theta \Big/ \underbrace{\int_0^a J_{2m}^2(\sqrt{\lambda_{mn}}\, r)\, r\,dr}_{\frac{a^2}{2} J_{2m+1}^2(\sqrt{\lambda_{mn}}\, a)}\right]}$$

7.7.4



$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \qquad$ let $u = f(r) g(\theta) h(t) \rightarrow h''(t) = -\lambda h(t)$

$g''(\theta) = -\mu g(\theta)$

$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0.$

a) $\underline{\mu > 0} \rightarrow g(\theta) = A\cos\sqrt{\mu}\,\theta + B\sin\sqrt{\mu}\,\theta \qquad g(0) = 0 = A$

$g(\pi/3) = 0 = B\sin\sqrt{\mu}\,\pi/3 \rightarrow \sqrt{\mu}\,\frac{\pi}{3} = m\pi$

$g(\theta) = B\sin(3m\theta) \qquad \mu = (3m)^2 \qquad m = 1, 2, 3, \ldots$

$\underline{\mu = 0} \rightarrow g(\theta) = A_0\theta + B_0 \qquad g(0) = 0 = B_0$

$g(\pi/3) = 0 = A_0\pi/3 \rightarrow A_0 = 0.$

Since $\lambda_{nm} > 0$, the solution for differential equation time dependent is:

$$h(t) = C_{nm} e^{-\lambda_{nm} K t}$$

$$\Rightarrow \boxed{u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} e^{-\lambda_{nm} K t} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)}$$

The initial value proble specify the temperature at $t=0$

$$u(x,y,0) = \alpha(x,y) = \sum_{n,m} C_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right).$$

Multiplying by $\sin(n'\pi x/L)\sin(m'\pi y/L)$ and integrating between 0 and L in x and 0 and H in y, we obtain:

$$\boxed{C_{nm} = \frac{4}{LH} \int_0^L \int_0^H \alpha(x,y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dx\, dy}$$

Where we used orthogonality of trigonometric functions.

Since $\mu=(3m)^2$, then:

$\underline{\lambda>0} \rightarrow f(r)=C_1 J_{3m}(\sqrt{\lambda}r)+C_2 Y_{3m}(\sqrt{\lambda}r)$ $\quad |f(0)|<\infty \rightarrow C_2=0$

$f'(a)=0=C_1 J'_{3m}(\sqrt{\lambda}a) \longrightarrow \sqrt{\lambda_{mn}}\,a=z_{mn} \rightarrow$ zeros of J'_{3m}

$\Rightarrow f(r)=C_1 J_{3m}(\sqrt{\lambda_{mn}}r)$ $\quad \lambda_{mn}=(z_{mn}/a)^2$

$\lambda>0 \rightarrow h(t)=\left\{\begin{matrix}\sin c\sqrt{\lambda_{mn}}t \\ \cos c\sqrt{\lambda_{mn}}t\end{matrix}\right\}$

$\underline{\lambda=0} \rightarrow f\sim r^{\pm 3m}$, $f'(a)=0 \rightarrow m=0 \rightarrow f(r)=C_0$

but m cannot be zero since $m=1,2,3,\ldots$

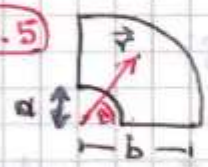
$$\Rightarrow \boxed{u(r,\theta,t)\sim J_{3m}(\sqrt{\lambda_{mn}}r)\sin(3m\theta)\left\{\begin{matrix}\cos c\sqrt{\lambda_{mn}}t \\ \sin c\sqrt{\lambda_{mn}}t\end{matrix}\right\}}$$

b) That is similar to part a) only change the boundary condition on $r=a$:

$f(a)=0=C_1 J_{3m}(\sqrt{\lambda}a) \longrightarrow \sqrt{\lambda_{mn}}\,a=z_{mn} \rightarrow$ zeros of J_{3m}.

The solution is the same above, only that z_{mn} here are the zeros of J_{3m} and not of J'_{3m}.

* 7.7.5



$u=0$ on the entire boundary

By separation of variables $\Rightarrow$ $h''(t)=-\lambda h(t)$, $g''(\theta)=-\mu g(\theta)$, $r\frac{d}{dr}\left(r\frac{df}{dr}\right)+(\lambda r^2-\mu)f=0$

$m=1,2,3\ldots$

Like in exercise 7.7.3: $g(\theta)=B\sin(2m\theta)$, by other hand for $\lambda>0$ we have: $f(r)=C_1 J_{2m}(\sqrt{\lambda}r)+C_2 Y_{2m}(\sqrt{\lambda}r)$

but now we don't have $r=0 \rightarrow C_2$ can be no null.

$f(a)=0=C_1 J_{2m}(\sqrt{\lambda}a)+C_2 Y_{2m}(\sqrt{\lambda}a)$

$f(b)=0=C_1 J_{2m}(\sqrt{\lambda}b)+C_2 Y_{2m}(\sqrt{\lambda}b)$

Nontrivial solutions of that system can be obtained for the coefficients c_1 and c_2 since the determinant

$$J_{2m}(\sqrt{\lambda}a)Y_{2m}(\sqrt{\lambda}b)-J_{2m}(\sqrt{\lambda}b)Y_{2m}(\sqrt{\lambda}a)=0 \rightarrow \lambda_{mn} \text{ are the solutions of this}$$

$\Rightarrow$ Since $\lambda=\lambda_{mn}>0 \rightarrow h(t)=\left\{\begin{matrix}\sin c\sqrt{\lambda_{mn}}t \\ \cos c\sqrt{\lambda_{mn}}t\end{matrix}\right\}$ $\quad n=1,2,3,\ldots$

$$\Rightarrow \boxed{u(r,\theta,t)\sim\left\{\begin{matrix}J_{2m}(\sqrt{\lambda_{mn}}r) \\ Y_{2m}(\sqrt{\lambda_{mn}}r)\end{matrix}\right\}\left\{\begin{matrix}\cos c\sqrt{\lambda_{mn}}t \\ \sin c\sqrt{\lambda_{mn}}t\end{matrix}\right\}\sin(2m\theta)}$$

The frequencies of oscillations are $c\sqrt{\lambda_{mn}}$

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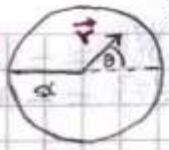
$u(a,\theta,t) = -\frac{\partial u}{\partial r}(a,\theta,t)$

See section 7.7.2

7.7.6



$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$ by separation of variables $\rightarrow$ $h''(t) = -\lambda h(t)$, $g''(\theta) = -\mu g(\theta)$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0.$$

a)

$\mu > 0 \rightarrow g(\theta) = \left\{ \begin{matrix} \sin \sqrt{\mu}\,\theta \\ \cos \sqrt{\mu}\,\theta \end{matrix} \right\}$ $\left.\begin{matrix} g(-\pi) = g(\pi) \\ g'(-\pi) = g(\pi) \end{matrix}\right\} \rightarrow \sqrt{\mu} = m = 1, 2, 3 \ldots$

$\mu = 0 \rightarrow g(\theta) = b\theta + d$ $\quad g(-\pi) = g(\pi) \rightarrow b = 0 \rightarrow g(\theta) = d = d\cos\sqrt{\mu}\,\theta$ if $\sqrt{\mu} = m = 0$.

$\Rightarrow g(\theta) = \left\{ \begin{matrix} \sin m\theta \\ \cos m\theta \end{matrix} \right\}$ $\quad \mu = m^2$, $m = 0, 1, 2, 3 \ldots$

There are not solutions for $\lambda < 0$ that satisfy the B.C.

$\lambda > 0 \rightarrow f(r) = C_1 J_m(\sqrt{\lambda}\,r) + C_2 Y_m(\sqrt{\lambda}\,r)$ $\quad |f(0)| < \infty \rightarrow C_2 = 0$.

$f(a) = -f'(a) \rightarrow J_m(\sqrt{\lambda}\,a) = -J_m'(\sqrt{\lambda}\,a)$

OR $J_m(\sqrt{\lambda}\,a) - J_m'(\sqrt{\lambda}\,a) = 0 \rightarrow$ λ_{mn} are the solutions of this equation.

Since $\lambda = \lambda_{mn} > 0 \rightarrow h(t) = \left\{ \begin{matrix} \sin c\sqrt{\lambda_{mn}}\,t \\ \cos c\sqrt{\lambda_{mn}}\,t \end{matrix} \right\}$

$\lambda = 0 \rightarrow f \sim r^{\pm m} \Rightarrow f = C_0 + C_1 r + C_2 r^2 + \cdots + C_m r^m + C_{m+1} r^{m+1} + \cdots$

$-f' = 0 - C_1 - 2C_2 r \cdots - mC_m r^{m-1} - (m+1)C_{m+1} r^m + \cdots$

$f(a) = -f'(a) \Rightarrow C_0 = -C_1 = 2C_2 = -6C_3 = \cdots = (-1)^m m!\, C_m = \cdots$

If someone $C_i = 0 \Rightarrow$ all other $C_j = 0$ for all j.
If not, then $C_0 = (-1)^m m!\, C_m$ if $C_m \neq 0$ and m can be arbitrary, m can be $m \to \infty \Rightarrow m! \to \infty \Rightarrow C_j = 0 \ \forall j$.

$$\rightarrow \boxed{u(r,\theta,t) \sim J_m(\sqrt{\lambda_{mn}}\,r) \left\{ \begin{matrix} \sin m\theta \\ \cos m\theta \end{matrix} \right\} \left\{ \begin{matrix} \sin c\sqrt{\lambda_{mn}}\,t \\ \cos c\sqrt{\lambda_{mn}}\,t \end{matrix} \right\}}$$

b) Thus the membrane only oscillates with frequencys $c\sqrt{\lambda_{mn}}$.
whit λ_{mn} solutions of $\boxed{J_m(\sqrt{\lambda}\,a) - J_m'(\sqrt{\lambda}\,a) = 0}$

c) $u(r,\theta,0) = 0 \rightarrow$ implies that all terms with $\cos c\sqrt{\lambda_{mn}}\,t$ vanish.

$\frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta \rightarrow$ that is indentically initial condition of problem 7.7.1 and the solution is the same, then:

$$\boxed{u(r,\theta,t) = \sum_{n=1}^{\infty} B_{3n} J_3(\sqrt{\lambda_{3n}}\,r)\sin(3\theta)\sin c\sqrt{\lambda_{3n}}\,t}$$

with $\boxed{B_{3n} = \frac{1}{c\sqrt{\lambda_{3n}}} \frac{\int_0^a \alpha(r) J_3(\sqrt{\lambda_{3n}}\,r)\, r\,dr}{\int_0^a J_3^2(\sqrt{\lambda_{3n}}\,r)\, r\,dr}}$

where λ_{3n} is determined by the solutions of:

$$J_3(\sqrt{\lambda}\,a) - J_3'(\sqrt{\lambda}\,a) = 0.$$

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$u(a,\theta,t)=0$

7.7.7

$$\frac{\partial u}{\partial t} = K\nabla^2 u \qquad \text{Separing variables } u = h(t)\phi(r,\theta) \Rightarrow \quad h'(t) = -\lambda K h(t), \quad \nabla^2\phi = -\lambda\phi(r,\theta)$$

The spatial problem is similar to exercise 7.7.1 with $\phi(a,\theta)=0$

$$\Rightarrow \phi(r,\theta) \sim J_m(\sqrt{\lambda_{mn}}\,r)\begin{Bmatrix}\cos m\theta\\ \sin m\theta\end{Bmatrix} \qquad m=0,1,2,\dots \quad n=1,2,3,\dots$$

$\lambda_{mn} = (z_{mn}/a)^2$ $\quad z_{mn}$ the zeros of J_m.

$$\lambda_{mn}>0 \to h'(t) = -\lambda_{mn}Kh(t) \Rightarrow h(t) = e^{-\lambda_{mn}Kt}$$

$$\Rightarrow \boxed{u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]e^{-\lambda_{mn}Kt}}$$

$$f(r,\theta) = u(r,\theta,0) = \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_{0n}}\,r)A_{0n} + \sum_{m=1}\sum_{n=1} J_m(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]$$

multiplying by $J_0(\sqrt{\lambda_{0n'}}\,r)$ and in $dA = r\,dr\,d\theta$:

$$\int_{-\pi}^{\pi}\int_0^a f(r,\theta)J_0(\sqrt{\lambda_{0n'}}\,r)\,r\,dr\,d\theta = \sum_{n=1}^{\infty} A_{0n}\underbrace{\int_0^a J_0(\sqrt{\lambda_{0n}}\,r)J_0(\sqrt{\lambda_{0n'}}\,r)\,r\,dr}_{0 \text{ if } n\neq n'}\int_{-\pi}^{\pi}d\theta$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^a J_m(\ )J_0(\ )\,r\,dr\left[A_{mn}\underbrace{\int_{-\pi}^{\pi}\cos m\theta\,d\theta}_{0} + B_{mn}\underbrace{\int_{-\pi}^{\pi}\sin m\theta\,d\theta}_{0}\right]$$

$$= A_{0n'}(2\pi)\int_0^a J_0^2(\sqrt{\lambda_{0n'}}\,r)\,r\,dr$$

$$\Rightarrow \boxed{A_{0n} = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi}\int_0^a f(r,\theta)J_0(\sqrt{\lambda_{0n'}}\,r)\,r\,dr\,d\theta \Big/ \underbrace{\int_0^a J_0^2(\sqrt{\lambda_{0n}}\,r)\,r\,dr}_{\frac{a^2}{2}J_1^2(\sqrt{\lambda_{mn}}\,a)}\right]}$$

Now, multiplying by $J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\cos(m'\theta)$ and integrating:

$$\int_{-\pi}^{\pi}\int_0^a f(r,\theta)J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\cos(m'\theta)\,r\,dr\,d\theta = \sum_{n=1}^{\infty}A_{0n}\int_0^a J_0(\ )J_{m'}(\ )\,r\,dr\cdot\underbrace{\int_{-\pi}^{\pi}\cos m'\theta\,d\theta}_{0}$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^a J_m(\ )J_{m'}(\ )\,r\,dr\left[A_{mn}\underbrace{\int_{-\pi}^{\pi}\cos m\theta\cos m'\theta\,d\theta}_{\pi\delta_{mm'}} + B_{mn}\underbrace{\int_{-\pi}^{\pi}\sin m\theta\cos m'\theta\,d\theta}_{0}\right]$$

$$= \sum_{n=1}^{\infty}A_{m'n}\,\pi\underbrace{\int_0^a J_{m'}(\sqrt{\lambda_{m'n}}\,r)J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr}_{0 \text{ if } n\neq n'} = A_{m'n'}\,\pi\int_0^a J_{m'}^2(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr$$

$m'\neq 0$ $\qquad$ $m'\neq 0$

$$\Rightarrow \boxed{A_{mn} = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\int_0^a f(r,\theta)J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)\,r\,dr\,d\theta \Big/ \int_0^a J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr\right]} \quad m\neq 0$$

Similarly

$$\boxed{B_{mn} = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\int_0^a f(r,\theta)J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\,r\,dr\,d\theta \Big/ \int_0^a J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr\right]} \quad m\neq 0$$

$\hookrightarrow \frac{a^2}{2}J_{m+1}^2(\sqrt{\lambda_{mn}}\,a)$

$B_{0n}=0$ or can be arbitrary, it is no matter.

Note that $\lim_{t\to\infty} u = 0$ since $u \sim e^{-\lambda Kt}$ with $\lambda>0$.

$\hookrightarrow$ the temperature will be zero at $t\to\infty$.

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$\frac{\partial u}{\partial r}(a,\theta,t)=0 \rightarrow$ entire boundary insulated.

* **7.7.8**

$$\frac{\partial u}{\partial t} = k\nabla^2 u \quad \text{by separation of variables } u = h(t)\phi(r,\theta) \left\{ \begin{array}{l} h'(t) = -k\lambda h \\ \nabla^2\phi = -\lambda\phi \end{array} \right.$$

The spatial problem is similar to exercise 7.7.2 and its solution is:

For $\lambda > 0 \rightarrow u(r,\theta) \sim J_m(\sqrt{\lambda_{mn}}\, r) \left\{ \begin{array}{l} \cos m\theta \\ \sin m\theta \end{array} \right\}$ $\quad m = 0,1,2,\ldots$, $n = 1,2,3,\ldots$

$\rightarrow \lambda_{mn} = (z_{mn}/a)^2 \quad z_{mn} \rightarrow$ zeros of J_m'

$\lambda > 0 \rightarrow h'(t) = -k\lambda_{mn} h(t) \rightarrow h(t) = e^{-\lambda_{mn} k t}$

For $\lambda = 0 \rightarrow f \sim r^{\pm m}$, but $f'(a) = 0 \rightarrow m = 0 \rightarrow f(r) = c_0$, $g(\theta) = \cos 0 = 1$

$\lambda = 0 \rightarrow h'(t) = 0 \rightarrow h(t) = cte$

$$\Rightarrow \boxed{u(r,\theta,t) = c_0 + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\, r)\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right] e^{-\lambda_{mn} k t}}$$

$$f(r,\theta) = u(r,\theta,0) = c_0 + A_{01}\underbrace{J_0(\sqrt{\lambda_{01}}\, r)}_{1} + \sum_{n=2}^{\infty} J_0(\sqrt{\lambda_{0n}}\, r) A_{0n} + \sum_{m=1}\sum_{n=1} J_m(\sqrt{\lambda_{mn}}\, r)\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]$$

since $\lambda_{01} = 0$ when $z_{01} = 0 \rightarrow$ first zero of J_0'.

$$\Rightarrow \int_{-\pi}^{\pi}\int_0^a f(r,\theta)\, r\,dr\,d\theta = (c_0 + A_{01})\pi a^2 + \sum_{n=2} A_{0n}\int_{-\pi}^{\pi} d\theta \underbrace{\int_0^a J_0(\sqrt{\lambda_{0n}}\, r)\, r\,dr}_{\frac{a}{\sqrt{\lambda_{0n}}} J_1(\sqrt{\lambda_{0n}}\, a) = \frac{-a}{\sqrt{\lambda_{0n}}} \cancelto{0}{J_0'(z_{01})}}$$

$$+ \sum_{m=1}\sum_{n=1}\int_0^a J_m(\,)\, r\,dr\left[A_{mn}\cancelto{0}{\int_{-\pi}^{\pi}\cos m\theta\, d\theta} + B_{mn}\cancelto{0}{\int_{-\pi}^{\pi}\sin m\theta\, d\theta}\right]$$

$$\Rightarrow \boxed{c_0 + A_{01} = \frac{1}{\pi a^2}\int_{-\pi}^{\pi}\int_0^a f(r,\theta)\, r\,dr\,d\theta} \rightarrow$$ average of the initial temperature distribution. We can choose $c_0 = 0$ rescaling cte.

Multiplying $f(r,\theta)$ by $J_0(\sqrt{\lambda_{0n'}})$ and integrating.

$$\int_{-\pi}^{\pi}\int_0^a f(r,\theta) J_0(\sqrt{\lambda_{0n'}}\, r)\, r\,dr\,d\theta = (c_0 + A_{01})\cancelto{0}{\int_0^a J_0(\sqrt{\lambda_{0n'}}\, r)\, r\,dr}\int_{-\pi}^{\pi} d\theta + \sum_{n=2} A_{0n}\underbrace{\int_0^a J_0(\sqrt{\lambda_{0n}}\, r) J_0(\sqrt{\lambda_{0n'}}\, r)\, r\,dr}_{0 \text{ if } n \neq n'}(2\pi)$$

$$+ \sum_{m=1}\sum_{n=1}\int_0^a J_m(\,) J_0(\,)\, r\,dr\left[A_{mn}\cancelto{0}{\int_{-\pi}^{\pi}\cos m\theta\, d\theta} + B_{mn}\cancelto{0}{\int_{-\pi}^{\pi}\sin m\theta\, d\theta}\right]$$

$$= A_{0n'}(2\pi)\int_0^a J_0^2(\sqrt{\lambda_{0n'}}\, r)\, r\,dr \quad n \geq 2 \quad \Rightarrow \boxed{A_{0n} = \frac{1}{2\pi}\,\frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta) J_0(\sqrt{\lambda_{0n}}\, r)\, r\,dr\,d\theta}{\int_0^a J_0^2(\sqrt{\lambda_{0n}}\, r)\, r\,dr}}$$

Now, multiply $f(r,\theta)$ by $J_{m'}(\sqrt{\lambda_{m'n'}}\, r)\cos m'\theta$ and integrate:

$$\int_{-\pi}^{\pi}\int_0^a f(r,\theta) J_{m'}(\sqrt{\lambda_{m'n'}}\, r)\cos m'\theta\,(r\,dr\,d\theta) = (c_0 + A_{01})\int_0^a J_{m'}(\sqrt{\lambda_{m'n'}}\, r)\, r\,dr\cancelto{0}{\int_{-\pi}^{\pi}\cos m'\theta\, d\theta}$$

$$+ \sum_{n=2} A_{0n}\int_0^a J_0(\,) J_{m'}(\,)\, r\,dr\cancelto{0}{\int_{-\pi}^{\pi}\cos m\theta\, d\theta} + \sum_{m=1}\sum_{n=1}\int_0^a J_m(\,) J_{m'}(\,)\, r\,dr\left[A_{mn}\cancelto{\pi\delta_{mm'}}{\int_{-\pi}^{\pi}\cos m\theta\cos m'\theta\, d\theta} + B_{mn}\cancelto{0}{\int_{-\pi}^{\pi}\sin m\theta\cos m'\theta\, d\theta}\right]$$

$$= \sum_{n=1}\pi A_{m'n}\underbrace{\int_0^a J_{m'}(\sqrt{\lambda_{m'n}}\, r) J_{m'}(\sqrt{\lambda_{m'n'}}\, r)\, r\,dr}_{0 \text{ if } n \neq n'}$$

$$= \pi A_{m'n'}\int_0^a J_{m'}^2(\sqrt{\lambda_{m'n'}}\, r)\, r\,dr \Rightarrow \boxed{A_{mn} = \frac{1}{\pi}\,\frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta) J_m(\sqrt{\lambda_{mn}}\, r)\cos m\theta\, r\,dr\,d\theta}{\int_0^a J_m^2(\sqrt{\lambda_{mn}}\, r)\, r\,dr}}$$

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Similarly $\boxed{B_{mn} = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\int_0^a f(r,\theta) J_m(\sqrt{\lambda_{mn}}\, r) \sin m\theta \, (r\,dr\,d\theta) \Big/ \int_0^a J_m^2(\sqrt{\lambda_{mn}}\, r)\, r\,dr\right]}$

For evaluate $\lim_{t\to\infty} u(r,\theta,t)$ not that as $t\to\infty$, $e^{-\lambda_{mn}Kt}\to 0$ except for $m=0$, $n=1$ since $\lambda_{01}=0 \Rightarrow \lim_{t\to\infty} u(r,\theta,t) \to (C_0 + A_{01})$

Thus the temperature will be the average of the initial temperature distribution as $t\to\infty$. Using physical reasoning, the equilibrium should be a constant. By conservation of thermal energy that constant should be the above value, since the boundaries are insulated.

7.7.9

$\frac{\partial^2 u}{\partial t} = K\nabla^2 u$ Let $u = f(r)g(\theta)h(t)$ by separation of variables (Sec 7.7.2) $\rightarrow \begin{cases} h'(t) = -\lambda K h(t) \\ g''(\theta) = -\mu g(\theta) \\ r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0 \end{cases}$

$\underline{\mu > 0} \rightarrow g(\theta) = A\cos\sqrt{\mu}\,\theta + B\sin\sqrt{\mu}\,\theta$

$\underline{\mu = 0} \rightarrow g(\theta) = b\theta + d$

$\underline{\lambda > 0} \rightarrow f(r) = c_1 J_{\sqrt{\mu}}(\sqrt{\lambda}\, r) + \cancel{c_2}^{\,0} Y_{\sqrt{\mu}}(\sqrt{\lambda}\, r)$ since $|f(0)| < \infty$

$h(t) = c e^{-\lambda K t}$

$\underline{\lambda = 0} \rightarrow f(r) \sim r^{\pm\sqrt{\mu}}$

$h(t) = cte$

The solutions are combinations of $f(r)g(\theta)h(t)$.

a) $g(0) = 0 \longrightarrow \begin{cases} A = 0 \\ d = 0 \end{cases}$; $g(\pi) = 0 \longrightarrow \begin{cases} B\sin\sqrt{\mu}\,\pi = 0 \\ b\pi = 0 \rightarrow b = 0. \end{cases} \rightarrow \sqrt{\mu} = m = 1, 2, 3, \ldots$

$f'(a) = 0 \longrightarrow \begin{cases} J_m'(\sqrt{\lambda}\, a) = 0 \rightarrow \sqrt{\lambda}\, a = z_{mn} \text{ OR } \lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2 \\ \frac{d}{dr}(r^{\pm m})\big|_{r=a} = 0 \rightarrow m = 0 \rightarrow \begin{cases} f = cte \\ g = \cos m\theta = 1 \end{cases} \end{cases}$ $z_{mn} \rightarrow$ zeros of $J_m'(z)$. $n = 1, 2, 3, \ldots$

$\rightarrow \boxed{u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} e^{-\lambda_{mn}Kt} J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)}$ $\lim_{t\to\infty} u = 0$

$f(r,\theta) = u(r,\theta,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)$ multiplying by $J_{m'}(\sqrt{\lambda_{m'n'}}\, r)\sin(m'\theta)$ and integrating.

$$\int_0^{\pi}\int_0^a f(r,\theta) J_{m'}(\sqrt{\lambda_{m'n'}}\, r)\sin m'\theta \,(r\,dr\,d\theta)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\int_0^a J_m(\sqrt{\lambda_{mn}}\, r) J_{m'}(\sqrt{\lambda_{m'n'}}\, r)\, r\,dr \underbrace{\int_0^{\pi}\sin(m\theta)\sin m'\theta\, d\theta}_{\frac{\pi}{2}\delta_{mm'}}$$

$$= \sum_{n=1}^{\infty} C_{m'n}\frac{\pi}{2}\underbrace{\int_0^a J_{m'}(\sqrt{\lambda_{m'n}}\, r) J_{m'}(\sqrt{\lambda_{m'n'}}\, r)\, r\,dr}_{0 \text{ if } n\neq n'}$$

$$= C_{m'n'}\frac{\pi}{2}\int_0^a J_{m'}^2(\sqrt{\lambda_{m'n'}}\, r)\, r\,dr$$

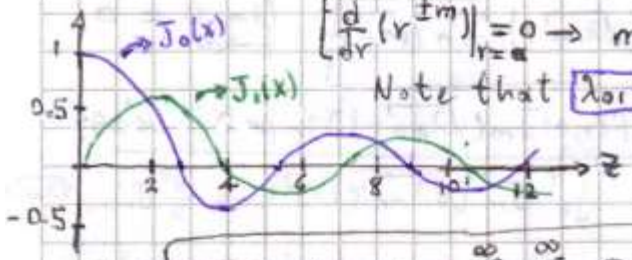
$$\rightarrow \boxed{C_{mn} = \frac{2}{\pi}\left[\int_0^{\pi}\int_0^a f(r,\theta) J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)\, r\,dr\,d\theta \Big/ \int_0^a J_m^2(\sqrt{\lambda_{mn}}\, r)\, r\,dr\right]}$$

$\frac{a^2}{2}\left[1 - \left(\frac{m}{\sqrt{\lambda_{mn}}\, a}\right)^2\right] J_m^2(\sqrt{\lambda_{mn}}\, a)$

b) $g'(0)=0 \rightarrow \begin{cases} B=0 \\ b=0 \end{cases}$; $g'(\pi)=0 \rightarrow \begin{cases} A\sqrt{\mu}\,\sin\sqrt{\mu}\,\pi = 0 \rightarrow \sqrt{\mu} = m = 1,2,3,\ldots \\ d \text{ arbitrary} \rightarrow b = A\cos(m\pi) \text{ since } m=0. \end{cases}$

$\Rightarrow g(\theta) = A\cos(m\theta) \quad m=0,1,2,\ldots$

$f'(a)=0 \rightarrow \begin{cases} J_m'(\sqrt{\lambda}\,a)=0 \rightarrow \sqrt{\lambda}\,a = z \text{ OR } \lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2 \\ \left[\frac{d}{dr}(r^{\pm m})\right]_{r=a} = 0 \rightarrow m=0 \rightarrow \begin{cases} f = cte \\ g = \cos m\theta \end{cases} \end{cases}$ $z_{mn} \rightarrow$ zeros of $J_m'(z)$ $n=1,2,3,\ldots$ when $\lambda=0$.



Note that $\boxed{\lambda_{01}=0}$ since $z_{01}=0 \rightarrow$ First zero of $J_0'(z)$

$J_0(\sqrt{\lambda_{01}}\,r) = J_0(0) = 1$

Remember too $\boxed{J_0'(z) = -J_1(x)}$

$$\Rightarrow \boxed{u(r,\theta,t) = c_0 + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{mn}\, e^{-\lambda_{mn}kt} J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)}$$

$\lim_{t\to\infty} u = c_0 + c_{01}$ since $\lambda_{01}=0$.

$$f(r,\theta) = u(r,\theta,0) = c_0 + C_{01}J_0(\sqrt{\lambda_{01}}\,r) + \sum_{n=2}^{\infty} C_{0n}J_0(\sqrt{\lambda_{0n}}\,r) + \sum_{m=1}\sum_{n=1} C_{mn}J_m(\sqrt{\lambda_{mn}}\,r)\cdot\cos(m\theta)$$

Like in exercise 7.7.8 we can found from here the coefficients manipulating $f(r,\theta)$ doing convenient for integrate but now between $0<\theta<\pi$, $0<r<a$; then:

$$\boxed{C_0 + C_{01} = \frac{2}{\pi a^2}\int_0^{\pi}\int_0^{a} f(r,\theta)\,r\,dr\,d\theta}$$

average of initial temperature distribution.

We can choose $c_0=0$ rescaling.

$$C_{0n} = \frac{1}{\pi}\left[\int_0^{\pi}\int_0^{a} f(r,\theta)J_0(\sqrt{\lambda_{0n}}\,r)\,r\,dr\,d\theta \Big/ \int_0^{a} J_0^2(\sqrt{\lambda_{0n}}\,r)\,r\,dr\right] \qquad n\neq 1$$

$$C_{mn} = \frac{2}{\pi}\left[\int_0^{\pi}\int_0^{a} f(r,\theta)J_m(\sqrt{\lambda_{mn}}\,r)\cos m\theta\,(r\,dr\,d\theta) \Big/ \int_0^{a} J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr\right] \qquad m\neq 0,\ n\neq 1.$$

c) That is similar to b) except that now $f(a)=0$:

$f(a)=0 \rightarrow \begin{cases} J_m(\sqrt{\lambda}\,a)=0 \rightarrow \lambda_{nm} = (z_{mn}/a)^2 \\ c_m a^{\pm m} = 0 \rightarrow c_m = 0. \end{cases}$ with z_{mn} the zeros of $J_m(z)$ $n=1,2,3,\ldots$

$$\rightarrow \boxed{u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{mn}\, e^{-\lambda_{mn}kt} J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)}$$

$\lim_{t\to\infty} u(r,\theta,t) = 0.$

$$f(r,\theta) = u(r,\theta,0) = \sum_{n=1}^{\infty} C_{0n}J_0(\sqrt{\lambda_{0n}}\,r) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)$$

$\rightarrow$ There are not ambiguity in C_{01} since $\lambda_{01}\neq 0$ because the first zero of $J_m(z)$ is $z_{01}\neq 0$

Like in part b):

$$C_{0n} = \frac{1}{\pi}\left[\int_0^{\pi}\int_0^{a} f(r,\theta)J_0(\sqrt{\lambda_{0n}}\,r)\,r\,dr\,d\theta \Big/ \int_0^{a} J_0^2(\sqrt{\lambda_{0n}}\,r)\,r\,dr\right]$$

and $$C_{mn} = \frac{2}{\pi}\left[\int_0^{\pi}\int_0^{a} f(r,\theta)J_m(\sqrt{\lambda_{mn}}\,r)\cos m\theta\,(r\,dr\,d\theta) \Big/ \int_0^{a} J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr\right]$$

Add $(1+\delta_{0m})$ for summarize for all $m\geq 0$

$\frac{a^2}{2}\left[1-\left(\frac{m}{\sqrt{\lambda_{mn}}\,a}\right)^2\right]J_m^2(\sqrt{\lambda_{mn}}\,a)$

d) That is similar to a) except that now $f(a)=0$:

$f(a)=0 \rightarrow f(r) \sim J_m(\sqrt{\lambda_{mn}}\, r)$ with $\lambda_{mn}=(z_{mn}/a)^2$ $\quad z_{mn} \rightarrow$ the zeros of $J_m(z)$

$n=1,2,3,\ldots$

just like in part b).

$$\Rightarrow \boxed{u(r,\theta,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\, e^{-\lambda_{mn}Kt}\, J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)} \qquad \lim_{t\to\infty} u = 0$$

$$C_{mn}=\frac{2}{\pi}\,\frac{\int_0^{\pi}\int_0^{a} f(r,\theta)J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)\, r\,dr\,d\theta}{\int_0^a J_m^2(\sqrt{\lambda_{mn}}\, r)\, r\,dr \rightarrow \frac{a^2}{2}J_{m+1}^2(\sqrt{\lambda_{mn}}\, a)}$$

* 7.7.10 $\dfrac{\partial u}{\partial t}=K\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) \longrightarrow$ Heat equation circularly symmetric.

Let $u=\phi(r)h(t) \rightarrow \phi\dfrac{dh}{dt}=Kh\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{d\phi}{dr}\right)$ dividing by $Kh\phi$:

$$\underbrace{\frac{1}{Kh}\frac{dh}{dt}}_{-\lambda}=\underbrace{\frac{1}{\phi}\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)}_{-\lambda} \Rightarrow \begin{array}{l} h'(t)=-\lambda K h(t) \\ \dfrac{d}{dr}\left(r\dfrac{d\phi}{dr}\right)+\lambda r\phi=0. \end{array}$$

Since $u(a,t)=0 \Rightarrow \phi(a)=0$.

The radial equation is a Sturm-Liouville problem like (5.6.1) with $p(r)=r$, $\sigma(r)=r$ and $q(r)=0$. By the Rayleigh quotient from eq (5.6.3):

Sin $\phi(a)=0 \rightarrow$
$$\lambda=\frac{\cancel{-r\phi(d\phi/dr)\big|_0^a}^{\,0}+\int_0^a (d\phi/dr)^2\, r\,dr}{\int_0^a \phi^2\, r\,dr}>0$$

$\lambda>0$ since $r>0$ and ϕ^2, $(d\phi/dr)>0$.

$\Rightarrow$ We may use the transformation $z=\sqrt{\lambda}\, r \rightarrow \dfrac{d}{dr}=\dfrac{d}{dz}\dfrac{dz}{dr}=\sqrt{\lambda}\dfrac{d}{dz}$

$$\Rightarrow \frac{d}{dr}\left(r\frac{d\phi}{dr}\right)+\lambda r\phi=\sqrt{\lambda}\frac{d}{dz}\left(r\sqrt{\lambda}\frac{d\phi}{dz}\right)+\lambda r\phi=0 \rightarrow \frac{d}{dz}\left(z\frac{d\phi}{dz}\right)+z\phi=0.$$

multiplying by z: $\boxed{z\dfrac{d}{dz}\left(z\dfrac{d\phi}{dz}\right)+(z^2-0^2)=0}$ Bessel equation of zero order.

$\Rightarrow \phi=C_1J_0(z)+C_2Y_0(z)$ but $|\phi(0)|<0 \rightarrow C_2=0$

$\rightarrow \phi=C_1J_0(\sqrt{\lambda}\, r)$ $\quad \phi(a)=0 \rightarrow J_0(\sqrt{\lambda}\, a)=0 \rightarrow \lambda_{0n}=\left(\dfrac{z_{0n}}{a}\right)^2$

with $z_{0n} \rightarrow$ zeros of $J_0(z)$. $\quad n=1,2,3,\ldots$

Since $\lambda>0 \rightarrow h'(t)=-\lambda_{0n}Kt \rightarrow h(t)=e^{-\lambda_{0n}Kt}$

$$\Rightarrow \boxed{u(r,t)=\sum_{n=1} C_n J_0(\sqrt{\lambda_{0n}}\, r)\, e^{-\lambda_{0n}Kt}} \qquad \lim_{t\to\infty} u = 0$$

Equilibrium solution since $u(a,t)=0$ just like in exercise (1.5.10)

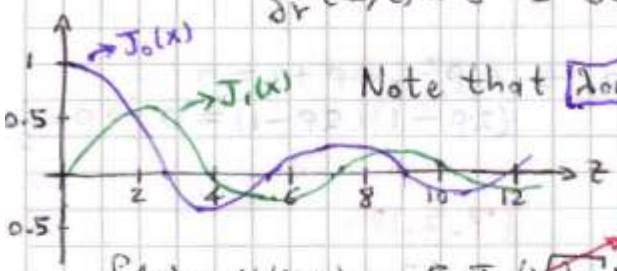
$f(r)=u(r,0)=\sum_{n=1} C_n J_0(\sqrt{\lambda_{0n}}\, r)$ Multiplying by $J_0(\sqrt{\lambda_{0m}}\, r)$

integrating and using orthogonality: $\boxed{C_n=\int_0^a f(r)J_0(\sqrt{\lambda_{0n}}\, r)\, r\,dr \Big/ \int_0^a J_0^2(\sqrt{\lambda_{0n}}\, r)\, r\,dr} \rightarrow \frac{a^2}{2}J_1^2(\sqrt{\lambda_{0n}}\, a)$

7.7.11 The solution is the same as in exercise **7.7.10**,

$$\boxed{u(r,t) = \sum_{n=1} C_n J_0(\sqrt{\lambda_{0n}}\, r)\, e^{-\lambda_{0n} K t}} \qquad \lim_{t\to\infty} u = C_1 \quad \text{since } \lambda_{01} = 0.$$

but now $\frac{\partial u}{\partial r}(a,t) = 0 \rightarrow J_0'(\sqrt{\lambda_{0n}}\, a) = 0 \rightarrow \lambda_{0n} = \left(\frac{z_{0n}}{a}\right)^2$ $\quad z_{0n} \rightarrow$ zeros of $J_0'(z)$.



Note that $\boxed{\lambda_{01} = 0}$ since $z_{01} = 0 \rightarrow$ first zero of $J_0'(z)$

$J_0(\sqrt{\lambda_{01}}\, r) = J_0(0) = 1$

Remember too $J_0'(x) = -J_1(x)$.

$$f(r) = u(r,0) = C_1 \cancel{J_0(\sqrt{\lambda_{01}}\, r)}^{\,1} + \sum_{n=2}^{\infty} C_n J_0(\sqrt{\lambda_{0n}}\, r) \qquad \text{integrating:}$$

$$\int_0^a f(r)\, r\, dr = C_1 \int_0^a r\, dr + \sum_{n=2}^{\infty} C_n \underbrace{\int_0^a J_0(\sqrt{\lambda_{0n}}\, r)\, r\, dr}$$

$$= C_1 \frac{a^2}{2} + \sum_{n=2}^{\infty} C_n \left(\frac{a}{\sqrt{\lambda_{0n}}} J_1(\sqrt{\lambda_{0n}}\, a)\right) = \frac{C_1 a^2}{2} + \sum_{n=2}^{\infty} \frac{C_n a}{\sqrt{\lambda_{0n}}} \left[-\cancel{J_0'(\sqrt{\lambda_{0n}}\, a)}^{\,0}\right]$$

$$\Rightarrow \boxed{C_1 = \frac{2}{a^2} \int_0^a f(r)\, dr}$$

Now, multiplying $f(r)$ by $J_0(\sqrt{\lambda_{0n'}}\, r)$ and integrating:

$$\int_0^a f(r) J_0(\sqrt{\lambda_{0n'}})\, r\, dr = C_1 \cancel{\int_0^a J_0(\sqrt{\lambda_{0n'}}\, r)\, r\, dr}^{\,0} + \sum_{n=2}^{\infty} C_n \underbrace{\int_0^a J_0(\sqrt{\lambda_{0n}}\, r) J_0(\sqrt{\lambda_{0n'}}\, r)\, r\, dr}_{0 \text{ if } n \neq n'}$$

$$= C_1 \int_0^a J_0^2(\sqrt{\lambda_{0n'}}\, r)\, r\, dr \rightarrow \frac{a^2}{2}\left[1 - \left(\frac{0}{\sqrt{\lambda_{0n}}\, a}\right)^2\right] J_0^2(\sqrt{\lambda_{0n}}\, a).$$

$$\Rightarrow \boxed{C_n = \left[\int_0^a f(r) J_0(\sqrt{\lambda_{0n}}\, r)\, r\, dr \Big/ \int_0^a J_0^2(\sqrt{\lambda_{0n}}\, r)\, r\, dr\right]} \qquad n \neq 1$$

7.7.12 * **a)** $x^2 y'' - 6y \approx 0$. Let $y = x^p \rightarrow p(p-1) - 6 = 0$

$p^2 - p - 6 = (p-3)(p+2) = 0$

$\Rightarrow p = -2, 3 \Rightarrow y = c_1 \cdot x^{-2} + c_2 x^3 + \cdots$

b) $x^2 y'' + \frac{3}{16} y \approx 0$ let $y = x^p \rightarrow p(p-1) + \frac{3}{16} = 0 \rightarrow 16p^2 - 16p + 3 = 0$

$\Rightarrow p = \frac{16 \pm \sqrt{256 - 4(16)\cdot 3}}{2(16)} = \frac{16 \pm 8}{16} = 1 \pm \frac{1}{2} \rightarrow y = c_1 x^{1/2} + c_2 x^{3/2} + \cdots$

* **c)** $x^2 y'' + x y' + 4y \approx 0$ let $y = x^p \rightarrow p(p-\cancel{1}) + \cancel{p} + 4 = 0 \rightarrow p = \pm 2i$

$\Rightarrow y = c_1 x^{2i} + c_2 x^{-2i} = c_1 e^{2i \ln x} + c_2 e^{-2i \ln x}$

if $c_2 = c_1 \Rightarrow y = 2c_1 \cos(2\ln x)$
$c_2 = -c_1 \Rightarrow y = 2i \sin(2\ln x)$ $\Big\} \Rightarrow y = A\cos(2\ln x) + B\sin(2\ln x) + \cdots$

d) $x^2y'' + xy' - 4y = 0$ let $y = x^\rho \rightarrow \rho(\rho-\cancel{1}) + \cancel{\rho} - 4 = 0 \rightarrow \rho = \pm 2$

$\rightarrow y = c_1 x^2 + c_2 x^{-2} + \cdots$

* e) $x^2y'' - 4xy' + 6y = 0$, let $y = x^\rho \rightarrow \rho(\rho-1) - 4\rho + 6 = 0$

$\rho^2 - 5\rho + 6 = (\rho-3)(\rho-2) = 0$

$\Rightarrow \rho = 2, 3 \rightarrow y = c_1 x^2 + c_2 x^3 + \cdots$

f) $x^2y'' + \frac{1}{4}y = 0 \rightarrow \rho(\rho-1) + \frac{1}{4} = 0 \rightarrow 4\rho^2 - 4\rho + 1 = 0$

$(2\rho-1)(2\rho-1) = 0 \rightarrow \rho = \frac{1}{2}$

$\Rightarrow y = c_1 x^{1/2} + \cdots$

7.7.13 $r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - m^2)f = 0$ (7.7.18)

$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)f = 0$ comparing this with (5.6.1) $p(r) = r$, $\sigma(r) = r$ and $q(r) = -\frac{m^2}{r}$

$\Rightarrow$ By the Rayleigh quotient (eq. (5.6.3)), we have:

$$\lambda = \frac{-r\phi(d\phi/dr)\Big|_0^a + \int_0^a \left[(d\phi/dr)^2 r + (m^2/r)\phi^2\right]dr}{\int_0^a \phi^2 r\,dr} > 0$$

since $f(a)=0$ 0

$r > 0$ and $m^2 > 0 \Rightarrow \boxed{\lambda > 0}$

By: Herminso Villarraga Gómez

Regular Sturm-Liouville problem

$1 \le r \le 2$

(7.8.1) $\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)f = 0 \qquad f(1)=0,\ f(2)=0 \quad m=0,1,2,\ldots$

a) $\lambda = \dfrac{\cancel{-pf\,df/dr\,\Big|_1^2}^{\,0} + \int_1^2 [p(df/dr)^2 - qf^2]\,dr}{\int_1^2 f^2\sigma\,dr} \rightarrow$ Rayleigh quotient Eq (5.3.8)

$f(1)=f(2)=0$

$p=r,\quad q=-\frac{m^2}{r} \quad \sigma = r \rightarrow$ according to eq (5.3.1)

$$\Rightarrow \lambda = \frac{\int_1^2 \left[r(df/dr)^2 + \frac{m^2}{r}f^2\right]dr}{\int_1^2 f^2 r\,dr} \ge 0 \quad \text{since } r>0$$

If $\lambda = 0 \rightarrow r\left(\frac{df}{dr}\right)^2 + \frac{m^2}{r}f^2 = 0 \rightarrow \frac{df}{dr} = i\frac{m}{r}f(r)$

$\Rightarrow \frac{df}{f} = im\frac{dr}{r} \rightarrow \ln f - \ln C = im\ln r$

$\rightarrow f(r) = C r^{im}$, but $f(1)=0 \rightarrow C=0$.

$\Rightarrow \boxed{\lambda > 0}$

*b) Since $\lambda > 0 \Rightarrow f(r) = C_1 J_m(\sqrt{\lambda}\,r) + C_2 Y_m(\sqrt{\lambda}\,r)$ eq (7.7.37)

$f(1)=0 \rightarrow \quad C_1 J_m(\sqrt{\lambda}) + C_2 Y_m(\sqrt{\lambda}) = 0$

$f(2)=0 \rightarrow \quad C_1 J_m(2\sqrt{\lambda}) + C_2 Y_m(2\sqrt{\lambda}) = 0$

by determinant condition for a nontrivial solution (or by elimination), the eigenvalues are determined by:

$$J_m(\sqrt{\lambda})Y_m(2\sqrt{\lambda}) - J_m(2\sqrt{\lambda})Y_m(\sqrt{\lambda}) = 0.$$

c) Using the Rayleigh quotient minimization principle (5.6.5) with $p=r$, $q=-m^2/r$ and $\sigma=r$:

$$\lambda_1 = \min \frac{\cancel{-rf\,df/dr\,\Big|_1^2}^{\,0} + \int_1^2 \left[r(df/dr)^2 + \frac{m^2}{r}f^2\right]dr}{\int_1^2 f^2 r\,dr}$$

$f(1)=f(2)=0$

Since $r>1 \rightarrow$ the smallest value occurs with $\underline{m=0}$ ✓

*d) Upper and lower bounds for the smallest eigenvalue can be obtained using the technique discussed in Section 5.7.

$$\lambda_1 = \min \frac{\int_1^2 r(df/dr)^2\,dr}{\int_1^2 f^2 r\,dr} \quad \text{when } \underline{m=0}$$

$$\Rightarrow \frac{r_{min}}{r_{max}}\left(\frac{\int_1^2 (df/dr)^2\,dr}{\int_1^2 f^2\,dr}\right)_{min} \le \lambda_1 \le \frac{r_{máx}}{r_{min}}\left(\frac{\int_1^2 (df/dr)^2\,dr}{\int_1^2 f^2\,dr}\right)_{min} \Rightarrow$$

By: Herminso Villarraga Gómez

7.3.1 a) According to solution of excercise 7.2.3 (c) the solution for the two-dimensional heat equation with zero temperature on all sides is:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right) e^{-\lambda_{nm} K t}$$

with $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

and $C_{nm} = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dx\,dy$

Since $\lambda_{nm} > 0$ for all $n,m \geq 1$ $\Rightarrow \lim_{t\to\infty} u(x,y,t) = 0$.

b) From excercise 7.2.2 separating variables in heat equation $\frac{dh}{dt} = -\lambda K h(t)$ and $\nabla^2\phi = -\lambda\phi(x,y)$. Let $\phi(x,y) = f(x)g(y)$

$\Rightarrow$ separating the spatial equation above $\Rightarrow$ $\frac{d^2f}{dx^2} = -\gamma f(x)$

with $\gamma + \delta = \lambda$ (See excercise 7.2.3(a)) $\frac{d^2g}{dy^2} = -\delta g(y)$

but now $f'(0) = f'(L) = 0$ and $g'(0) = g'(H) = 0$. These boundary conditions are homegeneus, then we expect oscillatory solution that exist when $\gamma, \delta > 0$:

$f(x) = A_1\cos\sqrt{\gamma}x + A_2\sin\sqrt{\gamma}x$, $\quad f'(0) = 0 = A_2\sqrt{\gamma} \rightarrow A_2 = 0$

$g(y) = B_1\cos\sqrt{\delta}y + B_2\sin\sqrt{\delta}y$, $\quad g'(0) = 0 = B_2\sqrt{\delta} \rightarrow B_2 = 0$

$\Rightarrow f'(L) = 0 = -A_1\sqrt{\gamma}\sin\sqrt{\gamma}L \longrightarrow \sqrt{\gamma}L = n\pi \rightarrow \gamma_n = (n\pi/L)^2$

$g'(H) = 0 = -B_1\sqrt{\delta}\sin\sqrt{\delta}H \longrightarrow \rightarrow \delta_m = (m\pi/H)^2$

$\Rightarrow \boxed{f_n(x) = A_n\cos\left(\frac{n\pi x}{L}\right)}$; $\boxed{g_m(y) = B_n\cos\left(\frac{m\pi y}{H}\right)}$ $\quad n,m = 1,2,3,\ldots$

There are not solution for $\gamma, \delta < 0$ that satisfy homogeneous boundary conditions.

Clearly $\lambda_{nm} > 0 \longrightarrow \frac{dh}{dt} = -\lambda_{nm}Kh(t) \rightarrow \boxed{h(t) = C_{nm}e^{-\lambda_{nm}Kt}}$

with $\lambda_{nm} = \gamma_n + \delta_m = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

But there are solution for $\gamma = 0, \delta = 0 \Rightarrow$

$$\Rightarrow \frac{1}{2}\min\left(\frac{\int_1^2 (df/dr)^2 dr}{\int_1^2 f^2 dr}\right) \le \lambda_1 \le \frac{2}{1}\min\left(\frac{\int_1^2 (df/dr)^2 dr}{\int_1^2 f^2 dr}\right)$$

We can evaluate the expressions above, since we recognize the minimum of $\int_a^b (du/dx)^2 dx / \int_a^b f^2 dx$ subject to $f(a)=0$ and $f(b)=0$ as the lowest eigenvalue of a different problem: namely, one with constant coefficients

$$\frac{d^2 f}{dx^2} + \bar{\lambda} f = 0 \qquad f(a)=0 \text{ and } f(b)=0.$$

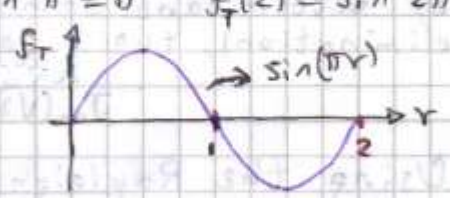
We alredy know that $\bar{\lambda} = [n\pi/(b-a)]^2$ (see section 5.3.3), and hence the lowest eigenvalue for this problem is $\bar{\lambda}_1 = [\pi/(b-a)]^2$. But the minimization property of the Rayleigh quotient implies that $\bar{\lambda}_1 = \min \dfrac{\int_a^b (df/dx)^2 dx}{\int_a^b f^2 dx}$

~~Finally, we have proved that the lowest eigenvalue of our problem~~ Since $x \to r$, $a=1$, $b=2$, $\Rightarrow \lambda_1 = \left(\frac{\pi}{2-1}\right)^2 = \min \dfrac{\int_1^2 (df/dr)^2 dr}{\int_1^2 f^2 dr}$

$$\Rightarrow \frac{1}{2}\left(\frac{\pi}{2-1}\right)^2 \le \lambda_1 \le \frac{2}{1}\left(\frac{\pi}{2-1}\right)^2 \rightarrow \boxed{\frac{1}{2}\pi^2 \le \lambda_1 \le 2\pi^2}$$

e) Let $\boxed{f_{trial} = \sin(r\pi)}$, $f_T(1) = \sin\pi = 0$ $\quad f_T(2) = \sin 2\pi = 0$

We choose trial functions with no zeros in the interior that satisfy the BC's $\hookrightarrow 1<r<2$



$\lambda_1 \le RQ[f_T]$ according to (5.5.6). $\quad m=0$ for lowest value.

$$\Rightarrow \lambda_1 \le \frac{\cancel{-r f_T\, df_T/dr\,\Big|_1^2}^{\,0} + \int_1^2 \left[r(df_T/dr)^2 + \frac{m^2}{r} f_T^2\right] dr}{\int_1^2 f_T^2\, r\, dr}$$

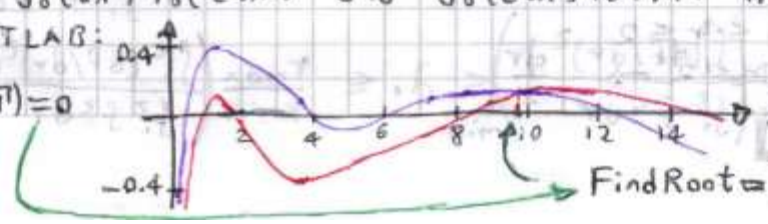
$f_T(1) = f_T(2) = 0$

$$\lambda_1 \le \frac{\int_1^2 r(\pi\cos\pi r)^2 dr}{\int_1^2 \sin^2(\pi r)\, r\, dr} = \frac{3\pi^2/2}{3/2} = \pi^2 = \boxed{9.8696}$$

This is between range found in d)

f) If you Plot $J_0(\sqrt{\lambda})Y_0(2\sqrt{\lambda})$ and $J_0(2\sqrt{\lambda})Y_0(\sqrt{\lambda})$ in Mathematica OR MATLAB:

$J_0(\sqrt{\lambda})Y_0(2\sqrt{\lambda}) - J_0(2\sqrt{\lambda})Y_0(\sqrt{\lambda}) = 0$



FindRoot = $\boxed{9.75332}$

By: Herminso Villarraga Gómez

7.8.2

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

$$u\big|_{\text{boundaries}} = 0$$

$$u(r,\theta,t) = G(r,\theta)$$

Let $u = f(r)g(\theta)h(t)$ by separation of variables: (See section 7.7.2)

$$h'(t) = -\lambda h(t)$$

$$g''(\theta) = -\mu g(\theta)$$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu)f = 0$$

a) Multiplying the last equation for $1/r$, we have:

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{\mu}{r}\right)f = 0$$

Since $u(a,\theta,t) = 0 \rightarrow f(a) = 0$

since $|u(0,\theta,t)| < \infty \rightarrow |f(0)| < \infty$

b) This is a regular Sturm-Liouville problem (See eq 5.3.1), with $p = r$, $q = -\mu/r$ and $\sigma = r$. Using Rayleight Quotient (eq 5.3.8):

$$\lambda = \frac{-rf(df/dr)\big|_0^a + \int_0^a \left[r(df/dr)^2 + \mu f^2/r\right]dr}{\int_0^a f^2 r\,dr}$$

Since $f(a) = 0 \rightarrow$

$$= \frac{\cancel{rf(0)(df/dr)\big|_{r=0}} + \int_0^a \left[r(df/dr)^2 + \mu f^2/r\right]dr}{\int_0^a f^2 r\,dr}$$

0 if there are not singularities at $r \to 0$

Thus if $\mu \geq 0$, $r > 0$ $\rightarrow$ then both numerator and denominator are positive, and hence $\lambda \geq 0$.

If $\lambda = 0 \rightarrow r\left(\frac{df}{dr}\right)^2 + \frac{\mu^2}{r}f^2 = 0 \rightarrow \frac{df}{dr} = i\frac{\mu}{r}f(r)$

$$\Rightarrow \frac{df}{f} = i\mu\frac{dr}{r} \rightarrow \ln f - \ln c = i\mu \ln r$$

$$\rightarrow f(r) = c\,r^{i\mu}, \text{ but } f(a) = 0 \rightarrow c = 0.$$

$\Rightarrow \boxed{\lambda > 0}$

c) Since this is a regular Sturm-Liouville, for each μ, the eigenfunction $f_1(r)$ corresponding to the smallest eigenvalue λ_1 has not zeros for $0 < r < a$.

$\mu > 0 \rightarrow g(\theta) = A\cos\sqrt{\mu}\,\theta + B\sin\sqrt{\mu}\,\theta$

$g(0) = 0 = A$

$g(\pi/2) = 0 = B\sin\sqrt{\mu}\,\pi/2 \rightarrow \frac{\sqrt{\mu}\,\pi}{2} = m\pi$

$g(\theta) = B\sin(2m\theta)$

$\rightarrow \mu = (2m)^2 \quad m = 1, 2, 3, \ldots$

$\mu = 0 \rightarrow g(\theta) = A_0\theta + B_0$

$g(0) = 0 = B_0$

$g(\pi/2) = 0 = A_0\pi/2 \rightarrow A = 0$ There are not Solution for $\mu = 0$

For $\mu < 0$ there are not solutions such that $g(0) = g(\pi/2) = 0$

I don't need exam $\lambda \leq 0$ since in b) I proved $\lambda > 0 \Rightarrow$

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Since $\mu=(2m)^2$ $\longrightarrow$ $f=C_1J_\mu(\sqrt{\lambda}r)+C_2Y_\mu(\sqrt{\lambda}r)$

$\underline{\lambda>0}$ $\rightarrow$ $f(r)=C_1J_{2m}(\sqrt{\lambda}r)+C_2Y_{2m}(\sqrt{\lambda}r)$ $\quad |f(0)|<\infty \rightarrow C_2=0$

$f(a)=0=C_1J_{2m}(\sqrt{\lambda}a) \rightarrow \sqrt{\lambda}a=z_{mn} \rightarrow$ zeros of J_{2m}

$\lambda_{mn}=(z_{mn}/a)^2 \quad n=1,2,3,\ldots$

$\rightarrow f(r)=CJ_{2m}(\sqrt{\lambda_{mn}}r)$

excluding $z=0$ for no trivial solutions. ($J_{2m}(0)=0$ for $m>0$)

If z_{mn} denote the zeros of J_{2m} in creciente order as n such that z_{m1} is the smallest value of z for which $J_{2m}(z)=0$, then $\lambda_{m1}=(z_{m1}/a)^2$ is the smallest value of λ ($\lambda_{m1}\leq\lambda_{mn}$ for all others eigenvalues). Now ~~the eigenfunction~~ we can show that the eigenfunction $J_{2m}(\sqrt{\lambda_{m1}}r)$ corresponding to the smallest eigenvalue λ_{m1} has no zeros for $0<r<a$:

$$J_{2m}(\sqrt{\lambda_{m1}}r)=0 \rightarrow \sqrt{\lambda_{m1}}r=z_{mn}=\sqrt{\lambda_{mn}}a \rightarrow r=\sqrt{\frac{\lambda_{mn}}{\lambda_{m1}}}a\geq a$$

since $\lambda_{m1}\leq\lambda_{mn}$ for all others λ_{mn}. $\Rightarrow f(r)=J_{2m}(\sqrt{\lambda_{m1}}r)$ has no zeros for $0<r<a$.

* d) The h problem is now $h'=-\lambda_{mn}Kh(t) \rightarrow h=e^{-\lambda_{mn}Kt}$

$$\Rightarrow \boxed{u(r,\theta,t)=\sum_{n=1}\sum_{m=1}C_{mn}J_{2m}(\sqrt{\lambda_{mn}}r)\sin(2m\theta)e^{-\lambda_{mn}Kt}}$$

$$G(r,\theta)=u(r,\theta,0)=\sum_{n=1}\sum_{m=1}C_{mn}J_{2m}(\sqrt{\lambda_{mn}}r)\sin(2m\theta)$$

multiplying by $J_{2m'}(\sqrt{\lambda_{m'n'}}r)\sin(2m'\theta)$ and integrating:

$$\int_0^{\pi/2}\int_0^a G(r,\theta)J_{2m'}(\sqrt{\lambda_{m'n'}}r)\sin(2m'\theta)(r\,dr\,d\theta)$$

$$=\sum_{n=1}\sum_{m=1}C_{mn}\int_0^a J_{2m}(\ )J_{2m'}(\ )\,r\,dr\underbrace{\int_0^{\pi/2}\sin(2m\theta)\sin(2m'\theta)\,d\theta}_{(\pi/4)\delta_{mm'}}$$

$$=\sum_{n=1}C_{mn}\frac{\pi}{4}\underbrace{\int_0^a J_{2m'}(\sqrt{\lambda_{mn}}r)J_{2m'}(\sqrt{\lambda_{m'n'}}r)\,r\,dr}_{0 \text{ if } n\neq n'}$$

$$=\frac{\pi}{4}\int_0^a J_{2m'}^2(\sqrt{\lambda_{mn'}}r)\,r\,dr$$

$$\Rightarrow \boxed{C_{mn}=\frac{4}{\pi}\left[\int_0^{\pi/2}\int_0^a G(r,\theta)J_{2m}(\sqrt{\lambda_{mn}}r)\sin(2m\theta)\,r\,dr\,d\theta\Big/\underbrace{\int_0^a J_{2m}^2(\sqrt{\lambda_{mn}}r)\,r\,dr}_{\frac{a^2}{2}J_{2m+1}^2(\sqrt{\lambda_{mn}}a)}\right]}$$

7.8.3 a) Similar to problem 7.8.2 a): $u(r,\theta,t)=f(r)g(\theta)h(t)$

$$\frac{d}{dr}\left(r\frac{df}{dr}\right)+\left(\lambda r-\frac{\mu}{r}\right)f=0 \qquad \begin{array}{l} u(a,\theta,t)=0 \rightarrow f(a)=0 \\ |u(0,\theta,t)|<\infty \rightarrow |f(0)|<\infty \end{array}$$

b) That is similar to 7.8.2 too.

c) $\mu > 0 \rightarrow g(\theta) = A\cos\sqrt{\mu}\,\theta + B\sin\sqrt{\mu}\,\theta$ $\quad g'(0) = 0 = B\sqrt{\mu} \rightarrow B = 0$

$g'(\pi/2) = 0 = -A\sqrt{\mu}\sin\sqrt{\mu}\,\pi/2$

$\rightarrow \mu = (2m)^2 \quad m = 1, 2, 3, \ldots$

$g(\theta) = A\cos(2m\theta)$

$\underline{\mu = 0} \rightarrow g(\theta) = B_0\theta + A_0 \quad \left.\begin{array}{l} g'(0) = 0 = B_0 \\ g'(\pi/2) = 0 = B_0 \end{array}\right\} \Rightarrow g(\theta) = A_0 = A_0\cos(2m\theta)$ if $m = 0$

$\rightarrow g(\theta) = A_m\cos(2m\theta) \quad m = 0, 1, 2, 3, \ldots$

For $\mu < 0$ there are not solutions such that $g'(0) = g'(\pi/2) = 0$

by part b) $\lambda > 0 \rightarrow f(r) = C J_{2m}(\sqrt{\lambda_{mn}}\,r)$ similar to exercise (7.8.2c)
In similar way like in (7.8.2c) $J_{2m}(\sqrt{\lambda_{mn}}\,r)$ has no zeros for $0 < r < a$.

d) Also $h = e^{-\lambda_{mn}\kappa t}$, then: $\quad \lambda_{mn} = (z_{mn}/a)^2 \quad n = 0, 1, 2, 3, \ldots$

$$u(r,\theta,t) = \sum_{n=1}\sum_{m=0} C_{mn} J_{2m}(\sqrt{\lambda_{mn}}\,r)\cos(2m\theta)\, e^{-\lambda_{mn}\kappa t}$$

$$G(r,\theta,0) = u(r,\theta,0) = \sum_{n=1} C_{0n} J_0(\sqrt{\lambda_{0n}}\,r) + \sum_{n=1}\sum_{m=1} C_{mn} J_{2m}(\sqrt{\lambda_{mn}}\,r)\cos(2m\theta)$$

multiplying by $J_0(\sqrt{\lambda_{0n'}}\,r)$ and integrating:

$$\int_0^{\pi/2}\int_0^a G(r,\theta) J_0(\sqrt{\lambda_{0n'}}\,r)\, r\,dr\,d\theta = \sum_{n=1} C_{0n}\frac{\pi}{2}\underbrace{\int_0^a J_0(\sqrt{\lambda_{0n}}\,r) J_0(\sqrt{\lambda_{0n'}}\,r)\, r\,dr}_{0 \text{ if } n \neq n'}$$

$$+ \sum_{n=1}\sum_{m=1} C_{mn}\cancel{\int_0^{\pi/2}\cos(2m\theta)\,d\theta}^{\,0}\int_0^a J_{2m}(\quad) J_0(\sqrt{\lambda_{0n'}}\,r)\, r\,dr$$

$$= C_{0n'}\frac{\pi}{2}\int_0^a J_0^2(\sqrt{\lambda_{0n'}}\,r)\, r\,dr \rightarrow \boxed{C_{0n} = \frac{2}{\pi}\,\frac{\int_0^{\pi/2}\int_0^a G(r,\theta) J_0(\sqrt{\lambda_{0n}}\,r)\, r\,dr\,d\theta}{\int_0^a J_0^2(\sqrt{\lambda_{0n}}\,r)\, r\,dr}}$$

Now multiplying by $J_{2m'}(\sqrt{\lambda_{m'n'}}\,r)\cos(2m'\theta)$ and integrating:

$\underline{m \geq 1}$ $$\boxed{C_{mn} = \frac{4}{\pi}\left[\int_0^{\pi/2}\int_0^a G(r,\theta) J_{2m}(\sqrt{\lambda_{mn}}\,r)\sin(2m\theta)\, r\,dr\,d\theta \Big/ \int_0^a J_{2m}^2(\sqrt{\lambda_{mn}}\,r)\, r\,dr\right]}$$

7.8.4 From section (7.7.7) $\rightarrow f(r) = C J_m(\sqrt{\lambda_{mn}}\,r)$

with $\lambda_{mn} = (z_{mn}/a)^2$, being $z_{mn} \rightarrow$ the zeros of $J_m(z) = 0$.
If we put the zeros z_{mn} in order as n such that

$z_{m1} < z_{m2} < \cdots < z_{m,n-1} < z_{m,n} < \cdots$; then, the nth eigenfunction

$J_m(\sqrt{\lambda_{mn}}\,r)$ has zeros when $J_m(\sqrt{\lambda_{mn}}\,r)=0$

$\Rightarrow \sqrt{\lambda_{mn}}\,r = z_{mn'}$, but $\sqrt{\lambda_{mn}} = (z_{mn}/a)$ then:

$$\frac{z_{mn}}{a}\,r = z_{mn'} \longrightarrow r = \frac{z_{mn'}}{z_{mn}}\,a,$$

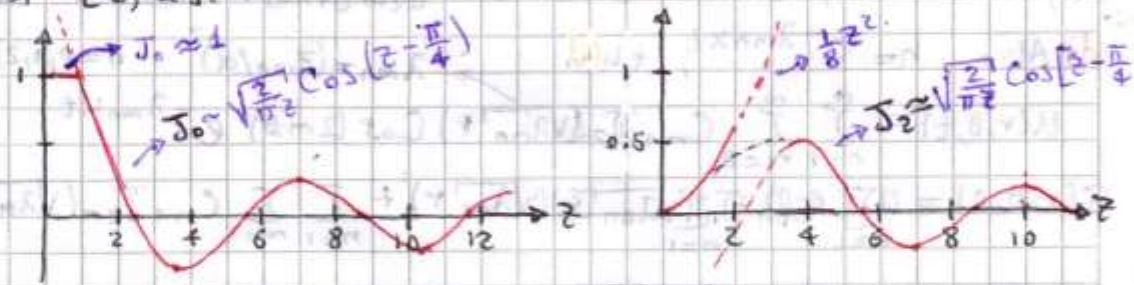
Now $\underline{0 < r < a}$ $\Rightarrow$ $0 < \dfrac{z_{mn'}}{z_{mn}}\,a < a$

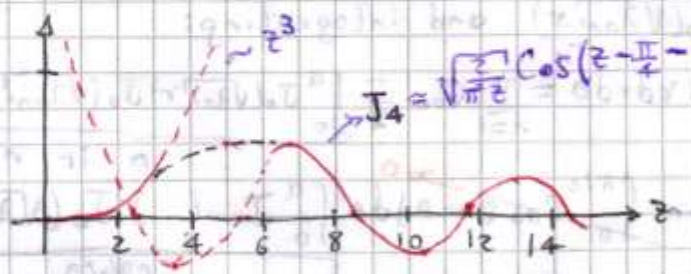
$$0 < \frac{z_{mn'}}{z_{mn}} < 1 \longrightarrow 0 < z_{mn'} < z_{mn}$$

This is true for n' from $n'=1$ to $n'=1$ (Remember $z_{m,n-1} < z_{m,n} < z_{m,n+1}$). Hence $J_m(\sqrt{\lambda_{mn}}\,r)$ has $(n-1)$ zeros on $[0,a]$.

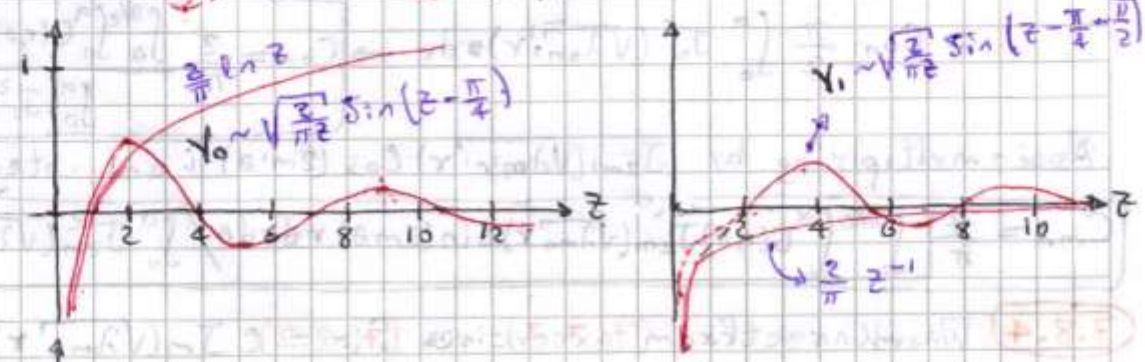
7.8.5

Using (7.8.2) and (7.8.3) $\Rightarrow$

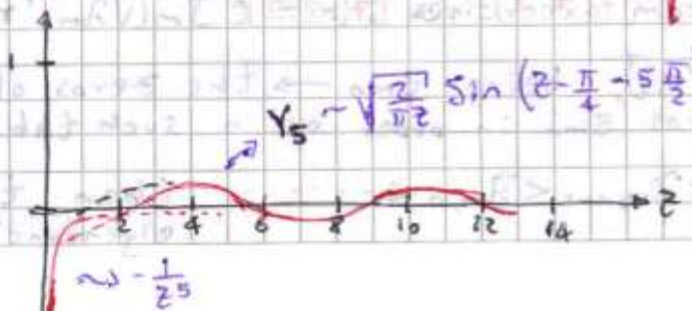




and (7.8.3)



use (7.7.33)



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7.8.6 The solution for a vibrating circular membrane is:

$$u(r,\theta,t) \sim J_m(\sqrt{\lambda_{mn}}\, r)\begin{Bmatrix}\cos m\theta \\ \sin m\theta\end{Bmatrix}\begin{Bmatrix}\cos c\sqrt{\lambda_{mn}}\, t \\ \sin c\sqrt{\lambda_{mn}}\, t\end{Bmatrix} \qquad \text{Eq (7.7.45)}$$

with $\lambda_{mn} = (z_{mn}/a)^2$, $z_{mn} \to$ zeros of $J_m(z)$

The frequencies of vibration are $c\sqrt{\lambda_{mn}}$, thus the large frequencies arise for large z_{mn}'s, then we can use:

$$J_m(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4} - m\frac{\pi}{4}\right) \qquad \text{for large } z. \quad \text{Eq (7.8.3)}$$

The zeros of this function satisfy $z - \frac{\pi}{4} - m\frac{\pi}{4} \approx \left(\frac{2n-1}{2}\right)\pi$

$$\Rightarrow z_{mn} \approx \left(\frac{2n-1}{2}\right)\pi + m\frac{\pi}{4} + \frac{\pi}{2} \qquad n = 1, 2, 3, \ldots$$

Thus, the large frequencies are $\omega_{mn} \approx c\sqrt{\lambda_{mn}} \approx c\, z_{mn}/a$

$$\boxed{\omega_{mn} \approx \frac{c}{a}\left[\left(\frac{2n-1}{2}\right)\pi + m\frac{\pi}{4} + \frac{\pi}{2}\right]}$$

7.8.7

$$\underbrace{z^2\frac{d^2f}{dz^2} + z\frac{df}{dz}} + (z^2 - m^2)f = 0$$

$$z\frac{d}{dz}\left(z\frac{df}{dz}\right) + (z^2 - m^2)f = 0 \qquad \text{Let } f = y/z^{1/2}$$

$$\Rightarrow \underbrace{z\frac{d}{dz}\left(z\left(\frac{dy}{dz}\frac{1}{z^{1/2}} - \frac{y}{2z^{3/2}}\right)\right)} + \left(z^{3/2} - \frac{m^2}{z^{1/2}}\right)y = 0$$

Now

$$= z\frac{d}{dz}\left(\frac{dy}{dz}z^{1/2} - \frac{y}{2z^{1/2}}\right) = z\left(\frac{d^2y}{dz^2}z^{1/2} + \cancel{\frac{dy}{dz}\frac{1}{2z^{1/2}}} - \cancel{\frac{dy}{dz}\frac{1}{2z^{1/2}}} + y/4z^{3/2}\right)$$

$$\Rightarrow \frac{d^2y}{dz^2}z^{3/2} + \frac{y}{4z^{1/2}} + \left(z^{3/2} - \frac{m^2}{z^{1/2}}\right)y = 0 \qquad \text{dividing by } z^{3/2}.$$

$$\frac{d^2y}{dz^2} + \frac{1}{4z^2}y + \left(1 - \frac{m^2}{z^2}\right)y = 0$$

oR

$$\boxed{\frac{d^2y}{dz^2} + y\left(1 + \frac{1}{4}z^{-2} - m^2 z^{-2}\right) = 0}$$

* **7.8.8** For $m = 1/2$, the differential equation above (based in exercise 7.8.7) is:

$$\frac{d^2y}{dz^2} + y\left(1 + \cancel{\frac{1}{4}z^{-2}} - \cancel{m^2 z^{-2}}\right) = 0 \rightarrow \frac{d^2y}{dz^2} + y = 0 \rightarrow \begin{matrix} y_1 = c_1 \sin z \\ y_2 = c_2 \cos z \end{matrix}$$

from exercise (7.8.7) $f = \dfrac{y}{z^{1/2}} \longrightarrow \begin{matrix} f_1 = (c_1/z^{1/2})\sin z \\ f_2 = (c_2/z^{1/2})\cos z \end{matrix}$

Note that $\lim_{z\to 0} f_1 = C_1 \lim_{z\to 0} \frac{\sin z}{z^{1/2}} = C_1 \lim_{z\to 0} \frac{\cos z}{\frac{1}{2} z^{-1/2}} =$

$$= 2C_1 \lim_{z\to 0} \sqrt{z}\cos z = 0.$$

and $\lim_{z\to 0} f_2 = C_2 \lim_{z\to 0} \frac{\cos z}{z^{1/2}} = \pm\infty$ (depending of sign of C_2)

Since $Y_{1/2}$ approaches $\pm\infty$ as $z\to 0$ and $J_{1/2}$ appraches to zero as $z\to 0$:

$$J_{1/2}(z) = f_1(z) = (C_1/z^{1/2})\sin z$$
$$Y_{1/2}(z) = f_2(z) = (C_2/z^{1/2})\cos z.$$

Eq. (7.8.3) says $J_{1/2} \sim \sqrt{\frac{2}{\pi z}}\sin(z-\pi/2)$ as $z\to\infty$

$$= \sqrt{\frac{2}{\pi z}}\cos(z-\pi/2) = \sqrt{\frac{2}{\pi z}}\sin z$$

and $Y_{1/2} \sim \sqrt{\frac{2}{\pi z}}\sin(z-\pi/2) = -\sqrt{\frac{2}{\pi z}}\cos z$ as $z\to\infty$

$\Rightarrow C_1 = \sqrt{2/\pi}$ and $C_2 = -\sqrt{2/\pi} \Rightarrow$ $\boxed{\begin{array}{l} J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z \\ Y_{1/2}(z) = -\sqrt{\frac{2}{\pi z}}\cos z \end{array}}$

Eq. (7.7.33) says for small z ($z\to 0$):

$J_{1/2}(z) \sim \frac{1}{z^{1/2}(1/2)!} z^{1/2}$ and $Y_{1/2}(z) \sim -\frac{2^{1/2}(-1/2)!}{\pi} z^{-1/2}$

Now $(1/2)! = \frac{1}{2}\sqrt{\pi}$ and $(-1/2)! = \sqrt{\pi}$. Thus for small z:

$J_{1/2}(z) \sim \sqrt{2/\ }$ and $Y_{1/2}(z) \sim -\sqrt{2/\pi z}$ for $z\to 0$.

Since $\sin z \sim z$ and $\cos z \sim 1$ for small z, our derived expressions for $J_{1/2}$ an $Y_{1/2}$ give for small z:

$J_{1/2}(z) \sim \sqrt{2z/\pi}$ and $Y_{1/2}(z) \sim -\sqrt{2/\pi z}$ This verifies the eq. (7.7.33)

7.8.9 By exercise (7.8.7) we can use $f = y/z^{1/2}$. Let

$$y(z) = \sqrt{\frac{2}{\pi}}\cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) \rightarrow y'(z) = -\sqrt{\frac{2}{\pi}}\sin\left(z - \frac{\pi}{4} - m\pi/2\right)$$

$$y''(z) = -\sqrt{\frac{2}{\pi}}\cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) = -y(z)$$

$$\Rightarrow \underbrace{\frac{d^2y}{dz^2} + y\left(1 + \frac{1}{4}z^{-2} - m^2 z^{-2}\right)}_{\text{Bessel's equation in form of excersice (7.8.7).}} = -y(z) + y(z)\left(1 + \cancel{\frac{1}{4z^2}}^{\,0 \text{ as } z\to\infty} - \cancel{\frac{m^2}{z^2}}^{\,0 \text{ as } z\to\infty}\right)$$

$\simeq 0$ Thus $y(z)$ satisfy the Bessel equation for $z\to\infty$

$\Rightarrow f(z) = \frac{1}{z^{1/2}}\sqrt{\frac{2}{\pi}}\cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)$ is solution of Bessel's eq verifying (7.8.3).

By similar way, let now $y(z) = \sqrt{\frac{z}{\pi}} \sin\left(z - \frac{\pi}{4} - m\pi/2\right)$

$$y'(z) = \sqrt{\frac{z}{\pi}} \cos\left(z - \pi/4 - m\pi/2\right)$$

$$y''(z) = -\sqrt{\frac{z}{\pi}} \sin\left(z - \pi/4 - m\pi/2\right) = -y(z).$$

$$\Rightarrow \quad y''(z) + y\left(1 - \frac{1}{4z^2} - \frac{m^2}{z^2}\right) = -y(z) + y\left(1 - \underbrace{\frac{1}{4z^2}}_{\to 0 \text{ as } z\to\infty} - \underbrace{\frac{m^2}{z^2}}_{\to 0 \text{ as } z\to\infty}\right)$$

≈ 0 Thus $y(z)$ satisfy the Bessel's eq for $z \to \infty$.

$\Rightarrow f(z) = \frac{1}{z^{1/2}} \sqrt{\frac{z}{\pi}} \sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)$ is solution of Bessel's eq too, verifying (7.8.3).

7.8.10 a) $\frac{d^2y}{dz^2} + y\left(1 + \frac{1}{4}z^{-2} - m^2 z^{-2}\right) = 0$ substituing in $y = e^{iz} w(z)$

$$\frac{d^2}{dz^2} e^{iz} w(z) = \frac{d}{dz}\frac{d}{dz} e^{iz} w(z) = \frac{d}{dz}\left[e^{iz}\frac{dw}{dz} + ie^{iz} w(z)\right]$$

$$= e^{iz}\frac{d^2w}{dz^2} + ie^{iz}\frac{dw}{dz} + ie^{iz}\frac{dw}{dz} + i\cdot i e^{iz} w(z)$$

$$= e^{iz}\frac{d^2w}{dz^2} + 2ie^{iz}\frac{dw}{dz} - e^{iz} w(z),$$ sustituin above:

$$e^{iz}\frac{d^2w}{dz^2} + 2ie^{iz}\frac{dw}{dz} - e^{iz} w(z) + e^{iz} w(z)\left(1 + \frac{1}{4}z^{-2} - m^2 z^{-2}\right) = 0.$$

dividing by e^{iz} $\Rightarrow$ $\boxed{\frac{d^2w}{dz^2} + 2i\frac{dw}{dz} + \frac{\gamma}{z^2} w = 0}$ where $\gamma = \frac{1}{4} - m^2$

b) Substituting $w = \sum_{n=0}^{\infty} \beta_n z^{-n}$ in the last equation:

$$\sum_{n=0}^{\infty}\left(-n(-n-1)\beta_n z^{-n-2}\right) + 2i\sum_{n=0}^{\infty}\left(-n\beta_n z^{-n-1}\right) + \gamma\sum_{n=0}^{\infty}\beta_n z^{-n-2} = 0.$$

$$\sum_{n=1}^{\infty} n(n+1)\beta_n z^{-n-2} - 2i\sum_{n=1}^{\infty} n\beta_n z^{-n-1} + \gamma\sum_{n=0}^{\infty}\beta_n z^{-n-2} = 0.$$

$$\sum_{n=1}^{\infty} n(n+1)\beta_n z^{-n-2} - 2i\sum_{n=0}^{\infty}(n+1)\beta_{n+1} z^{-n-2} + \gamma\sum_{n=0}^{\infty}\beta_n z^{-n-2} = 0$$

$$\left[\gamma\beta_0 - 2i\beta_1\right] z^{-2} + \sum_{n=1}^{\infty}\left[n(n+1)\beta_n - 2i(n+1)\beta_{n+1} + \gamma\beta_n\right] z^{-n-2} = 0$$

Assuming that $\beta_0 = 1$, we have $(\gamma - 2i\beta_1) = 0 \to \boxed{\beta_1 = -\frac{i}{2}\gamma = \frac{im^2}{2} - \frac{i}{8}}$

$1(1+1)\beta_1 - 2i(1+1)\beta_2 + \gamma\beta_1 = 0. \to (2+\gamma)\beta_1 - 4i\beta_2 = 0 \to \beta_2 = -\frac{i}{4}(2+\gamma)\beta_1$

$$\boxed{\beta_2 = -\frac{1}{8}(2+\gamma)\gamma}$$

$2(2+1)\beta_2 - 2i(2+1)\beta_3 + \gamma\beta_2 = 0$

$(6+\gamma)\beta_2 - 6i\beta_3 = 0 \Rightarrow \beta_3 = -\frac{i}{6}(6+\gamma)\beta_2 \to \boxed{\beta_3 = \frac{i}{48}(6+\gamma)(2+\gamma)\gamma}$

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c) The solution to Bessel's differential equations are: (as in exercise 7.8.7)

$$f = \frac{Y}{z^{1/2}} = \frac{e^{iz}}{z^{1/2}} W(z) = \frac{e^{iz}}{z^{1/2}} \sum_{n=0}^{\infty} \beta_n z^{-n} = e^{iz} \sum_{n=0}^{\infty} \beta_n z^{-n-1/2}$$

for large z, we may neglect all terms except β_0 and β_1.
Thus for large z: $\boxed{f \sim e^{iz} z^{-1/2} - e^{iz}\left(\frac{im^2}{2} - \frac{i}{8}\right) z^{-3/2}}$

d) Since we must have from part b)

$$n(n+1)\beta_n - 2i(n+1)\beta_{n+1} + \gamma\beta_1 = 0 \quad \text{for all } n \geq 1$$

$$\beta_{n+1} = -\frac{i}{2}\left(n + \frac{\gamma}{n+1}\right)\beta_n \quad \text{for all } n \geq 1$$

Then, the recursion formula is as follows:

$$\beta_0 = 1, \quad \beta_1 = -\frac{i}{2}\gamma \ldots \boxed{\beta_n = -\frac{i}{2}\left(n-1+\frac{\gamma}{n}\right)\beta_{n-1}} \quad \text{for } n \geq 2$$

where $\gamma = \frac{1}{4} - m^2$

Since $\gamma = \frac{1}{4} - m^2$, γ is some fixed finite number.
Thus $(n-1)n$ will be greater than $|\gamma|$ for some $n = N$ and hence for all $n \geq N$. Thus for all $n \geq N$ the absolute value of $-\frac{i}{2}(n-1+\gamma/n)$ will continue to increase.
Thus for large n, β_n will equal a number of large absolute value times β_{n-1} and β_{n+1} will equal a number of large absolute value times β_n and so on. Obviously then the series will diverge.

7.8.11 $\frac{d^2f}{dz^2} + \left(1 - \frac{m^2}{z^2}\right)f(z) = -\frac{1}{z}\frac{df}{dz}$ Bessel's eq in (7.8.1).

Let $x = 1/z \rightarrow \frac{df}{dz} = -\frac{1}{z^2}\frac{df}{dx}$, $\frac{d^2f}{dz^2} = \frac{1}{z^3}\frac{df}{dx} + \frac{1}{z^4}\frac{d^2f}{dx^2}$

$$\Rightarrow \cancel{\frac{1}{z^3}\frac{df}{dx}} + \frac{1}{z^4}\frac{d^2f}{dx^2} + \left(1 - \frac{m^2}{z^2}\right)f(z) = \cancel{-\frac{1}{z}\left(-\frac{1}{z^2}\frac{df}{dx}\right)}$$

$$z = 1/x \Rightarrow x^4\frac{d^2f}{dx^2} + (1 - x^2m^2)f(x) = 0 \rightarrow \boxed{\frac{d^2f}{dx^2} + \frac{(1/x^2 - m^2)}{x^2}f(x) = 0}$$

$\Rightarrow x = 0$ is a singular point, and according to (7.8.7)

$$b(x) = \frac{S(x)}{x^2} = \frac{1/x^2 - m^2}{x^2} \rightarrow S(x) = \frac{1}{x^2} - m^2, \text{ by Taylor series:}$$

$$S(x) = S(0) + S'(0)\cdot x + \frac{1}{2}S''(0)\cdot x^2 + \mathcal{O}(x^3)$$

but $S(0)$ and $S'(0)\cdot x$ are undefined $\Rightarrow$ $S(x)$ has not Taylor expansion around $x = 0 \Rightarrow$ Irregular singular point.

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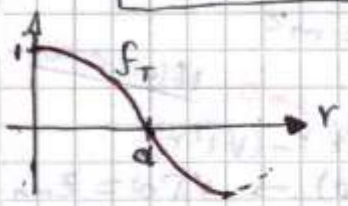
→ Regular Sturm-Liouville problem

7.8.12 $\frac{d}{dr}\left(r\frac{df}{dr}\right)+\left(\lambda r-\frac{m^2}{r}\right)f=0 \qquad f(a)=0, \quad |f(0)|<\infty$

according to (5.3.1) $p=r$, $q=-m^2/r$ and $\sigma=r$. Then by Rayleigh quotient (5.3.8):

$$\lambda=\frac{-pf\,df/dr\,\big|_0^a+\int_0^a\left[p(df/dr)^2-qf^2\right]dr}{\int_0^a f^2\sigma\,dr}$$

Let $\boxed{f_{Trial}=\cos\left(\frac{\pi}{2a}r\right)}$ such that $f_T(a)=\cos\frac{\pi}{2}=0$.



We choose trial function with no zeros between $0<r<a$ that satisfy BC's.

$\lambda_1\le RQ[f_T]$ according to (5.5.6)

$$\Rightarrow \lambda_1\le\frac{-rf_T\,df_T/dr\,\big|_0^a+\int_0^a\left[r(df_T/dr)^2+\cancel{\frac{m^2}{r}f_T^2}\right]dr}{\int_0^a f_T^2\,r\,dr}$$

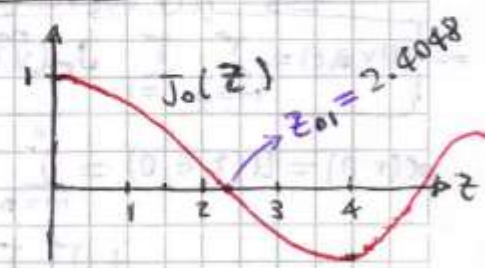
$m=0$ according to the excercise

$$\lambda_{01}\le\frac{\left\{\cancel{-r\cos\left(\frac{\pi}{2a}r\right)\left[-\frac{\pi}{2a}\sin\left(\frac{\pi r}{2a}\right)\right]}\right\}\Big|_0^a+\int_0^a r\left[-\frac{\pi}{2a}\sin\left(\frac{\pi r}{2a}\right)\right]^2dr}{\int_0^a r\left(\cos\frac{\pi r}{2a}\right)^2dx}$$

$$\lambda_{01}\le\frac{\frac{1}{16}(4+\pi^2)}{a^2\frac{(-4+\pi^2)}{4\pi^2}}=\frac{\pi^2(4+\pi^2)}{4a^2(\pi^2-4)}=\boxed{\frac{5.83035}{a^2}}$$

For compare with the exact answer,

$$\boxed{\lambda_{01}=\left(\frac{z_{01}}{a}\right)^2=\frac{5.78306}{a^2}}$$



7.8.13

If $\gamma=0,\ \delta=0$: $\rightarrow f''(x)=0 \rightarrow f(x)=A_2x+A_0,\quad f'(0)=0=A_2$

$g''(y)=0 \rightarrow g(y)=B_2x+B_0,\quad g'(0)=0=B_2$

$\Rightarrow \boxed{f_0(x)=A_0}\quad \boxed{g_0(y)=B_0}$ and $\lambda=\gamma+\delta=0$

$\lambda=0 \rightarrow \frac{dh}{dt}=0 \rightarrow \boxed{h(t)=C_0}$

but there are other posibilities:

If $\gamma=0,\ \delta>0 \rightarrow \boxed{f_0(x)=A_0},\ \boxed{g_m(y)=B_m\cos\left(\frac{m\pi y}{H}\right)}$

$\lambda=\delta_m=\left(\frac{m\pi}{H}\right)^2>0 \rightarrow \frac{dh}{dt}=-\delta_m K h(t) \rightarrow \boxed{h(t)=C_m e^{-\lambda_{0m}Kt}}\quad \lambda_{0m}=\left(\frac{m\pi}{H}\right)$

$m=1,2,3,\ldots$

If $\gamma>0,\ \delta=0 \rightarrow \boxed{f_n(x)=A_n\cos\left(\frac{n\pi x}{L}\right)}$ and $\boxed{g_0(y)=B_0}$

$\lambda=\gamma_n=\left(\frac{n\pi}{L}\right)^2>0 \rightarrow \frac{dh}{dt}=-\gamma_n K h(t) \rightarrow \boxed{h(t)=C_n e^{-\lambda_{n0}Kt}}\quad \lambda_{n0}=\left(\frac{n\pi}{L}\right)$

$n=1,2,3,\ldots$

The general solution is the superposition of solutions of all cases knowing that $u(x,y,t)=h(t)\phi(x,y)=h(t)f(x)g(y)$ in each case:

$$\Rightarrow u=A_0B_0C_0+\sum_{n=1}^{\infty}A_nB_0C_{n0}e^{-\lambda_{n0}Kt}\cos\left(\frac{n\pi x}{L}\right)+\sum_{m=1}^{\infty}A_0B_mC_{0m}e^{-\lambda_{0m}Kt}\cos\left(\frac{m\pi y}{H}\right)$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_nB_mC_{nm}e^{-\lambda_{nm}Kt}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$$

$$\boxed{u=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_{nm}e^{-\lambda_{nm}Kt}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)}$$

For $t\rightarrow\infty$ $u=C_{00}$

Since $f(x,y)=u(x,y,0)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_{nm}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{L}\right)$

OR $f(x,y)=C_{00}+\sum_{n=1}^{\infty}C_{n0}\cos\left(\frac{n\pi x}{L}\right)+\sum_{m=1}^{\infty}C_{0m}\cos\left(\frac{m\pi y}{H}\right)+\sum_{n,m=1}^{\infty}C_{nm}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$

Integrating $\int_0^H\int_0^L f(x,y)\,dx\,dy=C_{00}\underbrace{\int_0^H\int_0^L dx\,dy}_{LH}+\sum C_{n0}\cancel{\iint}^{\,0}+\sum\cancel{\iint}^{\,0}+\cancel{\ }^{\,0}$

$\Rightarrow \boxed{C_{00}=\frac{1}{LH}\int_0^H\int_0^L f(x,y)\,dx\,dy} \rightarrow$ The average of the initial temperature distribution.

Multiplying by $\cos\left(\frac{n'\pi x}{L}\right)$ and integrating

$$\int_0^H\int_0^L f(x,y)\cos\left(\frac{n'\pi x}{L}\right)dx\,dy=C_{00}\cancel{\int_0^L\cos\frac{n'\pi x}{L}dx}^{\,0}\int_0^H dy+\sum_{n=1}^{\infty}C_{n0}\underbrace{\int_0^L\cos\left(\frac{n\pi x}{L}\right)\cos\frac{n'\pi x}{L}}_{(L/2)\delta_{nn'}}\underbrace{\int_0^H dy}_{H}$$

$$+\sum_{m=1}^{\infty}C_{0m}\cancel{\int_0^L\cos\frac{n'\pi x}{L}dx}^{\,0}\underbrace{\int_0^H\cos\left(\frac{m\pi y}{L}\right)dy}_{0}+\sum_{n,m=1}^{\infty}C_{nm}\underbrace{\int_0^L\cos\left(\frac{n\pi x}{L}\right)\cos\frac{n'\pi x}{L}dx}_{(L/2)\delta_{nn'}}\cancel{\int_0^H\cos\frac{m\pi y}{H}dy}^{\,0}$$

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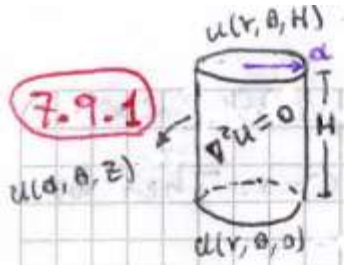
7.9.1



$\nabla^2 u = 0 \rightarrow$ Let $u = f(r)g(\theta)h(z) \Rightarrow$ $\boxed{h'' = \lambda h(z) \quad g'' = -\mu g(\theta) \quad r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu^2)f = 0}$

$\mu > 0 \rightarrow g(\theta) = \left\{ \begin{matrix} \sin \sqrt{\mu}\,\theta \\ \cos \sqrt{\mu}\,\theta \end{matrix} \right\}, \quad \begin{matrix} g(-\pi) = g(\pi) \\ g'(-\pi) = g'(\pi) \end{matrix} \Rightarrow \sqrt{\mu} = m = 1, 2, 3, \ldots$

$\mu = 0 \rightarrow g(\theta) = c_1\theta + c_0 \qquad g(-\pi) = g(\pi) \rightarrow g(\theta) = c_0 = \cos m\theta$ if $m = 0$

$\rightarrow \boxed{g(\theta) = \left\{ \begin{matrix} \sin m\theta \\ \cos m\theta \end{matrix} \right\}} \qquad m = 0, 1, 2, 3, \ldots \qquad \mu = m^2$

a) For Bessel's eq solution is $\lambda > 0$ $\quad f(r) = c_1 J_m(\sqrt{\lambda}\, r) + \cancel{c_2 Y_m(\sqrt{\lambda}\, r)} \rightarrow 0$ $\quad |f(0)| < \infty$

$u(a, \theta, z) = 0 \rightarrow \quad f(a) = 0 = c_1 J_m(\sqrt{\lambda}\, a) \rightarrow \sqrt{\lambda}\, a = z_{mn}$

$\Rightarrow \boxed{f(r) = C J_m(\sqrt{\lambda_{mn}}\, r)} \qquad \lambda_{mn} = (z_{mn}/a)^2, \quad z_{mn} \rightarrow$ zeros of $J_m(z)$

$n = 1, 2, 3, 4, \ldots$

if $\lambda > 0 \rightarrow h'' = \lambda h(z) \Rightarrow h = A\cosh[\sqrt{\lambda}(z - H)] + B\sinh[\sqrt{\lambda}(z - H)]$

$\therefore u(r, \theta, H) = 0 \rightarrow h(H) = 0 \rightarrow A = 0 \rightarrow \boxed{h(z) = B\sinh[\sqrt{\lambda}(z - H)]}$

For $\lambda = 0 \rightarrow f \sim r^{\pm m}$, but $f(a) = 0 \rightarrow f = 0 \rightarrow$ There are not solution

For $\lambda < 0$ there are not solution that satisfy the BC's.

OR $\mu < 0$

$$\rightarrow \boxed{u(r, \theta, z) = \sum_{m=0} \sum_{n=1} J_m(\sqrt{\lambda_{mn}}\, r) \sinh[\sqrt{\lambda_{mn}}(z - H)] [A_{mn} \sin m\theta + B_{mn} \cos m\theta]}$$

$$\alpha(r, \theta) = u(r, \theta, 0) = \sum_{n=1} J_0(\sqrt{\lambda_{0n}}\, r) \sinh(-\sqrt{\lambda_{0n}}\, H) B_{0n}$$

$$+ \sum_{m=1} \sum_{n=1} J_{mn}(\sqrt{\lambda_{mn}}\, r) \sinh(-\sqrt{\lambda_{mn}}\, H) [A_{mn} \sin m\theta + B_{mn} \cos m\theta]$$

Multiplying by $J_0(\sqrt{\lambda_{0n'}}\, r)$ and integrating:

$$\int_{-\pi}^{\pi} \int_0^a \alpha(r, \theta) J_0(\sqrt{\lambda_{0n'}}\, r)\, r\, dr\, d\theta = \sum_{n=1} B_{0n} \sinh(-\sqrt{\lambda_{0n}}\, H)(2\pi) \underbrace{\int_0^a J_0(\sqrt{\lambda_{0n}}\, r) J_0(\sqrt{\lambda_{0n'}}\, r)\, r\, dr}_{0 \text{ if } n \neq n'}$$

$$+ \sum_{m=1} \sum_{n=1} \sinh(-\sqrt{\lambda_{mn}}\, H) \int_0^a J_{mn}(\ )\, J_0(\ )\, r\, dr \left[A_{mn} \cancel{\int_{-\pi}^{\pi} \sin m\theta\, d\theta} + B_{mn} \cancel{\int_{-\pi}^{\pi} \cos m\theta\, d\theta} \right] \rightarrow 0$$

$$= B_{0n'} \sinh(-\sqrt{\lambda_{0n'}}\, H)(2\pi) \int_0^a J_0^2(\sqrt{\lambda_{0n'}}\, r)\, r\, dr$$

$$\boxed{B_{0n} = \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} \int_0^a \alpha(r, \theta) J_0(\sqrt{\lambda_{0n}}\, r)\, r\, dr\, d\theta \Big/ \sinh(-\sqrt{\lambda_{0n}}\, H) \int_0^a J_0^2(\sqrt{\lambda_{0n}}\, r)\, r\, dr \right]}$$

$\frac{a^2}{2} J_1^2(\sqrt{\lambda_{0n}}\, a)$

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Summarize: $$\boxed{B_{mn} = \frac{2/\sinh(-\sqrt{\lambda_{mn}}H)}{(1+\delta_{0m})\pi a^2 J_{m+1}^2(\sqrt{\lambda_{mn}}a)} \int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta) J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta)\, r\,dr\,d\theta}$$

Now multiplying by $J_{m'}(\sqrt{\lambda_{m'n'}}r)\sin m'\theta$ and integrating:

$$\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta) J_{m'}(\sqrt{\lambda_{m'n'}}r)\sin(m'\theta)\, r\,dr\,d\theta = \sum_{n=1} B_{0n}\sin(-\sqrt{\lambda_{0n}}H)\int_0^a J_0(\ )J_{m'}(\ )\,r\,dr \cdot \cancel{\int_{-\pi}^{\pi}\sin(m'\theta)\,d\theta} \to 0$$

$$+\sum_{m=1}\sum_{n=1}\sinh(-\sqrt{\lambda_{mn}}H)\int_0^a J_m(\ )J_{m'n'}(\ )\,r\,dr\left[A_{mn}\underbrace{\int_{-\pi}^{\pi}\sin m\theta\sin m'\theta\,d\theta}_{\pi\delta_{mm'}} + \cancel{B_{mn}\int_{-\pi}^{\pi}(\ )\,d\theta}\right] \to 0$$

$$= \sum_{n=1}\sinh(-\sqrt{\lambda_{m'n}}H)\,A_{m'n}\underbrace{\int_0^a J_{m'n}(\sqrt{\lambda_{m'n}}r)\,J_{m'n'}(\sqrt{\lambda_{m'n'}}r)\,r\,dr}_{0 \text{ if } n\neq n'}\cdot\pi$$

$$= A_{m'n'}\sinh(-\sqrt{\lambda_{m'n'}}H)\int_0^a J_{m'n'}^2(\sqrt{\lambda_{m'n'}}r)\,r\,dr$$

$$\Rightarrow \boxed{A_{mn} = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta)\,r\,dr\,d\theta \Big/ \sinh(-\sqrt{\lambda_{mn}}H)\underbrace{\int_0^a J_m^2(\sqrt{\lambda_{mn}}r)\,r\,dr}_{\to \frac{a^2}{2}J_{m+1}^2(\sqrt{\lambda_{mn}}a)}\right]}$$

Similarly $$\boxed{B_{mn} = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)J_m(\sqrt{\lambda_{mn}}r)\cos(m\theta)\,r\,dr\,d\theta \Big/ \sinh(-\sqrt{\lambda_{mn}}H)\int_0^a J_m^2(\sqrt{\lambda_{mn}}r)\,r\,dr\right]}$$

* b) Using part a) and evaluating for $z=0$, we have:

$$\alpha(r)\sin(7\theta) = u(r,\theta,0) = \sum_{m=0}\sum_{n=1} J_m(\sqrt{\lambda_{mn}}r)\sinh[-\sqrt{\lambda_{mn}}H]\,[A_{mn}\sin m\theta + B_{mn}\cos m\theta]$$

$$\Rightarrow \alpha(r)\cancel{\sin(7\theta)} = \sum_{n=1} J_7(\sqrt{\lambda_{7n}}r)\sinh[-\sqrt{\lambda_{7n}}H]\,A_{7n}\cancel{\sin(7\theta)} \qquad B_{mn}=0,\ A_{mn}=0 \text{ if } m\neq 7$$

multiplying by $J_7(\sqrt{\lambda_{7n'}}r)$ and integrating:

$$\int_0^a \alpha(r)J_7(\sqrt{\lambda_{7n'}}r)\,r\,dr = \sum_{n=1} A_{7n}\sinh[-\sqrt{\lambda_{7n}}H]\underbrace{\int_0^a J_7(\sqrt{\lambda_{7n}}r)J_7(\sqrt{\lambda_{7n'}}r)\,r\,dr}_{0 \text{ if } n\neq n'}$$

$$= A_{7n'}\sinh[-\sqrt{\lambda_{7n'}}H]\int_0^a J_7^2(\sqrt{\lambda_{7n'}}r)\,r\,dr$$

$$\Rightarrow \boxed{A_{7n} = \frac{1}{\sinh[-\sqrt{\lambda_{7n}}H]}\left[\int_0^a \alpha(r)J_7(\sqrt{\lambda_{7n}}r)\,r\,dr \Big/ \underbrace{\int_0^a J_7^2(\sqrt{\lambda_{7n}}r)\,r\,dr}_{\to \frac{a^2}{2}J_8^2(\sqrt{\lambda_{7n}}a)}\right]}$$

and $$\boxed{u(r,\theta,z) = \sum_{n=1} A_{7n}J_7(\sqrt{\lambda_{7n}}r)\sinh[\sqrt{\lambda_{7n}}(z-H)]\sin(7\theta)}$$

c) $f(r) = C J_m(\sqrt{\lambda_{mn}}r)$ with $\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$ but now $f'(a)=0$ implies $z_{mn} \to$ zeros of $J_m'(z)$.

$n = 1, 2, 3, \ldots$

Also $g(\theta) = \left\{\begin{matrix}\sin m\theta \\ \cos m\theta\end{matrix}\right\}$ $m = 0, 1, 2, 3, \ldots$

and $h(z) = A\cosh(\sqrt{\lambda}z) + B\sinh(\sqrt{\lambda}z)$ just like in part a)

$u(r,\theta,0) = 0 \to h(0) = 0 \to A = 0. \to h(z) = B\sinh(\sqrt{\lambda}z)$

$$\to u(r,\theta,z) = \sum_{m=0}\sum_{n=1} J_m(\sqrt{\lambda_{mn}}r)\sinh[\sqrt{\lambda_{mn}}z]\,[A_{mn}\sin m\theta + B_{mn}\cos m\theta]$$

Now evaluating the condition over $u(r,\theta,H)$:

$$\beta(r)\cos(3\theta) = \sum_{m=0}^{\infty}\sum_{n=1} J_m(\sqrt{\lambda_{mn}}\,r)\sinh[\sqrt{\lambda_{mn}}\,H]\,[A_{mn}\sin m\theta + B_{mn}\cos m\theta]$$

$$\Rightarrow \beta(r)\cancel{\cos(3\theta)} = \sum_{n=1} J_3(\sqrt{\lambda_{3n}}\,r)\sinh[\sqrt{\lambda_{3n}}\,H]\cdot B_{3n}\cancel{\cos 3\theta} \qquad A_{mn}=0,\quad B_{mn}=0 \text{ if } m\neq 3.$$

multiplying by $J_3(\sqrt{\lambda_{3n'}}\,r)$ and integrating:

$$\int_0^a \beta(r)\,J_3(\sqrt{\lambda_{3n'}}\,r)\,r\,dr = \sum_{n=1} B_{3n}\sinh[\sqrt{\lambda_{3n}}\,H]\underbrace{\int_0^a J_3(\sqrt{\lambda_{3n}}\,r)\,J_3(\sqrt{\lambda_{3n'}}\,r)\,r\,dr}_{0 \text{ if } n\neq n'}$$

$$= B_{3n'}\sinh[\sqrt{\lambda_{3n'}}\,H]\int_0^a J_3^2(\sqrt{\lambda_{3n'}}\,r)\,r\,dr$$

$$\Rightarrow \boxed{B_{3n} = \frac{1}{\sinh[\sqrt{\lambda_{3n}}\,H]}\left[\int_0^a \beta(r)\,J_3(\sqrt{\lambda_{3n}}\,r)\,r\,dr \Big/ \int_0^a J_3^2(\sqrt{\lambda_{3n}}\,r)\,r\,dr\right]}$$

$$\frac{a^2}{2}\left[1-\left(\frac{3}{\sqrt{\lambda_{3n}}\,a}\right)^2\right]J_3^2(\sqrt{\lambda_{3n}}\,a)$$

and $\boxed{u(r,\theta,z) = \sum_{n=1}^{\infty} B_{3n}\,J_3(\sqrt{\lambda_{3n}}\,r)\sinh[\sqrt{\lambda_{3n}}\,z]\cos 3\theta}$

d) Always $\boxed{g(\theta) = \left\{\begin{matrix}\sin m\theta \\ \cos m\theta\end{matrix}\right\}}$ $m = 0,1,2,\ldots$ since $0\le\theta\le 2\pi$ by periodic BC's.

Now we need evaluate $f(r)$ and $h(z)$ for the BC's:

$\underline{\lambda > 0} \rightarrow f(r) = C_1 J_m(\sqrt{\lambda}\,r) + \cancel{C_2 Y_m(\sqrt{\lambda}\,r)}$ $\to 0$ since $|f(0)|<\infty$

$\frac{\partial u}{\partial r}(a,\theta,z) = 0 \rightarrow f'(a) = 0 \rightarrow J_m'(\sqrt{\lambda}\,a) = 0 \rightarrow \sqrt{\lambda}\,a = z_{mn}$ zeros of $J_m'(z)$

$\Rightarrow \boxed{f(r) = C_1 J_m(\sqrt{\lambda_{mn}}\,r)}$ $\qquad \lambda_{mn} = (z_{mn}/a)^2 \qquad n = 1,2,3,\ldots$

if $\underline{\lambda > 0} \rightarrow h'' = \lambda h(z) \rightarrow h(z) = A\cosh[\sqrt{\lambda}(z-H)] + B\sinh[\sqrt{\lambda}(z-H)]$

$\frac{\partial u}{\partial z}(r,\theta,H) = 0 \rightarrow h'(H) = 0 \rightarrow B = 0 \rightarrow \boxed{h(z) = B\sinh[\sqrt{\lambda}(z-H)]}$

$\underline{\lambda = 0} \rightarrow f \sim r^{\pm m},\ f'(a) = 0 \rightarrow f = cte.$ or $\boxed{m=0}$

$h''(z) = 0 \rightarrow h(z) = A_0 z + B_0,\ h'(H) = 0 \rightarrow A_0 = 0 \rightarrow h(z) = B_0$

$$\Rightarrow u(r,\theta,z) = C_0 + \sum_{m=0}\sum_{n=1} J_m(\sqrt{\lambda_{mn}}\,r)\sinh[\sqrt{\lambda_{mn}}(z-H)]\,[A_{mn}\cos m\theta + B_{mn}\sin m\theta]$$

$$\frac{\partial u}{\partial z}(r,\theta,0) = \alpha(r)\sin 3\theta = \sum_{m=0}\sum_{n=1}\sqrt{\lambda_{mn}}\,J_m(\sqrt{\lambda_{mn}}\,r)\cosh(-\sqrt{\lambda_{mn}}\,H)\,[A_{mn}\cos m\theta + B_{mn}\sin m\theta]$$

$$\Rightarrow \alpha(r)\cancel{\sin 3\theta} = \sum_{n=1}\sqrt{\lambda_{3n}}\,J_3(\sqrt{\lambda_{3n}}\,r)\cosh(-\sqrt{\lambda_{3n}}\,H)\,A_{3n}\cancel{\cos 3\theta} \qquad B_{mn}=0,\quad A_{mn}=0 \text{ if } m\neq 3$$

multiplying by $J_3(\sqrt{\lambda_{3n'}}\,r)$ and integrating:

$$\int_0^a \alpha(r)\,J_3(\sqrt{\lambda_{3n'}}\,r)\,r\,dr = \sum_{n=1}\sqrt{\lambda_{3n}}\cosh(-\sqrt{\lambda_{3n}}\,H)\underbrace{\int_0^a J_3(\sqrt{\lambda_{3n}}\,r)\,J_3(\sqrt{\lambda_{3n'}}\,r)\,r\,dr}_{0 \text{ if } n\neq n'}\,A_{3n}$$

$$= \sqrt{\lambda_{3n'}}\cosh(-\sqrt{\lambda_{3n'}}\,H)\int_0^a J_3^2(\sqrt{\lambda_{3n'}}\,r)\,r\,dr\,A_{3n'}$$

$$\boxed{A_{3n} = \frac{1}{\sqrt{\lambda_{3n}}\cosh(-\sqrt{\lambda_{3n}}\,H)}\left[\int_0^a \alpha(r)\,J_3(\sqrt{\lambda_{3n}}\,r)\,r\,dr \Big/ \int_0^a J_3^2(\sqrt{\lambda_{3n}}\,r)\,r\,dr\right]}$$

and $\boxed{u(r,\theta,z) = \sum_{n=1}^{\infty} A_{3n}\,J_3(\sqrt{\lambda_{3n}}\,r)\sinh[\sqrt{\lambda_{3n}}(z-H)]\cos 3\theta} + C_0$

Summarize $\rightarrow$ $$\boxed{A_{nm} = \frac{2/\left[1-(m/\sqrt{\lambda_{mn}}\,a)^2\right] J_m^2(\sqrt{\lambda_{mn}}\,a)}{(1+\delta_{0m})\pi a^2 \sqrt{\lambda_{mn}}\cosh(-\sqrt{\lambda_{mn}}\,H)} \int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\, J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)\, r\,dr\,d\theta}$$

For $m \geq 0$.

e) From part d) we have:

$$\boxed{u(r,\theta,z) = C_0 + \sum_{m=0}\sum_{n=1} J_m(\sqrt{\lambda_{mn}}\,r)\sinh\left[\sqrt{\lambda_{mn}}(z-H)\right]\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]}$$

But now $\frac{\partial u}{\partial z}(r,\theta,0) = \alpha(r,\theta)$:

$$\rightarrow \alpha(r,\theta) = \sum_{m=0}\sum_{n=1} \sqrt{\lambda_{mn}}\, J_m(\sqrt{\lambda_{mn}}\,r)\cosh\left[\sqrt{\lambda_{mn}}(-H)\right]\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]$$

$$= \sum_{n=2} \sqrt{\lambda_{0n}}\, J_0(\sqrt{\lambda_{0n}}\,r)\cosh\left[\sqrt{\lambda_{0n}}(-H)\right] A_{0n}$$ since $J_0(\sqrt{\lambda_{01}}\,r)=1$ because $\lambda_{01}=0$ when $z_{01}=0$ is the first zero of J'

$$+ \sum_{m=1}\sum_{n=1} \sqrt{\lambda_{mn}}\, J_m(\sqrt{\lambda_{mn}}\,r)\cosh(-\sqrt{\lambda_{mn}}\,H)\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]$$

Multiplying by $J_0(\sqrt{\lambda_{0n'}}\,r)$ and integrating:

$$\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\, J_0(\sqrt{\lambda_{0n'}}\,r)\, r\,dr\,d\theta = \sum_{n=1} \sqrt{\lambda_{0n}}\, A_{0n}\cosh(-\sqrt{\lambda_{0n}}\,H)\,(2\pi)\underbrace{\int_0^a J_0(\sqrt{\lambda_{0n}}\,r)J_0(\sqrt{\lambda_{0n'}}\,r)\,r\,dr}_{0 \text{ if } n\neq n'}$$

$$+ \sum_{m=1}\sum_{n=1} \sqrt{\lambda_{mn}}\cosh(-\sqrt{\lambda_{mn}}\,H)\int_0^a J_m(\ )J_0(\ )\,r\,dr\left[A_{mn}\underbrace{\int_{-\pi}^{\pi}\cos m\theta}_{0} + B_{mn}\underbrace{\int_{-\pi}^{\pi}\sin m\theta}_{0}\right]$$

$$= \sqrt{\lambda_{0n'}}\, A_{0n'}\cosh(-\sqrt{\lambda_{0n'}}\,H)\,(2\pi)\int_0^a J_0^2(\sqrt{\lambda_{0n'}}\,r)\,r\,dr.$$

$$\Rightarrow \boxed{A_{0n} = \frac{1}{2\pi\sqrt{\lambda_{0n}}\cosh(-\sqrt{\lambda_{0n}}\,H)}\left[\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\, J_0(\sqrt{\lambda_{0n}}\,r)\,r\,dr\,d\theta \Big/ \int_0^a J_0^2(\sqrt{\lambda_{0n}}\,r)\,r\,dr\right]}$$

$n \geq 2$

Now multiplying $J_{m'n'}(\sqrt{\lambda_{m'n'}}\,r)\cos m'\theta$ and integrating, we obtain:

$$\boxed{A_{mn} = \frac{1}{\pi\sqrt{\lambda_{mn}}\cosh(-\sqrt{\lambda_{mn}}\,H)}\left[\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\, J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)\, r\,dr\,d\theta \Big/ \int_0^a J_{mn}^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr\right]}$$

$$\boxed{B_{mn} = \frac{1}{\pi\sqrt{\lambda_{mn}}\cosh(-\sqrt{\lambda_{mn}}\,H)}\left[\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\, J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\, r\,dr\,d\theta \Big/ \int_0^a J_{mn}^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr\right]}$$

Similarly

Note that since $J_0(\sqrt{\lambda_{01}}\,r)=1$ because $\lambda_{01}=0$ when $z_{01}=0$ is the first zero of $J'(z)$ then if:

$$u(r,\theta,0) = C_0 + A_{01}\,\cancel{J_0(\sqrt{\lambda_{01}}\,r)}^{\,1} + \sum_{n=2} J_0(\sqrt{\lambda_{0n}}\,r)\sinh\left[\sqrt{\lambda_{mn}}(-H)\right] A_{0n}$$

$$+ \sum_{m=1}\sum_{n=1} J_m(\sqrt{\lambda_{mn}}\,r)\sinh\left[\sqrt{\lambda_{mn}}(-H)\right]\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]$$

then integrating in $r\,dr\,d\theta$ and using orthogonality as above:

$$\int_{-\pi}^{\pi}\int_0^a u(r,\theta,0)\, r\,dr\,d\theta = (C_0 + A_{01})(\pi a^2) \rightarrow \boxed{C_0 + A_{01} = \frac{1}{\pi a^2}\int_{-\pi}^{\pi}\int_0^a u(r,\theta,0)\, r\,dr\,d\theta}$$

Mean value of u over the bottom (Condition for existence of the solution)

Remember that $\int_0^a J_{mn}^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr = \frac{a^2}{2}\left[1-\left(\frac{m}{\sqrt{\lambda_{mn}}\,a}\right)^2\right] J_m^2(\sqrt{\lambda_{mn}}\,a)$.

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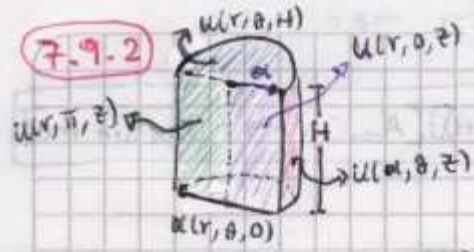
7.9.2



$\nabla^2 u = 0 \rightarrow$ Let $u = f(r)g(\theta)h(z) \rightarrow \boxed{\begin{array}{l} h'' = \lambda h(z) \\ g'' = -\mu g(\theta) \end{array}}$

$$\boxed{r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0}$$

$\mu > 0 \rightarrow g(\theta) = A\cos\sqrt{\mu}\,\theta + B\sin\sqrt{\mu}\,\theta$

$\mu = 0 \rightarrow g(\theta) = A_0\theta + B_0$

$\mu < 0 \rightarrow g(\theta) = A'\cosh(\sqrt{-\mu}\,\theta) + B'\sinh(\sqrt{-\mu}\,\theta)$

$\lambda > 0 \rightarrow f(r) = C_1 J_{\sqrt{\mu}}(\sqrt{\lambda}\,r) + \cancel{C_2 Y_{\sqrt{\mu}}(\sqrt{\lambda}\,r)}$ 0 since $|f(0)| < \infty$

$h(z) = D\cosh[\sqrt{\lambda}(z - z_0)] + E\cosh[\sqrt{\lambda}(z - z_0)]$

$\lambda < 0 \rightarrow f(r) = \cancel{C_1' K_{\sqrt{\mu}}(\sqrt{-\lambda}\,r)} + C_2' I_{\sqrt{\mu}}(\sqrt{-\lambda}\,r)$

since $|f(0)| < \infty$ 0

$h(z) = F\cos(\sqrt{-\lambda}\,z) + G\sin(\sqrt{-\lambda}\,z)$

$\lambda = 0 \rightarrow f(r) \sim (cte)\, r^{\pm\sqrt{\mu}}$; $h(z) = F_0 z + G_0$

The solutions are combinations of the form $f(r)g(\theta)h(z)$.

α) $g(\theta) = 0 \rightarrow \begin{cases} A = B_0 = 0 \\ A' = 0 \end{cases}$; $g(\pi) = 0 \rightarrow \begin{cases} B\sin\sqrt{\mu}\,\pi = 0 \rightarrow \sqrt{\mu} = m = 1, 2, 3, \ldots \\ B' = A_0 = 0 \end{cases}$ $\mu = m^2$

$f(\alpha) = 0 \rightarrow \begin{cases} J_m(\sqrt{\lambda}\,\alpha) = 0 \rightarrow \sqrt{\lambda}\,\alpha = z_{mn} \rightarrow \lambda_{mn} = \left(\frac{z_{mn}}{\alpha}\right)^2 \\ (cte)\alpha^{\pm m} = 0 \rightarrow cte = 0. \end{cases}$ z_{mn} the zeros of $J_m(z)$. $n = 1, 2, 3, \ldots$

There are not solutions for $\lambda < 0$ since $h(z)$ is not oscillatory according to BC's.

$h(0) = 0 \rightarrow \begin{cases} \text{Choising } z_0 = 0 \rightarrow D = 0 \\ G_0 = 0 \end{cases} \Rightarrow \boxed{h(z) = E\sinh(\sqrt{\lambda_{mn}}\,z)}$

$$\Rightarrow \boxed{u(r,\theta,z) = \sum_{m=1}\sum_{n=1} E_{mn} J_{mn}(\sqrt{\lambda_{mn}}\,r)\sinh(\sqrt{\lambda_{mn}}\,z)\sin(m\theta)}$$

$$\alpha(r,\theta) = u(r,\theta,H) = \sum_{m=1}\sum_{n=1} E_{mn} J_{mn}(\sqrt{\lambda_{mn}}\,r)\sinh(\sqrt{\lambda_{mn}}\,H)\sin m\theta$$

multiplying by $J_{m'n'}(\sqrt{\lambda_{m'n'}}\,r)\sin(m'\theta)$ and integrating:

$$\int_0^\pi\int_0^\alpha \alpha(r,\theta)[\;]\,r\,dr\,d\theta = \sum_{m=1}\sum_{n=1} E_{mn}\sinh(\sqrt{\lambda_{mn}}\,H)\int_0^\alpha J_{mn}J_{m'n'}\,r\,dr\underbrace{\int_0^\pi \sin m\theta\sin m'\theta\,d\theta}_{(\pi/2)\delta_{mm'}}$$

$$= \sum_{n=1} E_{m'n}\sinh(\sqrt{\lambda_{m'n}}\,H)\underbrace{\int_0^\alpha J_{m'n}(\sqrt{\lambda_{m'n}}\,r)J_{m'n'}(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr}_{0 \text{ if } n \neq n'} \cdot \frac{\pi}{2}$$

$$= E_{m'n'}\frac{\pi}{2}\sinh(\sqrt{\lambda_{m'n'}}\,H)\int_0^\alpha J_{m'n'}^2(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr$$

$$\boxed{E_{mn} = \frac{2}{\pi\sinh(\sqrt{\lambda_{mn}}\,H)}\,\frac{\int_0^\pi\int_0^\alpha \alpha(r,\theta)J_{mn}(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\,r\,dr\,d\theta}{\int_0^\alpha J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr}}$$

$\frac{\alpha^2}{2}J_{m+1}^2(\sqrt{\lambda_{mn}}\,\alpha)$

Summarizing: $\boxed{C_{nm} = \frac{4}{LH(1+\delta_{on})(1+\delta_{om})}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{L}\right)dx\,dy}$ for all $n,m \geq 0$.

$$\Rightarrow \int_0^H\int_0^L f(x,y)\cos\left(\frac{n'\pi x}{L}\right)dx\,dy = \sum_{n=1}^{\infty} C_{n0}\frac{L}{2}\delta_{nn'}H = C_{n0}\frac{LH}{2}$$

for $n>0$ $\Rightarrow \boxed{C_{n0} = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)dx\,dy}$

Similarly multiplying by $\cos\left(\frac{m'\pi y}{H}\right)$ and integrating between $0<x<L$ and $0<y<H$:

for $m>0$ $\boxed{C_{0m} = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{m\pi y}{H}\right)dx\,dy}$

Now, multiplying by $\cos\left(\frac{n'\pi x}{L}\right)\cos\left(\frac{m'\pi y}{H}\right)$ and integrating:

$$\int_0^H\int_0^L f(x,y)\cos\frac{n'\pi x}{L}\cos\frac{m'\pi y}{H}dx\,dy = C_{00}\cancelto{0}{\int_0^L\cos\frac{n'\pi x}{L}dx}\cancelto{0}{\int_0^H\cos\frac{m'\pi y}{H}dy}$$

$$+\sum_{n=1}^{\infty}C_{n0}\int_0^L\cos\frac{n\pi x}{L}\cos\frac{n'\pi x}{L}\cancelto{0}{\int_0^H\cos\frac{m'\pi y}{H}dy} + \sum_{m=1}^{\infty}C_{0m}\cancelto{0}{\int_0^L\cos\frac{n'\pi x}{L}dx}\int_0^H\cos\frac{m\pi y}{H}\cos\frac{m'\pi y}{H}dy$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\underbrace{\int_0^L\cos\frac{n\pi x}{L}\cos\frac{n'\pi x}{L}dx}_{(L/2)\delta_{nn'}}\underbrace{\int_0^H\cos\frac{m\pi y}{L}\cos\frac{m'\pi y}{L}dy}_{(H/2)\delta_{mm'}} = C_{n'm'}\frac{LH}{4}$$

$$\Rightarrow \boxed{C_{nm} = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)dx\,dy} \quad \text{for } n,m>0.$$

*c) Now $f'(0)=f'(L)=0$ and $g(0)=g(H)=0$, then:

If $\gamma>0,\ \delta>0$: $f_n(x) = A_n\cos\left(\frac{n\pi x}{L}\right)$ $\quad g_m(x) = B_n\sin\left(\frac{m\pi y}{H}\right)$

$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 > 0 \rightarrow \frac{dh}{dt} = -\lambda_{nm}k\,h(t) \rightarrow h(t) = C_{nm}e^{-\lambda_{nm}kt}$ $\quad n=1,2,3,\ldots$

If $\gamma=0,\ \delta>0$: $f(x) = A_0$ $\quad g_m(y) = B_n\sin\left(\frac{m\pi y}{H}\right)$

$\lambda_{0m} = \left(\frac{m\pi}{H}\right)^2 > 0 \rightarrow \frac{dh}{dt} = -\lambda_{0m}k\,h(t) \rightarrow h(t) = C_{0m}e^{-\lambda_{0m}kt}$ $\quad n=1,2,3\ldots$

If $\delta=0 \rightarrow g(x) = B_2x + B_0$ but $g(0)=0=B_0$ $\quad g(L)=0=B_2L \rightarrow B_2=0$

thus, there is not solution for $\delta=0$.

Similarly there are not solution for $\gamma,\delta<0$.

$$\Rightarrow \boxed{u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}C_{nm}e^{-\lambda_{nm}kt}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)}$$

For $t\rightarrow\infty$ $u=0$

$$f(x,y) = u(x,y,0) = \sum_{m=1}^{\infty}C_{0m}\sin\left(\frac{m\pi y}{H}\right) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

multiplying by $\sin\left(\frac{m'\pi y}{H}\right)$ and integrating between $0<x<L$, $0<y<H \Rightarrow$

Summarize: $\boxed{C_{nm} = \frac{4}{LH(1+\delta_{n0})}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dx\,dy}$ $n\geq 0,\ m>0.$

$$\boxed{C_{0m} = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\sin\left(\frac{m\pi y}{H}\right)dx\,dy} \quad \text{for } m>0$$

Now, multiplying by $\cos\left(\frac{n'\pi x}{L}\right)\sin\left(\frac{m'\pi y}{H}\right)$ and integrating again:

$$\boxed{C_{nm} = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dx\,dy} \quad \text{for } n,m>0.$$

d) Now $f''(x) = -\gamma f(x)$, $g''(y) = -\delta g(y)$ with $f(0)=0$, $f'(L)=0$, $g'(0)=g'(H)=0$

If $\gamma>0,\ \delta>0$: $f(x) = A_1\cos\sqrt{\gamma}\,x + A_2\sin\sqrt{\gamma}\,x$ $\quad f(0)=0=A_1$

$f'(L) = 0 = A_2\sqrt{\gamma}\cos\sqrt{\gamma}\,L \rightarrow \sqrt{\gamma}\,L = \left(\frac{2n-1}{2}\right)\pi$

$\Rightarrow f(x) = A_n\sin\left[\frac{(2n-1)\pi x}{2L}\right] \quad n=1,2,3,\ldots$

$g(y) = B_1\cos\sqrt{\delta}\,y + B_2\sin\sqrt{\delta}\,y$ $\quad g'(0)=0=B_2\sqrt{\delta} \rightarrow B_2=0$

$g'(L)=0=-B_1\sqrt{\delta}\sin\sqrt{\delta}\,H \rightarrow \delta = \left(\frac{m\pi}{H}\right)^2$

$\Rightarrow g(y) = B_n\cos\left(\frac{m\pi y}{H}\right) \quad m=1,2,3,\ldots$

$\lambda_{nm} = \left[\frac{(2n-1)\pi}{2L}\right]^2 + \left(\frac{m\pi}{H}\right)^2 > 0 \rightarrow \frac{dh}{dt} = -\lambda_{nm}Kh \rightarrow h(t) = C_{nm}e^{-\lambda_{nm}Kt}$

If $\gamma>0,\ \delta=0$: $f(x) = A_n\sin\left[\frac{(2n-1)\pi x}{2L}\right] \quad n=1,2,3\ldots$

$g''(y)=0 \rightarrow g(y) = B_2y + B_0 \quad g'(0)=0=B_2 \rightarrow \underline{g(y)=B_0}$

$\lambda_{n0} = \left[\frac{(2n-1)\pi}{2L}\right]^2 > 0 \rightarrow \frac{dh}{dt} = -\lambda_{n0}Kh(t) \rightarrow h(t) = C_{n0}e^{-\lambda_{n0}Kt}$

If $\gamma=0$: $f''(x)=0 \rightarrow f(x) = A_2x + A_0$ $\quad f(0)=0=A_0$, $f'(L)=0=A_2L \rightarrow A_2=0$

There are not solution for $\gamma=0$.
Also, don't exist solution for $\gamma,\delta<0$.

$$\Rightarrow \boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\cos\left(\frac{m\pi y}{H}\right)e^{-\lambda_{nm}Kt}}$$ For $t\to\infty$ $u\to 0$

$$f(x,y) = u(x,y,0) = \sum_{n=1}^{\infty} C_{n0}\sin\left[\frac{(2n-1)\pi x}{2L}\right] + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\cos\left(\frac{m\pi y}{H}\right)$$

multiplying by $\sin\left[\frac{(2n'-1)\pi x}{2L}\right]$ and integrating:

$$\boxed{C_{n0} = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\sin\left[\frac{(2n-1)\pi x}{2L}\right]dx\,dy}$$

Multiplyin by $\sin[(2n'-1)\pi x/L]\cos(m'\pi y/H)$ and integrating again:

$$\Rightarrow \boxed{C_{nm} = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\sin\left[\frac{(2n-1)\pi x}{L}\right]\cos\left(\frac{m\pi y}{H}\right)dx\,dy}$$

Add $(1+\delta_{0m})$ for summarize for all $n\geq 0,\ m\geq 0$.

e) $f''(x) = -\gamma f(x)$ with $f(0) = f(L) = 0$

$g''(y) = -\delta g(y)$ $\quad g(0) = 0 \quad g'(H) + h g(H) = 0 \quad h > 0.$

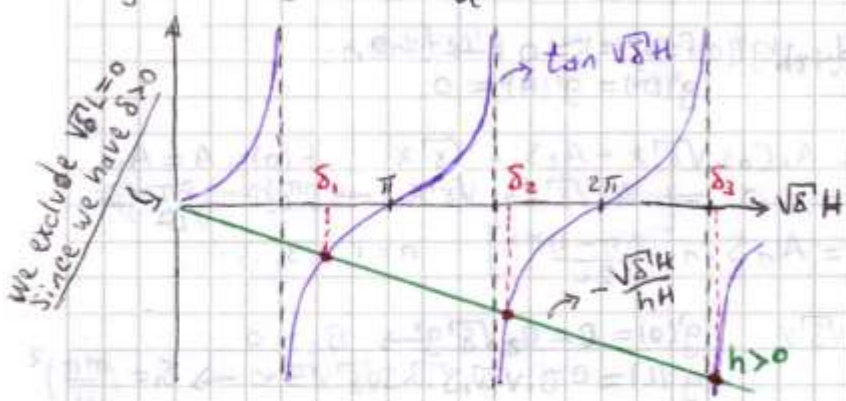
If $\gamma > 0,\ \delta > 0$: $f(x) = A_1 \cos\sqrt{\gamma}\,x + A_2 \sin\sqrt{\gamma}\,x \qquad f(0) = 0 = A_1$

$f(L) = 0 = A_2 \sin\sqrt{\gamma}\,L \rightarrow \gamma = (n\pi/L)^2 \quad n = 1, 2, 3$

$\rightarrow \boxed{f(x) = A_n \sin(n\pi x/L)}$

$g(x) = B_1 \cos\sqrt{\delta}\,y + B_2 \sin\sqrt{\delta}\,y \qquad g(0) = 0 = B_1$

$g'(H) + h g(H) = 0 = B_2(\sqrt{\delta}\cos\sqrt{\delta}H + h \sin\sqrt{\delta}H) \rightarrow \boxed{\tan\sqrt{\delta}H = -\frac{\sqrt{\delta}H}{hH}}$



There is an infinite number of intersections, each corresponds to a positive eigenvalue

$$\boxed{g_m(y) = B_m \sin\sqrt{\delta_m}\,y}$$

where the allowable δ_m are determined graphically.

If $\gamma = 0$: $f''(x) = 0 \rightarrow f(x) = A_1 x + A_0 \qquad f(0) = 0 = A_0$

$f(L) = 0 = A_1 L \rightarrow A_1 = 0$

There are not solutions for $\gamma = 0$. Similarly we can see that there are not solutions for $\gamma < 0$.

If $\delta = 0$: $g''(y) = 0 \rightarrow g(y) = B_1 y + B_0 \qquad g(0) = 0 = B_0$

$g'(H) + h g(H) = 0 = B_1(1 + hH) \rightarrow hH = -1$ OR $B_1 = 0$

Since $h > 0 \rightarrow B_1 = 0$.

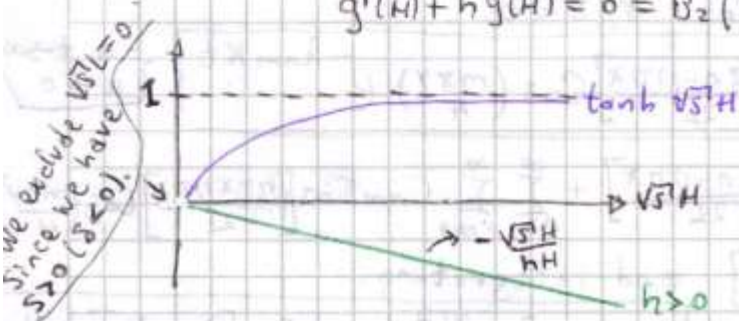
There are not solution for $\delta = 0$.

If $\delta < 0$: let $s = -\delta \Rightarrow g''(y) = s g(y) \rightarrow g(y) = B_1 \cosh\sqrt{s}\,y + B_2 \sinh\sqrt{s}\,y$

$g(0) = 0 = B_1$

$g'(H) + h g(H) = 0 = B_2(\sqrt{s}\cosh\sqrt{s}H + h \sinh\sqrt{s}H) = 0$

$\Rightarrow \tanh\sqrt{s}H = -\frac{\sqrt{s}H}{hH}$



There are not intersections for $h > 0$ when $\sqrt{s}L > 0$, then there are no negative eigenvalues for $g(y)$.

For $\gamma > 0,\ \delta > 0$ $\quad \boxed{\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \delta_m}$ where $\delta_m > 0$ are determined by the graphics above.

$\frac{dh}{dt} = -\lambda_{nm} k h(t) \rightarrow \boxed{h(t) = C_{nm} e^{-\lambda_{nm} k t}}$ $\quad n = 1, 2, 3, \ldots$ $\quad m = 1, 2, 3, \ldots$

$$\Rightarrow \boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} e^{-\lambda_{nm} k t} \sin\left(\frac{n\pi x}{L}\right) \sin\sqrt{\delta_m}\, y}$$

$$f(x,y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\sqrt{\delta_m}\, y$$

multiplying by $\sin(n'\pi x/L)\sin\sqrt{\delta_{m'}}\, y$ and integrating:

$$\int_0^H\int_0^L f(x,y)\sin\left(\frac{n'\pi x}{L}\right)\sin\left(\sqrt{\delta_{m'}}\, y\right)dx\,dy$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}} \underbrace{\int_0^H \sin\sqrt{\delta_m}\, y \sin\sqrt{\delta_{m'}}\, y\, dy}_{0 \text{ if } m=m'}$$

by orthogonality of eigenfunctions of an Sturm-Liouville problem

$$= C_{nm}\frac{L}{2}\int_0^H \sin^2\sqrt{\delta_m}\, y\, dy$$

$$\Rightarrow \boxed{C_{nm} = \frac{2}{L}\,\frac{\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\sqrt{\delta_m}\, y\right)dx\,dy}{\int_0^H \sin^2\sqrt{\delta_m}\, y\, dy}}$$

where the allowable δ_m are determined graphycally. (See above).

7.3.2 $\boxed{\frac{\partial u}{\partial t} = k\nabla^2\phi}$ with $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

Let $u(x,y,z,t) = \phi(x,y,z)h(t) \rightarrow \phi(x,y,z)\frac{dh}{dt} = kh(t)\nabla^2\phi$ dividing by $kh\phi$

$$\underbrace{\frac{1}{kh}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{\phi}\nabla^2\phi}_{-\lambda} \quad \text{separing variables} \Rightarrow \boxed{\begin{aligned}\frac{dh}{dt} &= -\lambda k h(t)\\ \nabla^2\phi &= -\lambda\phi(x,y,z)\end{aligned}}$$

Let $\phi(x,y,z) = f(x)g(y)p(z)$

$$\Rightarrow g(y)p(z)\frac{d^2f}{dx^2} + f(x)p(z)\frac{d^2g}{dy^2} + f(x)g(y)\frac{d^2p}{dz^2} = -\lambda f(x)g(y)p(z)$$

dividing by fgp: $\underbrace{\frac{1}{f}\frac{d^2f}{dx^2}}_{-\gamma} + \underbrace{\frac{1}{g}\frac{d^2g}{dy^2}}_{-\delta} + \underbrace{\frac{1}{p}\frac{d^2p}{dz^2}}_{-\Gamma} = -\lambda \rightarrow \boxed{\begin{aligned} f''(x) &= -\gamma f(x)\\ g''(y) &= -\delta f(y)\\ p''(z) &= -\Gamma p(z)\end{aligned}}$

a) $f(0) = f(L) = 0 \rightarrow f(x) = A_n \sin(n\pi x/L)$ $\quad \gamma_n = (n\pi/L)^2 \quad n = 1,2,3..$

There are not solutions for $\gamma \le 0$.

$g'(0) = g'(H) = 0 \rightarrow g(y) = B_n\cos(m\pi y/H)$ $\quad \delta_m = (m\pi/H)^2 \quad m = 1,2,3..$

$g_0(y) = B_0$ when $\delta_0 = 0 \quad m = 0$

No solutions for $\delta < 0$.

$p(z) = C_1\cos\sqrt{\Gamma}\, z + C_2\sin\sqrt{\Gamma}\, z$, $\quad p'(0) = 0 = C_2\sqrt{\Gamma} \rightarrow C_2 = 0$

$p(W) = 0 = C_1\cos\sqrt{\Gamma}\, W \rightarrow \sqrt{\Gamma}\, W = \left(\frac{2q-1}{2}\right)\pi$

$\rightarrow p(z) = C_q\cos\left[\frac{(2q-1)\pi z}{2W}\right]$ $\quad \Gamma_q = \left[\frac{(2q-1)\pi}{2W}\right]^2 \quad q = 1,2,3..$

There are not solutions for $\Gamma \le 0$

$\lambda = \lambda_{nmq} = \gamma_n + \delta_m + \Gamma_q > 0 \rightarrow \boxed{h(t) = D_{nmq}e^{-\lambda_{nmq}kt}}$ $\quad n,q = 1,2,3\ldots$ $\quad m = 0,1,2,3\ldots$

* b) Like in part a) $g(\theta) = B\sin(m\theta)$ $m = 1, 2, 3, \ldots$ $\mu = m^2$.

Now, $h(0) = 0$ and $h'(H) = 0$ imply oscillatory solutions in z according to the BC's. (There are not solutions for $\lambda > 0$)

For $\lambda < 0$: $h(0) = 0 \to F = 0$

$h'(H) = 0 = G\sqrt{\lambda}\cos(\sqrt{-\lambda}H) \to \sqrt{-\lambda}H = \left(\frac{2n-1}{2}\right)\pi \to \lambda_n = -\left(\frac{2n-1}{2}\frac{\pi}{H}\right)^2$ $n = 1, 2, 3, \ldots$

If $\lambda = 0$ $h(0) = 0 \to G_0 = 0$; $h'(H) = 0 \to F_0 = 0 \to$ There are not solution if $\lambda = 0$

$$\Rightarrow \boxed{u(r,\theta,z) = \sum_{m=1}\sum_{n=1} G_{mn} I_m\left[\frac{(2n-1)\pi}{2H} r\right] \sin\left[\frac{(2n-1)\pi}{2H} z\right] \sin(m\theta)}$$

$$\beta(\theta,z) = u(a,\theta,z) = \sum_{m=1}\sum_{n=1} G_{mn} I_m\left[\frac{(2n-1)\pi a}{2H}\right] \sin\left[\frac{(2n-1)\pi}{2H} z\right] \sin(m\theta)$$

Multiplying by $\sin[(2n'-1)\pi z/2H]\sin(m'\theta)$ and integrating:

$$\int_0^{\pi}\int_0^{H} \beta(\theta,z)\sin[(2n'-1)\pi z/H]\sin(m'\theta)\,dz\,d\theta =$$

$$\sum_{m=1}\sum_{n=1} G_{mn} I_m\left[\frac{(2n-1)\pi a}{2H}\right] \underbrace{\int_0^H \sin\frac{(2n-1)\pi z}{2H}\sin\frac{(2n'-1)\pi z}{2H}\,dz}_{(H/2)\delta_{nn'}} \underbrace{\int_0^{\pi}\sin m\theta \sin m'\theta\,d\theta}_{(\pi/2)\delta_{mm'}}$$

$$= G_{m'n'} I_{m'}\left[\frac{(2n'-1)\pi a}{2H}\right] \frac{H}{2}\cdot\frac{\pi}{2}$$

$$\Rightarrow \boxed{G_{mn} = \frac{4}{\pi H I_m[(2n'-1)\pi a/2H]} \int_0^{\pi}\int_0^{H} \beta(\theta,z)\sin\left[\frac{(2n-1)\pi z}{H}\right]\sin(m\theta)\,dz\,d\theta}$$

c) $g'(0) = 0 \to \begin{cases} B = A_0 = 0 \\ B' = 0 \end{cases}$ $g'(\pi) = 0 \to \begin{cases} -A\sqrt{\mu}\sin\sqrt{\mu}\pi = 0 \\ A' = 0 \end{cases} \to \sqrt{\mu} = m = 1, 2, 3 \ldots$ $\mu = m^2$

For $\mu = 0 \to g(\theta) = B_0 = B\cos m\theta$ since $m = 0$

$\boxed{g(\theta) = A_m\cos(m\theta)}$ $m = 0, 1, 2, \ldots$

$h'(0) = h'(H) = 0 \longrightarrow$ implies oscillatory solutions in z according to BC's, then there are not solutions for $\lambda > 0$.

For $\lambda < 0$: $h'(0) = 0 \to G = 0$; $h'(H) = \sqrt{\lambda}F\sin(\sqrt{-\lambda}H) = 0 \to \sqrt{-\lambda}H = n\pi$

$\lambda_n = -\left(\frac{n\pi}{H}\right)^2$ $n = 1, 2, 3, \ldots$

If $\lambda = 0$: $h'(0) = 0 \to F_0 = 0$, $h'(H) = 0 \Rightarrow h(z) = cte = F_0\cos\frac{n\pi}{H}z$ with $n = 0$.

$\boxed{h(z) = F_n\cos(n\pi z/H)}$ $n = 0, 1, 2, \ldots$

$$\Rightarrow \boxed{u(r,\theta,z) = \sum_{m=0}\sum_{n=0} F_{mn} I\left[\frac{n\pi r}{H}\right]\cos\left(\frac{n\pi z}{H}\right)\cos m\theta}$$

For evaluate the coefficients use $\frac{\partial u}{\partial r}(a,\theta,z) = \beta(\theta,z) \longrightarrow$

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Since the problem is independent of θ

7.9.4* a) $h(0)=h(H)=0$, $f(a)=0$, $g(\theta)=1=A'\cos\sqrt{\mu}\,\theta+B'\sin\sqrt{\mu}\,\theta$ with $\boxed{\mu=0}$

We note that this problem is the same as 7.9.3(a) with $m=\sqrt{\mu}/2=0\rightarrow m=0$ and replacing $\sin(2m\theta)$ by 1 throghout, to get:

$$\boxed{u(r,z,t)=\sum_{l=1}\sum_{n=1}A_{ln}\sin\left(\frac{l\pi z}{H}\right)J_0\left(\sqrt{\lambda'_{0n}}\,r\right)e^{-\lambda_{ln}kt}}\quad \lim_{t\to\infty}u=0.$$

with $\lambda'_{0n}=(z_{0n}/a)^2$ being $z_{0n}\rightarrow$ zeros of $J_0(z)$ $\quad\lambda_{ln}=\lambda'_{0n}+\left(\frac{l\pi}{H}\right)^2$.

Likewise, to get the coefficients, we remove the integral over θ in the answer to 7.9.3(a), along with the concominant normalization factor $\pi/4$, and set $m=0$ to get:

$$\boxed{A_{lm}=\frac{4}{H}\left[\int_0^H\int_0^a f(r,z)\sin\left(\frac{l\pi z}{H}\right)J_0\left(\sqrt{\lambda'_{0n}}\,r\right)r\,dr\,dz\Big/\int_0^a J_0^2\left(\sqrt{\lambda'_{0n}}\,r\right)r\,dr\right]}$$

b) $h'(0)=h'(H)=0$, $f'(a)=0$, $g(\theta)=1=A'\cos\sqrt{\mu}\,\theta+B'\sin\sqrt{\mu}\,\theta$ with $\boxed{\mu=0}$ for not θ-dependence.

Just as we did above, we notice that this problem is identical to 7.9.3(b), except for the θ dependence. This time we replace $\cos(2m\theta)$ by 1 and set $\mu=0$ (and hence $m=0$) throughout to get:

$$\boxed{u(r,z,t)=\sum_{l=0}\sum_{n=1}B_{ln}\cos\left(\frac{l\pi z}{H}\right)J_0\left(\sqrt{\lambda'_{0n}}\,r\right)e^{-\lambda_{ln}kt}}$$

Now $\lambda'_{0n}=(z_{0n}/a)^2$ with $z_{0n}\rightarrow$ zeros of $J_0'(z)$ and $\lambda_{ln}=\lambda'_{0n}+\left(\frac{l\pi}{H}\right)^2$

Likewise, the coefficients are given by:

$$\boxed{B_{01}=\frac{2}{Ha^2}\int_0^H\int_0^a f(r,z)\,r\,dr\,dz}\rightarrow \text{Average value of initial distribution } u$$

Note that $\lim_{t\to\infty}u=B_{01}$

$$B_{0n\,(n\neq1)}=\frac{1}{H}\left[\int_0^H\int_0^a f(r,z)J_0\left(\sqrt{\lambda'_{0n}}\,r\right)r\,dr\,dz\Big/\int_0^a J_0^2\left(\sqrt{\lambda'_{0n}}\,r\right)r\,dr\right]$$

$$B_{ln\,(l\neq1,\,n\neq1)}=\frac{4}{H}\left[\int_0^H\int_0^a f(r,z)\cos\left(\frac{l\pi z}{H}\right)J_0\left(\sqrt{\lambda'_{0n}}\,r\right)r\,dr\,dz\Big/\int_0^a J_0^2\left(\sqrt{\lambda'_{0n}}\,r\right)r\,dr\right]$$

We can summarize like in 7.9.3(b)

c) $h(0)=h(H)=0$, $f'(a)=0$ and $g(\theta)=1=A'\cos\sqrt{\mu}\,\theta+B'\sin\sqrt{\mu}\,\theta$. with $\boxed{\mu=0}$ for not θ-dependence

Just as we did above, we notice that this problem is identical to 7.9.3(c), except for the θ-dependence. This time we replace $\cos(2m-1)\theta$ by 1 and set $\mu=0$, which is tantamount taking $m=1/2$ to get:

$$\boxed{u(r,z,t)=\sum_{l=1}\sum_{n=1}C_{ln}\sin\left(\frac{l\pi z}{H}\right)J_0\left(\sqrt{\lambda'_{0n}}\,r\right)e^{-\lambda_{ln}kt}}$$

with $\lambda'_{0n}=(z_{0n}/a)^2$ and $z_{0n}\rightarrow$ zeros of $J_0'(z)$. $\lambda_{ln}=\lambda'_{0n}+(l\pi/H)^2$.

$$\boxed{C_{ln}=\frac{4}{H}\left[\int_0^H\int_0^a f(r,z)\sin\left(\frac{l\pi z}{H}\right)J_0\left(\sqrt{\lambda'_{0n}}\,r\right)r\,dr\,dz\Big/\int_0^a J_0^2\left(\sqrt{\lambda'_{0n}}\,r\right)r\,dr\right]}$$

7.9.5 $\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)+\frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}=0$

Let $u(\rho,\theta,\phi)=f(\rho)q(\theta)g(\phi)$, substituing and dividing by $fgq/\rho^2\sin^2\phi$:

$$\frac{\sin^2\phi}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)+\frac{\sin\phi}{g}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\underbrace{\frac{1}{q}\frac{d^2q}{d\theta^2}}_{-\lambda}=0 \longrightarrow \boxed{\frac{d^2q}{d\theta^2}=-\lambda q(\theta)}$$

Now, divide by $\sin^2\phi$:

$$\underbrace{\frac{1}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)}_{\mu}+\underbrace{\frac{1}{g\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)}_{-\mu}-\frac{\lambda}{\sin^2\phi}=0$$

$$\Rightarrow \boxed{\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)-\mu f(\rho)=0} \text{ and } \boxed{\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\left(\mu\sin\phi-\frac{\lambda}{\sin\phi}\right)g=0}$$

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7.10.1 $\dfrac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$ Let $u(\rho,\theta,\phi,t) = w(\rho,\theta,\phi)h(t)$

$$\Rightarrow \underbrace{\frac{1}{c^2 h}\frac{d^2 u}{dt^2}}_{-\lambda} = \underbrace{\frac{1}{w}\nabla^2 w}_{-\lambda} \Rightarrow \nabla^2 w + \lambda w = 0 \quad \text{and} \quad \boxed{\frac{d^2 h}{dt^2} = -\lambda c^2 h(t)}$$

Using the eq for the Laplacian in spherical coordinates:

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial w}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial w}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 w}{\partial\theta^2} + \lambda w = 0$$

we seek solutions of the form $w(\rho,\theta,\phi) = f(\rho)q(\theta)g(\phi)$. Sustituing and dividing by $fqg/\rho^2\sin^2\phi$

$$\frac{\sin^2\phi}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + \frac{\sin\phi}{g}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \underbrace{\frac{1}{q}\frac{d^2 q}{d\theta^2}}_{-m^2} + \lambda\rho^2\sin^2\phi = 0 \rightarrow \boxed{\frac{d^2 q}{d\theta^2} = -m^2 q(\theta)}$$

Now, divide by $\sin^2\phi$:

$$\underbrace{\frac{1}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + \lambda\rho^2}_{\mu} + \underbrace{\frac{1}{g\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) - \frac{m^2}{\sin^2\phi}}_{-\mu} = 0$$

$$\Rightarrow \boxed{\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + (\lambda\rho^2 - \mu)f(\rho) = 0} \quad \text{and} \quad \boxed{\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \left(\mu\sin\phi - \frac{m^2}{\sin\phi}\right)g = 0}$$

Since we expect vibrations $\Rightarrow$ $\boxed{h(t) \sim \left\{\begin{matrix}\cos c\sqrt{\lambda}\,t \\ \sin c\sqrt{\lambda}\,t\end{matrix}\right\}}$

For periodic BC's $\begin{matrix} q(-\pi) = q(\pi) \\ q'(-\pi) = q'(\pi)\end{matrix} \Rightarrow \boxed{g(\theta) \sim \left\{\begin{matrix}\cos m\theta \\ \sin m\theta\end{matrix}\right\}}$ $m = 0, 1, 2, 3, \ldots$

For bounded solutions at $\phi = 0$ (North Pole), $\phi = \pi$ (South pole) $\rightarrow$ $\boxed{g(\phi) = P_n^m(\cos\phi)}$ $\rightarrow$ Exist only for $n \geq m$

$\rightarrow$ and $\boxed{\mu = n(n+1)}$

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + (\lambda\rho^2 - n(n+1))f = 0 \rightarrow \text{For bounded solutions at } \rho = 0 \rightarrow \boxed{f(\rho) = \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda}\rho)} \quad n \geq m$$

$$\boxed{u(\rho,\theta,\phi,t) \sim \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda}\rho)\, P_n^m(\cos\phi) \left\{\begin{matrix}\cos m\theta \\ \sin m\theta\end{matrix}\right\} \left\{\begin{matrix}\cos c\sqrt{\lambda}\,t \\ \sin c\sqrt{\lambda}\,t\end{matrix}\right\}}$$

- If $u(a,\theta,\phi,t) = 0 \rightarrow J_{n+\frac{1}{2}}(\sqrt{\lambda}a) = 0 \rightarrow \sqrt{\lambda}a = z_{n\ell} \rightarrow \boxed{\lambda_{n\ell} = (z_{n\ell}/a)^2}$

  $\ell = 1, 2, 3, \ldots$ $\hookrightarrow$ Zeros of $J_{n+\frac{1}{2}}(z)$

a) $\dfrac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0 \rightarrow$ this restriction eliminates the $\sin c\sqrt{\lambda}\,t$ terms.

$$\Rightarrow u = \sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda_{n\ell}}\rho) P_n^m(\cos\phi)\left[A_{mn\ell}\cos m\theta + B_{mn\ell}\sin m\theta\right]\cos(c\sqrt{\lambda}\,t)$$

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$P_0^0(x) = 1$
$P_1^0(x) = x$

Now $u(\rho,\theta,\phi,0) = F(\rho,\theta,\phi)$, then:

$$F(\rho,\theta,\phi) = \sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda_{n\ell}}\rho) P_n^m(\cos\phi)\left[A_{mn\ell}\cos m\theta + B_{mn\ell}\sin m\theta\right]$$

$$= \sum_{\ell=1}^{\infty} \rho^{-1/2} J_{1/2}(\sqrt{\lambda_{0\ell}}\rho) A_{00\ell} + \sum_{\ell=1}^{\infty}\sum_{n=1} \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda_{n\ell}}\rho) P_n(\cos\phi) A_{0n\ell}$$

$$+ \sum_{\ell=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1} \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda_{n\ell}}\rho) P_n^m(\cos\phi)\left[A_{mn\ell}\cos m\theta + B_{mn\ell}\sin m\theta\right]$$

Multiplying by $\rho^{-1/2} J_{n'+\frac{1}{2}}(\sqrt{\lambda_{n'\ell'}}\rho) P_{n'}^{m'}(\cos\phi)\cos(m'\theta)$ and integrating in dv:

$$\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{a} F(\rho,\theta,\phi)\rho^{-1/2} J_{n'+1/2}(\sqrt{\lambda_{n'\ell'}}\rho) P_{n'}^{m'}(\cos\phi)\cos(m'\theta)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \sum_{\ell,n,m}\int_0^a J_{n+1/2}(\,)J_{n'+1/2}(\,)\rho\, d\rho \underbrace{\int_{-\pi}^{\pi} P_n^m P_{n'}^{m'}\sin\phi\, d\phi}\left[A_{mn\ell}\underbrace{\int_{-\pi}^{\pi}\cos m\theta\cos m'\theta\, d\theta}_{\pi(1+\delta_{0m})\delta_{mm'}} + \cancel{B_{\ell mn}(0)}\right]$$

$$= \sum_{\ell,n}\int_0^a J_{n+1/2}(\,)J_{n'+1/2}(\,)\rho\, d\rho \underbrace{\int_{-\pi}^{\pi} P_n^{m'}(\cos\phi)P_{n'}^{m'}(\cos\phi)\sin\phi\, d\phi}_{\frac{2}{2n+1}\frac{(n+m')!}{(n-m')!}\delta_{nn'}} \cdot A_{m'n\ell}\pi(1+\delta_{0m'})$$

$$= \sum_{\ell}\underbrace{\int_0^a J_{n'+1/2}(\sqrt{\lambda_{n'\ell}}\rho)J_{n'+1/2}(\sqrt{\lambda_{n'\ell'}}\rho)\rho\, d\rho}_{\frac{a^2}{2}J^2_{n'+3/2}(\sqrt{\lambda_{n'\ell'}}a)\delta_{\ell\ell'}} \cdot \frac{2}{2n'+1}\frac{(n'+m')!}{(n'-m')!} A_{m'n'\ell}\pi(1+\delta_{0m'})$$

$$= \frac{a^2}{\cancel{2}} J^2_{n'+3/2}(\sqrt{\lambda_{n'\ell'}}a)\frac{\cancel{2}\pi}{2n'+1}\frac{(n'+m')!}{(n'-m')!}(1+\delta_{0m'})A_{m'n'\ell'}$$

$$\boxed{A_{mn\ell} = \frac{\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{a} F(\rho,\theta,\phi)\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\rho)P_n^m(\cos\phi)\cos(m\theta)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}{\pi a^2(1+\delta_{0m})J^2_{n+3/2}(\sqrt{\lambda_{n\ell}}a)(n+m)!/[(2n+1)(n-m)!]}}$$

Similarly for $B_{mn\ell}$ replacing $\cos m\theta$ by $\sin(m\theta)$.

b) Now, $u(\rho,\theta,\phi,0) = 0 \rightarrow$ vanish the $\cos c\sqrt{\lambda}t$ terms, thus:

$$u = \sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda_{n\ell}}\rho)P_n^m(\cos\phi)\left[A_{mn\ell}\cos m\theta + B_{mn\ell}\sin m\theta\right]\sin c\sqrt{\lambda}t.$$

and using $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = G(\rho,\theta,\phi)$, we have

$$G(\rho,\theta,\phi) = \sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda_{n\ell}}\rho)P_n^m(\cos\phi)\left[A_{mn\ell}\cos m\theta + B_{mn\ell}\sin m\theta\right] c\sqrt{\lambda}$$

Likewise, the coefficients now are given by:

$$\boxed{\left\{\begin{matrix}A_{mn\ell}\\ B_{mn\ell}\end{matrix}\right\} = \frac{\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{a} G(\rho,\theta,\phi)\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\rho)P_n^m(\cos\phi)\left\{\begin{matrix}\cos m\theta\\ \sin m\theta\end{matrix}\right\}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}{c\sqrt{\lambda_{n\ell}}\,\pi a^2(1+\delta_{0m})J^2_{n+3/2}(\sqrt{\lambda_{n\ell}}a)(n+m)!/[(n-m)!(2n+1)]}}$$

Method 1

c) Using the answer of part (a) with $F(\rho,\phi)$ replacing $F(\rho,\theta,\phi)$, the integral that determines the coefficient lead to:

$$\int_{-\pi}^{\pi}\begin{Bmatrix}\cos m\theta\\ \sin m\theta\end{Bmatrix} d\theta \int_0^{\pi}\int_0^{a} F(\rho,\phi)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\,\rho)\,P_n^m(\cos\phi)\,\rho^2 \sin\phi\, d\rho\, d\phi$$

$$= \begin{Bmatrix}2\pi\\ 0\end{Bmatrix}\delta_{0m} \int_0^{\pi}\int_0^{a} F(\rho,\phi)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\,\rho)\,P_n^m(\cos\phi)\,\rho^2 \sin\phi\, d\rho\, d\phi$$

$$\Rightarrow \boxed{\begin{Bmatrix}A_{mn\ell}\\ B_{mn\ell}\end{Bmatrix} = \frac{\begin{Bmatrix}2\pi\\ 0\end{Bmatrix}\delta_{0m}\int_0^{\pi}\int_0^{a} F(\rho,\theta)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\,\rho)\,P_n^m(\cos\phi)\,\rho^2 \sin\phi\, d\rho\, d\phi}{\pi a^2(1+\delta_{0m})\,J_{n+3/2}^2(\sqrt{\lambda_{n\ell}}\,a)\,(n+m)!\,/\,[(n-m)!\,(2n+1)]}}$$

Method 2: Using the expansión found in part (a) for $u(\rho,\theta,\phi,t)$ at $t=0$:

$$F(\rho,\phi) = \sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\,\rho)\,P_n^m(\cos\phi)\,[A_{mn\ell}\cos(m\theta) + B_{mn\ell}\sin(m\theta)]$$

Since $F(\rho,\phi)$ has no dependence of θ $\rightarrow$ $\boxed{\begin{array}{l} B_{mn\ell}=0 \quad \text{and} \\ A_{mn\ell}=0 \ \text{except for } m=0.\end{array}}$

$$\Rightarrow F(\rho,\phi) = \sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\,\rho)\,P_n^0(\cos\phi)\,A_{0n\ell}$$

Now multiplying by $\rho^{-1/2} J_{n'+1/2}(\sqrt{\lambda_{n'\ell'}}\,\rho)\,P_{n'}(\cos\phi)$ and integrating in $\rho^2\sin\phi\, d\rho\, d\phi$:

$$\int_0^{\pi}\int_0^{a} F(\rho,\theta)\,\rho^{-1/2} J_{n'+1/2}(\sqrt{\lambda_{n'\ell'}}\,\rho)\,P_{n'}(\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi =$$

$$\sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty} A_{0n\ell}\int_0^{a} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\,\rho)\,J_{n'+1/2}(\sqrt{\lambda_{n'\ell'}}\,\rho)\,\rho\, d\rho \underbrace{\int_0^{\pi} P_n(\cos\phi)\,P_{n'}(\cos\phi)\,\sin\phi\, d\phi}_{\frac{2}{2n+1}\delta_{nn'}}$$

$$= \sum_{\ell=1}^{\infty} A_{0n'\ell}\,\frac{2}{2n'+1}\underbrace{\int_0^{a} J_{n'+1/2}(\sqrt{\lambda_{n'\ell}}\,\rho)\,J_{n'+1/2}(\sqrt{\lambda_{n'\ell'}}\,\rho)\,\rho\, d\rho}_{\frac{a^2}{2}J_{n'+3/2}^2(\sqrt{\lambda_{n'\ell'}}\,a)\,\delta_{\ell\ell'}}$$

$$= A_{0n'\ell'}\,\frac{\cancel{2}}{2n'+1}\,\frac{a^2}{\cancel{2}}\,J_{n'+3/2}^2(\sqrt{\lambda_{n'\ell'}}\,a)$$

$$\Rightarrow \boxed{A_{0n\ell} = \frac{\int_0^{\pi}\int_0^{a} F(\rho,\theta)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\,\rho)\,P_n(\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi}{a^2 J_{n+3/2}^2(\sqrt{\lambda_{n\ell}}\,a)\,/\,(2n+1)}}$$

This is the same found by method 1 when $m=0$!. Then:

$$\boxed{u(\rho,\theta,\phi,t) = \sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\,\rho)\,P_n(\cos\phi)\cdot A_{0n\ell}\cdot\cos c\sqrt{\lambda_{n\ell}}\,t}$$

with $\lambda_{n\ell} = (z_{n\ell}/a)^2$ $\quad z_{n\ell} \rightarrow$ Zeros of $J_{n+1/2}(z)$.

d) Just as we did above, we can use the two methods:

Method 1: Replacing $G(\rho,\theta,\phi)$ in part (b) by $G(\rho,\phi)$:

$$\begin{Bmatrix} A_{mnl} \\ B_{mnl} \end{Bmatrix} = \frac{\int_{-\pi}^{\pi} \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} d\theta \int_0^{\pi}\int_0^{a} G(\rho,\phi)\, \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n^m(\cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}{c\sqrt{\lambda_{nl}}\,\pi a^2 (1+\delta_{om}) J_{n+3/2}^2(\sqrt{\lambda_{nl}}a)\,(n+m)!/[(n-m)!\,(2n+1)]}$$

$$\boxed{\begin{Bmatrix} A_{mnl} \\ B_{mnl} \end{Bmatrix} = \frac{\begin{Bmatrix} 2\pi\delta_{om} \\ 0 \end{Bmatrix} \int_0^{\pi}\int_0^{a} G(\rho,\phi)\, \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n^m(\cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}{c\sqrt{\lambda_{nl}}\,\pi a^2 (1+\delta_{om}) J_{n+3/2}^2(\sqrt{\lambda_{nl}}a)\,(n+m)!/[(n-m)!\,(2n+1)]}}$$

Method 2: Using $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = G(\rho,\phi)$ in the expansion found in (b):

$$G(\rho,\phi) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n^m(\cos\phi)\left[A_{mnl}\cos m\theta + B_{mnl}\sin m\theta\right] c\sqrt{\lambda_{nl}}$$

but $G(\rho,\phi)$ has no dependence of θ $\rightarrow$ $\boxed{\begin{matrix} B_{mnl}=0 \text{ and} \\ A_{mnl}=0 \text{ except for } m=0 \end{matrix}}$

$$\Rightarrow G(\rho,\phi) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n(\cos\phi)\, A_{onl}\, c\sqrt{\lambda_{nl}}$$

Likewise, multiplying by $\rho^{-1/2} J_{n'+1/2}(\sqrt{\lambda_{n'l'}}\rho) P_{n'}(\cos\phi)$ and integrating in $\rho^2\sin\phi\, d\rho\, d\phi$, by use of orthogonality relations:

$$\boxed{A_{onl} = \frac{\int_0^{\pi}\int_0^{a} G(\rho,\phi)\, \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n(\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi}{c\sqrt{\lambda_{nl}}\, a^2 J_{n+3/2}^2(\sqrt{\lambda_{nl}}a)/(2n+1)}}$$

just the same answer found by method 1 when $m=0$.

$$\Rightarrow \boxed{u(\rho,\theta,\phi,t) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n(\cos\phi)\, A_{onl} \sin c\sqrt{\lambda_{nl}}\, t}$$

e) Method 1: Using the answer of part (a):

$$\begin{Bmatrix} A_{mnl} \\ B_{mnl} \end{Bmatrix} = \frac{\int_{-\pi}^{\pi} \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \cos(3\theta)\, d\theta \int_0^{\pi}\int_0^{a} F(\rho,\phi)\, \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n^m(\cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi}{\pi a^2 (1+\delta_{om}) J_{n+3/2}^2(\sqrt{\lambda_{nl}}a)\,(n+m)!/[(n-m)!\,(2n+1)]}$$

$$\boxed{\begin{Bmatrix} A_{mnl} \\ B_{mnl} \end{Bmatrix} = \frac{\begin{Bmatrix} \pi\delta_{3m} \\ 0 \end{Bmatrix} \int_0^{\pi}\int_0^{a} F(\rho,\phi)\, \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n^m(\cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi}{\pi a^2 (1+\delta_{om}) J_{n+3/2}^2(\sqrt{\lambda_{nl}}a)\,(n+m)!/[(n-m)!\,(2n+1)]}}$$

Method 2: Using the expansion found in (a) for u in $t=0$:

$$F(\rho,\phi)\cos 3\theta = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n^m(\cos\phi)\left[A_{mnl}\cos m\theta + B_{mnl}\sin(m\theta)\right]$$

That series only kept terms as $\cos 3\theta$ $\rightarrow$ $\boxed{\begin{matrix} B_{mnl}=0 \text{ and} \\ A_{mnl}=0 \text{ except for } m=3 \end{matrix}}$

$$\Rightarrow F(\rho,\phi)\cancel{\cos(3\theta)} = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho)\, P_n^3(\cos\phi)\cdot A_{3nl}\,\cancel{\cos(3\theta)}$$

Now, multiplying by $\rho^{-1/2} J_{n'+1/2}(\sqrt{\lambda_{n'l'}}\,\rho)\, P_{n'}^3(\cos\phi)$ and integrating in $\rho^2 \sin\phi\, d\rho\, d\phi$ and using orthogonality we have:

$$\boxed{A_{3nl} = \frac{\int_0^{\pi}\int_0^{a} F(\rho,\phi)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho)\, P_n^3(\cos\phi)\,\rho^2 \sin\phi\, d\rho\, d\phi}{a^2 J_{n+3/2}^2(\sqrt{\lambda_{nl}}\,a)\,(n+3)!\,/\,[(n-3)!\,(2n+1)]}}$$

just the same answer found for method 1 when $m=3$. Then:

$$\boxed{u(\rho,\theta,\phi,t) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho)\, P_n^3(\cos\phi)\cdot A_{3nl}\cos(3\theta)\cos c\sqrt{\lambda_{nl}}\,t}$$

f) Method 1: Using the answer of part (a):

$$\left\{\begin{matrix} A_{mnl} \\ B_{mnl}\end{matrix}\right\} = \frac{\int_{-\pi}^{\pi}\left\{\begin{matrix}\cos m\theta \\ \sin m\theta\end{matrix}\right\}\sin(2\theta)\,d\theta \int_0^{\pi} P_n^m(\cos\phi)\sin\phi\, d\phi \int_0^{a} F(\rho)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho)\,\rho^2 d\rho}{\pi a^2 (1+\delta_{0m})\, J_{n+3/2}^2(\sqrt{\lambda_{nl}}\,a)\,(n+m)!\,/\,[(2n+1)(n-m)!]}$$

$$\boxed{\left\{\begin{matrix} A_{mnl} \\ B_{mnl}\end{matrix}\right\} = \frac{\left\{\begin{matrix} 0 \\ \pi\delta_{2m}\end{matrix}\right\}\int_0^{\pi} P_n^m(\cos\phi)\sin\phi\, d\phi \int_0^{a} F(\rho)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho)\,\rho^2 d\rho}{\pi a^2 (1+\delta_{0m})\, J_{n+3/2}^2(\sqrt{\lambda_{nl}}\,a)\,(n+m)!\,/\,[(2n+1)(n-m)!]}}$$

Method 2: Using the series for $u(\rho,\theta,\phi,t)$ from (a), evaluating in $t=0$

$$F(\rho)\sin 2\theta = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho)\, P_n^m(\cos\phi)\,[A_{mnl}\cos(m\theta) + B_{mnl}\sin(m\theta)]$$

The series kept only terms as $\sin(2\theta) \longrightarrow \boxed{\begin{matrix} A_{mnl} = 0 \\ B_{mnl} = 0 \text{ except for } m=2\end{matrix}}$

$$\Rightarrow F(\rho)\cancel{\sin(2\theta)} = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho)\, P_n^2(\cos\phi)\, B_{2nl}\,\cancel{\sin(2\theta)}$$

multiplying by $\rho^{-1/2} J_{n'+1/2}(\sqrt{\lambda_{n'l'}}\,\rho)\, P_{n'}^2(\cos\phi)$ and integrating in $\rho^2 \sin\phi\, d\rho\, d\phi$, by use of orthogonality:

$$\boxed{B_{2nl} = \frac{\int_0^{\pi} P_n^2(\cos\phi)\sin\phi\, d\phi \int_0^{a} F(\rho)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho)\,\rho^2 d\rho}{a^2 J_{n+3/2}^2(\sqrt{\lambda_{nl}}\,a)\,(n+2)!\,/\,[(2n+1)(n-2)!}}$$

just as we obtained by method 1 with $m=2$. Finally:

$$\boxed{u(\rho,\theta,\phi,t) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho)\, P_n^2(\cos\phi)\, B_{2nl}\sin(2\theta)\cos c\sqrt{\lambda_{nl}}\,t}$$

g) Method 1: Likewise from part (a):

$$\left\{\begin{matrix} A_{mnl} \\ B_{mnl}\end{matrix}\right\} = \frac{\int_{-\pi}^{\pi}\left\{\begin{matrix}\cos m\theta \\ \sin m\theta\end{matrix}\right\} d\theta \int_0^{\pi} P_n^m(\cos\phi)\sin\phi\, d\phi \int_0^{a} F(\rho)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho)\,\rho^2 d\rho}{\pi a^2 (1+\delta_{0m})\, J_{n+3/2}^2(\sqrt{\lambda_{nl}}\,a)\,(n+m)!\,/\,[(2n+1)(n-m)!]}$$

$P_0(\cos\phi) = 1$

$$\Rightarrow \begin{Bmatrix} A_{mnl} \\ B_{mnl} \end{Bmatrix} = \frac{\begin{Bmatrix} 2\pi\delta_{om} \\ 0 \end{Bmatrix} \int_0^{\pi} P_n^m(\cos\phi)\sin\phi\, d\phi \int_0^a F(\rho)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho)\,\rho^2 d\rho}{\pi a^2(1+\delta_{om}) J^2_{n+3/2}(\sqrt{\lambda_{nl}}a)\,(n+m)!/[(n-m)!\,(2n+1)]}$$

$$= \frac{\begin{Bmatrix} 2\pi\delta_{om} \\ 0 \end{Bmatrix} \int_0^{\pi} 1\cdot P_n^0(\cos\phi)\sin\phi\, d\phi \int_0^a F(\rho)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho)\,\rho^2 d\rho}{2\pi a^2 J^2_{n+3/2}(\sqrt{\lambda_{nl}}a)\,/\,(2n+1)}$$

$$\boxed{\begin{Bmatrix} A_{mnl} \\ B_{mnl} \end{Bmatrix} = \frac{\begin{Bmatrix} \delta_{om} \\ 0 \end{Bmatrix} \frac{2}{2(0)+1}\delta_{on} \int_0^a F(\rho)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho)\,\rho^2 d\rho}{a^2 J^2_{n+3/2}(\sqrt{\lambda_{nl}}a)/(2n+1)}}$$

Method 2: Evaluating in $t=0$ the series for $u(\rho,\theta,\phi,t)$ from (a):

$$F(\rho) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho)\, P_n^m(\cos\phi)\,[A_{mnl}\cos(m\theta) + B_{mnl}\sin m\theta]$$

the series has ~~no dependence of θ~~ neither θ-dependence or ϕ-dependence $\Rightarrow$ $\boxed{\begin{array}{l} B_{mnl} = 0 \\ A_{mnl} = 0 \text{ except when } m=0,\ n=0 \end{array}}$

$\Rightarrow F(\rho) = \sum_{l=1}^{\infty} \rho^{-1/2} J_{1/2}(\sqrt{\lambda_{0l}}\rho)\cdot A_{00l}$ where we used $P_0^0(x) = 1$.

Multiplying by $\rho^{-1/2} J_{1/2}(\sqrt{\lambda_{0l'}}\rho)$ and integrating in $\rho^2 d\rho$:

$$\int_0^a F(\rho)\,\rho^{-1/2} J_{1/2}(\sqrt{\lambda_{0l'}}\rho)\,\rho^2 d\rho = \sum_{l=1}^{\infty} A_{00l} \underbrace{\int_0^a J_{1/2}(\sqrt{\lambda_{0l}}\rho)\, J_{1/2}(\sqrt{\lambda_{0l'}}\rho)\,\rho\, d\rho}_{\frac{a^2}{2} J_{3/2}(\sqrt{\lambda_{0l'}}a)\,\delta_{ll'}}$$

$$= A_{00l'}\,\frac{a^2}{2} J_{3/2}(\sqrt{\lambda_{0l'}}a)$$

$$\Rightarrow \boxed{A_{00l} = \frac{2}{a^2 J^2_{3/2}(\sqrt{\lambda_{0l}}a)} \int_0^a F(\rho)\,\rho^{-1/2} J_{1/2}(\sqrt{\lambda_{0l}}\rho)\,\rho^2 d\rho} \rightarrow$$ This is the same answer by method 1

$$\Rightarrow \boxed{u(\rho,\theta,\phi,t) = \sum_{l=1}^{\infty} \rho^{-1/2} J_{1/2}(\sqrt{\lambda_{0l}}\rho)\cdot A_{00l}\cos c\sqrt{\lambda_{0l}}\,t}$$

h) Method 1: Proceed similar to part (g), but now use answer of part (b) replacing $G(\rho,\theta,\phi)$ with $G(\rho)$, then:

$$\begin{Bmatrix} A_{mnl} \\ B_{mnl} \end{Bmatrix} = \frac{\begin{Bmatrix} 2\pi\delta_{om} \\ 0 \end{Bmatrix} \frac{2}{2(0)+1}\delta_{on} \int_0^a G(\rho)\,\rho^{-1/2} J_{1/2}(\sqrt{\lambda_{0l}}\rho)\,\rho^2 d\rho}{2\pi a^2 J^2_{3/2}(\sqrt{\lambda_{0l}}a)\cdot c\sqrt{\lambda_{0l}}}$$

Method 2: Similar to (g), evaluate $t=0$ in $u(\rho,\theta,\phi,t)$ from (b):

$$G(\rho) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho)\, P_n^m(\cos\phi)\,[A_{mnl}\cos m\theta + B_{mnl}\sin m\theta]\, c\sqrt{\lambda}$$

This series has neither θ-dependence or ϕ-dependence $\Rightarrow$

$$\boxed{\begin{array}{l} B_{mnl} = 0 \\ A_{mnl} = 0 \rightarrow \text{except for } m=n=0 \end{array}}$$

$$\rightarrow G(\rho) = \sum_{l=1}^{\infty} \rho^{-1/2} J_{1/2}(\sqrt{\lambda_{0l}}\,\rho) \cdot A_{00l}\; c\sqrt{\lambda_{0l}}$$

Again, multiplying by $\rho^{-1/2} J_{1/2}(\sqrt{\lambda_{0l}}\,\rho)$ and integrating in $\rho^2 d\rho$ by use of orthogonality:

$$\boxed{A_{00l} = \frac{1}{c\sqrt{\lambda_{0l}}\, J_{3/2}^2(\sqrt{\lambda_{0l}}\,a)} \int_0^a G(\rho)\, J_{1/2}(\sqrt{\lambda_{0l}}\,\rho)\, \rho^{3/2} d\rho}$$

This is the same answer as by method 1 $\leftarrow$

$$\Rightarrow \boxed{u(\rho,\theta,\phi,t) = \sum_{l=0}^{\infty} \rho^{-1/2} J_{1/2}(\sqrt{\lambda_{0l}}\,\rho)\, A_{00l} \sin c\sqrt{\lambda_{0l}}\, t}$$

7.10.2 $\dfrac{\partial u}{\partial t} = K \nabla^2 u$ Let $u(\rho,\theta,\phi,t) = W(\rho,\theta,\phi)\, h(t)$ $\rightarrow$ $n = 0,1,2,\ldots$ $m = 0,1,2,\ldots,n$.

$$\Rightarrow \nabla^2 W + \lambda W = 0 \rightarrow W \sim \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda}\,\rho)\, P_n^m(\cos\phi) \left\{ \begin{matrix} \cos m\theta \\ \sin m\theta \end{matrix} \right\}$$

$$h'(t) = -\lambda K h(t) \rightarrow h(t) \sim \exp(-\lambda K t)$$

- If $u(a,\theta,\phi,t) = 0 \rightarrow J_{n+1/2}(\sqrt{\lambda}\,a) = 0 \rightarrow \sqrt{\lambda}\,a = z_{nl} \rightarrow \boxed{\lambda_{nl} = \left(\frac{z_{nl}}{a}\right)^2}$

$l = 1,2,3,\ldots$ $\hookrightarrow$ zeros of $J_{n+1/2}(z)$

$$\boxed{u(\rho,\theta,t) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} e^{-\lambda_{nl} K t} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho) P_n^m(\cos\phi)\left[A_{mnl}\cos m\theta + B_{mnl}\sin m\theta\right]}$$

Evaluating $e^{-\lambda_{nl} K t}\big|_{t=0} = 1$, thus using the initial condition:

a) $$F(\rho,\theta,\phi) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho) P_n^m(\cos\phi)\left[A_{mnl}\cos(m\theta) + B_{mnl}\sin(m\theta)\right]$$

This is identical to one of the initial conditions for the first problem of this section (exercice 7.10.1 (a)), and so the solution for the coefficients will be the same:

$$\boxed{\left\{\begin{matrix} A_{mnl} \\ B_{mnl} \end{matrix}\right\} = \frac{\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{a} F(\rho,\theta,\phi)\, \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho) P_n^m(\cos\phi) \left\{\begin{matrix}\cos m\theta \\ \sin m\theta\end{matrix}\right\} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}{\pi a^2 (1+\delta_{0m})\, J_{n+3/2}^2(\sqrt{\lambda_{nl}}\,a)\, (n+m)! \,/\, [(n-m)!\,(2n+1)]}}$$

b) If $u(\rho,\theta,\phi,0) = F(\rho,\phi)$, it is independent of θ, then the only value of m that survives in the series is $m=0$. Thus the general solution is:

$$u(\rho,\theta,\phi,t) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty} e^{-\lambda_{nl} K t} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho) P_n(\cos\phi) \cdot A_{0nl}$$

where A_{0nl} can be found setting $F(\rho,\phi)$ in place of $F(\rho,\theta,\phi)$ in answer of 7.10.2 (a), just as we did in exercise 7.10.1 (c):

$$A_{0nl} = \frac{\int_0^{\pi}\int_0^{a} F(\rho,\theta)\, \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\,\rho) P_n(\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi}{a^2 J_{n+3/2}^2(\sqrt{\lambda_{nl}}\,a)/(2n+1)}$$

See also Method 2 in solution of excercise 7.10.1 (c).

c) $u(\rho,\theta,\phi,0) = F(\rho,\phi)\cos\theta$ Setting this in the answer of part (a):

$$\begin{Bmatrix} A_{mnl} \\ B_{mnl} \end{Bmatrix} = \frac{\int_{-\pi}^{\pi} \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \cos\theta\, d\theta \int_0^{\pi}\int_0^{a} F(\rho,\phi)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n^m(\cos\phi)\,\rho^2 \sin\phi\, d\rho\, d\phi}{\pi a^2 (1+\delta_{0m}) J_{n+3/2}^2(\sqrt{\lambda_{nl}}a)\,(n+m)!/[(n-m)!(2n+1)]}$$

$$= \frac{\begin{Bmatrix} \pi\delta_{1m} \\ 0 \end{Bmatrix} \int_0^{\pi}\int_0^{a} (\quad \cdots \quad)\,\rho^2 \sin\phi\, d\rho\, d\phi}{\pi a^2 (\quad) \cdots \quad / [\quad]}$$

$$\boxed{\begin{Bmatrix} A_{mnl} \\ B_{mnl} \end{Bmatrix} = \frac{\begin{Bmatrix} \delta_{1m} \\ 0 \end{Bmatrix} \int_0^{\pi}\int_0^{a} F(\rho,\phi)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n^1(\cos\phi)\,\rho^2 \sin\phi\, d\rho\, d\phi}{a^2 J_{n+3/2}^2(\sqrt{\lambda_{nl}}a)\,(n+1)!/[(n-1)!(2n+1)]}}$$

In this case, then the only terms that survive are the A_{1nl}. Thus the general solution is:

$$\boxed{u(\rho,\theta,\phi,t) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n^1(\cos\phi)\cdot A_{1nl}\cos\theta\, e^{-\lambda_{nl}Kt}}$$

Also, the coefficient A_{1nl} can be found by Method 2, equalling that series to $F(\rho,\phi)\cos\theta$ at $t=0$ (See excersise 7.10.1 (e))

d) $u(\rho,\theta,\phi,0) = F(\rho)$ → This is a special case of part (b), for which $n=0$ (Recall that $P_0(x)=1$). It is independent of both θ and ϕ, then the only values of m and n that survive in the series are $m=0$, $n=0$. Thus:

$$u(\rho,\theta,\phi,t) = \sum_{l=0}^{\infty} e^{-\lambda_{nl}Kt}\,\rho^{-1/2} J_{1/2}(\sqrt{\lambda_{nl}}\rho)\cdot A_{00l}$$

where A_{00l} can be found from answer for (b) setting $n=0$:

$$\boxed{A_{00l} = \frac{\int_0^{\pi} 1\cdot P_0(\cos\phi)\sin\phi\, d\phi \int_0^{a} F(\rho)\,\rho^{-1/2} J_{1/2}(\sqrt{\lambda_{0l}}\rho)\,\rho^2 d\rho}{a^2 J_{3/2}^2(\sqrt{\lambda_{0l}}a)}}$$

where $\int_0^{\pi} \underbrace{P_0(\cos\phi)}_{1} \sin\phi\, d\phi = 2$

→ This also can be found as in exercise 7.10.1 (g).

7.10.3 $\frac{\partial u}{\partial t} = K\nabla^2 u$. Let $u(\rho,\theta,\phi,t) = W(\rho,\theta,\phi)\,h(t)$

→ $\nabla^2 W + \lambda W = 0 \longrightarrow W \sim \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda}\rho) P_n^m(\cos\phi) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix}$ $n=0,1,2,\ldots$ $m=0,1,2,\ldots,n.$

$h'(t) = -\lambda K h(t) \longrightarrow h(t) \sim \exp(-\lambda K t)$

- If $\frac{\partial u}{\partial \rho}(a,\theta,\phi,t) = 0 \rightarrow J'_{n+1/2}(\sqrt{\lambda}a) = 0 \rightarrow \sqrt{\lambda}a = z_{nl} \rightarrow \lambda_{nl} = (z_{nl}/a)^2$ $l = 1,2,3,\ldots$ ↳ zeros of $J'_{n+1/2}(z)$

$$\rightarrow \boxed{u(\rho,\theta,t) = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} e^{-\lambda_{nl}Kt}\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{nl}}\rho) P_n^m(\cos\phi)\left[A_{mnl}\cos m\theta + B_{mnl}\sin m\theta\right]}$$

It is the same as in 7.10.2 exercise, but now z_{nl} are the zeros of the derivative of $J_{n+1/2}$

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$$\frac{\partial u}{\partial r}(a,\theta,z) = \sum_{m=0}\sum_{n=0} F_{mn}\left(\frac{n\pi}{H}\right) I_m'\left[\frac{n\pi a}{H}\right] \cos\left(\frac{n\pi z}{H}\right)\cos(m\theta) = \quad (\theta,z)$$

$$\Rightarrow \beta(\theta,z) = F_{00}\left(\frac{0\pi}{H}\right) I_0'[0] + \sum_{m=1} F_{m0}\left(\frac{0\pi}{H}\right) I_m'[0]\cos m\theta + \sum_{n=1} F_{0n}\left(\frac{n\pi}{H}\right) I_0'\left[\frac{n\pi a}{H}\right]\cos\left(\frac{n\pi z}{H}\right)$$

$$+ \sum_{m=1}\sum_{n=1} F_{mn}\left(\frac{n\pi}{H}\right) I_m'\left[\frac{n\pi a}{H}\right]\cos\left(\frac{n\pi z}{H}\right)\cos(m\theta).$$

Multiplying by $\cos(n'\pi z/H)$ and integrating:

$$\int_0^{\pi}\int_0^{H} \beta(\theta,z)\cos\left(\frac{n'\pi z}{H}\right) dz\, d\theta = \sum_{n=1} F_{0n}\left(\frac{n\pi}{H}\right) I'\left[\frac{n\pi a}{H}\right]\cdot \underbrace{\pi\int_0^{H}\cos\frac{n\pi z}{H}\cos\frac{n'\pi z}{H}\, dz}_{(H/2)\,\delta_{nn'}}$$

$$+ \sum_{m=1}\sum_{n=1} F_{mn}\left(\frac{n\pi}{H}\right) I_m'\left[\frac{n\pi a}{H}\right]\int_0^{\pi}\cos m\theta\, d\theta \cdot \int_0^{H}\cos\frac{n\pi z}{H}\cos\frac{n'\pi z}{H}\, dz.$$

$$= F_{0n'}\left(\frac{n'\pi}{H}\right) I'[n\pi a/H]\,\frac{\pi H}{2}$$

$$\Rightarrow \boxed{F_{0n} = \frac{2}{n\pi^2 I'[n\pi a/H]}\int_0^{\pi}\int_0^{H}\beta(\theta,z)\cos\left(\frac{n\pi z}{H}\right) dz\, d\theta} \qquad n\neq 0$$

Now, multiplying by $\cos(n'\pi z/H)\cos(m'\theta)$ and integrating in similar way:

$$\int_0^{\pi}\int_0^{H}\beta(\theta,z)\cos\left(\frac{n'\pi z}{H}\right)\cos(m\theta)\, dz\, d\theta = 0 + \sum_{m=1}\sum_{n=1} F_{mn}\left(\frac{n\pi}{H}\right) I'\left[\frac{n\pi a}{H}\right]\frac{\pi}{2}\delta_{mm'}\frac{H}{2}\delta_{nn'}$$

$$= F_{m'n'}\left(\frac{n'\pi}{H}\right) I'[n'\pi a/H]\,\frac{\pi H}{4}$$

$$\boxed{F_{mn} = \frac{4}{n\pi^2 I'[n\pi a/H]}\int_0^{\pi}\int_0^{H}\beta(\theta,z)\cos\left(\frac{n\pi z}{H}\right)\cos(m\theta)\, dz\, d\theta} \qquad m,n\neq 0$$

Add $(1+\delta_{0m})$ for summarize for $n\geq 0$

Note that we don't need F_{m0} when $m\neq 0$ since $I_m[0]=0$ for all $m\geq 1$. then the terms with $n=0$, $m\neq 0$ no contribute.

But we need find F_{00} knowing $u(r,\theta,0)$, then:

$$u(r,\theta,0) = F_{00} + \sum_{\substack{m,n \\ \text{No both zero}}} F_{mn} I_m[n\pi a/H]\cos\left(\frac{n\pi z}{H}\right)\cos(m\theta)$$

integrating in $dz\,d\theta$ and using orthogonality:

$$\int_0^{\pi}\int_0^{H} u(r,\theta,0)\, dz\, d\theta = F_{00}(\pi H) \rightarrow \boxed{F_{00} = \frac{1}{\pi H}\int_0^{\pi}\int_0^{H} u(r,\theta,0)\, dz\, d\theta}$$

Mean value of u over the bottom. (Condition for existence of the solution)

See Orthogonality for Bessel functions in Arfken.

Recall that since $z_{n\ell} = \sqrt{\lambda_{n\ell}}\,a \rightarrow$ are the zeros of $J'_\nu(z)$

$$\rightarrow \boxed{\int_0^a J_\nu(\sqrt{\lambda_{n\ell}}\rho)\, J_\nu(\sqrt{\lambda_{n\ell'}}\rho)\,\rho\, d\rho = \frac{a^2}{2}\left[1-\left(\frac{\nu}{\sqrt{\lambda_{n\ell}}\,a}\right)^2\right] J_\nu^2(\sqrt{\lambda_{n\ell}}\,a)\,\delta_{\ell\ell'}}$$

Now, since $e^{-\lambda_{n\ell}kt}\big|_{t=0} = 1$, the initial value for each problem gives:

a) $$F(\rho,\theta,\phi) = \sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\rho)\, P_n^m(\cos\phi)\left[A_{mn\ell}\cos m\theta + B_{mn\ell}\sin m\theta\right]$$

multiplying by $\rho^{-1/2} J_{n'+1/2}(\sqrt{\lambda_{n'\ell'}}\rho)\, P_{n'}^{m'}(\cos\phi)\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\}$ and integrating in dV

$$\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{a} F(\rho,\theta,\phi)\,\rho^{-1/2} J_{n'+1/2}(\sqrt{\lambda_{n'\ell'}}\rho)\, P_{n'}^{m'}(\cos\phi)\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \sum_{\ell,n,m}\int_0^a J_{n+1/2}(\cdot)\, J_{n'+1/2}(\cdot)\,\rho\, d\rho \int_{-\pi}^{\pi} P_n^m P_{n'}^{m'}\sin\phi\, d\phi\left[A_{mn\ell}\underbrace{\int_{-\pi}^{\pi}\cos m\theta\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\}d\theta}_{\left\{\begin{matrix}\pi(1+\delta_{0m})\delta_{mm'}\\ 0\end{matrix}\right\}} + B\underbrace{\int_{-\pi}^{\pi}\sin m\theta\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\}d\theta}_{\left\{\begin{matrix}0\\ \pi(1-\delta_{0m})\delta_{mm'}\end{matrix}\right\}}\right]$$

$$= \sum_{\ell,n}\int_0^a J_{n+1/2}(\cdot)\, J_{n'+1/2}(\cdot)\,\rho\, d\rho \underbrace{\int_{-\pi}^{\pi} P_n^{m'} P_{n'}^{m'}\sin\phi\, d\phi}_{\frac{2}{2n+1}\frac{(n+m')!}{(n-m')!}\delta_{nn'}}\,\pi\left\{\begin{matrix}(1+\delta_{0m'})\\ (1-\delta_{0m'})\end{matrix}\right.\left.\begin{matrix}A_{m'n\ell}\\ B_{m'n\ell}\end{matrix}\right\}.$$

$$= \sum_{\ell}\underbrace{\int_0^a J_{n'+1/2}(\sqrt{\lambda_{n\ell}}\rho)\, J_{n'+1/2}(\sqrt{\lambda_{n'\ell'}}\rho)\,\rho\, d\rho}_{\frac{a^2}{2}\left[1-\left(\frac{n'+1/2}{\sqrt{\lambda_{n'\ell'}}\,a}\right)^2\right] J^2_{n'+1/2}(\sqrt{\lambda_{n'\ell'}}\,a)\,\delta_{\ell\ell'}}\cdot\frac{2}{2n'+1}\frac{(n'+m')!}{(n'-m')!}\,\pi\left\{\begin{matrix}(1+\delta_{0n'})\\ (1-\delta_{0n'})\end{matrix}\right.\left.\begin{matrix}A_{m'n'\ell'}\\ B_{m'n'\ell'}\end{matrix}\right\}$$

$$= \frac{a^2}{\cancel{2}}\left[1-\left(\frac{n'+1/2}{\sqrt{\lambda_{n'\ell'}}\,a}\right)^2\right] J^2_{n'+1/2}(\sqrt{\lambda_{n'\ell'}}\,a)\cdot\frac{\cancel{2}}{2n'+1}\frac{(n'+m')!}{(n'-m')!}\,\pi\left\{\begin{matrix}(1+\delta_{0n'})\\ (1-\delta_{0n'})\end{matrix}\right.\left.\begin{matrix}A_{m'n'\ell'}\\ B_{m'n'\ell'}\end{matrix}\right\}$$

$$\boxed{\left\{\begin{matrix}(1+\delta_{0n})A_{mn}\\ (1-\delta_{0n})B_{mn}\end{matrix}\right\} = \frac{\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{a} F(\rho,\theta,\phi)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\rho)\, P_n^m(\cos\phi)\left\{\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}\right\}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}{\pi a^2\left[1-\left(\frac{n+1/2}{\sqrt{\lambda_{n\ell}}\,a}\right)^2\right] J^2_{n+1/2}(\sqrt{\lambda_{n\ell}}\,a)\,(n+m)!\,/\left[(n-m)!\,(2n+1)\right]}}$$

Note the similarity with answer of exercise 7.10.1(a) OR 7.10.2(a).

b) Using the answer of part a) with $F(\rho,\phi)$ replacing $F(\rho,\theta,\phi)$:

$$\left\{\begin{matrix}A_{mn\ell}\\ B_{mn\ell}\end{matrix}\right\} = \frac{\int_{-\pi}^{\pi}\left\{\begin{matrix}\cos m\theta\\ \sin m\theta\end{matrix}\right\}d\theta\int_0^{\pi}\int_0^{a} F(\rho,\phi)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\rho)\, P_n^m(\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi}{\pi a^2(1+\delta_{0n})\left[\;\cdots\;\right] J^2_{n+1/2}(\cdots)\,(n+m)!\,/\left[\;\cdots\;\right]}$$

$$\boxed{\left\{\begin{matrix}A_{mn\ell}\\ B_{mn\ell}\end{matrix}\right\} = \frac{\left\{\begin{matrix}2\pi\delta_{0m}\\ 0\end{matrix}\right\}\int_0^{\pi}\int_0^{a} F(\rho,\phi)\,\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n\ell}}\rho)\, P_n(\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi}{\pi a^2(1+\delta_{0n})\left[1+\left(\frac{n+1/2}{\sqrt{\lambda_{n\ell}}\,a}\right)^2\right] J^2_{n+1/2}(\sqrt{\lambda_{n\ell}}\,a)\,/\,(2n+1)}}$$

Then note that $\boxed{\begin{array}{l} B_{mnl}=0 \\ A_{mnl}=0 \text{ except for } m=0 \end{array}}$

$$\Rightarrow \boxed{u(\rho,\theta,\phi,t)=\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{nl}}\rho)P_n(\cos\phi)\,A_{0nl}\,e^{-\lambda_{nl}Kt}}$$

This also can be found by Method 2 (See exercise 7.10.1(c))

c) Using the answer of part (a):

$$\left\{\begin{matrix}A_{mnl}\\B_{mnl}\end{matrix}\right\}=\frac{\int_{-\pi}^{\pi}\left\{\begin{matrix}\cos m\theta\\ \sin m\theta\end{matrix}\right\}\cos(3\theta)\,d\theta\int_0^{\pi}\int_0^{\alpha}F(\rho,\phi)\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{nl}}\rho)P_n^m(\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi}{\pi\alpha^2(1+\delta_{0n})\left[1-\left(\frac{n+1/2}{\sqrt{\lambda_{nl}}\alpha}\right)^2\right]J_{n+1/2}^2(\sqrt{\lambda_{nl}}\alpha)\,(n+m)!/[(n-m)!\,(2n+1)]}$$

$$=\frac{\left\{\begin{matrix}\pi\delta_{3m}\\0\end{matrix}\right\}\int_0^{\pi}\int_0^{\alpha}F(\rho,\phi)\rho^{-1/2}\cdots\cdots\cdots\rho^2\sin\phi\,d\rho\,d\phi}{\pi\alpha^2(\cdots)[\cdots\cdots]J_{n+1/2}^2(\cdots)\,(n+m)!/[\cdots\cdots]}$$

$$\boxed{\left\{\begin{matrix}A_{mnl}\\B_{mnl}\end{matrix}\right\}=\frac{\int_0^{\pi}\int_0^{\alpha}F(\rho,\phi)\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{nl}}\rho)P_n^3(\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi}{\alpha^2(1+\delta_{0n})\left[1-\left(\frac{n+1/2}{\sqrt{\lambda_{nl}}\alpha}\right)^2\right]J_{n+1/2}^2(\sqrt{\lambda_{nl}}\alpha)\,(n+3)!/[(2n+1)(n-3)!]}\left\{\begin{matrix}\delta_{3m}\\0\end{matrix}\right\}}$$

Then the series only kept terms as $\cos 3\theta \rightarrow \boxed{\begin{array}{l} B_{mnl}=0 \\ A_{mnl}=0 \text{ except if } m=3 \end{array}}$

$$\Rightarrow u(\rho,\theta,\phi,t)=\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{nl}}\rho)P_n^3(\cos\phi)\cdot A_{3nl}\cos(3\theta)\,e^{-\lambda_{mnl}Kt}$$

This can be found by a second method (see exercise 7.10.1(e))

d) Evaluating in $t=0$ the series for $u(\rho,\theta,\phi,t)$ above, we have:

$$F(\rho)=\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{nl}}\rho)P_n^m(\cos\phi)\left[A_{mnl}\cos m\theta+B_{mnl}\sin m\theta\right]$$

since this has not dependence of θ or ϕ $\Rightarrow$ $\boxed{\begin{array}{l} B_{mnl}=0 \\ A_{mnl}=0 \text{ except when } m=0,\ n=0. \end{array}}$

$$\Rightarrow F(\rho)=\sum_{l=1}^{\infty}\rho^{-1/2}J_{1/2}(\sqrt{\lambda_{0l}}\rho)\cdot A_{00l}.$$

Multiplying by $\rho^{-1/2}J_{1/2}(\sqrt{\lambda_{0l'}}\rho)$ and integrating in $\rho^2 d\rho$:

$$\int_0^{\alpha}F(\rho)\rho^{-1/2}J_{1/2}(\sqrt{\lambda_{0l'}}\rho)\rho^2\,d\rho=\sum_{l=1}^{\infty}A_{00l}\underbrace{\int_0^{\alpha}J_{1/2}(\sqrt{\lambda_{0l}}\rho)J_{1/2}(\sqrt{\lambda_{0l'}}\rho)\rho\,d\rho}_{\frac{\alpha^2}{2}\left[1-\left(\frac{1/2}{\sqrt{\lambda_{0l}}\alpha}\right)^2\right]J_{1/2}^2(\sqrt{\lambda_{0l}}\alpha)}$$

$$=A_{00l'}\frac{\alpha^2}{2}\left[1-\frac{1}{4\lambda_{0l'}\alpha^2}\right]J_{1/2}^2(\sqrt{\lambda_{0l'}}\alpha)$$

$$\Rightarrow \boxed{A_{00l}=\frac{2}{\alpha^2\left[1-1/4\lambda_{0l}\alpha^2\right]J_{1/2}^2(\sqrt{\lambda_{0l}}\alpha)}\int_0^{\alpha}F(\rho)J_{1/2}(\sqrt{\lambda_{0l}}\rho)\rho^{3/2}\,d\rho}$$

$$\Rightarrow u(\rho,\theta,\phi,t)=\sum_{l=1}^{\infty}\rho^{-1/2}J_{1/2}(\sqrt{\lambda_{0l}}\rho)\,A_{00l}\,e^{-\lambda_{0l}Kt}.$$

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7.10.4 $\frac{d}{d\phi}\left(\sin\phi\,\frac{dg}{d\phi}\right)+\left(\mu\sin\phi-\frac{m^2}{\sin\phi}\right)g=0$

According to eq (5.3.1), here $p=\sin\phi$, $q=-\frac{m^2}{\sin\phi}$, $\sigma=\sin\phi$

$$\Rightarrow \mu=\frac{\cancel{-\sin\phi\, g\, dg/d\phi\,\Big|_0^\pi}^{\,0}+\int_0^\pi\left[\sin\phi\,(dg/d\phi)^2+m^2g^2/\sin\phi\right]}{\int_0^\pi[g(\phi)]^2\sin\phi\,d\phi}$$ → Rayleigh Quotient Eq (5.3.8)

$\sin\phi\big|_0^\pi=0$

Since $0\le\phi\le\pi \rightarrow 0\le\sin\phi\le 1 \rightarrow \mu\ge 0$.

If $\mu=0 \rightarrow \sin\phi(dg/d\phi)^2+m^2g^2/\sin\phi=0$. Since both terms are positive, then: $dg/d\phi=0$ and $mg=0$

$g=cte$ and $m=0$ or $g=0$.

For no trivial solution $\rightarrow$ $g=cte=1$ and $m=0$. Note that it is an eigenfunction, just: $P_0^0(\cos\phi)=1$ with eigenvalue $\mu=0$. $(n=0,\ m=0)$. (Recall $\mu=n(n+1)$)

7.10.5 $\frac{d}{dx}\left[(1-x^2)\frac{dg}{dx}\right]+\left(\mu-\frac{m^2}{1-x^2}\right)g=0$

Now $p=(1-x^2)$, $q=-m^2/(1-x^2)$ and $\sigma=1$ according to eq (5.3.1)

$$\Rightarrow \mu=\frac{\cancel{-(1-x^2)\,g\,dg/dx\,\Big|_{-1}^{1}}^{\,0}+\int_{-1}^{1}\left[(1-x^2)(dg/dx)^2+m^2g^2/(1-x^2)\right]}{\int_{-1}^{1}[g(x)]^2\,dx}$$ → RQ Eq (5.3.8)

$(1-x^2)\big|_{-1}^{1}=0$

since $-1\le x\le 1 \rightarrow 0\le(1-x^2)\le 1 \rightarrow \mu\ge 0$ since both the numerator and denominator are nonnegative.

If $\mu=0 \rightarrow (1-x^2)(dg/dx)^2+m^2g^2/(1-x^2)=0$. Since both terms are nonnegative: $\rightarrow dg/dx^2=0$ and $m=0$ or $g=0$.

For no trivial solution $\rightarrow$ $g=cte=1$ and $m=0$. Just the same answer above. $g=1=P_0^0(x)$. and $\mu=n(n+1)=0$ $(n=0)$.

7.10.6 $\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)+(\lambda\rho^2-\mu)f=0 \rightarrow p=\rho^2,\ q=-\mu,\ \sigma=\rho^2$ according to eq (5.3.1).

$$\Rightarrow \lambda=\frac{\cancel{-\rho^2 f\,(df/d\rho)\Big|_0^a}^{\,0}+\int_0^a\left[\rho^2(df/d\rho)^2+\mu f^2\right]d\rho}{\int_0^a[f(\rho)]\,\rho^2\,d\rho}$$ → RQ Eq (5.3.8)

$\rho\big|_0=0 \rightarrow$
$f(a)=0$

Since $\mu\ge 0$ (as it was showed in exercise 7.10.4 - 7.10.5), and the others terms are nonnegative $\Rightarrow \lambda\ge 0$

If $\lambda=0 \rightarrow \rho^2(df/d\rho)^2+\mu f^2=0 \rightarrow df/d\rho=0$ and $\mu=0$ OR $f=0$

For no trivial solution $\rightarrow$ $f=cte$, but $f(a)=0 \rightarrow cte=0 \rightarrow \boxed{\lambda>0}$

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This integration is over the surface of an sfera of radius a.

7.10.7 $\nabla^2 W + \lambda W = 0 \rightarrow \lambda = \dfrac{-\cancel{\oint W \vec{\nabla} W \cdot \hat{n}\, ds}^{\,0} + \int_V |\nabla W|^2 dV}{\int_V W^2 dV}$ Rayleigh Quotient Eq (7.6.5).

$W(a) = 0 \rightarrow \oint_S W \vec{\nabla} W \cdot d\vec{s} = 0$

$W|_S = W(a, \theta, \phi) = 0 \quad \Rightarrow \quad \lambda \geq 0$

If $\lambda = 0 \rightarrow \nabla W = 0 \rightarrow \dfrac{\partial W}{\partial \rho}\hat{\rho} + \dfrac{1}{\rho}\dfrac{\partial W}{\partial \phi}\hat{\phi} + \dfrac{1}{\rho \sin\phi}\dfrac{\partial W}{\partial \theta}\hat{\theta} = 0.$

$\Rightarrow \dfrac{\partial W}{\partial \rho} = \dfrac{\partial W}{\partial \phi} = \dfrac{\partial W}{\partial \theta} = 0$ since $\rho > 0$ and $\Rightarrow \sin\phi > 0$ $(0 < \phi < \pi)$

then $W = cte$, but $W(\rho = a) = 0 \rightarrow cte = 0 \rightarrow \lambda = 0$ is not an eigenvalue $\Rightarrow \boxed{\lambda > 0}$.

7.10.8

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Inside a Sphere $(\rho < \alpha)$

7.10.9 $\nabla^2 u = 0 \longrightarrow \boxed{u(\rho,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \rho^n [A_{mn}\cos m\theta + B_{mn}\sin m\theta] P_n^m(\cos\phi)}$

a) $F(\phi)\cos 4\theta = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \alpha^n [A_{mn}\cos m\theta + B_{mn}\sin m\theta] P_n^m(\cos\phi)$

Since the θ-dependence on the boundary is explicitly given as $\cos 4\theta$, all other $\cos m\theta$ and $\sin m\theta$ terms are elimated.

$\Rightarrow F(\phi)\cancel{\cos 4\theta} = \sum_{n=4}^{\infty} \alpha^n A_{4n} \cancel{\cos(4\theta)} P_n^4(\cos\phi)$ and $\boxed{\begin{array}{l} B_{mn}=0 \\ A_{mn}=0 \text{ if } m\neq 4\end{array}}$

multiplying by $P_{n'}^4(\cos\phi)$ and integrating in $\sin\phi\, d\phi$:

$$\int_0^{\pi} F(\phi) P_{n'}^4(\cos\phi)\sin\phi\, d\phi = \sum_{n=4}^{\infty} \alpha^n A_{4n} \underbrace{\int_0^{\pi} P_n^4(\cos\phi) P_{n'}^4(\cos\phi)\sin\phi\, d\phi}_{\frac{2}{2n+1}\frac{(n+4)!}{(n-4)!}\delta_{nn'}}$$

$$= \alpha^{n'} A_{4n'} \frac{2}{2n'+1}\frac{(n'+4)!}{(n'-4)!}$$

$$\Rightarrow \boxed{A_{4n} = \frac{(2n+1)(n-4)!}{2\alpha^n (n+4)!}\int_0^{\pi} F(\phi) P_n^4(\cos\phi)\sin\phi\, d\phi}$$

and $u(\rho,\theta,\phi) = \sum_{n=4}^{\infty} A_{4n}\rho^n \cos(4\theta) P_n^4(\cos\phi)$.

b) $F(\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \alpha^n [A_{mn}\cos m\theta + B_{mn}\sin m\theta] P_n^m(\cos\phi)$

Since this has not θ-dependence, then $\boxed{\begin{array}{l} B_{mn}=0 \\ A_{mn}=0 \text{ except for } m=0\end{array}}$

$\Rightarrow F(\phi) = \sum_{n=0}^{\infty} \alpha^n A_{0n} P_n^0(\cos\phi)$ multiplying by $P_{n'}(\cos\phi)$ and integrating:

$$\int_0^{\pi} F(\phi) P_{n'}(\cos\phi)\sin\phi\, d\phi = \sum_{n=0}^{\infty} \alpha^n A_{0n} \underbrace{\int_0^{\pi} P_n^0(\cos\phi) P_{n'}(\cos\phi)\sin\phi\, d\phi}_{\frac{2}{2n+1}\delta_{nn'}}$$

$$= \alpha^{n'} A_{0n'} \frac{2}{2n'+1}$$

$$\Rightarrow \boxed{A_{0n} = \frac{(2n+1)}{2\alpha^n}\int_0^{\pi} F(\phi) P_n(\cos\phi)\sin\phi\, d\phi}$$

$$\Rightarrow u(\rho,\theta,\phi) = \sum_{n=0}^{\infty} A_{0n}\rho^n P_n(\cos\phi)$$

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$$\boxed{u(\rho,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \rho^n\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]P_n^m(\cos\phi)}$$

c) $$\frac{\partial u}{\partial \rho}(\rho,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} n\rho^{n-1}\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]P_n^m(\cos\phi)$$

$$\Rightarrow F(\phi)\cos 4\theta = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} n a^{n-1}\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]P_n^m(\cos\phi)$$

Since the θ-dependence is only given as $\cos 4\theta$,

$$\Rightarrow F(\phi)\cancel{\cos 4\theta} = \sum_{n=4}^{\infty} n a^{n-1} A_{4n}\cancel{\cos(4\theta)}P_n^4(\cos\phi)$$ and $\boxed{B_{mn}=0,\ A_{mn}=0 \text{ when } m\neq 4}$

multiplying by $P_{n'}^4(\cos\phi)$ and integrating:

$$\int_0^{\pi} F(\phi)P_{n'}^4(\cos\phi)\sin\phi\, d\phi = \sum_{n=4}^{\infty} n a^{n-1} A_{4n}\underbrace{\int_0^{\pi} P_n^4(\cos\phi)P_{n'}^4(\cos\phi)\sin\phi\, d\phi}_{\frac{2}{2n+1}\frac{(n+4)!}{(n-4)!}\delta_{nn'}}$$

$$= n' a^{n'-1} A_{4n'}\frac{2}{(2n'+1)}\frac{(n'+4)!}{(n'-4)!}$$

$$\Rightarrow A_{4n} = \frac{(2n+1)(n-4)!}{2n a^{n-1}(n+4)!}\int_0^{\pi} F(\phi)P_n^4(\cos\phi)\sin\phi\, d\phi$$

$$\boxed{u(\rho,\theta,\phi) = \sum_{n=4}^{\infty} A_{4n}\rho^n\cos(4\theta)P_n^4(\cos\phi)}$$

d) Now $$F(\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} n a^{n-1}\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]P_n^m(\cos\phi).$$

Since this has not θ-dependence $\Rightarrow$ $\boxed{B_{mn}=0,\ A_{mn}=0 \text{ except for } m=0}$

$$\Rightarrow F(\phi) = \sum_{n=0}^{\infty} n a^{n-1} A_{0n}P_n^0(\cos\phi)$$ multiplying by $P_{n'}(\cos\phi)$ and integrating:

$$\int_0^{\pi} F(\phi)P_{n'}(\cos\phi)\sin\phi\, d\phi = \sum_{n=1}^{\infty} n a^{n-1} A_{0n}\underbrace{\int_0^{\pi} P_n(\cos\phi)P_{n'}(\cos\phi)\sin\phi\, d\phi}_{\frac{2}{2n+1}\delta_{nn'}}$$

$$= n' a^{n'-1} A_{0n'}\frac{2}{2n'+1}$$

$$\Rightarrow \boxed{A_{0n} = \frac{(2n+1)}{2n a^{n-1}}\int_0^{\pi} F(\phi)P_n(\cos\phi)\sin\phi\, d\phi}$$

This is singular when $n=0$, but $\int_0^{\pi} F(\phi)\sin\phi\, d\phi = 0 \rightarrow \boxed{A_{00}=0}$
(Recall $P_0(\cos\phi)=1$).

$$\Rightarrow \boxed{u(\rho,\theta,\phi) = \sum_{n=1}^{\infty} A_{0n}\rho^n P_n(\cos\phi)}$$

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As in previous exercises: $\boxed{U(\rho,\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\rho^{n}\left[A_{mn}\cos m\theta+B_{mn}\sin m\theta\right]P_n^m(\cos\phi)}$

e) $F(\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} n a^{n-1}\left[A_{nm}\cos m\theta+B_{mn}\sin m\theta\right]P_n^m(\cos\phi)$.

multiplying by $\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\}P_{n'}^{m'}(\cos\phi)$ and integrating in $\sin\phi\, d\phi\, d\theta$

$$\int_0^{2\pi}\int_0^{\pi}F(\theta,\phi)\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\}P_{n'}^{m'}(\cos\phi)\sin\phi\, d\phi\, d\theta=$$

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} n a^{n-1}\left[A_{nm}\underbrace{\int_0^{2\pi}\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\}\cos m\theta\, d\theta}_{\left\{\begin{matrix}\pi(1+\delta_{0m})\delta_{mm'}\\ 0\end{matrix}\right\}}+B_{mn}\underbrace{\int_0^{2\pi}\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\}\sin m\theta\, d\theta}_{\left\{\begin{matrix}0\\ \pi(1-\delta_{0m})\delta_{mm'}\end{matrix}\right\}}\right]\int_0^{\pi}P_n^m P_{n'}^{m'}\sin\phi\, d\phi$$

$$=\sum_{n=m}^{\infty} n a^{n-1}\pi\cdot\left\{\begin{matrix}(1+\delta_{0m})A_{nm'}\\ (1-\delta_{0m})B_{nm'}\end{matrix}\right\}\underbrace{\int_0^{\pi}P_n^{m'}(\cos\phi)P_{n'}^{m'}(\cos\phi)\sin\phi\, d\phi}_{\frac{2}{2n+1}\frac{(n+m')!}{(n-m')!}\delta_{nn'}}$$

$$=n' a^{n'-1}\pi\left\{\begin{matrix}1+\delta_{0m'}\\ 1-\delta_{0m'}\end{matrix}\right\}\frac{2}{2n'+1}\frac{(n'+m')!}{(n'-m')!}\left\{\begin{matrix}A_{n'm'}\\ B_{n'm'}\end{matrix}\right\}$$

$$\boxed{\left\{\begin{matrix}(1+\delta_{0m})A_{mn}\\ (1-\delta_{0m})A_{mn}\end{matrix}\right\}=\frac{(2n+1)(n-m)!}{2n a^{n-1}\pi(1+\delta_{0m})(n+m)!}\int_0^{2\pi}\int_0^{\pi}F(\theta,\phi)\left\{\begin{matrix}\cos m\theta\\ \sin m\theta\end{matrix}\right\}P_n^m(\cos\phi)\sin\phi\, d\phi\, d\theta}$$

They are singular when $n=0$, but since

$$\int_0^{\pi}\int_0^{2\pi}F(\theta,\phi)\sin\phi\, d\theta\, d\phi=0\rightarrow\boxed{A_{00}=B_{00}=0}$$

Recall that $P_0^0(\cos\phi)=1$ and $m\leq n$ for define Legendre Polynomios.

7.10.10 As in the text we know that the angular solution are

$$q(\theta)g(\phi)\sim\left\{\begin{matrix}\cos m\theta\\ \sin m\theta\end{matrix}\right\}P_n^m(\cos\phi)$$

The radial equation will be: $\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)-n(n+1)f=0$ Eq (7.10.28) in the text.

the solutions to this were found in the text; they are $f\sim\rho^n$ or $f\sim\rho^{-(n+1)}$. If we assume that the potential is bounded at $\rho\to\infty$, we reject the unbounded solution ρ^n. Thus the final ~~product~~ the solution is:

$$\boxed{U(\rho,\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\rho^{-(n+1)}\left[A_{mn}\cos m\theta+B_{mn}\sin m\theta\right]P_n^m(\cos\phi)}$$

Solution outside of the sphere $(\rho>a)$

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Using the potential given on the sphere at $\rho=a$. surface

a) $$F(\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} a^{-(n+1)}\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]P_n^m(\cos\phi).$$

Multiplying by $\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\} P_{n'}^{m'}(\cos\phi)$ and integrating:

$$\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\} P_{n'}^{m'}(\cos\phi)\sin\phi\, d\phi\, d\theta$$

$$=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} a^{-(n+1)}\left[A_{mn}\underbrace{\int_0^{2\pi}\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\}\cos m\theta\, d\theta}_{\left\{\begin{matrix}\pi(1+\delta_{0m})\delta_{mm'}\\ 0\end{matrix}\right\}} + B_{mn}\underbrace{\int_0^{\pi}\left\{\begin{matrix}\cos m'\theta\\ \sin m'\theta\end{matrix}\right\}\sin m\theta\, d\theta}_{\left\{\begin{matrix}0\\ \pi(1-\delta_{0m})\delta_{mm'}\end{matrix}\right\}}\right]\int_0^{\pi} P_n^m P_{n'}^{m'}\sin\phi\, d\phi$$

$$=\sum_{n=m'}^{\infty} a^{-(n+1)}\pi\left\{\begin{matrix}(1+\delta_{0m})A_{m'n}\\ (1-\delta_{0m})B_{m'n}\end{matrix}\right\}\underbrace{\int_0^{\pi} P_n^{m'}(\cos\phi)P_{n'}^{m'}(\cos\phi)\sin\phi\, d\phi}_{\frac{2}{2n+1}\frac{(n+m')!}{(n-m')!}\delta_{nn'}}$$

$$=a^{-(n'+1)}\pi\left\{\begin{matrix}1+\delta_{0m'}\\ 1-\delta_{0m'}\end{matrix}\right\}\frac{2}{2n'+1}\frac{(n'+m')!}{(n'-m')!}\left\{\begin{matrix}A_{m'n'}\\ B_{m'n'}\end{matrix}\right\}$$

$$\boxed{\left\{\begin{matrix}(1+\delta_{0m})A_{mn}\\ (1-\delta_{0m})B_{mn}\end{matrix}\right\}=\frac{(2n+1)(n-m)!\,a^{n+1}}{2\pi\,(n+m)!}\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\left\{\begin{matrix}\cos m\theta\\ \sin m\theta\end{matrix}\right\}P_n^m(\cos\phi)\sin\phi\, d\phi\, d\theta}$$

b) $$F(\phi)=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} a^{-(n+1)}\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]P_n^m(\cos\phi)$$

Since there is not θ-dependence $\rightarrow$ $\boxed{\begin{matrix}B_{mn}=0\\ A_{mn}=0 \text{ except for } m=0\end{matrix}}$

$\Rightarrow$ $$F(\phi)=\sum_{n=0}^{\infty} a^{-(n+1)}A_{0n}P_n^0(\cos\phi)$$ Multiplying by $P_{n'}(\cos\phi)$ and integrating:

$$\int_0^{\pi} F(\phi)P_{n'}(\cos\phi)\sin\phi\, d\phi=\sum_{n=0} a^{-(n+1)}A_{0n}\underbrace{\int_0^{\pi} P_n(\cos\phi)P_{n'}(\cos\phi)\sin\phi\, d\phi}_{\frac{2}{2n+1}\delta_{nn'}}$$

$$=a^{-(n'+1)}A_{0n'}\frac{2}{2n'+1}$$

$$\Rightarrow \boxed{A_{0n}=\frac{(2n+1)a^{n+1}}{2}\int_0^{\pi} F(\phi)P_n(\cos\phi)\sin\phi\, d\phi}$$

and $$\boxed{u(\rho,\theta,\phi)=\sum_{n=0}^{\infty}\rho^{-(n+1)}A_{0n}P_n(\cos\phi)}$$

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c) $u(a,\theta,\phi) = \begin{cases} V & \text{if } 0<\phi<\pi/2 \\ -V & \text{if } \pi/2<\phi<\pi. \end{cases}$ Azimuthal symmetry (No θ-dependence).

Note that $u(a,\theta,\phi) = F(\phi)$ since we define $F(\phi) = \begin{cases} V & \text{if } 0<\phi<\pi/2 \\ -V & \text{if } \pi/2<\phi<\pi \end{cases}$

then we can use the answer of part b):

$$A_{0n} = \frac{2n+1}{2} a^{n+1} \int_0^{\pi} F(\phi) P_n(\cos\phi) \sin\phi \, d\phi$$

$$= \frac{2n+1}{2} a^{n+1} \left[\int_0^{\pi/2} V P_n(\cos\phi) \sin\phi \, d\phi - \int_{\pi/2}^{\pi} V P_n(\cos\phi) \sin\phi \, d\phi \right]$$

$$\boxed{A_{0n} = \frac{(2n+1)}{n} a^{n+1} V \left[\int_0^{\pi/2} P_n(\cos\phi) \sin\phi \, d\phi - \int_{\pi/2}^{\pi} P_n(\cos\phi) \sin\phi \, d\phi \right]}$$

and $u(\rho,\phi,\theta) = \sum_{n=0}^{\infty} \rho^{-(n+1)} A_{0n} P_n(\cos\phi)$

7.10.11 $\nabla^2 u = 0 \rightarrow \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi \frac{\partial u}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} = 0$

Let $u(\rho,\theta,\phi) = f(\rho) q(\theta) g(\phi)$, substituing and dividing by $fgq/\rho^2\sin^2\phi$.

$$\frac{\sin^2\phi}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + \frac{\sin\phi}{g}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \underbrace{\frac{1}{q}\frac{d^2q}{d\theta^2}}_{-\lambda} = 0 \rightarrow \boxed{\frac{d^2q}{d\theta^2} = -\lambda\, q(\theta)}$$

Now, divide by $\sin^2\phi$:

$$\underbrace{\frac{1}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)}_{\mu} + \underbrace{\frac{1}{g\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) - \frac{\lambda}{\sin^2\phi}}_{-\mu} = 0$$

For periodic BC's we expect oscillations and assume separation $\lambda \geq 0$.

$$\boxed{\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) - \mu f(\rho) = 0} \quad \text{and} \quad \boxed{\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \left(\mu\sin\phi - \frac{\lambda}{\sin\phi}\right) g = 0}$$

Since $\lambda > 0 \rightarrow q(\theta) = A\cos\sqrt{\lambda}\,\theta + B\sin\sqrt{\lambda}\,\theta$ $\quad q(0) = 0 \rightarrow A = 0$

$q(\pi/2) = 0 = B\sin\sqrt{\lambda}\,\pi/2 = 0 \rightarrow \sqrt{\lambda}\,\pi/2 = m\pi$

$\lambda_m = (2m)^2 \quad m = 1,2,3,\ldots$

$\Rightarrow \boxed{q(\theta) = B\sin(2m\theta)}$

For $\lambda = 0 \rightarrow q(\theta) = A_0\theta + B_0 \quad q(0) = B_0 = 0$

$q(\pi/2) = A_0(\pi/2) = 0 \rightarrow A_0 = 0$

There are not solutions for $\lambda \leq 0$ that satisfy the BC's.

For bounded solutions at $\phi = 0$ and $\phi = \pi \rightarrow \boxed{g(\phi) = P_n^{2m}(\cos\phi)}$

and $\boxed{\mu = n(n+1)}$ $\quad n = 0,1,2,\ldots$ $\quad m \leq n.$

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) - n(n+1) f = 0 \rightarrow \boxed{f(r) = C_1\rho^n + \cancel{C_2}\,\rho^{-(n+1)}}$$

0 since $\rho < a$ and $|f(0)| < \infty$

$$\Rightarrow \boxed{u(\rho,\theta,\phi) = \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} C_{mn}\rho^n \sin(2m\theta) P_n^{2m}(\cos\phi)}$$ Recall $m = 1,2,3,\ldots$

Now using $U(a,\theta,\phi) = F(\theta,\phi)$ we have:

$$F(\theta,\phi) = \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} C_{mn}\, a^n \sin(2m\theta)\, P_n^{2m}(\cos\phi).$$

multiplying by $\sin(2m'\theta)\, P_{n'}^{2m'}(\cos\phi)$ and integrating in $\sin\phi\, d\phi\, d\theta$

$$\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin(2m'\theta)\, P_{n'}^{2m'}(\cos\phi)\sin\phi\, d\phi\, d\theta$$

$$= \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} C_{mn}\, a^n \underbrace{\int_0^{2\pi}\sin(2m\theta)\sin(2m'\theta)\, d\theta}_{\pi\delta_{mm'}} \int_0^{\pi} P_n^{2m} P_{n'}^{2m'} \sin\phi\, d\phi$$

$$= \sum_{n=0}^{\infty} C_{m'n}\, a^n \pi \underbrace{\int_0^{\pi} P_n^{2m'}(\cos\phi)\, P_{n'}^{2m'}(\cos\phi)\sin\phi\, d\phi}_{\frac{2}{2n+1}\frac{(m'+n)!}{(n-m')!}\delta_{nn'}}$$

$$= C_{m'n'}\, a^{n'} \pi \frac{2}{2n'+1}\frac{(2m'+n')!}{(n'-2m')!}$$

$$\rightarrow \boxed{C_{mn} = \frac{(n-2m)!\,(2n+1)}{2\pi a^n (2m+n)!}\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin(2m\theta)\, P_n^{2m}(\cos\phi)\sin\phi\, d\phi\, d\theta}$$

7.10.12 Let $U(a,\theta,\phi) = \begin{cases} F(\theta,\phi) & \text{if } 0<\phi<\pi/2 \\ 0 & \text{if } \pi/2<\phi<\pi. \end{cases}$

We know from separation of variables of the Laplace's equation in spherical coordinates that the solution inside a sphere is:

$$U(\rho,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \rho^n\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right] P_n^m(\cos\phi) \quad \text{Eq (7.10.31)}$$

We need that $u=0$ at $z=0=\rho\cos\phi \rightarrow$ at $\phi=\pi/2$ $u=0$

$\Rightarrow P_n^m(\cos\pi/2) = P_n^m(0) = 0 \rightarrow n+m = \text{odd}.$

Recall that $\boxed{P_n^m(0)=0 \text{ if } n+m=\text{odd}}$ $\rightarrow$ See Arfken, Weber.

$$\rightarrow \boxed{U(\rho,\theta,\phi) = \sum_{m}\sum_{\substack{n \\ n+m=\text{odd}}} \rho^n\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right] P_n^m(\cos\phi)}$$

Now, using the potential given on the sphere surface at $\rho=a$:

$$U(a,\theta,\phi) = \sum_{m}\sum_{\substack{n \\ m+n=\text{odd}}} a^n\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right] P_n^m(\cos\phi)$$

multiplying by $\left\{\begin{matrix}\cos m'\theta \\ \sin m'\theta\end{matrix}\right\} P_{n'}^{m'}(\cos\phi)$ and integrating as in exercise 7.10.10(a): we obtain:

* b) Like in part a) $g(\theta) = B\sin(m\theta)$ $m = 1, 2, 3, \ldots$ $\mu = m^2$.

Now, $h(0) = 0$ and $h'(H) = 0$ imply oscillatory solutions in z according to the BC's.

For $\lambda < 0$: $h(0) = 0 \rightarrow F = 0$ (There are not solutions for $\lambda > 0$)

$h'(H) = 0 = G\sqrt{\lambda}\cos(\sqrt{-\lambda}H) \rightarrow \sqrt{-\lambda}H = \left(\frac{2n-1}{2}\right)\pi \rightarrow \lambda_n = -\left(\frac{2n-1}{2}\frac{\pi}{H}\right)^2$ $n = 1, 2, 3, \ldots$

If $\lambda = 0$ $h(0) = 0 \rightarrow G_0 = 0$; $h'(H) = 0 \rightarrow F_0 = 0 \rightarrow$ There are not solution if $\lambda = 0$

$$\Rightarrow \boxed{u(r,\theta,z) = \sum_{m=1}\sum_{n=1} G_{mn} I_m\left[\frac{(2n-1)\pi}{2H} r\right] \sin\left[\frac{(2n-1)\pi}{2H} z\right] \sin(m\theta)}$$

$$\beta(\theta,z) = u(a,\theta,z) = \sum_{m=1}\sum_{n=1} G_{mn} I_m\left[\frac{(2n-1)\pi a}{2H}\right] \sin\left[\frac{(2n-1)\pi}{2H} z\right] \sin(m\theta)$$

Multiplying by $\sin[(2n'-1)\pi z/2H]\sin(m'\theta)$ and integrating:

$$\int_0^{\pi}\int_0^{H} \beta(\theta,z)\sin[(2n'-1)\pi z/H]\sin(m'\theta)\,dz\,d\theta =$$

$$\sum_{m=1}\sum_{n=1} G_{mn} I_m\left[\frac{(2n-1)\pi a}{2H}\right] \underbrace{\int_0^H \sin\frac{(2n-1)\pi z}{2H}\sin\frac{(2n'-1)\pi z}{2H}\,dz}_{(H/2)\delta_{nn'}} \underbrace{\int_0^{\pi}\sin m\theta \sin m'\theta\,d\theta}_{(\pi/2)\delta_{mm'}}$$

$$= G_{m'n'} I_{m'}\left[\frac{(2n'-1)\pi a}{2H}\right]\frac{H}{2}\cdot\frac{\pi}{2}$$

$$\Rightarrow \boxed{G_{mn} = \frac{4}{\pi H I_m[(2n'-1)\pi a/2H]}\int_0^{\pi}\int_0^{H}\beta(\theta,z)\sin\left[\frac{(2n-1)\pi z}{H}\right]\sin(m\theta)\,dz\,d\theta}$$

c) $g'(0) = 0 \rightarrow \begin{cases} B = A_0 = 0 \\ B' = 0 \end{cases}$ $g'(\pi) = 0 \rightarrow \begin{cases} -A\sqrt{\mu}\sin\sqrt{\mu}\pi = 0 \\ A' = 0 \end{cases} \rightarrow \sqrt{\mu} = m = 1, 2, 3 \ldots$ $\mu = m^2$

For $\mu = 0 \rightarrow g(\theta) = B_0 = B\cos m\theta$ since $m = 0$

$\boxed{g(\theta) = A_m\cos(m\theta)}$ $m = 0, 1, 2, \ldots$

$h'(0) = h'(H) = 0 \longrightarrow$ implies oscillatory solutions in z according to BC's, then there are not solutions for $\lambda > 0$.

For $\lambda < 0$: $h'(0) = 0 \rightarrow G = 0$; $h'(H) = \sqrt{\lambda}F\sin(\sqrt{-\lambda}H) = 0 \rightarrow \sqrt{-\lambda}H = n\pi$

$\lambda_n = -\left(\frac{n\pi}{H}\right)^2$ $n = 1, 2, 3, \ldots$

If $\lambda = 0$: $h'(0) = 0 \rightarrow F_0 = 0$, $h'(H) = 0 \Rightarrow h(z) = cte = F_0\cos\frac{n\pi}{H}z$ with $n = 0$.

$\boxed{h(z) = F_n\cos(n\pi z/H)}$ $n = 0, 1, 2, \ldots$

$$\Rightarrow \boxed{u(r,\theta,z) = \sum_{m=0}\sum_{n=0} F_{mn} I\left[\frac{n\pi r}{H}\right]\cos\left(\frac{n\pi z}{H}\right)\cos m\theta}$$

For evaluate the coefficients use $\frac{\partial u}{\partial r}(a,\theta,z) = \beta(\theta,z) \longrightarrow$

$$\begin{Bmatrix}(1+\delta_{0m})A_{mn}\\(1-\delta_{0m})B_{mn}\end{Bmatrix} = \frac{(2n+1)(n-m)!}{2\pi\alpha^n(n+m)!}\int_0^{2\pi}\int_0^{\pi} u(\alpha,\theta,\phi)\begin{Bmatrix}\cos m\theta\\ \sin m\theta\end{Bmatrix} P_n^m(\cos\phi)\sin\phi\, d\phi$$

But since $u(\alpha,\theta,\phi) = \begin{cases} F(\theta,\phi) & \text{for } 0<\phi<\pi/2 \\ 0 & \text{for } \pi/2<\phi\le\pi \end{cases}$ then we have:

$$\boxed{\begin{Bmatrix}(1+\delta_{0m})A_{mn}\\(1-\delta_{0m})B_{mn}\end{Bmatrix} = \frac{(2n+1)(n-m)!}{2\pi\alpha^n(n+m)!}\int_0^{2\pi}\int_0^{\pi/2} F(\theta,\phi)\begin{Bmatrix}\cos m\theta\\ \sin m\theta\end{Bmatrix} P_n^m(\cos\phi)\sin\phi\, d\phi}$$

for $\underline{m+n = \text{odd}}$

Remember that only survive terms that $n+m = \text{odd}$, then

$\boxed{A_{mn} = B_{mn} = 0 \quad \text{if} \quad m+n = \text{even number}}$

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$$\frac{\partial u}{\partial r}(a,\theta,z) = \sum_{m=0}\sum_{n=0} F_{mn}\left(\frac{n\pi}{H}\right) I'_m\left[\frac{n\pi a}{H}\right] \cos\left(\frac{n\pi z}{H}\right)\cos(m\theta) = \quad (\theta,z)$$

$$\Rightarrow \beta(\theta,z) = F_{00}\left(\frac{0\pi}{H}\right)\underbrace{I'_0[0]}_{0} + \sum_{m=1} F_{m0}\left(\frac{0\pi}{H}\right) I'_m[0]\cos m\theta + \sum_{n=1} F_{0n}\left(\frac{n\pi}{H}\right) I'_0\left[\frac{n\pi a}{H}\right]\cos\left(\frac{n\pi z}{H}\right)$$

$$+ \sum_{m=1}\sum_{n=1} F_{mn}\left(\frac{n\pi}{H}\right) I'_m\left[\frac{n\pi a}{H}\right]\cos\left(\frac{n\pi z}{H}\right)\cos(m\theta).$$

Multiplying by $\cos(n'\pi z/H)$ and integrating:

$$\int_0^{\pi}\int_0^{H} \beta(\theta,z)\cos\left(\frac{n'\pi z}{H}\right) dz\, d\theta = \sum_{n=1} F_{0n}\left(\frac{n\pi}{H}\right) I'\left[\frac{n\pi a}{H}\right]\cdot \underbrace{\pi\int_0^H \cos\frac{n\pi z}{H}\cos\frac{n'\pi z}{H}\, dz}_{(H/2)\,\delta_{nn'}}$$

$$+ \sum_{m=1}\sum_{n=1} F_{mn}\left(\frac{n\pi}{H}\right) I'_m\left[\frac{n\pi a}{H}\right]\int_0^{\pi}\cos m\theta\, d\theta \cdot \int_0^H \cos\frac{n\pi z}{H}\cos\frac{n'\pi z}{H}\, dz.$$

$$= F_{0n'}\left(\frac{n'\pi}{H}\right) I'[n\pi a/H]\,\frac{\pi H}{2}$$

$$\Rightarrow \boxed{F_{0n} = \frac{2}{n\pi^2 I'[n\pi a/H]}\int_0^{\pi}\int_0^{H}\beta(\theta,z)\cos\left(\frac{n\pi z}{H}\right) dz\, d\theta} \qquad n\neq 0$$

Now, multiplying by $\cos(n'\pi z/H)\cos(m'\theta)$ and integrating in similar way:

$$\int_0^{\pi}\int_0^{H}\beta(\theta,z)\cos\left(\frac{n'\pi z}{H}\right)\cos(m\theta)\, dz\, d\theta = 0 + \sum_{m=1}\sum_{n=1} F_{mn}\left(\frac{n\pi}{H}\right) I'\left[\frac{n\pi a}{H}\right]\frac{\pi}{2}\delta_{mm'}\frac{H}{2}\delta_{nn'}$$

$$= F_{m'n'}\left(\frac{n'\pi}{H}\right) I'[n'\pi a/H]\,\frac{\pi H}{4}$$

$$\boxed{F_{mn} = \frac{4}{n\pi^2 I'[n\pi a/H]}\int_0^{\pi}\int_0^{H}\beta(\theta,z)\cos\left(\frac{n\pi z}{H}\right)\cos(m\theta)\, dz\, d\theta} \qquad m,n\neq 0$$

Add $(1+\delta_{0m})$ for summarize for $n\geq 0$

Note that we don't need F_{m0} when $m\neq 0$ since $I_m[0]=0$ for all $m\geq 1$. then the terms with $n=0$, $m\neq 0$ no contribute.

But we need find F_{00} knowing $u(r,\theta,0)$, then:

$$u(r,\theta,0) = F_{00} + \sum_{\substack{m,n\\ \text{No both zero}}} F_{mn}\, I_m[n\pi a/H]\cos\left(\frac{n\pi z}{H}\right)\cos(m\theta)$$

integrating in $dz\,d\theta$ and using orthogonality:

$$\int_0^{\pi}\int_0^{H} u(r,\theta,0)\, dz\, d\theta = F_{00}(\pi H) \rightarrow \boxed{F_{00} = \frac{1}{\pi H}\int_0^{\pi}\int_0^{H} u(r,\theta,0)\, dz\, d\theta}$$

Mean value of u over the bottom. (Condition for existence of the solution)

$\rightarrow u(r,\theta,0) = u(r,\theta,H) = 0.$

d) The periodic BC's in z demand oscillatory solutions in z, thus

For $\lambda < 0$ $\quad h(0) = 0 = F$; $h(H) = G \sin(\sqrt{-\lambda}\, H) = 0 \rightarrow \lambda = -\left(\frac{m\pi}{H}\right)^2$

$\Rightarrow h(z) = G_n \sin(m\pi z/H) \qquad m = 1, 2, 3, \ldots$

$\lambda = 0 \rightarrow h(0) = 0 = G_0$ and $h(H) = 0 = F_0 H \rightarrow F_0 = 0$

There are not oscillatory solutions in $h(z)$ that satisfy the BC's for $\lambda > 0$.

~~On other hand for $g(\theta)$, we don't have oscillatory solutions since we don't have periodic BC's for θ, then there are not solutions here also.~~

~~For $\lambda = 0$ $g(\theta) = a_0 + b_0 \theta$ as $g(\theta) = B \sin(\sqrt{-\lambda}\,\theta)$~~

~~For $\lambda > 0$ $g(\theta) = a_0 \cos$ as $g(\theta) = A \cos$~~

Since $\lambda_m < 0 \Rightarrow f(r) = C_2' I_{\sqrt{\mu}}(\sqrt{-\lambda_m}\, r)$ with $\lambda_n = -(m\pi/H)^2$

$u(a,\theta,z) = 0 \rightarrow f(a) = C_2' I_{\sqrt{\mu}}(\sqrt{-\lambda_m}\, a) = 0$

OR $\quad I_{\sqrt{\mu}}\left(\frac{m\pi a}{H}\right) = 0 \rightarrow \mu_{mn} = \nu_{mn}^2 \qquad n = 1, 2, 3, \ldots$

OR $\mu_{mn} = \pm \nu_{mn} \rightarrow$ Values for what $(m\pi a/H)$ is one zero of $I_{\pm\nu_{mn}}$ $(\nu_{mn} > 0)$

See Abramowitz-Stegun 9.6 Pg 374.

$I_\nu(\alpha r)$ is bounded for $r \to 0$, but $|I_{-\nu}(0)| \to \infty$, then $\mu_{mn} = -\nu_{mn}$ cannot be a solution.

$I_\nu(\alpha r)$ is positive for $\nu \geq 0$ and $\alpha r > 0$, then don't have zeros for α fixed $\alpha r = (m\pi a/H)$ and someone ν.

$I_\nu(\alpha r) = 0$ for someone ν only when $r = 0 \rightarrow m = 0$, but $m = 1, 2, 3 \ldots$; so m cannot be zero.

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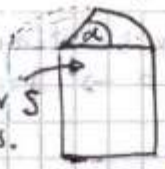
$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}.$$

$\alpha = \pi/2$

7.9.3



$u=0$ over S in all sides.

$$\frac{\partial u}{\partial t} = K\nabla^2 u \quad \text{let } u = T(t)\phi(r,\theta,z) \text{ and divide by } T\phi:$$

$$\underbrace{\frac{1}{KT}\frac{dT}{dt}}_{-\lambda} = \underbrace{\frac{1}{\phi}\nabla^2\phi}_{-\lambda} \quad \text{separating} \Rightarrow \boxed{\frac{dT}{dt} = -\lambda K T(t)}$$

$$\nabla^2\phi + \lambda\phi = 0$$

Let $\phi(r,\theta,z) = f(r)g(\theta)h(z)$ and divide by $\phi = fgh$:

$$\Rightarrow \underbrace{\frac{1}{rf}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{r^2 g}\frac{d^2g}{d\theta^2}}_{-\lambda'} + \underbrace{\frac{1}{h}\frac{d^2h}{dz^2} + \lambda}_{\lambda'} = 0 \rightarrow \boxed{\frac{d^2h}{dz^2} = -(\lambda-\lambda')h(z)}$$

and

$$\underbrace{\frac{r}{f}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \lambda' r^2}_{\mu} + \underbrace{\frac{1}{g}\frac{d^2g}{d\theta^2}}_{-\mu} = 0 \rightarrow \boxed{\frac{d^2g}{d\theta^2} = -\mu g(\theta)}$$

$$\boxed{r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda' r^2 - \mu)f(r) = 0}$$

a) $h(0) = h(H) = 0 \rightarrow h(z) = \cancel{A}\cos\sqrt{\lambda-\lambda'}\,z + B\sin\sqrt{\lambda-\lambda'}\,z$ $\quad h(0)=0$

$h(H)=0 \rightarrow \lambda-\lambda' = (\ell\pi/H)^2 \quad \ell = 1,2,3,\ldots$

There are not solutions for $\lambda-\lambda' \le 0$ that satisfy this BC's.

$g(0) = g(\pi/2) = 0 \rightarrow g(\theta) = \cancel{A'}\cos\sqrt{\mu}\,\theta + B'\sin\sqrt{\mu}\,\theta$ $\quad g(0)=0$

$g(\pi/2)=0 \rightarrow \mu = (2m)^2 \quad m = 1,2,3,\ldots$

There are not solutions for $\mu \le 0$ that satisfy the BC's.

$|f(0)| < \infty$

Since $\mu > 0$, $\lambda' > 0 \rightarrow f(r) = C_1 J_{\sqrt{\mu}}(\sqrt{\lambda'}\,r) + \cancel{C_2} Y_{\sqrt{\mu}}(\sqrt{\lambda'}\,r)$, $\sqrt{\mu} = 2m$

For $\lambda' = 0 \rightarrow f \sim (cte)r^{\pm\sqrt{\mu}}$ but $f(a)=0 \rightarrow cte = 0$

$f(a) = 0 \rightarrow J_{2m}(\sqrt{\lambda'}\,a) = 0 \rightarrow \lambda' a = z_{mn} \rightarrow$ Zeros of $J_{2m}(z)$

$\lambda'_{mn} = (z_{mn}/a)^2 \quad n = 1,2,3,\ldots$

$T' = -\lambda K T(t) \rightarrow T(t) = C e^{-\lambda K t}$

with $\lambda = \lambda' + (\ell\pi/H)^2 = (z_{mn}/a)^2 + (\ell\pi/H)^2 = \lambda_{\ell mn}$.

$$\Rightarrow \boxed{u(r,\theta,z,t) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{\ell mn}\sin\left(\frac{\ell\pi z}{H}\right)\sin(2m\theta)\,J_{2m}(\sqrt{\lambda'_{mn}}\,r)\,e^{-\lambda_{\ell mn}Kt}}$$

$$f(r,\theta,z) = u(t=0) = \sum_{\ell=1}\sum_{m=1}\sum_{n=1} A_{\ell mn}\sin\left(\frac{\ell\pi z}{H}\right)\sin(2m\theta)\,J_{2m}(\sqrt{\lambda'_{mn}}\,r)$$

multiplying by $\sin(\ell'\pi z/H)\sin(2m'\theta)\,J_{2m'}(\sqrt{\lambda'_{m'n'}}\,r)$ and integrating in $r\,dr\,d\theta\,dz$ between $0<z<H$, $0<\theta<\pi/2$ and $0<r<a$ and using orthogonality:

$$\boxed{A_{\ell mn} = \frac{8}{\pi H}\left[\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\sin\left(\frac{\ell\pi z}{H}\right)\sin(2m\theta)\,J_{2m}(\sqrt{\lambda'_{mn}}\,r)\,r\,dr\,d\theta\,dz\right] \Big/ \int_0^a J_{2m}^2(\sqrt{\lambda'_{mn}}\,r)\,r\,dr}$$

$\frac{a^2}{2}J_{2m+1}^2(\sqrt{\lambda'_{mn}}\,a)$

Since $\lambda_{\ell mn} > 0$ for all $\ell, m, n \Rightarrow e^{-\lambda_{\ell mn}Kt} \rightarrow 0$ as $t \rightarrow \infty$, then $u(r,\theta,z,t) \rightarrow 0$ as $t \rightarrow \infty$.

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$h'(0)=0.$

* b) $h'(0)=h'(H) \rightarrow h(z) = A\cos\sqrt{\lambda-\lambda'}\,z + \cancel{B}\sin\sqrt{\lambda-\lambda'}\,z$ $\quad \ell=0$ include case $\lambda-\lambda'=0$

$h'(H)=0 \rightarrow \lambda-\lambda' = (\ell\pi/H)^2 \qquad \ell = 0,1,2,3,\ldots$

↳ There are not solutions for $\lambda-\lambda'<0$ that satisfy BC's.

$g'(0)=g'(\pi/2) \rightarrow g(\theta) = A'\cos\sqrt{\mu}\,\theta + \cancel{B'}\sin\sqrt{\mu}\,\theta$ $\quad m=0$ include case $\mu=0$

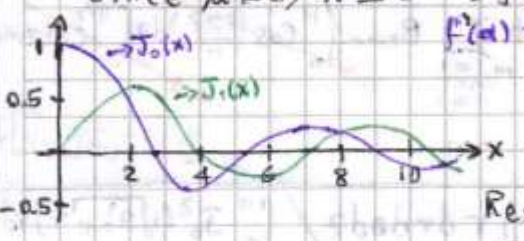
$g'(\pi/2)=0 \rightarrow \mu = (2m)^2 \qquad m = 0,1,2,3,\ldots$

↳ There are not solutions for $\mu<0$ that satisfy BC's.

Since $\mu\geq 0$, $\lambda'\geq 0 \rightarrow f(r) = C_1 J_{\sqrt{\mu}}(\sqrt{\lambda'}\,r) + \cancel{C_2} Y_{\sqrt{\mu}}(\sqrt{\lambda'}\,r) \qquad \sqrt{\mu}=2m$

$f'(a)=0 \rightarrow J'_{2m}(\sqrt{\lambda'}\,a)=0 \rightarrow \lambda' a = z_{mn} \rightarrow$ Zeros of $J'_{2m}(z)$

$\lambda'_{mn} = (z_{mn}/a)^2 \qquad n=1,2,3,\ldots$



Remember $\int_0^a J_0(\alpha r) = \frac{a}{\alpha} J_1(\alpha a)$

Note $\boxed{\lambda'_{01}=0}$ since $\boxed{z_{01}=0} \rightarrow$ first zero of $J'_{2m}(z)$

Remember too that $\boxed{J_0'(x) = -J_1(x)}$

If $\lambda'=0 \rightarrow f \sim (cte)\, r^{\pm\sqrt{\mu}}$, but $f'(a)=0 \rightarrow \mu=0 \rightarrow f = cte \rightarrow$ that is the case included above when $\lambda'_{01}=0$ since $J_0(0)=1=cte$.

$T' = -\lambda K T(t) \rightarrow T(t) = e^{-\lambda K t}$ with $\lambda = \lambda_{\ell mn} = \lambda'_{mn} + (\ell\pi/H)^2$

$$\Rightarrow \boxed{u(r,\theta,z,t) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{\ell mn}\cos\left(\frac{\ell\pi z}{H}\right)\cos(2m\theta)\,J_{2m}(\sqrt{\lambda'_{mn}}\,r)\,e^{-\lambda_{\ell mn}Kt}}$$

$$u(t=0) = f(r,\theta,z) = B_{001}\underbrace{\cancel{J_0(\sqrt{\lambda'_{01}}\,r)}}_{1} + \sum_{n=2} B_{00n} J_0(\sqrt{\lambda'_{0n}}\,r) + \sum_{\ell=1} B_{\ell 01}\cos\left(\frac{\ell\pi z}{H}\right)\underbrace{\cancel{J_0(\sqrt{\lambda'_{01}}\,r)}}_{1}$$

$$+ \sum_{m=1} B_{0m1}\cos(2m\theta)\,J_{2m}(\sqrt{\lambda'_{m1}}\,r) + \sum_{\ell=1}\sum_{m=1}\sum_{n=1} B_{\ell mn}\cos\left(\frac{\ell\pi z}{H}\right)\cos(2m\theta)\,J_{2m}(\sqrt{\lambda'_{mn}}\,r)$$

Integrating for $r\,dr\,d\theta\,dz$ by use of orthogonality:

$$\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\,r\,dr\,d\theta\,dz = B_{001}\left(\frac{H\pi a^2}{4}\right) + \sum_{n=2} B_{00n}\left(\frac{H\pi}{2}\right)\underbrace{\int_0^a J_0(\sqrt{\lambda'_{0n}}\,r)}_{\frac{a}{\sqrt{\lambda'_{0n}}}J_1(\sqrt{\lambda'_{0n}}\,a) = -\frac{a\,\cancel{J_0(z_{0n})}}{\sqrt{\lambda'_{0n}}}}$$

z_{0n} → zeros of J_0'

$$+ \sum_{\ell=1} B_{\ell 01}\left(\frac{\pi a^2}{4}\right)\cancel{\int_0^H \cos\left(\frac{\ell\pi z}{H}\right)dz} + \sum_{m=1} B_{0m1} H \cancel{\int_0^{\pi/2}\cos 2m\theta\,d\theta}\int_0^a J_{2m}(\quad)\,r\,dr$$

$$+ \sum_{\ell=1}\sum_{m=1}\sum_{n=1} B_{\ell mn}\cancel{\int_0^H \cos\left(\frac{\ell\pi z}{H}\right)dz}\cancel{\int_0^{\pi/2}\cos(2m\theta)\,d\theta}\int_0^a J_{2m}(\sqrt{\lambda'_{mn}}\,r)\,r\,dr$$

$$\Rightarrow \boxed{B_{001} = \frac{4}{H\pi a^2}\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\,r\,dr\,d\theta\,dz} \rightarrow$$ Mean or average value of the initial distribution of u

Note that $\lambda_{\ell mn} > 0$ for all ℓmn except when $\ell=m=0$ and $n=1$. Since $\lambda_{001}=0 \Rightarrow e^{-\lambda_{\ell mn}Kt} \rightarrow 0$ for all ℓ, m, n except when $\ell=m=0$ and $n=1$. Thus $\boxed{\lim_{t\to\infty} u = B_{001}}$

Now, multiplying by $J_0(\sqrt{\lambda'_{0n'}}\,r)$ and integrating:

$$\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\,J_0(\sqrt{\lambda'_{0n'}}\,r)\,r\,dr\,d\theta\,dz = B_{001}\left(\frac{H\pi}{2}\right)\underbrace{\int_0^a J_0(\sqrt{\lambda'_{0n'}}\,r)\,r\,dr}_{\frac{a}{\sqrt{\lambda'_{0n'}}}J_1(\sqrt{\lambda'_{0n'}}\,r)=-\frac{a\,J_0'(z_{0n'})}{\sqrt{\lambda'_{0n'}}}\to 0}$$

$$+\sum_{n=2} B_{00n}\left(\frac{H\pi}{2}\right)\underbrace{\int_0^a J_0(\sqrt{\lambda'_{0n}}\,r)\,J_0(\sqrt{\lambda'_{0n'}}\,r)\,r\,dr}_{0 \text{ if } n\neq n'} + \sum_{\ell=1} B_{\ell 01}\frac{\pi}{2}\int_0^H \cos\left(\frac{\ell\pi z}{H}\right)dz\int_0^a J_0\,r\,dr$$

$$+\sum_{m=1} B_{0m1}\,H\int_0^{\pi/2}\cos(2m\theta)\,d\theta\int_0^a J_{2m}(\,)\,J_0(\,)\,r\,dr + \sum_{\substack{\ell=1\\m=1\\n=1}} B_{\ell mn}\int_0^H \cos\left(\frac{\ell\pi z}{H}\right)dz\int_0^{\pi/2}\cos 2m\theta\int_0^a J_{2m}J_0$$

$$= B_{00n'}\left(\frac{H\pi}{2}\right)\int_0^a J_0^2(\sqrt{\lambda'_{0n'}}\,r)\,r\,dr$$

$$\frac{a^2}{2}\left[1-\left(\frac{0}{\sqrt{\lambda'_{0n}}\,a}\right)^2\right]J_0^2(\sqrt{\lambda'_{0n}}\,a)$$

$$\Rightarrow \boxed{B_{00n\,(n\neq 1)} = \frac{2}{\pi H}\left[\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\,J_0(\sqrt{\lambda'_{0n}}\,r)\,r\,dr\,d\theta\,dz\Big/\int_0^a J_0^2(\sqrt{\lambda'_{0n}}\,r)\,r\,dr\right]}$$

Multiplying by $\cos(\ell'\pi z/H)$ and integrating

$$\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\cos\left(\frac{\ell'\pi z}{H}\right)r\,dr\,d\theta\,dz = B_{001}\left(\frac{\pi a^2}{4}\right)\int_0^H\cos\left(\frac{\ell'\pi z}{H}\right)dz$$

$$+\sum_{n=2} B_{00}\,\frac{\pi}{2}\int_0^a J_0(\sqrt{\lambda'_{0n}}\,r)\,r\,dr\int_0^H\cos\left(\frac{\ell'\pi z}{H}\right)dz + \sum_{\ell=1} B_{\ell 01}\left(\frac{\pi a^2}{4}\right)\underbrace{\int_0^H\cos\left(\frac{\ell\pi z}{H}\right)\cos\left(\frac{\ell'\pi z}{H}\right)dz}_{(H/2)\delta_{\ell\ell'}}$$

$$+\sum_{m=1} B_{0m1}\int_0^{\pi/2}\cos(2m\theta)\,d\theta\int_0^a J_{2m}(\sqrt{\lambda'_{m1}}\,r)\,r\,dr\int_0^H\cos\left(\frac{\ell'\pi z}{H}\right)dz$$

$$+\sum_{\ell=1}\sum_{m=1}\sum_{n=1} B_{\ell mn}\int_0^H\cos\left(\frac{\ell\pi z}{H}\right)\cos\left(\frac{\ell'\pi z}{H}\right)dz\int_0^{\pi/2}\cos(2m\theta)\,d\theta\int_0^a J_{2m}(\sqrt{\lambda'_{mn}}\,r)\,r\,dr$$

$$= B_{\ell' 01}\left(\frac{\pi a^2}{4}\right)\frac{H}{2} \rightarrow \boxed{B_{\ell 01\,(\ell\neq 0)} = \frac{8}{H\pi a^2}\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\cos\left(\frac{\ell'\pi z}{H}\right)r\,dr\,d\theta\,dz}$$

Similarly, multiplying by $\cos(2m'\theta)\,J_{2m'}(\sqrt{\lambda'_{m'1}}\,r)$ and integrating, only hold the term with B_{0m1}. Thus:

$$\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\cos(2m'\theta)\,J_{2m'}(\sqrt{\lambda'_{m'1}}\,r)\,r\,dr\,d\theta\,dz =$$

$$\sum_{m=1} B_{0m1}\,H\underbrace{\int_0^{\pi/2}\cos(2m\theta)\cos(2m'\theta)\,d\theta}_{(\pi/4)\delta_{mm'}}\int_0^a J_{2m}(\sqrt{\lambda'_{m1}}\,r)\,J_{2m'}(\sqrt{\lambda'_{m'1}}\,r)\,r\,dr$$

$$= B_{0m'1}\left(\frac{H\pi}{4}\right)\int_0^a J_{2m'}^2(\sqrt{\lambda'_{m'1}}\,r)\,r\,dr$$

$$\Rightarrow \boxed{B_{0m1\,(m\neq 0)} = \frac{4}{H\pi}\left[\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\cos(2m\theta)\,J_{2m}(\sqrt{\lambda'_{m1}}\,r)\,r\,dr\,d\theta\,dz\Big/\int_0^a J_{2m}^2(\sqrt{\lambda'_{m1}}\,r)\,r\,dr\right]}$$

$$\frac{a^2}{2}\left[1-\left(\frac{2m}{\sqrt{\lambda'_{m1}}\,a}\right)^2\right]J_{2m}^2(\sqrt{\lambda'_{m1}}\,a).$$

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Summarize:

$$\boxed{B_{\ell mn} = \frac{16 \Big/ \left[1-\left(\frac{z_m}{z_{mn}}\right)^2\right] J_{2m}^2(z_{mn})}{(1+\delta_{0\ell})(1+\delta_{0m})\, H\pi a^2} \int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\cos\left(\frac{\ell\pi z}{H}\right)\cos(2m\theta)\, J_{2m}(z_{mn}r)\, r\,dr\,d\theta\,dz}$$

$\ell, m \geq 0,$

Finally multiply by $\cos\left(\frac{\ell'\pi z}{H}\right)\cos(2m'\theta)\, J_{2m'}\left(\sqrt{\lambda'_{m'n'}}\, r\right)$ and similarly by convenient integration and use of orthogonality:

$$\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\cos\left(\frac{\ell'\pi z}{H}\right)\cos(2m'\theta)\, J_{2m'}\left(\sqrt{\lambda'_{m'n'}}\, r\right) r\,dr\,d\theta\,dz =$$

$$\sum_{\ell=1}\sum_{m=1}\sum_{n=1} B_{\ell mn}\left(\frac{H}{2}\delta_{\ell\ell'}\right)\left(\frac{\pi}{4}\delta_{mm'}\right)\int_0^a J_{2m}\left(\sqrt{\lambda'_{mn}}\, r\right) J_{2m'}\left(\sqrt{\lambda'_{m'n'}}\, r\right) r\,dr$$

$$= \sum_{n=1} B_{\ell' m' n}\frac{H\pi}{8}\underbrace{\int_0^a J_{2m'}\left(\sqrt{\lambda'_{m'n}}\, r\right) J_{2m'}\left(\sqrt{\lambda'_{m'n'}}\, r\right) r\,dr}_{0 \text{ if } n\neq n'}$$

$$= B_{\ell' m' n'}\left(\frac{H\pi}{8}\right)\int_0^a J_{2m'}^2\left(\sqrt{\lambda'_{m'n'}}\, r\right) r\,dr \qquad \sqrt{\lambda'_{mn}} = \left(\frac{z_{mn}}{a}\right)$$

$$\Rightarrow \boxed{B_{\ell mn}\ \ell\neq 0,\ m\neq 0 = \frac{8}{H\pi}\,\frac{\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\cos\left(\frac{\ell\pi z}{H}\right)\cos(2m\theta)\, J_{2m}\left(\sqrt{\lambda'_{mn}}\, r\right) r\,dr\,d\theta\,dz}{\int_0^a J_{2m}^2\left(\sqrt{\lambda'_{mn}}\, r\right) r\,dr = \frac{a^2}{2}\left[1-\left(\frac{2m}{z_{mn}}\right)^2\right] J_{2m}^2(z_{mn})}}$$

~~This problem is a combination of a) and b), thus~~

~~$h(z) = B\sin(\ell\pi z/H)$ $\ell = 1,2,3,\ldots$; $g(\theta) = A'\cos(2m\theta)$ $m = 0,1,2,\ldots$~~

~~$f(r) = J_{2m}(\sqrt{\lambda'_{mn}}\, r)$ with $\lambda'_{mn} = (z_{mn}/a)^2$ and z_{mn} zeros of J'_{2m} $n = 1,2,3,\ldots$~~

~~$T(t) = e^{-\lambda_{\ell mn}\kappa t}$ with $\lambda_{\ell mn} = \lambda'_{mn} + (\ell\pi/H)^2$~~

c) $h(0) = h(H) = 0 \longrightarrow h(z) = B\sin(\ell\pi z/H)$ $\ell = 1,2,3,\ldots$ just like in a)

$g'(0) = g(\pi/2) = 0 \longrightarrow g(\theta) = A'\cos\sqrt{\mu}\,\theta + \cancel{B'}\sin\sqrt{\mu}\,\theta$ $\to 0$ since $g'(0) = 0$

$g(\pi/2) = 0 \to \cos(\sqrt{\mu}\,\pi/2) = 0 \to \mu = (2m-1)^2$ $m = 1,2,3,4,\ldots$

$\hookrightarrow$ There are not solutions for $\mu \leq 0$ that satisfy BC's.

Since $\mu > 0$, $\lambda' \geq 0 \to f(r) = C_1 J_{\sqrt{\mu}}(\sqrt{\lambda'}\, r)$ just like in b).

$f'(a) = 0 \longrightarrow \lambda' = \lambda'_{mn} = (z_{mn}/a)^2 \longrightarrow$ zeros of $J'_{2m-1}(z)$

$n = 1,2,3,\ldots$

If $\lambda' = 0 \to f \sim (cte)\, r^{\pm\sqrt{\mu}}$

$f(a) = 0 \to \mu = 0 \to f = cte$, that is the case included above since $\lambda'_{01} = 0$ (look graph above in b) $\Rightarrow J_0(0) = 1 = cte$.

$T(t) = e^{-\lambda_{\ell mn}\kappa t}$ like in a) and b) with $\lambda_{\ell mn} = \lambda'_{mn} + (\ell\pi/H)^2$

$$\Rightarrow \boxed{u(r,\theta,t) = \sum_{\ell=1}\sum_{m=1}\sum_{n=1} C_{\ell mn}\sin\left(\frac{\ell\pi z}{H}\right)\cos(2m-1)\theta\, J_{2m-1}\left(\sqrt{\lambda'_{mn}}\, r\right) e^{-\lambda_{\ell mn}\kappa t}}$$

Since $\lambda_{\ell mn} > 0$ for all $\ell, m, n \Rightarrow \lim_{t\to\infty} u = 0$ And similarly to part b) by orthogonal

$$\boxed{C_{\ell mn} = \frac{8}{H\pi}\,\frac{\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)\sin(\ell\pi z/H)\cos(2m-1)\theta\, J_{2m-1}\left(\sqrt{\lambda'_{mn}}\, r\right) r\,dr\,d\theta\,dz}{\int_0^a J_{2m-1}^2\left(\sqrt{\lambda'_{mn}}\, r\right) r\,dr}}$$

Clearly $\lambda_{nm} > 0$, then the solution of differential equation time-dependet is:

$$h(t) = C_{nm}\cos c\sqrt{\lambda_{nm}}\,t + D_{nm}\sin c\sqrt{\lambda_{nm}}\,t$$

then by superposition principle the general solution of the wave equation given by product of $f_{nm}\, g_{nm}\, h_{nm}$ solutions is:

$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\left[C_{nm}\cos c\sqrt{\lambda_{nm}}\,t + D_{nm}\sin c\sqrt{\lambda_{nm}}\,t\right]}$$

The initial value problem is determined specificanding the initial position and velocity forms:

$$u(x,y,0) = \alpha(x,y)$$
$$\frac{\partial u}{\partial t}(x,y,0) = \beta(x,y)$$

$$\Rightarrow \alpha(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

multiplying by $\sin(n'\pi x/L)\sin(m'\pi y/H)$ and integrating:

$$\int_0^L\int_0^H \alpha(x,y)\sin\left(\frac{n'\pi x}{L}\right)\sin\left(\frac{m'\pi y}{H}\right)dx\,dy$$
$$= \sum_{n,m} C_{nm}\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}}\underbrace{\int_0^H \sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{m'\pi y}{H}\right)dy}_{(H/2)\delta_{mm'}}$$
$$= C_{n'm'}\left(\frac{L}{2}\right)\left(\frac{H}{2}\right) \Rightarrow \boxed{C_{nm} = \frac{4}{LH}\int_0^H\int_0^L \alpha(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dx\,dy}$$

The other coefficient D_{nm} can be determined with the initial condition for the velocity:

$$\beta(x,y) = \frac{\partial u}{\partial t}(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}\,D_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

multiplying by $\sin(n'\pi x/L)\sin(m'\pi y/H)$ and integrating:

$$\int_0^L\int_0^H \beta(x,y)\sin\left(\frac{n'\pi x}{L}\right)\sin\left(\frac{m'\pi y}{L}\right)dx\,dy = c\sqrt{\lambda_{n'm'}}\,D_{n'm'}\,\frac{LH}{4}$$

$$\Rightarrow \boxed{D_{nm} = \frac{4}{c\sqrt{\lambda_{nm}}\,LH}\int_0^L\int_0^H \beta(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dx\,dy}$$

c) The time-dependent differential equation from separation of variables in heat equation is:

$$\frac{dh}{dt} = -\lambda_{nm}k\,h(t) \quad \text{where} \quad \lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

from spatial separation of variables

Since $\lambda_{nm} > 0$, the solution for differential equation time dependent is:

$$h(t) = C_{nm} e^{-\lambda_{nm} K t}$$

$$\Rightarrow \boxed{u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} e^{-\lambda_{nm} K t} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)}$$

The initial value proble specify the temperature at $t=0$

$$u(x,y,0) = \alpha(x,y) = \sum_{n,m} C_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right).$$

Multiplying by $\sin(n'\pi x/L)\sin(m'\pi y/L)$ and integrating between 0 and L in x and 0 and H in y, we obtain:

$$\boxed{C_{nm} = \frac{4}{LH} \int_0^L \int_0^H \alpha(x,y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dx\, dy}$$

Where we used orthogonality of trigonometric functions.

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7.3.1* a) According to solution of excercise 7.2.3 (c) the solution for the two-dimensional heat equation with zero temperature on all sides is:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right) e^{-\lambda_{nm} K t}$$

with $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

and $C_{nm} = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dx\,dy$

Since $\lambda_{nm} > 0$ for all $n,m \geq 1 \Rightarrow \lim_{t\to\infty} u(x,y,t) = 0$.

b) From excercise 7.2.2 separating variables in heat equation $\frac{dh}{dt} = -\lambda K h(t)$ and $\nabla^2\phi = -\lambda\phi(x,y)$. Let $\phi(x,y) = f(x)g(y)$

$\Rightarrow$ separating the spatial equation above $\Rightarrow$ $\frac{d^2f}{dx^2} = -\gamma f(x)$

with $\gamma + \delta = \lambda$ (See excercise 7.2.3(a)) $\frac{d^2g}{dy^2} = -\delta g(y)$

but now $f'(0) = f'(L) = 0$ and $g'(0) = g'(H) = 0$. These boundary conditions are homegeneus, then we expect oscillatory solution that exist when $\gamma, \delta > 0$:

$f(x) = A_1\cos\sqrt{\gamma}x + A_2\sin\sqrt{\gamma}x$, $\quad f'(0) = 0 = A_2\sqrt{\gamma} \rightarrow A_2 = 0$

$g(y) = B_1\cos\sqrt{\delta}y + B_2\sin\sqrt{\delta}y$, $\quad g'(0) = 0 = B_2\sqrt{\delta} \rightarrow B_2 = 0$

$\Rightarrow f'(L) = 0 = -A_1\sqrt{\gamma}\sin\sqrt{\gamma}L \longrightarrow \sqrt{\gamma}L = n\pi \rightarrow \gamma_n = (n\pi/L)^2$

$g'(H) = 0 = -B_1\sqrt{\delta}\sin\sqrt{\delta}H \longrightarrow \rightarrow \delta_m = (m\pi/H)^2$

$\Rightarrow \boxed{f_n(x) = A_n\cos\left(\frac{n\pi x}{L}\right)}$; $\boxed{g_n(y) = B_n\cos\left(\frac{m\pi y}{H}\right)}$ $\quad n,m = 1,2,3,\ldots$

There are not solution for $\gamma, \delta < 0$ that satisfy homogeneous boundary conditions.

Clearly $\lambda_{nm} > 0 \longrightarrow \frac{dh}{dt} = -\lambda_{nm}Kh(t) \rightarrow \boxed{h(t) = C_{nm}e^{-\lambda_{nm}Kt}}$

with $\lambda_{nm} = \gamma_n + \delta_m = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

But there are solution for $\gamma = 0, \delta = 0 \Rightarrow$

If $\gamma=0,\ \delta=0$: $\rightarrow f''(x)=0 \rightarrow f(x)=A_2x+A_0,\quad f'(0)=0=A_2$

$g''(y)=0 \rightarrow g(y)=B_2x+B_0,\quad g'(0)=0=B_2$

$\Rightarrow \boxed{f_0(x)=A_0}\quad \boxed{g_0(y)=B_0}$ and $\lambda=\gamma+\delta=0$

$\lambda=0 \rightarrow \frac{dh}{dt}=0 \rightarrow \boxed{h(t)=C_0}$

but there are other posibilities:

If $\gamma=0,\ \delta>0 \rightarrow \boxed{f_0(x)=A_0},\ \boxed{g_m(y)=B_m\cos\left(\frac{m\pi y}{H}\right)}$

$\lambda=\delta_m=\left(\frac{m\pi}{H}\right)^2>0 \rightarrow \frac{dh}{dt}=-\delta_m k h(t) \rightarrow \boxed{h(t)=C_m e^{-\lambda_{0m}kt}}\quad \lambda_{0m}=\left(\frac{m\pi}{H}\right)$

$m=1,2,3,\ldots$

If $\gamma>0,\ \delta=0 \rightarrow \boxed{f_n(x)=A_n\cos\left(\frac{n\pi x}{L}\right)}$ and $\boxed{g_0(y)=B_0}$

$\lambda=\gamma_n=\left(\frac{n\pi}{L}\right)^2>0 \rightarrow \frac{dh}{dt}=-\gamma_n k h(t) \rightarrow \boxed{h(t)=C_n e^{-\lambda_{n0}kt}}\quad \lambda_{n0}=\left(\frac{n\pi}{L}\right)$

$n=1,2,3,\ldots$

The general solution is the superposition of solutions of all cases knowing that $u(x,y,t)=h(t)\phi(x,y)=h(t)f(x)g(y)$ in each case:

$$\Rightarrow u=A_0B_0C_0+\sum_{n=1}^{\infty}A_nB_0C_{n0}e^{-\lambda_{n0}kt}\cos\left(\frac{n\pi x}{L}\right)+\sum_{m=1}^{\infty}A_0B_mC_{0m}e^{-\lambda_{0m}kt}\cos\left(\frac{m\pi y}{H}\right)$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_nB_mC_{nm}e^{-\lambda_{nm}kt}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$$

$$\boxed{u=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_{nm}e^{-\lambda_{nm}kt}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)}$$

For $t\rightarrow\infty$ $u=C_{00}$

Since $f(x,y)=u(x,y,0)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_{nm}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{L}\right)$

OR $f(x,y)=C_{00}+\sum_{n=1}^{\infty}C_{n0}\cos\left(\frac{n\pi x}{L}\right)+\sum_{m=1}^{\infty}C_{0m}\cos\left(\frac{m\pi y}{H}\right)+\sum_{n,m=1}^{\infty}C_{nm}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$

Integrating $\int_0^H\int_0^L f(x,y)\,dx\,dy=C_{00}\underbrace{\int_0^H\int_0^L dx\,dy}_{LH}+\sum C_{n0}\cancel{\iint}^{\,0}+\sum\cancel{\iint}^{\,0}+\cancel{\ }^{\,0}$

$\Rightarrow \boxed{C_{00}=\frac{1}{LH}\int_0^H\int_0^L f(x,y)\,dx\,dy}$ → The average of the initial temperature distribution.

Multiplying by $\cos\left(\frac{n'\pi x}{L}\right)$ and integrating

$$\int_0^H\int_0^L f(x,y)\cos\left(\frac{n'\pi x}{L}\right)dx\,dy=C_{00}\cancel{\int_0^L\cos\frac{n'\pi x}{L}dx}^{\,0}\int_0^H dy+\sum_{n=1}^{\infty}C_{n0}\underbrace{\int_0^L\cos\left(\frac{n\pi x}{L}\right)\cos\frac{n'\pi x}{L}}_{(L/2)\delta_{nn'}}\underbrace{\int_0^H dy}_{H}$$

$$+\sum_{m=1}^{\infty}C_{0m}\cancel{\int_0^L\cos\frac{n'\pi x}{L}dx}^{\,0}\underbrace{\int_0^H\cos\left(\frac{m\pi y}{L}\right)dy}_{0}+\sum_{n,m=1}^{\infty}C_{nm}\underbrace{\int_0^L\cos\left(\frac{n\pi x}{L}\right)\cos\frac{n'\pi x}{L}dx}_{(L/2)\delta_{nn'}}\cancel{\int_0^H\cos\frac{m\pi y}{H}dy}^{\,0}$$

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Summarizing: $\boxed{C_{nm} = \frac{4}{LH(1+\delta_{0n})(1+\delta_{0m})}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{L}\right)dx\,dy}$ for all $n,m \geq 0$.

$$\Rightarrow \int_0^H\int_0^L f(x,y)\cos\left(\frac{n'\pi x}{L}\right)dx\,dy = \sum_{n=1}^{\infty} C_{n0}\frac{L}{2}\delta_{nn'}H = C_{n0}\frac{LH}{2}$$

for $n>0$ $\Rightarrow \boxed{C_{n0} = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)dx\,dy}$

Similarly multiplying by $\cos\left(\frac{m'\pi y}{H}\right)$ and integrating between $0<x<L$ and $0<y<H$:

for $m>0$ $\boxed{C_{0m} = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{m\pi y}{H}\right)dx\,dy}$

Now, multiplying by $\cos\left(\frac{n'\pi x}{L}\right)\cos\left(\frac{m'\pi y}{H}\right)$ and integrating:

$$\int_0^H\int_0^L f(x,y)\cos\frac{n'\pi x}{L}\cos\frac{m'\pi y}{H}dx\,dy = C_{00}\cancel{\int_0^L\cos\frac{n'\pi x}{L}dx}^{\,0}\cancel{\int_0^H\cos\frac{m'\pi y}{H}dy}^{\,0}$$

$$+\sum_{n=1}^{\infty}C_{n0}\int_0^L\cos\frac{n\pi x}{L}\cos\frac{n'\pi x}{L}\cancel{\int_0^H\cos\frac{m'\pi y}{H}dy}^{\,0} + \sum_{m=1}^{\infty}C_{0m}\cancel{\int_0^L\cos\frac{n'\pi x}{L}dx}^{\,0}\int_0^H\cos\frac{m\pi y}{H}\cos\frac{m'\pi y}{H}dy$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\underbrace{\int_0^L\cos\frac{n\pi x}{L}\cos\frac{n'\pi x}{L}dx}_{(L/2)\delta_{nn'}}\underbrace{\int_0^H\cos\frac{m\pi y}{L}\cos\frac{m'\pi y}{L}dy}_{(H/2)\delta_{mm'}} = C_{n'm'}\frac{LH}{4}$$

$$\Rightarrow \boxed{C_{nm} = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)dx\,dy}$$ for $n,m>0$.

* c) Now $f'(0)=f'(L)=0$ and $g(0)=g(H)=0$, then:

If $\gamma>0,\ \delta>0$: $f_n(x) = A_n\cos\left(\frac{n\pi x}{L}\right)$ $\quad g_m(x) = B_n\sin\left(\frac{m\pi y}{H}\right)$

$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 > 0 \rightarrow \frac{dh}{dt} = -\lambda_{nm}k\,h(t) \rightarrow h(t) = C_{nm}e^{-\lambda_{nm}kt}$ $\quad n=1,2,3,\ldots$

If $\gamma=0,\ \delta>0$: $f(x) = A_0$ $\quad g_m(y) = B_n\sin\left(\frac{m\pi y}{H}\right)$

$\lambda_{0m} = \left(\frac{m\pi}{H}\right)^2 > 0 \rightarrow \frac{dh}{dt} = -\lambda_{0m}k\,h(t) \rightarrow h(t) = C_{0m}e^{-\lambda_{0m}kt}$ $\quad n=1,2,3\ldots$

If $\delta=0 \rightarrow g(x) = B_2x + B_0$ but $g(0)=0=B_0$ $\quad g(L)=0=B_2L \rightarrow B_2=0$

thus, there is not solution for $\delta=0$.

Simillarly there are not solution for $\gamma,\delta<0$.

$$\Rightarrow \boxed{u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}C_{nm}e^{-\lambda_{nm}kt}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)}$$ For $t\rightarrow\infty$ $u=0$

$$f(x,y) = u(x,y,0) = \sum_{m=1}^{\infty}C_{0m}\sin\left(\frac{m\pi y}{H}\right) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

multiplying by $\sin\left(\frac{m'\pi y}{H}\right)$ and integrating between $0<x<L,\ 0<y<H \Rightarrow$

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Summarize: $\boxed{C_{nm} = \frac{4}{LH(1+\delta_{n0})}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dx\,dy}$

$n \geq 0,\ m > 0.$

$$\boxed{C_{0m} = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\sin\left(\frac{m\pi y}{H}\right)dx\,dy} \qquad \text{for } m>0$$

Now, multiplying by $\cos\left(\frac{n'\pi x}{L}\right)\sin\left(\frac{m'\pi y}{H}\right)$ and integrating again:

$$\boxed{C_{nm} = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dx\,dy} \qquad \text{for } n,m>0.$$

d) Now $f''(x) = -\gamma f(x)$ with $f(0)=0$ $f'(L)=0$
$g''(y) = -\delta g(y)$ $g'(0) = g'(H) = 0$

If $\gamma>0,\ \delta>0$: $f(x) = A_1\cos\sqrt{\gamma}\,x + A_2\sin\sqrt{\gamma}\,x$ $\quad f(0)=0=A_1$

$f'(L) = 0 = A_2\sqrt{\gamma}\cos\sqrt{\gamma}\,L \rightarrow \sqrt{\gamma}\,L = \left(\frac{2n-1}{2}\right)\pi$

$\Rightarrow f(x) = A_n\sin\left[\frac{(2n-1)\pi x}{2L}\right] \qquad n=1,2,3,\ldots$

$g(y) = B_1\cos\sqrt{\delta}\,y + B_2\sin\sqrt{\delta}\,y \qquad g'(0) = 0 = B_2\sqrt{\delta} \rightarrow B_2 = 0$

$g'(L) = 0 = -B_1\sqrt{\delta}\sin\sqrt{\delta}\,H \rightarrow \delta = \left(\frac{m\pi}{H}\right)^2$

$\Rightarrow g(y) = B_n\cos\left(\frac{m\pi y}{H}\right) \qquad m=1,2,3,\ldots$

$\lambda_{nm} = \left[\frac{(2n-1)\pi}{2L}\right]^2 + \left(\frac{m\pi}{H}\right)^2 > 0 \rightarrow \frac{dh}{dt} = -\lambda_{nm}Kh \rightarrow h(t) = C_{nm}e^{-\lambda_{nm}Kt}$

If $\gamma>0,\ \delta=0$: $f(x) = A_n\sin\left[\frac{(2n-1)\pi x}{2L}\right] \qquad n=1,2,3\ldots$

$g''(y) = 0 \rightarrow g(y) = B_2y + B_0 \qquad g'(0) = 0 = B_2 \rightarrow \underline{g(y) = B_0}$

$\lambda_{n0} = \left[\frac{(2n-1)\pi}{2L}\right]^2 > 0 \rightarrow \frac{dh}{dt} = -\lambda_{n0}Kh(t) \rightarrow h(t) = C_{n0}e^{-\lambda_{n0}Kt}$

If $\gamma=0$: $f''(x) = 0 \rightarrow f(x) = A_2x + A_0 \qquad f(0) = 0 = A_0$

$f'(L) = 0 = A_2L \rightarrow A_2 = 0$

There are not solution for $\gamma=0$.
Also, don't exist solution for $\gamma, \delta < 0$.

$$\Rightarrow \boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\cos\left(\frac{m\pi y}{H}\right)e^{-\lambda_{nm}Kt}}$$

For $t\to\infty$ $u\to 0$

$$f(x,y) = u(x,y,0) = \sum_{n=1}^{\infty} C_{n0}\sin\left[\frac{(2n-1)\pi x}{2L}\right] + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\cos\left(\frac{m\pi y}{H}\right)$$

multiplying by $\sin\left[\frac{(2n'-1)\pi x}{2L}\right]$ and integrating:

$$\boxed{C_{n0} = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\sin\left[\frac{(2n-1)\pi x}{2L}\right]dx\,dy}$$

Multiplyin by $\sin[(2n'-1)\pi x/L]\cos(m'\pi y/H)$ and integrating again:

$$\Rightarrow \boxed{C_{nm} = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\sin\left[\frac{(2n-1)\pi x}{L}\right]\cos\left(\frac{m\pi y}{H}\right)dx\,dy}$$

Add $(1+\delta_{0m})$ for summarize for all $n>0,\ m\geq 0$.

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e) $f''(x) = -\gamma f(x)$ with $f(0) = f(L) = 0$

$g''(y) = -\delta g(y)$ $\quad g(0) = 0 \quad g'(H) + h g(H) = 0 \quad h > 0.$

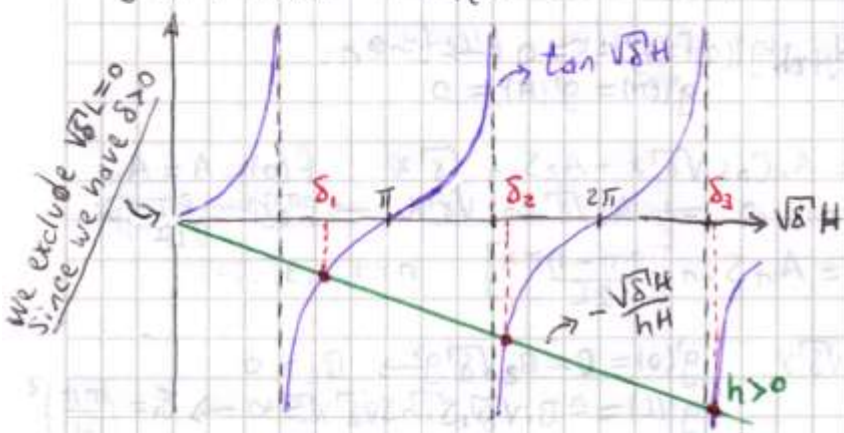
If $\gamma > 0,\ \delta > 0$: $f(x) = A_1 \cos\sqrt{\gamma}\,x + A_2 \sin\sqrt{\gamma}\,x \qquad f(0) = 0 = A_1$

$f(L) = 0 = A_2 \sin\sqrt{\gamma}\,L \rightarrow \gamma = (n\pi/L)^2 \quad n = 1,2,3$

$\rightarrow \boxed{f(x) = A_n \sin(n\pi x/L)}$

$g(x) = B_1 \cos\sqrt{\delta}\,y + B_2 \sin\sqrt{\delta}\,y \qquad g(0) = 0 = B_1$

$g'(H) + h\,g(H) = 0 = B_2(\sqrt{\delta}\cos\sqrt{\delta}H + h\sin\sqrt{\delta}H) \rightarrow \boxed{\tan\sqrt{\delta}H = -\frac{\sqrt{\delta}H}{hH}}$



There is an infinite number of intersections, each corresponds to a positive eigenvalue

$$\boxed{g_m(y) = B_m \sin\sqrt{\delta_m}\,y}$$

where the allowable δ_m are determined graphically.

If $\gamma = 0$: $f''(x) = 0 \rightarrow f(x) = A_1 x + A_0 \qquad f(0) = 0 = A_0$

$f(L) = 0 = A_1 L \rightarrow A_1 = 0$

There are not solutions for $\gamma = 0$. Similarly we can see that there are not solutions for $\gamma < 0$.

If $\delta = 0$: $g''(y) = 0 \rightarrow g(y) = B_1 y + B_0 \qquad g(0) = 0 = B_0$

$g'(H) + h\,g(H) = 0 = B_1(1 + hH) \rightarrow hH = -1$ OR $B_1 = 0$

Since $h > 0 \rightarrow B_1 = 0$.

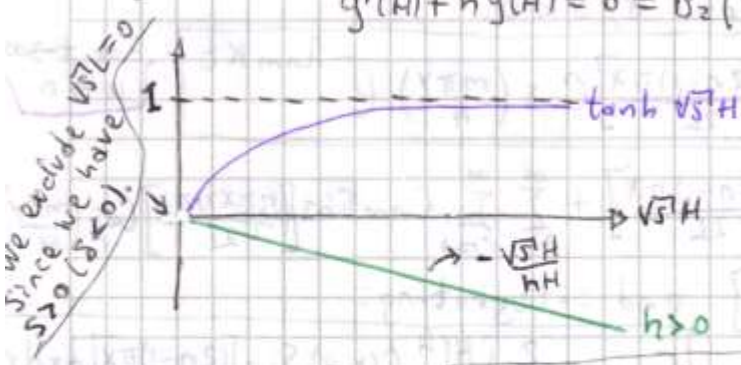
There are not solution for $\delta = 0$.

If $\delta < 0$: let $s = -\delta \Rightarrow g''(y) = s\,g(y) \rightarrow g(y) = B_1 \cosh\sqrt{s}\,y + B_2 \sinh\sqrt{s}\,y$

$g(0) = 0 = B_1$

$g'(H) + h\,g(H) = 0 = B_2(\sqrt{s}\cosh\sqrt{s}H + h\sinh\sqrt{s}H) = 0$

$\Rightarrow \tanh\sqrt{s}H = -\frac{\sqrt{s}H}{hH}$



There are not intersections for $h > 0$ when $\sqrt{s}L > 0$, then there are no negative eigenvalues for $g(y)$.

For $\gamma > 0,\ \delta > 0$ $\quad \boxed{\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \delta_m}$ where $\delta_m > 0$ are determined by the graphics above.

$\frac{dh}{dt} = -\lambda_{nm} k\,h(t) \rightarrow \boxed{h(t) = C_{nm}\, e^{-\lambda_{nm} k t}}$ $\quad n = 1,2,3,\ldots \quad m = 1,2,3,\ldots$

$$\Rightarrow \boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} e^{-\lambda_{nm} k t} \sin\left(\frac{n\pi x}{L}\right) \sin\sqrt{\delta_m}\, y}$$

$$f(x,y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\sqrt{\delta_m}\, y$$

multiplying by $\sin(n'\pi x/L)\sin\sqrt{\delta_{m'}}\,y$ and integrating:

$$\int_0^H\int_0^L f(x,y)\sin\left(\frac{n'\pi x}{L}\right)\sin\left(\sqrt{\delta_{m'}}\,y\right)dx\,dy$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}} \underbrace{\int_0^H \sin\sqrt{\delta_m}\,y \sin\sqrt{\delta_{m'}}\,y\,dy}_{0 \text{ if } m=m'}$$

by orthogonality of eigenfunctions of an Sturm-Liouville problem

$$= C_{nm}\frac{L}{2}\int_0^H \sin^2\sqrt{\delta_m}\,y\,dy$$

$$\Rightarrow \boxed{C_{nm} = \frac{2}{L}\,\frac{\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\sqrt{\delta_m}\,y\right)dx\,dy}{\int_0^H \sin^2\sqrt{\delta_m}\,y\,dy}}$$

where the allowable δ_m are determined graphycally. (See above).

7.3.2 $\boxed{\frac{\partial u}{\partial t} = k\nabla^2\phi}$ with $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

Let $u(x,y,z,t) = \phi(x,y,z)h(t) \rightarrow \phi(x,y,z)\frac{dh}{dt} = kh(t)\nabla^2\phi$ dividing by $kh\phi$

$$\underbrace{\frac{1}{kh}\frac{dh}{dt}}_{-\lambda} = \underbrace{\frac{1}{\phi}\nabla^2\phi}_{-\lambda}$$ separing variables $\Rightarrow$ $\boxed{\begin{aligned}\frac{dh}{dt} &= -\lambda k h(t)\\ \nabla^2\phi &= -\lambda\phi(x,y,z)\end{aligned}}$

Let $\phi(x,y,z) = f(x)g(y)p(z)$

$$\Rightarrow g(y)p(z)\frac{d^2f}{dx^2} + f(x)p(z)\frac{d^2g}{dy^2} + f(x)g(y)\frac{d^2p}{dz^2} = -\lambda f(x)g(y)p(z)$$

dividing by fgp: $\underbrace{\frac{1}{f}\frac{d^2f}{dx^2}}_{-\gamma} + \underbrace{\frac{1}{g}\frac{d^2g}{dy^2}}_{-\delta} + \underbrace{\frac{1}{p}\frac{d^2p}{dz^2}}_{-\Gamma} = -\lambda \rightarrow \boxed{\begin{aligned}f''(x) &= -\gamma f(x)\\ g''(y) &= -\delta f(y)\\ p''(z) &= -\Gamma p(z)\end{aligned}}$

a) $f(0) = f(L) = 0 \rightarrow f(x) = A_n \sin(n\pi x/L)$ $\quad \gamma_n = (n\pi/L)^2 \quad n = 1,2,3..$

There are not solutions for $\gamma \le 0$.

$g'(0) = g'(H) = 0 \rightarrow g(y) = B_n\cos(m\pi y/H)$ $\quad \delta_m = (m\pi/H)^2 \quad m = 1,2,3..$

$g_0(y) = B_0$ when $\delta_0 = 0 \quad m = 0$

No solutions for $\delta < 0$.

$p(z) = C_1\cos\sqrt{\Gamma}\,z + C_2\sin\sqrt{\Gamma}\,z$, $\quad p'(0) = 0 = C_2\sqrt{\Gamma} \rightarrow C_2 = 0$

$p(W) = 0 = C_1\cos\sqrt{\Gamma}\,W \rightarrow \sqrt{\Gamma}\,W = \left(\frac{2q-1}{2}\right)\pi$

$\rightarrow p(z) = C_q\cos\left[\frac{(2q-1)\pi z}{2W}\right]$ $\quad \Gamma_q = \left[\frac{(2q-1)\pi}{2W}\right]^2 \quad q = 1,2,3..$

There are not solutions for $\Gamma \le 0$

$\lambda = \lambda_{nmq} = \gamma_n + \delta_m + \Gamma_q > 0 \rightarrow \boxed{h(t) = D_{nmq}e^{-\lambda_{nmq}kt}}$ $\quad n,q = 1,2,3\ldots$ $\quad m = 0,1,2,3\ldots$

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Solutions to exercises of Haberman Chap 8

8.2.1 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x)$ $\qquad u(x,0) = f(x)$

here $Q(x)=0$

a) At the equilibrium $\frac{\partial u}{\partial t} = 0 = k\frac{\partial^2 u_E}{\partial x^2} + \cancel{Q(x)}^{\,0}$

$\Rightarrow \frac{\partial u_E}{\partial x^2} = 0$ with $u_E(0) = A$ and $\frac{\partial u_E}{\partial x} = B \rightarrow u_E(x) = Bx + A$

Let $v(x,t) \equiv u(x,t) - u_E(x)$, we can check that

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \quad \text{with} \quad \left.\begin{aligned} v(0,t) &= u(0,t) - u_E(0) = A - A = 0 \\ \frac{\partial v(L,t)}{\partial x} &= \frac{\partial u}{\partial x}(L,t) - \frac{\partial u_E}{\partial x}(L) = B - B = 0 \end{aligned}\right\} \text{BC's for } v(x,t)$$

Now let $v(x,t) = X(x)h(t) \Rightarrow X(x)\frac{dh}{dt} = k\,h(t)\,X''(x)$ dividing by khX

$$\frac{h'(t)}{kh} = \frac{X''(x)}{X} = -\lambda \Rightarrow \begin{aligned} X''(x) + \lambda X(x) &= 0 \rightarrow X = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x \\ h'(t) &= -\lambda k h(t) \end{aligned} \quad \text{for } \lambda > 0.$$

The BC's for v implies $X(0) = 0$ and $X'(L) = 0$ that are homogeneous BC's, then the solution are oscillatory and $\lambda > 0$ (There are not oscillatory solutions for $\lambda \leq 0$).

$X(0) = 0 = c_1$ and $X'(L) = 0 = c_2\sqrt{\lambda}\cos\sqrt{\lambda}L \rightarrow \lambda = \left(\frac{2n-1}{2L}\pi\right)^2 \quad n = 1, 2, 3,$

Thus $X_n(x) = \sin\left(\frac{2n-1}{2L}\pi x\right)$.

From $h'(t) = -\lambda h(t) \rightarrow h(t) = e^{-k\lambda t} = e^{-k\left(\frac{2n-1}{2L}\pi\right)^2 t}$

$$\Rightarrow v(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{2n-1}{2L}\pi\right)^2 t}\sin\left(\frac{2n-1}{2L}\pi x\right)$$

$$\underbrace{v(x,0)}_{u(x,0)-u_E(x)} = \sum_{n=1}^{\infty} c_n \sin\left(\frac{2n-1}{2L}\pi x\right) \quad \text{where} \quad \begin{aligned} u(x,0) &= f(x) \\ u_E(x) &= Bx + A \end{aligned}$$

Let $u(x,0) - u_E(x) = f(x) - Bx - A \equiv g(x)$.

Then multiplying above by $\sin\left(\frac{2n'-1}{2L}\pi x\right)$ and integrating:

$$\int_0^L g(x)\sin\left(\frac{2n'-1}{2L}\pi x\right)dx = \sum_{n=1}^{\infty} c_n \underbrace{\int_0^L \sin\left(\frac{2n-1}{2L}\pi x\right)\sin\left(\frac{2n'-1}{2L}\pi x\right)dx}_{(L/2)\delta_{nn'}}$$

$$= c_{n'}\frac{L}{2}$$

$$\Rightarrow c_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{2n-1}{2L}\pi x\right)dx \rightarrow \boxed{c_n = \frac{2}{L}\int_0^L (f(x) - Bx - A)\sin\left(\frac{2n-1}{2L}\pi x\right)dx}$$

Finally $v(x,t) + u_E(x) = \boxed{u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{2n-1}{2L}\pi\right)^2 t}\sin\left(\frac{2n-1}{2L}\pi x\right) + Bx + A}$

Note that as $t \rightarrow \infty \Rightarrow u \rightarrow u_E = Ax + B$

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b) At the equilibrium $\frac{\partial u}{\partial t} = 0 = k\frac{\partial^2 u_E}{\partial x^2} + \cancel{Q(x)}^{\,0} \rightarrow u_E(x) = C_1 x + C_0$

$\frac{\partial u_E}{\partial x}(0) = 0 = C_1$ and $\frac{\partial u_E}{\partial x}(L) = B \neq 0$ but $\frac{\partial u_E}{\partial x} = \frac{\partial}{\partial x}(\cancel{C_1}^{\,0} x + C_0) = 0$

thus there is not a stady state solution.

Let $r(x,t)$ be a reference temperature distribution. We would select the simplest $r(x,t)$ that satisfies the BC's. We need

$\frac{\partial r}{\partial x}(0,t) = 0$ and $\frac{\partial r}{\partial x}(L,t) = B \neq 0$

Suppose we let $r(x,t) \equiv r(x) = \frac{Bx^2}{2L}$, then $\frac{\partial r}{\partial x} = \frac{Bx}{L}$ thus:

$\frac{\partial r}{\partial x}(0,t) = 0$ and $\frac{\partial r}{\partial x}(L,t) = \frac{BL}{L} = B$

Now we can let $v(x,t) = u(x,t) - r(x,t)$

$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} - 0 = k\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 r}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{B}{L}$

$\Rightarrow \boxed{\frac{\partial v}{\partial t} = k\left(\frac{\partial^2 v}{\partial x^2} + \frac{B}{L}\right)}$ with $\frac{\partial v}{\partial x}(0,t) = \underbrace{\frac{\partial u}{\partial x}(0,t)}_{0 \text{ by BC's}} - \cancel{\frac{\partial r}{\partial x}(0,t)}^{\,0} = 0$

with $\frac{\partial v}{\partial x}(0,t) = \frac{\partial v}{\partial x}(L,t) = 0$ $\Leftarrow$ and $\frac{\partial v}{\partial x}(L,t) = \underbrace{\frac{\partial u}{\partial x}(L,t)}_{B} - \underbrace{\frac{\partial r}{\partial x}(L,t)}_{B} = 0$

and $v(x,0) = u(x,0) - r(x)$

$v(x,0) = f(x) - \frac{Bx^2}{2}$ $\rightarrow$ initial condition!!

c) At equilibrium $\frac{\partial u}{\partial t} = 0 = k\frac{\partial^2 u_E}{\partial x^2} \rightarrow u_E(x) = C_1 x + d$

$\frac{\partial u_E}{\partial x}(0) = C_1 = A \neq 0$ and $\frac{\partial u_E}{\partial x}(L) = C_1 = A \Rightarrow u_E(x) = Ax + d$.

Let $v(x,t) = u(x,t) - u_E(x) \rightarrow \frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} - \cancel{\frac{\partial u_E}{\partial t}}^{\,0} = k\frac{\partial^2 u}{\partial x^2} + \cancel{Q(x)}^{\,0}$

but $\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \cancel{\frac{\partial^2 u_E}{\partial x^2}}^{\,0} \rightarrow \boxed{\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}}$ thus v satisfy the heat equation.

with $\frac{\partial v}{\partial x}(0,t) = \underbrace{\frac{\partial u}{\partial x}(0,t)}_{A} - \underbrace{\frac{\partial u_E}{\partial x}(0)}_{A} = 0$

$\frac{\partial v}{\partial x}(L,t) = \underbrace{\frac{\partial u}{\partial x}(L,t)}_{A} - \underbrace{\frac{\partial u_E}{\partial x}(L)}_{A} = 0$

$\Rightarrow \boxed{\frac{\partial v}{\partial x}(0,t) = \frac{\partial v}{\partial x}(L,t) = 0}$ periodic BC's.

For separate variables let $v(x,t) = X(x)h(t) \rightarrow X(x)h'(t) = kh(t)X''(x)$

dividing by khX: $\frac{1}{kh}h'(t) = \frac{1}{X}X''(x) = -\lambda \rightarrow X = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$

$h = e^{-\lambda k t}$

$\frac{\partial v}{\partial x}(0,t) = 0 \rightarrow X'(0) = C_2\sqrt{\lambda} = 0 \rightarrow C_2 = 0$

$\frac{\partial v}{\partial x}(L,t) = 0 \rightarrow X'(L) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0 \rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 0, 1, 2, \ldots$

$\Rightarrow v(x,t) = \sum_{n=0}^{\infty} C_n e^{-k(n\pi/L)^2 t}\cos\left(\frac{n\pi x}{L}\right)$

Now $v(x,0) = u(x,0) - u_E(x) = f(x) - Ax - d \equiv g(x)$

$\Rightarrow \quad g(x) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)$ multiply by $\cos\left(\frac{n'\pi x}{L}\right)$ and integrate:

$$\int_0^L g(x)\cos\left(\frac{n'\pi x}{L}\right)dx = \sum_{n=0}^{\infty} C_n \underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}(1+\delta_{0n})}$$

$$= C_{n'}\frac{L}{2}(1+\delta_{0n'})$$

$$\Rightarrow C_n = \frac{2}{L(1+\delta_{0n})}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx \rightarrow \boxed{C_n = \frac{2}{L(1+\delta_{0n})}\int_0^L \left(f(x) - Ax - d\right)\cos\left(\frac{n\pi x}{L}\right)dx}$$

and $v(x,t) + u_E(x) = u(x,t) = \sum_{n=0}^{\infty} C_n e^{-k(n\pi/L)^2 t}\cos\left(\frac{n\pi x}{L}\right) + Ax + d$

For $t \to \infty \Rightarrow u = C_0 + Ax + d$, but $u_E = Ax + d$, that imposse $C_0 = 0$, then the solution exists if $C_0 = \frac{1}{L}\int_0^L (f(x) - Ax - d)\,dx = 0$.

d) At the equilibrium $\frac{\partial u}{\partial t} = 0 = k\frac{\partial^2 u_E}{\partial x^2} + k \Rightarrow \frac{\partial^2 u_E}{\partial x^2} = -1 \rightarrow u_E = -\frac{1}{2}x^2 + \alpha x + \beta$

To satisfy BC's: $u_E(0) = \boxed{\beta = A}$ $\quad u_E(L) = -\frac{1}{2}L^2 + \alpha L + A = B$

$\alpha L = B - A + \frac{1}{2}L^2$

$\boxed{\alpha = \frac{B-A}{L} + \frac{L}{2}}$

$\Rightarrow u_E(x) = -\frac{1}{2}x^2 + \left(\frac{B-A}{L} + \frac{L}{2}\right)x + A$

Let $v(x,t) = u(x,t) - u_E(x) \rightarrow$ $\left.\begin{array}{l} v(0,t) = u(0,t) - u_E(0) = A - A = 0 \\ v(L,t) = u(L,t) - u_E(L) = B - B = 0 \end{array}\right\}$ BC's for $v(x,t)$

$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t}$ and $\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_E}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + 1 \rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} - 1$

thus $\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + 1\right) = k\frac{\partial^2 v}{\partial x^2} \Rightarrow \boxed{\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}}$ with $\underline{v(0,t) = v(L,t) = 0}$

Separating variables and applying BC's $\Rightarrow v(x,t) = \sum_{n=1}^{\infty} C_n e^{-k(n\pi/L)^2 t}\sin\left(\frac{n\pi x}{L}\right)$

where by orthogonality $C_n = \frac{2}{L}\int_0^L v(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$

$v(x,0) = u(x,0) - u_E(x)$
$= f(x) + \frac{1}{2}x^2 - \alpha x - \beta \rightarrow \boxed{C_n = \frac{2}{L}\int_0^L \left(f(x) + \frac{1}{2}x^2 - \alpha x - \beta\right)\sin\left(\frac{n\pi x}{L}\right)dx}$

Since $v(x,t) + u_E(x) = \boxed{u(x,t) = \sum_{n=1}^{\infty} C_n e^{-k(n\pi/L)^2 t}\sin\left(\frac{n\pi x}{L}\right) - \frac{1}{2}x^2 + \alpha x + \beta}$

As $t \to \infty \rightarrow u(x,t) \rightarrow u_E(x)$.

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e) At equilibrium $\frac{\partial u}{\partial t} = 0 = K\frac{\partial^2 u_E}{\partial x^2} + K \rightarrow \frac{\partial^2 u_E}{\partial x^2} = -1 \rightarrow u_E = -\frac{1}{2}x^2 + \alpha x + \beta$

$\frac{\partial u_E}{\partial x}(0) = 0 = \alpha$ and $\frac{\partial u_E}{\partial x}(L) = 0 = -L + \cancel{\alpha}_{0}$, but $L \neq 0 \rightarrow$ there is not equilibrium solution

Let $r(x,t)$ be a reference temperature distribution. We would select the simplest $r(x,t)$ that satisfies the BC's. We need

$\frac{\partial r}{\partial x}(0,t) = 0$ and $\frac{\partial r}{\partial x}(L,t) \Rightarrow$ Suppose we let $r(x,t) = \frac{x^3}{3L} - \frac{1}{2}x^2$

$\frac{\partial r}{\partial x} = \frac{x^2}{L} - x \Rightarrow \frac{\partial r}{\partial x}(0,t) = \frac{\partial r}{\partial x}(L,t) = 0.$ ✓

Now we can let $v(x,t) = u(x,t) - r(x,t)$

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} - 0 = K\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + 1\right)}_{\frac{\partial^2 v}{\partial x^2} + \frac{2x}{L}}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 r}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{2x}{L} + 1$$

$$\Rightarrow \boxed{\frac{\partial v}{\partial t} = K\left(\frac{\partial^2 v}{\partial x^2} + \frac{2x}{L}\right)}$$

with
$$\left.\begin{aligned} \frac{\partial v}{\partial x}(0,t) &= \underbrace{\cancel{\frac{\partial u}{\partial x}(0,t)}}_{0} - \cancel{\frac{\partial r}{\partial x}(0,t)}^{\,0} = 0 \\ \frac{\partial v}{\partial x}(L,t) &= \underbrace{\frac{\partial u}{\partial x}(L,t)}_{0} - \cancel{\frac{\partial r}{\partial x}(L,t)}^{\,0} = 0 \end{aligned}\right\} \text{BC's}$$

$v(x,0) = u(x,0) - r(x)$

$\quad\quad\quad\quad \parallel$

$v(x,0) = f(x) - \frac{x^3}{3L} + \frac{1}{2}x^2 \rightarrow$ Initial condition

f) At equilibrium $\frac{\partial u}{\partial t} = 0 = K\frac{\partial^2 u_E}{\partial x^2} + \sin\frac{2\pi x}{L} \rightarrow \frac{\partial^2 u_E}{\partial x^2} = -\frac{1}{K}\sin\frac{2\pi x}{L}$

$$\boxed{u_E = \frac{L^2}{(2\pi)^2 K}\sin\frac{2\pi x}{L} + \alpha x + \beta}$$

$\frac{\partial u_E}{\partial x} = \frac{L}{2\pi K}\cos\left(\frac{2\pi x}{L}\right) + \alpha$

$\frac{\partial u_E}{\partial x}(0) = 0 = \frac{L}{2\pi K} + \alpha \rightarrow \boxed{\alpha = -\frac{L}{2\pi K}}$ $\quad \frac{\partial u_E}{\partial x}(L) = 0 = \frac{L}{2\pi K} + \alpha \rightarrow \boxed{\beta \text{ is free}}$

Let define $v(x,t) = u(x,t) - u_E(x) \rightarrow$
$$\left.\begin{aligned} \frac{\partial v}{\partial x}(0,t) &= \underbrace{\frac{\partial u}{\partial x}(0,t)}_{0} - \cancel{\frac{\partial u_E}{\partial x}(0)}^{\,0} = 0 \\ \frac{\partial v}{\partial x}(L,t) &= \underbrace{\frac{\partial u}{\partial x}(L,t)}_{0} - \cancel{\frac{\partial u_E}{\partial x}(L)}^{\,0} = 0 \end{aligned}\right\} \text{BC's for } v(x,t).$$

$\frac{\partial v}{\partial t} = \frac{\partial u(x,t)}{\partial t} - \cancel{\frac{\partial u_E}{\partial t}}^{\,0}$

$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_E}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{K}\sin\frac{2\pi x}{L}$

$K\frac{\partial^2 v}{\partial x^2} = \underbrace{K\frac{\partial^2 u}{\partial x^2} + \sin\frac{2\pi x}{L}}_{\partial u/\partial t} \Rightarrow \longrightarrow \boxed{\frac{\partial v}{\partial t} = K\frac{\partial^2 v}{\partial t^2}}$ with $\frac{\partial v}{\partial x}(0,t) = \frac{\partial v}{\partial x}(L,t) = 0$

Separating variables $v(x,t) = \sum_{n=1}^{\infty} C_n e^{-K\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi}{L}x\right)$

Using the IC and orthogonality $C_n = \frac{2}{L(1+\delta_{n0})}\int_0^L v(x,0)\cos\left(\frac{n\pi x}{L}\right)dx$

with $v(x,0) = u(x,0) - u_E(x)$
$\quad\quad\quad\quad = f(x) - u_E(x) \Rightarrow \boxed{C_n = \frac{2}{L(1+\delta_{n0})}\int_0^L (f(x) - u_E)\cos\frac{n\pi x}{L}\,dx}$

and $\boxed{v(x,t) + u_E(x) = u(x,t)}$

As $t \rightarrow \infty \Rightarrow u(x,t) = C_0 + u_E(x) \rightarrow$ for satisfy equilibrium condition $\Rightarrow C_0 = 0$

$\Rightarrow C_0 = \frac{1}{L}\int_0^L (f(x) - u_E)\,dx = 0 \rightarrow$ Condition for solution to exists

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8.2.2 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad u(x,0) = f(x)$

a) We want to find a reference temperature distribution $r(x,t)$ wich satisfies the BC's

$$\frac{\partial r}{\partial x}(0,t) = A(t) \quad \text{and} \quad \frac{\partial r}{\partial x}(L,t) = B(t)$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{1}{L}\left(B(t)\cdot x + (L-x)A(t)\right) \rightarrow r(x,t) = \frac{B(t)}{2L}x^2 + (Lx - x^2/2)\frac{A(t)}{L}$$

OR $r(x,t) = \frac{1}{2L}\left(B(t)x^2 + (2Lx - x^2)A(t)\right)$

Now let $v(x,t) = u(x,t) - r(x,t) \rightarrow \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{1}{2L}\left(B'(t)x^2 + (2Lx - x^2)A'(t)\right)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + \frac{1}{L}\left(B(t)x + (L-x)A(t)\right)$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{1}{L}\left(B(t)x + (L-x)A(t)\right)$$

$$\frac{\partial v}{\partial t} + \frac{1}{2L}\left(B'(t)x^2 + (2Lx - x^2)A'(t)\right) = k\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{L}\left(B(t)x + (L-x)A(t)\right)\right) + Q(x,t)$$

$$\boxed{\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} - \frac{1}{2L}\left(B'(t)x^2 + (2Lx - x^2)A'(t)\right) + \frac{k}{L}\left(B(t) - A(t)\right) + Q(x,t)}$$

with BC's: $\frac{\partial v}{\partial x}(0,t) = \frac{\partial v}{\partial x}(L,t) = 0$

and the IC: $v(x,0) = f(x) - \frac{1}{2L}\left(B(0)x^2 + (2Lx - x^2)A(0)\right)$

b) We want to find the simplest reference temperature distribution which satisfies the BC's:

We need $r(0,t) = A(t)$ and $\frac{\partial r}{\partial x}(L,t) = B(t)$

Let suppose $r(x,t) = A(t) + B(t)x$, that satisfy BC's.

and let define $v(x,t) = u(x,t) - r(x,t) \rightarrow \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + A'(t) + B'(t)x$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial v}{\partial t} + A'(t) + B'(t)x = k\frac{\partial^2 v}{\partial x^2} + Q(x,t)$$

$$\boxed{\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + Q(x,t) - A'(t) - B'(t)x}$$

with BC's: $\frac{\partial v}{\partial x}(L,t) = 0$ and $v(0,t) = 0$

and IC: $v(x,0) = f(x) - A(0) - B(0)x$.

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c) A reference temperature distribution $r(x,t)$ must satisfy the BC's

$$\frac{\partial r}{\partial x}(0,t) = A(t) \quad \text{and} \quad r(L,t) = B(t)$$

The simplest example is the linear function $r(x,t) = A(t)x + B(t) - LA(t)$

Let now define: $v(x,t) = u(x,t) - r(x,t) \rightarrow \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + A'(t)x + B'(t) - LA'(t)$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \cancel{\frac{\partial^2 r}{\partial x^2}}^{\,0}$$

$$\frac{\partial v}{\partial t} + A'(t)(x-L) + B'(t) = k\frac{\partial^2 v}{\partial x^2} + Q(x,t)$$

$$\boxed{\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + Q(x,t) - A'(t)(x-L) - B'(t)}$$

with BC's: $\frac{\partial v}{\partial x}(0,t) = \frac{\partial v}{\partial x}(L,t) = 0$

and IC $\rightarrow v(x,0) = f(x) - r(x,0)$.

d)

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8.2.3 We begin by looking for an equilibrium solution $u_E(r)$ to the equation

$$\frac{\partial u_E}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_E}{\partial r}\right) + Q(r) \quad \text{with} \quad u_E(a) = T$$

At equilibrium $\rightarrow 0$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\frac{r}{K}Q(r) \rightarrow \underbrace{r'\frac{\partial u}{\partial r'}\Big|_0^r}_{r\frac{\partial u}{\partial r} = r\frac{du_E}{dr}} = -\frac{1}{K}\int_0^r r'Q(r')\,dr' + C_1$$

$$\Rightarrow \frac{du_E}{dr} = -\frac{1}{Kr}\int_0^r r'Q(r')\,dr'$$

Now integrating between r to a by convenience:

$$\underbrace{\int_r^a \frac{du_E}{dr}\,dr}_{-u_E(r)+\underbrace{u_E(a)}_{T}} = -\frac{1}{K}\int_r^a \frac{1}{r'}\int_0^{r'} r''Q(r'')\,dr''\,dr' + C_1 \ln$$

$$\Rightarrow \boxed{u_E(r) = T + \frac{1}{K}\int_r^a \frac{1}{s}\int_0^s w\,Q(w)\,dw\,ds}$$

Now let $v(r,t) = u(r,t) - u_E(r) \rightarrow \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}$, $\frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} + \frac{du_E}{dr}$

$$\frac{\partial u}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r)$$

$$\rightarrow \frac{\partial v}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r} - \frac{1}{K}\int_0^r r'Q(r')\,dr'\right) + Q(r)$$

$$= \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{1}{r}\underbrace{\frac{\partial}{\partial r}\int_0^r r'Q(r')\,dr'}_{rQ(r)} + Q(r) = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + 0$$

$$\Rightarrow \boxed{\frac{\partial v}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial t}\right)} \quad \text{with BC} \rightarrow v(a,t) = 0 \quad \text{and IC} \rightarrow v(r,0) = u(r,0) - u_E = f(r) - u_E(r)$$

We will use separation of variables to solve this equation

Let $v(r,t) = R(r)\theta(t) \Rightarrow R(r)\theta'(t) = \frac{K}{r}\frac{\partial}{\partial r}\left(rR'(r)\theta(t)\right)$

$$= \frac{K}{r}\left(R'(r) + rR''(r)\right)\theta(t)$$

Divide both sides by $KR\theta$

$$\rightarrow \underbrace{\frac{\theta'(t)}{K\theta(t)}}_{-\lambda} = \underbrace{\frac{1}{R(r)}\frac{1}{r}\left(R'(r) + rR''(r)\right)}_{-\lambda}$$

$$\Rightarrow R'(r) + rR''(r) = -\lambda r R(r)$$

$$\boxed{r^2R''(r) + rR'(r) + (\lambda r^2 - 0^2)R(r) = 0} \rightarrow \text{Bessel eq of order } 0.$$

$$R(r) = c_1 J_0(\sqrt{\lambda}r) + \cancel{c_2} N_0(\sqrt{\lambda}r)$$

Since $|v(0,t)| < \infty \rightarrow 0$

$v(a,t) = 0 \rightarrow R(a) = 0 \rightarrow J_0(\sqrt{\lambda}a) = 0 \rightarrow \sqrt{\lambda_n} = \frac{z_n}{a}$ with z_n the zeros of $J_0(z)$

$$\Rightarrow \boxed{R_n(r) = J_0\left(\frac{z_n}{a}r\right)} \rightarrow \text{the eigenfunction in } r.$$

~~There are no eigenfunction~~

Therefore $\dfrac{\theta'(t)}{K\theta(t)} = -\lambda = -\left(\dfrac{z_n}{a}\right)^2 \rightarrow \boxed{\theta(t) = e^{-K(z_n/a)^2 t}}$

$$\Rightarrow \quad v(r,t) = \sum_{n=1}^{\infty} A_n e^{-K(z_n/a)^2 t} J_0(z_n r/a)$$

$$v(r,0) = \sum_{n=1}^{\infty} A_n J_0(z_n r/a)$$ multiplying by $J_0(z_{n'} r/a)$ and integrating in $r\,dr$.

$$\int_0^a v(r,0) J_0(z_{n'} r/a)\, r\,dr = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^a J_0(z_n r/a) J_0(z_{n'} r/a)\, r\,da}_{0 \text{ if } n \neq n'}$$

$$= A_{n'} \int_0^a J_0^2(z_n r/a)\, r\,dr$$

$$v(r,0) = f(r) - u_E(r) \longrightarrow \boxed{A_n = \int_0^a [f(r) - u_E(r)] J_0\left(\frac{z_n}{a} r\right) r\,dr \Big/ \int_0^a J_0^2\left(\frac{z_n}{a} r\right) r\,dr}$$

Now $u(r,t) = v(r,t) + u_E(r)$

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} A_n e^{-K(z_n/a)^2 t} J_0\left(\frac{z_n}{a} r\right) + T + \frac{1}{K}\int_0^a \frac{1}{s}\int_0^s w\, Q(w)\, dw\, ds}$$

8.2.4 we first solve the equilibrium equation $\cancel{\dfrac{\partial u_E}{\partial t}} = K\left(\dfrac{\partial^2 u_E}{\partial x^2} + \dfrac{\partial^2 u_E}{\partial y^2}\right)$

At equilibrium: 0

$$\Rightarrow \quad \frac{\partial^2 u_E}{\partial x^2} + \frac{\partial^2 u_E}{\partial y^2} = 0 \quad \text{with} \quad u_E(0,y) = u_E(L,y) = 0 \quad \text{and} \quad \frac{\partial u_E}{\partial y}(x,0) = 0, \quad u_E(x,H) = g(x)$$

Let $u_E(x,y) = h(x)\phi(y) \rightarrow$ separating variables as usual:

$$\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = -\lambda \rightarrow \frac{d^2h}{dx^2} + \lambda h(x) = 0 \quad \text{with} \quad h(0) = h(L) = 0$$

$$\Rightarrow h(x) = A_n \sin(n\pi x/L) \quad n = 1, 2, 3, \ldots$$

$$\frac{d^2\phi}{dy^2} - \lambda\phi(y) = 0 \quad \text{with} \quad \phi'(0) = 0$$

$$\Rightarrow \phi(y) = \cosh(n\pi y/L)$$

$$\Rightarrow \boxed{u_E(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right)}$$

$$u_E(x,H) = g(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi H}{L}\right)$$ Multiplying by $\sin(n'\pi x/L)$ and integrating as usual:

$$\boxed{A_n = \frac{2}{L\cosh(n\pi H/L)} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

you can check!!

Now let $u(x,y,t) - u_E(x,y) \equiv v(x,y,t)$, thus $v(x,y,t)$ satisfy

$$\frac{\partial v}{\partial t} = K\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad \text{with} \quad \left.\begin{array}{l} v(0,y,t) = v(L,y,t) = 0 \\ \dfrac{\partial v}{\partial y}(x,0,t) = v(x,H,t) = 0 \end{array}\right\} \text{Periodic BC's} \Rightarrow$$

and the IC $\rightarrow v(x,y,0) = f(x,y) - u_E(x,y)$

Let $v(x,y,t) = h(t)\,\varphi(x,y)$ then separating variables: after substitute in diff eq and divide by $Kh\varphi$ we have:

$$\frac{1}{Kh}\frac{dh}{dt} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = -\lambda \Rightarrow \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = -\lambda\varphi \quad \text{with } \varphi(0,y)=\varphi(L,y)=0$$

$$\frac{\partial\varphi}{\partial y}(x,0) = \varphi(x,H) = 0$$

$$\frac{dh}{dt} = -\lambda k h(t) \rightarrow \boxed{h(t) = e^{-\lambda k t}}$$

Let now $\varphi(x,y) = f(x)g(y)$, then separating variables:

$$\underbrace{\frac{1}{f}\frac{d^2f}{dx^2}}_{-\mu} + \underbrace{\frac{1}{g}\frac{d^2g}{dy^2}}_{\mu} - \lambda = 0 \Rightarrow \frac{d^2f}{dx^2} = -\mu f(x) \quad \text{with } f(0)=f(L)=0$$

$$\mu = (n\pi/L)^2 \rightarrow f(x) = \sin(n\pi x/L) \quad n=1,2,3..$$

$$\frac{d^2g}{dy^2} = -(\lambda-\mu)g(y) \quad \text{with } g'(0)=0 \text{ and } g(H)=0$$

$g(H)=0 \rightarrow m\neq 0$

$$\Rightarrow g(y) = \cos\left(\frac{2m-1}{2H}\pi y\right) \quad m=1,2,\ldots$$

With $\lambda-\mu = \left(\frac{2m-1}{2H}\pi\right)^2 \rightarrow \lambda = \mu + (\ )^2$

$$\boxed{\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{2m-1}{2H}\pi\right)^2}$$

$$\boxed{v(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{2m-1}{2H}\pi y\right)e^{-\lambda_{nm}kt}}$$

$$v(x,y,0) = \sum_{n,m} A_{nm}\sin\frac{n\pi x}{L}\cos\left(\frac{2m-1}{2H}\pi y\right)$$

$\parallel$

$f(x,y) - u_E(x,y)$

multiplyin by $\sin\left(\frac{n'\pi x}{L}\right)\cos\left(\frac{2m'-1}{2H}\pi y\right)$ and integrating.

$$\boxed{A_{nm} = \frac{4}{HL}\int_0^H\int_0^L \left(f(x,y) - u_E(x,y)\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{2m-1}{2H}\pi y\right)dx\,dy}$$

$$\boxed{u(x,y,t) = v(x,y,t) + u_E(x,y)}$$

Note that $v(x,y,t) \rightarrow 0$ as $t\rightarrow\infty$, thus $u(x,y,t) \rightarrow u_E(x,y)$ as $t\rightarrow\infty$.

8.2.5 At the equilibrium $\cancel{\frac{\partial u_E}{\partial t}}^{\,0} = K\nabla^2 u_E \rightarrow \boxed{\nabla^2 u_E(r,\theta) = 0 \text{ with } u_E(a,\theta) = g(\theta)}$

That problem is solved by separation of variables see section (2.5.2)

$$\rightarrow \nabla^2 u_E = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_E}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0 \qquad \text{letting } u_E(r,\theta) = \phi(\theta)G(r)$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}G(r)\frac{d^2\phi}{d\theta} = 0 \rightarrow \underbrace{\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right)}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}}_{-\lambda} = 0$$

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periodic BC's

Since $u_E(r,-\pi) = u_E(r,\pi)$, $\frac{\partial u_E}{\partial \theta}(r,-\pi) = \frac{\partial u_E}{\partial \theta}(r,\pi)$ $\Rightarrow$ $\phi(-\pi)=\phi(\pi)$, $\phi'(-\pi)=\phi'(\pi)$ $\Rightarrow$ $\phi \sim \left\{ \begin{matrix} \sin n\theta \\ \cos n\theta \end{matrix} \right\}$ $n = 0,1,2,3$

and $r^2 \frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0 \rightarrow G = C_1 r^n + C_2 r^{-n}$ $\quad n \neq 0$

$G = \bar{C}_1 + \bar{C}_2 \ln r$ if $n=0$.

But if $|u_E(0,\theta)| < \infty \rightarrow |G(0)| < \infty \rightarrow \bar{C}_2 = C_2 = 0$.

$$\Rightarrow \boxed{u_E(r,\theta) = \sum_{n=0}^{\infty} [A_n \cos n\theta + B_n \sin n\theta] r^n}$$

Since $u_E(a,\theta) = g(\theta)$, then using the orthogonality formulas or the Fourier coeficient are (See eqs 2.5.47):

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta, \quad A_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\,d\theta \quad B_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta.$$

Now let $v(r,\theta,t) = u(r,\theta,t) - u_E(r,\theta) \rightarrow \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \cancel{\frac{\partial u_E}{\partial t}}^{\,0}$

$\nabla^2 u = \nabla^2 v + \cancel{\nabla^2 u_E}^{\,0}$

Then since $\frac{\partial u}{\partial t} = k\nabla^2 u$

$\frac{\partial v}{\partial t} = k\nabla^2 v$ with $v(a,\theta,t) = \underbrace{u(a,\theta,t)}_{g(\theta)} - \underbrace{u_E(a,\theta)}_{g(\theta)} = 0$

and the initial condition is $v(r,\theta,0) = u(r,\theta,0) - u_E(r,\theta)$

$v(r,\theta,0) = f(r,\theta) - u_E(r,\theta)$

Summarizing the initial-value problem for $v(r,\theta,0)$ is

$$\boxed{\frac{\partial v}{\partial t} = k\nabla^2 v \quad \text{with} \quad \begin{matrix} v(a,\theta,t) = 0 \\ v(r,\theta,0) = f(r,\theta) - u_E(r,\theta) \end{matrix}}$$

$\rightarrow$ In 2-D inside a circle of radious a.

This problem is solved in exercise 7.7.7 and the solution is:

$$\boxed{v(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\,r)[A_{mn}\cos m\theta + B_{mn}\sin m\theta]\, e^{-\lambda_{mn} k t}}$$

with $\lambda_{mn} = (z_{mn}/a)^2$ and z_{mn} the zeroes of $J_m(z)$.

and the coefficients found using orthogonality relations:

$$A_{mn} = \frac{1}{\pi(1+\delta_{m0})}\left[\int_{-\pi}^{\pi}\int_0^a (f(r,\theta) - u_E(r,\theta)) J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)\, r\,dr\,d\theta \Big/ \left(\frac{a^2}{2} J_{m+1}^2(\sqrt{\lambda_{mn}}\,a)\right)\right]$$

$$B_{mn} = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\int_0^a (f(r,\theta) - u_E(r,\theta)) J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\, r\,dr\,d\theta \Big/ \left(\frac{a^2}{2} J_{m+1}^2(\sqrt{\lambda_{mn}}\,a)\right)\right]$$

$$\boxed{u(r,\theta,t) = v(r,\theta,t) + u_E(r,\theta)}$$

Note that $\lim_{t\to\infty} v(r,\theta,t) = 0$ because $u \sim e^{-\lambda_{mn} k t}$ $\lambda_{mn} > 0$

$\Rightarrow \lim_{t\to\infty} u(r,\theta,t) = u_E(r,\theta)$ ✓

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8.2.6 $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} + Q(x)$ $\qquad u(x,0) = f(x)$, $\quad \dfrac{\partial u}{\partial t}(x,0) = g(x)$

a) Suppose an "equilibrium solution" exists. Then $\dfrac{\partial^2 u_E}{\partial t^2} = 0$

$\rightarrow \quad 0 = c^2 \dfrac{\partial^2 u_E}{\partial x^2} + \cancel{Q(x)}^{\,0 \text{ here!}} \quad \rightarrow \quad u_E(x) = C_1 x + C_0$

$u_E(0) = A = C_0$ and $u_E(L) = B = C_1 L + C_0 = C_1 L + A \rightarrow C_1 = \dfrac{B-A}{L}$

$\Rightarrow \boxed{u_E(x) = \dfrac{B-A}{L} x + A}$

Now let $v(x,t) = u(x,t) - u_E(x) \rightarrow \begin{cases} \dfrac{\partial^2 v}{\partial t^2} = c^2\left(\dfrac{\partial^2 u}{\partial x^2} - \cancel{\dfrac{\partial^2 u_E}{\partial t^2}}^{\,0}\right) \\ \dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial^2 u}{\partial x^2} - \cancel{\dfrac{\partial^2 u_E}{\partial x^2}}^{\,0} \end{cases}$

Since $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

$\boxed{\dfrac{\partial^2 v}{\partial t^2} = c^2 \dfrac{\partial^2 v}{\partial x^2}}$ with IC's $\begin{cases} \dfrac{\partial v}{\partial t}(x,0) = \dfrac{\partial u}{\partial t}(x,0) - \cancel{\dfrac{\partial u_E}{\partial t}}^{\,0}(x,0) = g(x) \\ v(x,0) = u(x,0) - u_E(x) = f(x) - u_E(x) \end{cases}$

and $\begin{aligned} v(0,t) &= u(0,t) - u_E(0,t) = A - A = 0 \\ v(L,t) &= u(L,t) - u_E(L,t) = B - B = 0 \end{aligned} \Rightarrow \boxed{v(0,t) = v(L,t) = 0} \rightarrow$ BC's

That problem is solved in section 4.4 and the solution is:

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L}\right) \sin \frac{n\pi x}{L}}$$

where $A_n = \dfrac{2}{L}\displaystyle\int_0^L \left(f(x) - u_E(x)\right) \sin \frac{n\pi x}{L}\, dx$

$B_n = \dfrac{2}{n\pi c}\displaystyle\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$

Finally $\boxed{u(x,t) = v(x,t) + u_E(x)}$

b) Let $Q(x) = 1$ and consider the BC's $u(0,t) = u(L,t) = 0$

Suppose there exists an "equilibrium" solution, $u_E(x) \rightarrow \dfrac{\partial^2 u_E}{\partial t^2} = 0$

$\Rightarrow \quad 0 = c^2 \dfrac{d^2 u_E}{\partial x^2} + 1 \rightarrow \dfrac{d^2 u_E}{dt^2} = -\dfrac{1}{c^2} \rightarrow \dfrac{du_E}{dx} = -\dfrac{1}{c^2} x + C_1$

$u_E(x) = -\dfrac{x^2}{2c^2} + C_1 x + C_0$

We need $u_E(0) = 0 \rightarrow C_0 = 0$

$u_E(L) = 0 \rightarrow -\dfrac{L^2}{2c^2} + C_1 L = 0 \rightarrow C_1 L = \dfrac{L^2}{2c^2} \rightarrow C_1 = \dfrac{L}{2c^2}$

$\Rightarrow u_E(x) = -\dfrac{x^2}{2c} + \dfrac{Lx}{2c^2}$ OR $\boxed{u_E(x) = \dfrac{x}{2c^2}(L - x)}$

Now let $v(x,t) = u(x,t) - u_E(x) \longrightarrow \dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial^2 u}{\partial t^2} - \cancel{\dfrac{\partial^2 u_E}{\partial t^2}}^{\,0}$ and

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$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{c^2}, \quad \text{then since} \rightarrow \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} - \frac{1}{c^2}$$

Thus since $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 1 \rightarrow \frac{\partial^2 v}{\partial t^2} = c^2\left(\frac{\partial^2 v}{\partial x^2} - \cancel{\frac{1}{c^2}}\right) + \cancel{1} = c^2 \frac{\partial^2 v}{\partial x^2}$

Hence $v(x,t)$ satisfy $\boxed{\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}}$ with BC's $v(0,t) = v(L,t) = 0$ and IC's $v(x,0) = f(x) - u_E(x)$

you can check $\rightarrow$ $\frac{\partial v}{\partial t}(x,0) = g(x)$

That problem is solved in section (4.4) and the solution is:

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi c}{L} t + B_n \sin \frac{n\pi c}{L} t\right) \sin \frac{n\pi x}{L}}$$

where $A_n = \frac{2}{L}\int_0^L \left(f(x) - u_E(x)\right) \sin \frac{n\pi x}{L}\, dx$

$B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$

Finally $\boxed{u(x,t) = v(x,t) + u_E(x)}$

c) Let's find an equilibrium solution when $\frac{\partial^2 u_E}{\partial t^2} = 0$

$\Rightarrow 0 = c^2 \frac{\partial^2 u_E}{\partial x^2} + 1 \rightarrow u_E'' = -\frac{1}{c^2} \rightarrow u_E(x) = -\frac{1}{2c^2}x^2 + \alpha x + \beta$

Since $u_E(0) = A = \beta$ and $u_E(L) = B = -\frac{L^2}{2c^2} + \alpha L + A$

$\alpha L = B - A + \frac{L^2}{2c^2}$

$$\Rightarrow \boxed{u_E(x) = -\frac{1}{2c^2}x^2 + \left(\frac{B-A}{L} + \frac{L}{2c^2}\right)x + A}$$

Now define $v(x,t) = u(x,t) - u_E(x) \rightarrow \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} - \cancel{\frac{\partial^2 u_E}{\partial t^2}}^{\,0}$

$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_E}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{c^2}$

using from these eqs $\partial^2 u/\partial t^2 = \partial^2 v/\partial t^2$ and $\partial^2 u/\partial x^2 = \partial^2 v/\partial x^2 - 1/c^2$ in heat eq: $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} c^2 + 1 \rightarrow \frac{\partial^2 v}{\partial t^2} = c^2\left(\frac{\partial^2 v}{\partial x^2} - \cancel{\frac{1}{c^2}}\right) + \cancel{1}$

$\Rightarrow \boxed{\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}}$ with $\left.\begin{array}{l} v(0,t) = u(0,t) - u_E(0) = A - A = 0 \\ v(L,t) = u(L,t) - u_E(L) = B - B = 0 \end{array}\right\}$ BC's

and $\left.\begin{array}{l} v(x,0) = u(x,0) - u_E(x) = f(x) - u_E(x) \\ \frac{\partial v}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,0) - \cancel{\frac{\partial u_E}{\partial t}(x,0)}^{\,0} = g(x) \end{array}\right\}$ IC's

This problem is solved in section (4.4) and the solution is:

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$$\boxed{v(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L} \right) \sin \frac{n\pi x}{L}}$$

where $A_n = \frac{2}{L}\int_0^L \left(f(x) - u_E(x)\right) \sin \frac{n\pi x}{L}\, dx$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

Thus $\boxed{u(x,t) = v(x,t) + u_E(x)}$

Note that solution of this problem is feactible obtain by addition of problems of parts (a) and (b).

d) First we look for an equilibrium solution where $\frac{\partial u_E}{\partial t^2} = 0$

$$\Rightarrow \quad 0 = c^2 \frac{\partial^2 u_E}{\partial x^2} + \sin(\pi x/L) \rightarrow \frac{\partial^2 u_E}{\partial x^2} = -\frac{1}{c^2}\sin(\pi x/L)$$

$$\frac{\partial u_E}{\partial x} = \frac{L}{c^2\pi}\cos\frac{\pi x}{L} + \alpha \rightarrow u_E(x) = \frac{L^2}{c^2\pi^2}\sin(\pi x/L) + \alpha x + \beta$$

Since $u_E(0) = 0 = \beta$ and $u_E(L) = 0 = \frac{L^2}{c^2\pi^2}\sin\pi + \alpha L \Rightarrow \alpha = 0$

$$\Rightarrow \boxed{u_E(x) = \frac{L^2}{c^2\pi^2}\sin\left(\frac{\pi x}{L}\right)}$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} - \cancel{\frac{\partial^2 u_E}{\partial t^2}}^{\,0}$$

Now let define $v(x,t) = u(x,t) - u_E(x) \rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_E}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{c^2}\sin\frac{\pi x}{L}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} - \frac{1}{c^2}\sin(\pi x/L)$$

Thus since $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + \sin\frac{\pi x}{L}$

$$\frac{\partial^2 v}{\partial t^2} = c^2\left(\frac{\partial^2 v}{\partial x^2} - \cancel{\frac{1}{c^2}\sin\frac{\pi x}{L}}\right) + \cancel{\sin\frac{\pi x}{L}}$$

$$\Rightarrow \boxed{\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2}} \quad \text{with} \quad \left.\begin{aligned} v(0,t) &= u(0,t) - u_E(0) = 0 - 0 = 0 \\ v(L,t) &= u(L,t) - u_E(L) = 0 - 0 = 0 \end{aligned}\right\} \text{BC's}$$

and
$$\left.\begin{aligned} v(x,0) &= u(x,0) - u_E(x) = f(x) - u_E(x) \\ \frac{\partial v}{\partial t}(x,0) &= \frac{\partial u}{\partial t}(x,0) - \cancel{\frac{\partial u_E}{\partial t}(x,0)}^{\,0} = g(x) \end{aligned}\right\} \text{IC's}$$

Thus as before: $\boxed{v(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L} \right) \sin \frac{n\pi x}{L}}$

where $A_n = \frac{2}{L}\int_0^L \left(f(x) - u_E(x)\right) \sin \frac{n\pi x}{L}$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

You can check: $A_1 = -\frac{L^2}{c^2\pi^2} + \frac{2}{L}\int_0^L f(x)\sin\frac{\pi x}{L}\,dx$

and for $n > 1$ $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

Finally $\boxed{u(x,t) = v(x,t) + u_E(x)}$

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8.3.1 $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} + Q(x,t)$ $\qquad u(x,0) = f(x)$

a) $u(0,t) = \dfrac{\partial u}{\partial x}(L,t) = 0 \rightarrow \varphi_n(x) = \sin\left(\dfrac{2n-1}{2L}\pi x\right)$ are the eigenfunctions of the related homogeneous equation.

Let $\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\dfrac{(2n-1)\pi x}{2L}}$ If we differentiate an substitute in the PDE.

$$\Rightarrow \sum_{n=1}^{\infty} a_n'(t)\sin\frac{(2n-1)\pi x}{2L} = \sum_{n=1}^{\infty} -k\left(\frac{(2n-1)\pi}{2L}\right)^2 a_n(t)\sin\frac{(2n-1)\pi x}{2L} + Q(x,t)$$

$$\text{or} \quad \sum_{n=1}^{\infty}\left(a_n'(t) + k\left(\frac{(2n-1)\pi}{2L}\right)^2 a_n(t)\right)\sin\frac{(2n-1)\pi x}{2L} = Q(x,t)$$

multiplying by $\sin[(2n'-1)\pi x/L]$ an integrating by use the orthogonality:

$$a_n'(t) + k\left(\frac{(2n-1)\pi}{2L}\right)^2 a_n(t) = \underbrace{\frac{2}{L}\int_0^L Q(x,t)\sin\frac{(2n-1)\pi x}{2L}\,dx}_{q_n(t)} \qquad q_n(t) = \frac{2}{L}\int_0^L Q(x,t)\sin(\ )\,dx$$

since $Q = \sum_{n=1}^{\infty} q_n(t)\sin(\ )$

Using the integrating factor $e^{k\left(\frac{2n-1}{2L}\pi\right)^2 t}$:

$$\underbrace{a_n'(t)\,e^{k\left(\frac{2n-1}{2L}\pi\right)^2 t} + k\left(\frac{(2n-1)\pi}{2L}\right)^2 a_n(t)\,e^{k\left(\frac{2n-1}{2L}\pi\right)^2 t}}_{\frac{d}{dt}\left(a_n(t)\,e^{k(\ )^2 t}\right)} = q_n(t)\,e^{k(\ )^2 t}$$

$$\Rightarrow a_n(t)\,e^{k(\ )^2 t} - a_n(0) = \int_0^t q_n(s)\,e^{k(\ )^2 s}\,ds$$

$$\boxed{a_n(t) = a_n(0)\,e^{-k\left(\frac{2n-1}{2L}\pi\right)^2 kt} + \int_0^t q_n(s)\,e^{k\left(\frac{2n-1}{2L}\pi\right)^2 (s-t)}\,ds}$$

where $a_n(0)$ is found of the initial condition:

$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n(0)\sin\dfrac{(2n-1)\pi x}{2L}$ Multiplying by $\sin\dfrac{2n'-1}{2L}\pi x$ and integrating by use of orthogonality

$$\boxed{a_n(0) = \frac{2}{L}\int_0^L f(x)\sin\left[\frac{(2n-1)\pi}{2L}x\right]}$$

b) We need the eigenfunctions/eigenvalues of the related homogeneous equation to do the expansion. For the homogeneous problem $Q(x,t)=0$

$\Rightarrow \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ and separing variables with $u(x,t) = G(t)\phi(x)$

we obtain $\underbrace{\dfrac{1}{kG}\dfrac{dG}{dt}}_{-\lambda} = \underbrace{\dfrac{1}{\phi}\dfrac{d^2\phi}{dx^2}}_{-\lambda} = 0 \rightarrow \dfrac{d^2\phi}{dx^2} + \lambda\phi(x) = 0$

$u(0,t) = 0 \rightarrow \phi(0) = 0$

$u(L,t) + 2\dfrac{\partial u}{\partial x}(L,t) = 0 \rightarrow \phi(L) + 2\phi'(L) = 0 \Rightarrow$

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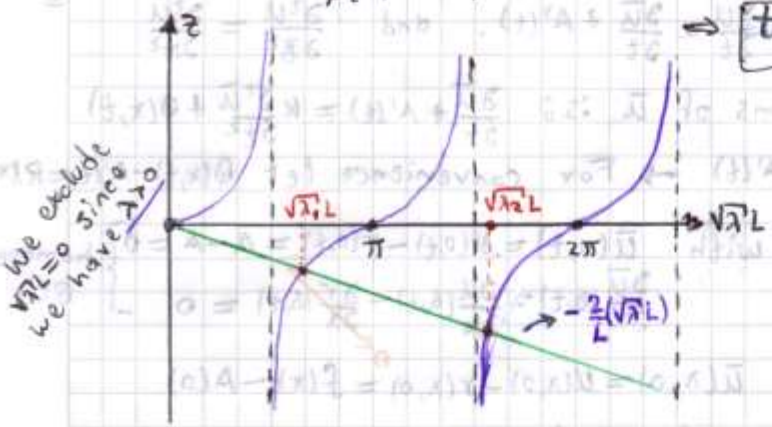
For $\lambda > 0$: $\phi(x) = C_1 \cos\sqrt{\lambda}\,x + C_2 \sin\sqrt{\lambda}\,x$ $\quad \phi(0) = 0 = C_1$

$$\phi(L) + 2\phi'(L) = 0 \rightarrow C_2 \sin\sqrt{\lambda}L + 2C_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$$

$$\Rightarrow \boxed{\tan(\sqrt{\lambda}L) = -\frac{2\sqrt{\lambda}L}{L}}$$



There is an infinite number of intersections, each corresponds to a positive eigenvalue. The eigenfunctions are:

$$\boxed{\varphi_n(x) = \sin\sqrt{\lambda_n}\,x}$$

This eigenfunctions are orthogonals according to the Sturm-Liouville problem because the PDE above have the form of (5.3.2).

There are not solutions for $\lambda \leq 0$ that satisfy the BC's, you can check that.

Let now, $\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x)}$ Differentiating and substituting in the PDE:

$$\sum_{n=1}^{\infty} a_n'(t)\varphi_n(x) = \sum_{n=1}^{\infty} -k\lambda_n^2 a_n(t)\varphi_n(x) + Q(x,t)$$

OR $$\sum_{n=1}^{\infty}\left(a_n'(t) + k\lambda_n a_n(t)\right)\varphi_n(x) = Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\varphi_n(x)$$

$$\Rightarrow \left(a_n'(t) + k\lambda_n a_n(t)\right) = q_n(t)$$ Multiplying by $e^{\lambda_n t}$ at both sides

$$\underbrace{a_n'(t)e^{\lambda_n tk} + k\lambda_n a_n(t)e^{\lambda_n tk}} = q_n(t)e^{\lambda_n tk}$$

$$\frac{d}{dt}\left(a_n(t)e^{\lambda_n tk}\right) = q_n(t)e^{\lambda_n tk} \qquad q_n(t) = \frac{\int_0^L Q(x,t)\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx}$$

$$a_n(t)e^{\lambda_n tk} - a_n(0) = \int_0^t q_n(s)e^{k\lambda_n s}\,ds$$

$$\boxed{a_n(t) = a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(s)e^{\lambda_n(s-t)k}\,ds}$$

where $a_n(0)$ can be found of the initial condition

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n(0)\varphi_n(x)$$ Multiplying by $\varphi_{n'}(x)$ and integrating by use the orthogonality (5.3.6) for the eigenfunctions of Sturm-Liouville

$$\int_0^L f(x)\varphi_{n'}(x)\,dx = \sum_{n=1}^{\infty} a_n(0)\underbrace{\int_0^L \varphi_n(x)\varphi_{n'}(x)\,dx}_{0 \text{ if } n\neq n'}$$

$$= a_{n'}(0)\int_0^L \varphi_{n'}^2(x)\,dx$$

$$\boxed{a_n(0) = \int_0^L f(x)\varphi_n(x)\,dx \Big/ \int_0^L \varphi_n^2(x)\,dx}$$

c) Let the reference temperature distribution be $r(x,t) = A(t)$
and $\bar{u}(x,t) = u(x,t) - r(x,t)$

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial u}{\partial t} - A'(t) \rightarrow \frac{\partial u}{\partial t} = \frac{\partial \bar{u}}{\partial t} + A'(t). \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \bar{u}}{\partial x^2}$$

Thus the original PDE in terms of $\bar{u}$ is: $\frac{\partial \bar{u}}{\partial t} + A'(t) = k\frac{\partial^2 \bar{u}}{\partial x^2} + Q(x,t)$

OR $\frac{\partial \bar{u}}{\partial t} = k\frac{\partial^2 \bar{u}}{\partial x^2} + Q(x,t) - A'(t) \rightarrow$ For convenience let $Q(x,t) - A'(t) = R(x,t)$

$$\Rightarrow \boxed{\frac{\partial \bar{u}}{\partial t} = k\frac{\partial^2 \bar{u}}{\partial x^2} + R(x,t)} \quad \text{with} \quad \left.\begin{aligned} &\bar{u}(0,t) = u(0,t) - r(0,t) = A - A = 0 \\ &\frac{\partial \bar{u}}{\partial x}(L,t) = \frac{\partial u}{\partial x}(L,t) - \cancel{\frac{\partial r}{\partial x}(L,t)}^{\,0} = 0 \end{aligned}\right\} \text{homogeneous BC's}$$

and the Initial condition: $\bar{u}(x,0) = u(x,0) - r(x,0) = f(x) - A(0)$

By the work done in part (a) we have:

$$\boxed{\bar{u}(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{2n-1}{2L}\pi x\right)} \quad \text{with} \quad a_n(0) = \frac{2}{L}\int_0^L (f(x) - A(0)) \sin\left(\frac{2n-1}{2L}\pi x\right) dx$$

and $a_n(t) = a_n(0)\, e^{-\left(\frac{2n-1}{2L}\pi\right)^2 tk} + \int_0^t r_n(s)\, e^{-\left(\frac{(2n-1)\pi}{2L}\right)^2 (t-s)k} ds$

being $r_n(t)$ the Fourier coeficient of the expansion in eigenfunctions for R:

$$R(x,t) = \sum_{n=1}^{\infty} r_n(t) \sin\left(\frac{2n-1}{2L}\pi x\right) \rightarrow r_n(t) = \frac{2}{L}\int_0^L R(x,t) \sin\left(\frac{2n-1}{2L}\pi x\right) dx$$

Hence the solution of the original initial boundary value problem becomes:

$$\boxed{u(x,t) = \bar{u}(x,t) + A(t)} = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{2n-1}{2L}\pi x\right) + A(t).$$

d) Let the reference temperature distribution $r(x,t) = \frac{A}{L}(L - x)$
It because we need to have $r(0,t) = A$ and $r(L,t) = 0$.

Now let $\bar{u}(x,t) = u(x,t) - r(x,t)$

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial u}{\partial t} - \cancel{\frac{\partial r}{\partial t}}^{\,0} \quad \text{and} \quad \frac{\partial^2 \bar{u}}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \cancel{\frac{\partial^2 r}{\partial x^2}}^{\,0} \quad \text{using this in PDE}$$

$$\Rightarrow \boxed{\frac{\partial \bar{u}}{\partial t} = k\frac{\partial^2 \bar{u}}{\partial x^2} + Q(x,t)} \quad \text{with} \quad \left.\begin{aligned} &\bar{u}(0,t) = u(0,t) - r(0,t) = A - A = 0 \\ &\bar{u}(L,t) = u(L,t) - r(L,t) = 0 - 0 = 0 \end{aligned}\right\} \text{BC's}$$

and $\bar{u}(x,0) = u(x,0) - r(x,0) = f(x) - \frac{A}{L}(L - x)$

We can suppose an expansion for $\bar{u}(x,t)$ in eigenfunctions of the related homogeneous equation with the same BC's:

$$\boxed{\bar{u}(r,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi}{L}x\right)}$$

If we differentiate and substitute in the PDE as in part (a), we ~~do~~ can find a differential eq for $a_n(t)$ and solve giving:

$$\boxed{a_n(t) = a_n(0)\, e^{-\left(\frac{n\pi}{L}\right)^2 tk} + \int_0^t q_n(s)\, e^{-\left(\frac{n\pi}{L}\right)^2 (t-s)k}\, ds}$$

where $a_n(0) = \frac{2}{L}\int_0^L \left[f(x) - \frac{A}{L}(L-x)\right] \sin\left(\frac{n\pi x}{L}\right) dx$

$q_n(t)$ is the coefficient of the Fourier expansion for $Q(x,t)$ also in eigenfunctions.

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin\left(\frac{n\pi}{L}x\right) \rightarrow q_n = \frac{2}{L}\int_0^L Q \sin(\,)\,dx$$

Hence the solution of the original problem is:

$$\boxed{u(x,t) = \bar{u}(x,t) + r(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right) + \frac{A}{L}(L-x)}$$

e) Now we need a reference temperatur distribution $r(x,t)$ such that $\frac{\partial r}{\partial x}(0,t) = A(t)$ and $\frac{\partial r}{\partial x}(L,t) = B(t)$

the more simplest election is $r(x,t) = A(t)\cdot x + \frac{B(t)-A(t)}{2L}x^2$.

Let now, $\bar{u}(x,t) = u(x,t) - r(x,t)$

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial u}{\partial t} - A'(t)\,x - \frac{B'(t)-A'(t)}{2L}x^2 \rightarrow \frac{\partial u}{\partial t} = \frac{\partial \bar{u}}{\partial t} + A'(t)x + \frac{B'(t)-A'(t)}{2L}x^2$$

$$\frac{\partial^2 \bar{u}}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{B(t)-A(t)}{L} \rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{B(t)-A(t)}{L}$$

substituting $\partial u/\partial t$ and $\partial^2 u/\partial x^2$ in ~~terms~~ expressed in terms of $\bar{u}$, the PDE become:

$$\frac{\partial \bar{u}}{\partial t} + A'(t)x + \frac{B'(t)-A'(t)}{2L}x^2 = k\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{B(t)-A(t)}{L}\right) + Q(x,t)$$

OR $\boxed{\frac{\partial \bar{u}}{\partial t} = k\frac{\partial^2 \bar{u}}{\partial x^2} + R(x,t)}$ if $R(x,t) \equiv Q(x,t) + k\left(\frac{B(t)-A(t)}{L}\right) - A'(t)x - \frac{B'(t)-A'(t)}{2L}x^2$

with $\frac{\partial \bar{u}}{\partial x}(0,t) = \frac{\partial u}{\partial x}(0,t) - \frac{\partial r}{\partial x}(0,t) = A(t) - A(t) = 0$

$\frac{\partial \bar{u}}{\partial x}(L,t) = \frac{\partial u}{\partial x}(L,t) - \frac{\partial r}{\partial x}(L,t) = B(t) - B(t) = 0$ } homogeneous BC's for the derivative.

and $\bar{u}(x,0) = u(x,0) - r(x,0) = f(x) - A(0)x + \frac{B(0)-A(0)}{2L}x^2 \equiv g(x)$

Let suppose an expansion for $\bar{u}(x,t)$ in eigenfunctions of the related homogeneous eq with homogeneous BC's for the derivative:

$$\boxed{\bar{u}(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos\left(\frac{n\pi x}{L}\right)}$$

This can be differentiated term-by-term since both $\bar{u}(x,t)$ and $\cos n\pi x/L$ satisfy the same homogeneous BC's. Thus substituting in the PDE:

$$\Rightarrow \sum_{n=0}^{\infty} a_n'(t) \cos\frac{n\pi x}{L} = \sum_{n=0}^{\infty} -k\left(\frac{n\pi}{L}\right)^2 a_n(t) \cos\frac{n\pi x}{L} + R(x,t) \Rightarrow$$

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where we use an expansion for $R(x,t)$ in eigenfunctions.

$$\sum_{n=0}^{\infty}\left(a_n'(t)+k\left(\frac{n\pi}{L}\right)^2 a_n(t)\right)\cos\frac{n\pi}{L}x=\sum_{n=0}^{\infty}\gamma_n(t)\cos\left(\frac{n\pi}{L}x\right)$$

$\Rightarrow a_n'(t)-k\left(\frac{n\pi}{L}\right)^2 a_n(t)=\gamma_n(t)$ multiplying by $e^{\left(\frac{n\pi}{L}\right)^2 t}$ at both sides:

$\Rightarrow \frac{d}{dt}\left(a_n(t)e^{k(n\pi/L)^2 t}\right)=\gamma_n(t)e^{(n\pi/L)^2 tk}$ and integrating:

$$a_n(t)e^{(\ )^2 t}-a_n(0)=\int_0^t \gamma_n(s)e^{(\ )^2 s}\,ds$$

$$\boxed{a_n(t)=a_n(0)e^{-(n\pi/L)^2 tk}+\int_0^t \gamma_n(s)\,e^{-(n\pi/L)^2(t-s)k}\,ds}$$

$\gamma_n(t)=\frac{2}{L(1+\delta_{0n})}\int_0^L R(x,t)\cos\frac{n\pi x}{L}dx$

where $a_n(0)$ can be found from the initial condition:

$g(x)=\bar{u}(x,0)=\sum_{n=0}^{\infty}a_n(0)\cos(n\pi/L)x$ Multiplying by $\cos(n\pi x/L)$ and integrating by use of orthogonality

$$\boxed{a_n(0)=\frac{2}{L(1+\delta_{0n})}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

where $g(x)=f(x)-A(0)x+\frac{B(0)-A(0)}{2L}x^2$ as was defined before.

Hence $u(x,t)=\bar{u}(x,t)+r(x,t)$ with $r(x,t)$ defined above.

f) Here the related homogeneous eigenfunctions yield a Fourier cosine series. Thus, by the method of eigenfunction expansion let:

$$\boxed{u(x,t)=\sum_{n=0}^{\infty}a_n(t)\cos(n\pi x/L)}$$

By the work done in part (e) above for PDE in $\bar{u}(x,t)$ we conclude that here we have:

$$a_n(t)=a_n(0)e^{-(n\pi/L)^2 tk}+\int_0^t q_n(s)\,e^{-(n\pi/L)^2(t-s)k}\,ds$$

and $a_n(0)=\frac{2}{L(1+\delta_{0n})}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ with $q_n(t)=\frac{2}{L(1+\delta_{0n})}\int_0^L Q(x,t)\cos(\ )dx$

g) If $Q(x,t)=Q(x)=\sum_{n=0}^{\infty}q_n\cos(n\pi x/L)$ $\rightarrow$ q_n is independent of t

then $\int_0^t q_n(s)\,e^{-(n\pi/L)^2(t-s)k}\,ds=q_n e^{-(n\pi/L)^2 t}\int_0^t e^{(n\pi/L)^2 sk}\,ds$

with $q_n=\frac{2}{L(1+\delta_{0n})}\int_0^L Q(x)\cos\frac{n\pi x}{L}dx$ $\quad=q_n e^{-(n\pi/L)^2 t}\left(e^{(n\pi/L)^2 t}-1\right)/(n\pi/L)^2 k$

$$=q_n\left(\frac{L}{n\pi}\right)^2\left(1-e^{-(n\pi/L)^2 t}\right)\frac{1}{k}$$

$$\Rightarrow\boxed{u(x,t)=\sum_{n=0}^{\infty}\left[a_n(0)e^{-(n\pi/L)^2 tk}+\frac{q_n}{k}\left(\frac{L}{n\pi}\right)^2\left(1-e^{-(n\pi/L)^2 tk}\right)\right]\cos(n\pi x/L)}$$

As $t\to\infty \Rightarrow u(x,t\to\infty)=\sum_{n=0}^{\infty}q_n\left(\frac{L}{n\pi}\right)^2\cos(n\pi x/L)$ $\rightarrow$ Equilibrium temperature due to the presence of the external source $Q(x)$!

with q_n

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8.3.2 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x)$ $\quad u(0,t) = u(L,t) = 0$ $\quad u(x,0) = f(x)$

Let $\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x/L)}$ substitutin in the PDE:

$$\sum_{n=1}^{\infty} a_n'(t)\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} -\left(\frac{n\pi}{L}\right)^2 a_n(t)\sin\left(\frac{n\pi x}{L}\right) + Q(x)$$

we can expand $Q(x)$ in eigenfunctions

$$\sum_{n=1}^{\infty}\left(a_n'(t) + \left(\frac{n\pi}{L}\right)^2 a_n(t)\right)\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} q_n \sin\frac{n\pi x}{L}$$

where q_n is independent of t.

$a_n'(t) + \left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n$ multiplying by $e^{(n\pi/L)^2 tk}$

$\frac{d}{dt}\left(a_n(t)e^{(n\pi/L)^2 tk}\right) = q_n e^{(n\pi/L)^2 tk}$ and integrating:

$$a_n(t)e^{(n\pi/L)^2 tk} - a_n(0) = \int_0^t q_n e^{(n\pi/L)^2 sk}\,ds$$

$$= q_n\left(\frac{L}{n\pi}\right)^2\left(e^{(n\pi/L)^2 tk} - 1\right)$$

$$\Rightarrow \boxed{a_n(t) = a_n(0)e^{-(n\pi/L)^2 tk} + \frac{q_n}{k}\left(\frac{L}{n\pi}\right)^2\left(1 - e^{-k(n\pi/L)^2 t}\right)}$$

where $a_n(0)$ can be found from the initial condition

$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n(0)\sin(n\pi x/L)$ multiplying by $\sin(n'\pi x/L)$ and integrating:

$$\boxed{a_n(0) = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx}$$

Similarly since $Q(x) = \sum_{n=1}^{\infty} q_n \sin(n\pi x/L)\,dx$ multiplying by $\sin\frac{n'\pi x}{L}$ and integrating:

$$q_n = \frac{2}{L}\int_0^L Q(x)\sin(n\pi x/L)\,dx$$

8.3.3 $c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + qu + f(x,t)$ $\quad u(0,t) = u(L,t) = 0$, $\quad u(x,0) = g(x)$

the homogeneous problem is $c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + qu$

Let $u = h(t)\phi(x) \rightarrow$ Separating variables $\Rightarrow \underbrace{\frac{1}{h}\frac{dh}{dt}}_{\lambda} = \underbrace{\frac{1}{c\rho\phi}\left\{\frac{\partial}{\partial x}\left(K_0\frac{\partial\phi}{\partial x}\right) + q\phi\right\}}_{-\lambda}$

thus calling $\mathcal{L} = \frac{\partial}{\partial x}\left(K_0\frac{\partial}{\partial x}\right) + q \Rightarrow \mathcal{L}[\phi] = -\lambda\sigma\phi$ with $\sigma \equiv c\rho$

and $u(0,t) = u(L,t) = 0 \rightarrow \phi(0) = \phi(L) = 0$

Then the related homogeneous eigenfunctions are $\phi_n(x)$. They are orthogonalas since $\mathcal{L}$ is a Sturm-Liouville operator according to section (5.3.2). The we can expand $u(x,t)$ as:

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$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)}$$ $\phi_n(x)$ eigenfunctions of with $\phi(0)=\phi(L)=0$.

$$\frac{\partial}{\partial x}\left(K_0 \frac{\partial \phi}{\partial x}\right) + q\phi + \lambda c\rho\,\phi = 0$$

This can be differentiated term-by-term since both $u(x,t)$ and $\phi_n(x)$ satisfy the same set of homogeneous BC's. Thus substituting in PDE:

$$c\rho \sum_{n=1}^{\infty} a_n'(t)\,\phi_n(x) = \sum_{n=1}^{\infty} a_n(t)\,\underbrace{\mathcal{L}[\phi_n(x)]}_{-\lambda_n c\rho\,\phi_n} + f(x,t)$$

using expansion in eigenfunctions for $f(x,t)$

$$\Rightarrow c\rho \sum_{n=1}^{\infty} \left(a_n'(t) + \lambda_n a_n(t)\right)\phi_n(x) = f(x,t) = \sum_{n=1}^{\infty} f_n(t)\,\phi_n(x)$$

$$\Rightarrow \quad \frac{da_n}{dt} + \lambda_n a_n = f_n(t)$$ Multiplying by $e^{\lambda_n t}$ at both sides:

$$\frac{d}{dt}\left(a_n(t)\,e^{\lambda_n t}\right) = f_n(t)\,e^{\lambda_n t}$$ and integrating:

$$a_n(t)\,e^{\lambda_n t} - a_n(0) = \int_0^t f_n(s)\,e^{\lambda_n s}\,ds$$

$$\Rightarrow \boxed{a_n(t) = a_n(0)\,e^{-\lambda_n t} + \int_0^t f_n(s)\,e^{-\lambda_n (t-s)}\,ds}$$

where $a_n(0)$ can be found of the IC:

$$g(x) = u(x,t) = \sum_{n=1}^{\infty} a_n(0)\,\phi_n(x)$$ Multiplying by $\phi_n(x)$ an integrating by use of the orthogonality (5.3.6)

$$\boxed{a_n(0) = \int_0^L g(x)\,\phi_n(x)\,\sigma\,dx \Big/ \int_0^L \phi_n^2(x)\,\sigma\,dx}$$ with $\sigma = c\rho$

Similarly if $f(x,t) = \sum_{n=1}^{\infty} f_n(t)\,\phi_n(x)$ by orthogonality

$$f_n(t) = \int_0^L f(x,t)\,\phi_n(x)\,\sigma\,dx \Big/ \int_0^L \phi_n^2\,\sigma\,dx$$

8.3.4 $\dfrac{\partial u}{\partial t} = \dfrac{1}{\sigma(x)}\dfrac{\partial}{\partial x}\left[K_0(x)\dfrac{\partial u}{\partial x}\right]$ $\quad K_0 > 0,\ \sigma > 0 \quad u(0,t) = A \quad u(L,t) = B \quad u(x,0) = g(x)$

a) Suppose there exists a time-independent solution $u_0(x)$, then:

$$0 = \frac{1}{\sigma(x)}\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u_0}{\partial x}\right] \rightarrow K_0(x)\frac{\partial u_0}{\partial x} = C_1 \rightarrow \frac{\partial u_0}{\partial x} = \frac{C_1}{K_0(x)}$$

$$u_0(x) - \underbrace{u_0(0)}_{A} = \int_0^x \frac{C_1}{K_0(s)}\,ds$$

$$\Rightarrow u_0(x) = A + \int_0^x \frac{C_1}{K_0(s)}\,ds$$

$$u_0(L) = B = A + \int_0^L \frac{C_1}{K_0(s)}\,ds \rightarrow C_1 = \frac{B-A}{\int_0^L (1/K_0(s))\,ds}$$

$$\Rightarrow \boxed{u_0(x) = A + \frac{B-A}{\int_0^L (1/K_0(s))\,ds}\int_0^x \frac{1}{K_0(s)}\,ds}$$

b) Let $v(x,t) = u(x,t) - u_0(x) \rightarrow \frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} - \cancel{\frac{\partial u_0}{\partial t}}^{\,0}$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - \frac{\partial u_0}{\partial x} \rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial u_0}{\partial x}$$ thus the PDE becomes:

$$\frac{\partial v}{\partial t} = \frac{1}{\sigma(x)}\frac{\partial}{\partial x}\left[K_0(x)\left(\frac{\partial v}{\partial x} + \frac{\partial u_0}{\partial x}\right)\right] = \frac{1}{\sigma(x)}\frac{\partial}{\partial x}\left[K_0(x)\left(\frac{\partial v}{\partial x} + \frac{B-A}{\int_0^L (1/K_0(s))\,ds}\,\frac{1}{K_0(x)}\right)\right]$$

$$= \frac{1}{\sigma(x)}\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial v}{\partial x} + \frac{B-A}{\int_0^L (\quad)\,ds}\right]$$

You can check

$$\boxed{\frac{\partial v}{\partial t} = \frac{1}{\sigma(x)}\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial v}{\partial x}\right]}$$ with $v(0,t) = v(L,t) = 0$ and $v(x,0) = g(x) - u_0(x)$

Thus $v(x,t)$ solves the linear, homogeneous heat equation with homogeneous BC's.

Hence $\lim_{t\to\infty} v(x,t) = 0 \iff \lim_{t\to\infty} u(x,t) = u_0(x)$.

8.3.5 ~~Since there is no angular dependence in the PDE~~

$$\frac{\partial u}{\partial t} = K\nabla^2 u + f(r,t) \quad \text{with} \quad u(r=a) = 0 \quad \text{and} \quad u(t=0) = 0$$

Since there is no angular dependence in f or in the initial or boundary conditions then $u(r,\theta,t) \equiv u(r,t)$ independent of θ. Hence using the Laplacian in polar coordinates (See eq 2.5.30):

$$\frac{\partial u}{\partial t} = K\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) + f(r,t) = \frac{K}{r}\left(\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right) + f(r,t)$$

$$\frac{\partial u}{\partial t} = \frac{K}{r}\frac{\partial u}{\partial r} + K\frac{\partial^2 u}{\partial r^2} + f(r,t)$$

Look for the eigenvalues/eigenfunctions of the associated homogeneous problem: $\frac{\partial \bar{u}}{\partial t} = \frac{K}{r}\frac{\partial \bar{u}}{\partial r} + K\frac{\partial^2 \bar{u}}{\partial r^2}$

Let $\bar{u}(r,t) = R(r)h(t) \rightarrow R(r)h'(t) = \frac{K}{r}R'(r)h(t) + KR''(r)h(t)$

OR $$\underbrace{\frac{h'(t)}{Kh(t)}}_{-\lambda} = \underbrace{\frac{1}{r}\frac{R'(r)}{R(r)} + \frac{R''(r)}{R(r)}}_{-\lambda}$$ by separation of variables.

$$\Rightarrow \frac{1}{r}R'(r) + R''(r) = -\lambda R(r)$$

$$r^2R''(r) + rR'(r) + \lambda r^2 R(r) = 0$$

OR $r^2R''(r) + rR'(r) + (\lambda r^2 - 0^2)R(r) = 0 \rightarrow$ Bessel's eq of order 0.

$$\Rightarrow R(r) = C_1 J_0(\sqrt{\lambda}\,r) + \cancel{C_2 N_0(\sqrt{\lambda}\,r)}^{\,0}$$ since $|R(0)| < \infty$.

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Now $\bar{u}(r=a)=0 \rightarrow R(a)=J_0(\sqrt{\lambda}\,a)=0 \rightarrow \boxed{\lambda_n=(z_n/a)^2}$

with z_n the n-th zero of $J_0(z)$.

Thus $J_0(\sqrt{\lambda_n}\,r)$ are the related homogeneous eigenfunctions

Hence we can suppose an expansion for $u(x,t)$ as:

$$\boxed{u(r,t)=\sum_{n=1}^{\infty} a_n(t)\,J_0(\sqrt{\lambda_n}\,r)}$$

Differentiate and substitute in the nonhomogeneous PDE:

$$\sum_{n=1}^{\infty} a_n'(t)\,J_0(\sqrt{\lambda_n}\,r)=\frac{K}{r}\sum_{n=1}^{\infty} a_n(t)\,J_0'(\sqrt{\lambda_n}\,r)\cdot\sqrt{\lambda_n}+K\sum_{n=1}^{\infty} a_n(t)\,\lambda_n J_0''(\sqrt{\lambda_n}\,r)+f(r,t)$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n'(t)\,J_0(\sqrt{\lambda_n}\,r)-\underbrace{a_n(t)\frac{K}{r}\sqrt{\lambda_n}\,J_0'(\sqrt{\lambda_n}\,r)-a_n(t)K\lambda_n J_0''(\sqrt{\lambda_n}\,r)}_{-a_n(t)K\left(\frac{\sqrt{\lambda_n}\,r}{r^2}J_0'(\sqrt{\lambda}\,r)+\frac{\lambda_n r^2}{r^2}J_0''(\sqrt{\lambda}\,r)\right)}=f(r,t)$$

$x^2J_0''(x)+xJ_0'(x)=-\lambda x^2J_0(x) \Rightarrow$ $\lambda_n J_0(\sqrt{\lambda}\,r)$ According to Bessel equation

$$\Rightarrow \sum_{n=1}^{\infty}\left(a_n'(t)+K\lambda_n a_n(t)\right)J_0(\sqrt{\lambda_n}\,r)=f(r,t)\equiv\sum_{n=1}^{\infty} f_n(t)\,J_0(\sqrt{\lambda_n}\,r)$$

$\Rightarrow a_n'(t)+K\lambda_n a_n(t)=f_n(t)$ Multiply by $e^{K\lambda_n t}$

$\frac{d}{dt}\left(a_n(t)e^{K\lambda_n t}\right)=f_n(t)e^{K\lambda_n t}$ and integrate to gives:

$$a_n(t)e^{K\lambda_n t}-a_n(0)=\int_0^t f_n(s)e^{K\lambda_n s}\,ds$$

$$\Rightarrow a_n(t)=a_n(0)e^{-K\lambda_n t}+e^{-K\lambda_n t}\int_0^t f_n(s)e^{K\lambda_n s}\,ds$$

where $a_n(0)$ can be found of the IC: $u(t=0)=0=\sum_{n=1}^{\infty} a_n(0)\,J_0(\sqrt{\lambda_n}\,r)$

Since $J_0(\sqrt{\lambda_n}\,r)$ can take arbitrary value depending of the r then $\Rightarrow u(t=0)=0 \rightarrow \boxed{a_n(0)=0}$

$$\Rightarrow \boxed{a_n(t)=e^{-K\lambda_n t}\int_0^t f_n(s)e^{K\lambda_n s}\,ds}$$

where f_n is the Fourier coefficient of $f(x,t)=\sum_{n=1}^{\infty} f_n(t)\,J_0(\sqrt{\lambda_n}\,r)$

multiplying by $J_0(\sqrt{\lambda_{n'}}\,r)$ and using orthogonality:

$$\int_0^a f(x,t)\,J_0(\sqrt{\lambda_{n'}}\,r)\,r\,dr=\sum_{n=1}^{\infty} f_n\underbrace{\int_0^a J_0(\sqrt{\lambda_n}\,r)\,J_0(\sqrt{\lambda_{n'}}\,r)\,r\,dr}_{0 \text{ if } n\neq n'}$$

$$=f_{n'}(t)\int_0^a J_0^2(\sqrt{\lambda_{n'}}\,r)\,r\,dr$$

$$\Rightarrow \boxed{f_n(t)=\int_0^a f(r,t)\,J_0(\sqrt{\lambda_n}\,r)\,r\,dr\Big/\int_0^a J_0^2(\sqrt{\lambda_n}\,r)\,r\,dr}$$

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8.3.6 $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin(5x)\, e^{-2t}$ $\quad u(0,t)=1$, $u(\pi,t)=0$ $\quad u(x,0)=0$.

Choose $r(x,t)$ to make BC's homogeneous.

Let $r(x,t) = 1 + c_1 x \longrightarrow r(0,t)=1$ satisfying the first BC.

$r(\pi,t) = 1 + c_1\pi = 0 \rightarrow c_1 = -1/\pi$

$\Rightarrow \boxed{r(x,t) = 1 - x/\pi}$

Now write $v(x,t) = u(x,t) - r(x,t)$ which gives $v(x,0) = \cancel{u(x,0)}^{\,0} - r(x,0)$

$v(x,0) = -(1 - x/\pi)$

$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} - \cancel{\frac{\partial r}{\partial t}}^{\,0}$

$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \cancel{\frac{\partial^2 r}{\partial x^2}}^{\,0}$ - Thus using that in the PDE:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \underbrace{\sin(5x)\, e^{-2t}}_{Q(x,t)}$$

$\Rightarrow \boxed{\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + Q(x,t)}$ with $v(0,t) = v(\pi,t) = 0$ $\rightarrow$ You can check!

and $v(x,0) = -(1 - x/\pi)$.

We will solve for $v(x,t)$ by the method of eigenfunction expansion. To do so, consider the related homogeneous problem.

$\frac{\partial u_H}{\partial t} = \frac{\partial^2 u_H}{\partial x^2}$ with $u(0,t) = u(\pi,t) = 0$ $\longrightarrow$

After separing variables, the x-eq is $\frac{d^2\phi}{dx^2} = -\lambda\phi$ $\quad \phi(0) = \phi(\pi) = 0$

$\rightarrow \phi_n(x) = \sin(nx)$

Expand $v(x,t)$ in a series of these eigenfunctions:

$\boxed{v(x,t) = \sum_{n=1}^{\infty} d_n(t) \sin(nx)}$ Differentiate and substitute in the PDE:

$$\sum_{n=1}^{\infty} d_n'(t) \sin nx = \sum_{n=1}^{\infty} -n^2 d_n(t) \sin nx + Q(x,t)$$

$$\sum_{n=1}^{\infty} \left(d_n'(t) + n^2 d_n(t)\right) \sin nx = Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin(nx)$$

$\Rightarrow d_n'(t) + n^2 d_n(t) = q_n(t)$ multiply by $e^{n^2 t}$ $\rightarrow$ integrating factor

$\frac{d}{dt}\left(d_n(t) e^{n^2 t}\right) = q_n(t) e^{n^2 t}$ upon integrating we will get:

$d_n(t) e^{n^2 t} - d_n(0) = \int_0^t q_n(\tau) e^{n^2 \tau}\, d\tau$

$$\boxed{d_n(t) = d_n(0) e^{-n^2 t} + e^{-n^2 t} \int_0^t q_n(\tau) e^{n^2 \tau}\, d\tau}$$

To determine $a_n(0)$ apply the IC:

$$v(x,0) = -(1 - x/\pi) = \sum_{n=1}^{\infty} a_n(0)\sin(nx)$$

multiply both sides by $\sin(n'x)$ and integrate applying orthogonality

$$\Rightarrow \quad a_n(0) = \frac{2}{\pi}\int_0^{\pi} -(1-x/\pi)\sin(nx)\,dx$$

$$= -\frac{2}{\pi}\int_0^{\pi}\left[\sin nx - \frac{x}{\pi}\sin nx\right]dx \qquad u = x \quad dv = \sin nx, \quad du = dx, \quad v = -\frac{1}{n}\cos nx$$

$$= -\frac{2}{\pi}\int_0^{\pi}\left[-\frac{\cos nx}{n}\Big|_0^{\pi} - \frac{1}{\pi}\left\{-\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\right\}\right]$$

$$= -\frac{2}{\pi}\left[-\left(\frac{\cancel{\cos n\pi} - 1}{n}\right) - \frac{1}{\pi}\left(-\frac{\pi}{n}\cancel{\cos n\pi} - 0\right)\right] = -\frac{2}{n\pi}$$

$$\boxed{a_n(0) = -2/n\pi}$$

Similarly we can find $q_n(t)$ since $Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin(nx)$.

$$\Rightarrow \quad q_n(t) = \frac{2}{\pi}\int_0^{\pi} Q(x,t)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin(5x)e^{-2t}\sin(nx)\,dx$$

$$= \frac{2}{\pi}e^{-2t}\int_0^{\pi}\sin(5x)\sin(nx)\,dx = \cancel{\frac{2}{\pi}}e^{-2t}\cdot\cancel{\frac{\pi}{2}}\delta_{n5}$$

$$\boxed{q_n(t) = e^{-2t}\delta_{n5}}$$

with this we can calculate the integral that appears in $a_n(t)$ above:

$$\int_0^t q_n(\tau)e^{n^2\tau}d\tau = \int_0^t e^{-2\tau}\delta_{n5}e^{n^2\tau}d\tau \neq 0 \text{ only if } n=5 \text{ thus,}$$

$$= \delta_{n5}\int_0^t e^{-2\tau}e^{25\tau}d\tau = \delta_{n5}\int_0^t e^{23\tau}d\tau = \frac{e^{23t}-1}{23}\delta_{n5}$$

$$\Rightarrow \boxed{\int_0^t q_n(\tau)e^{n^2\tau}d\tau = \left(\frac{e^{23t}-1}{23}\right)\delta_{n5}}$$

Substituting these expresions in $a_n(t)$ we have:

$$a_n(t) = -\frac{2}{n\pi}e^{-n^2t} + e^{-n^2t}\left(\frac{e^{23t}-1}{23}\right)\delta_{n5} = \left[\left(\frac{e^{23t}-1}{23}\right)\delta_{n5} - \frac{2}{n\pi}\right]e^{-n^2t}$$

Now, $u(x,t) = v(x,t) + r(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx) + \left(1 - \frac{x}{\pi}\right)$

$$\boxed{u(x,t) = \left[\left(\frac{e^{23t}-1}{23}\right) - \frac{2}{n\pi}\right]e^{-25t}\sin(5x) + \sum_{\substack{n=1\\ n\neq 5}}^{\infty}\left(-\frac{2}{n\pi}\right)e^{-n^2t}\sin(nx) + [1 - x/\pi]}$$

As $t \to \infty$, $u(x, t\to\infty) \to [1 - x/\pi] = r(x,t)$.

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8.3.7 $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ with $u(0,t)=0$ $u(L,t)=t$ $u(x,0)=0$.

First let us find a reference temperature distribution $r(x,t)$ that satisfies

$$r(0,t)=0 \text{ and } r(L,t)=t \rightarrow \boxed{r(x,t)=\frac{t}{L}x}$$

Now define $v(x,t)=u(x,t)-r(x,t)$

$$\left.\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial u}{\partial t} - \frac{x}{L} \rightarrow \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{x}{L} \\ \frac{\partial^2 v}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2} - \cancel{\frac{\partial^2 r}{\partial x^2}}^{\,0} \end{aligned}\right\} \text{ using this in the P.D.E:}$$

$$\Rightarrow \frac{\partial v}{\partial t} + \frac{x}{L} = \frac{\partial^2 v}{\partial x^2} \rightarrow \boxed{\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - \frac{x}{L}}$$

with $v(0,t)=v(L,t)=0$ (you can check) and $v(x,0)=\cancel{u(x,0)}^{\,0}-r(x,0)=0$

we will assume $\boxed{v(x,t)=\sum_{n=1}^{\infty} a_n(t)\sin\left(\frac{n\pi x}{L}\right)}$

As we did before (see exercises 8.3.1(d) and 8.3.2) substituting that in the PDE we can obtain:

$$a_n(t) = a_n(0)\,e^{-(n\pi/L)^2 t} + e^{-(n\pi/L)^2 t}\int_0^t q_n(\tau)\,e^{(n\pi/L)^2\tau}\,d\tau$$

Since $u(x,0)=0=\sum_{n=1}^{\infty} a_n(0)\sin\frac{n\pi x}{L} \rightarrow \boxed{a_n(0)=0}$

Now $Q(x,t) = -\frac{x}{L} = \sum_{n=1}^{\infty} q_n(t)\sin\frac{n\pi x}{L}$ then multiplying by $\sin\frac{n\pi x}{L}$ and integrating by use of the orthogonality.

$$q_n(t) = \frac{2}{L}\int_0^L -\frac{x}{L}\sin\frac{n\pi x}{L}\,dx$$

$u = x$ $\quad dv = \sin n\pi x/L$

$dx = dx$ $\quad v = -\frac{L}{n\pi}\cos n\pi x/L$

$$= -\frac{2}{L^2}\left[-\frac{xL}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L + \frac{L}{n\pi}\cancel{\int_0^L \cos(n\pi x/L)\,dx}^{\,0}\right] \text{ since } n\neq 0$$

$$= -\frac{2}{\cancel{L^2}}\left[-\frac{\cancel{L^2}}{n\pi}(-1)^n\right] = \frac{2}{n\pi}(-1)^n$$

$$\Rightarrow \int_0^t q_n(\tau)\,e^{(n\pi/L)^2\tau}\,d\tau = \frac{2}{n\pi}(-1)^n\int_0^t e^{(n\pi/L)^2\tau}\,d\tau = \frac{2}{n\pi}(-1)^n\,\frac{e^{(n\pi/L)^2 t}-1}{(n\pi/L)^2}$$

substituing this above in the expresion for $a_n(t)$ and recalling $a_n(0)=0$:

$$\boxed{a_n(t) = \frac{2}{n\pi}(-1)^n\left(\frac{L}{n\pi}\right)^2\left(1-e^{-(n\pi/L)^2 t}\right)}$$

Finally $u(x,t) = v(x,t) + r(x,t)$

$$\boxed{u(x,t) = \frac{2L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left(1-e^{-(n\pi/L)^2 t}\right)\sin\left(\frac{n\pi x}{L}\right) + \frac{xt}{L}}$$

As $t\rightarrow\infty$, $u \rightarrow \frac{2L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi x}{L} + \frac{xt}{L} = \left(\frac{x^3}{6L} - \frac{xL}{6}\right) + \frac{xt}{L}$ you can verify that!

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8.4.1 $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad u(x,0) = f(x)$

a) We need the eigenfunctions of the related homogeneous problem $\phi_n'' + \lambda_n \phi_n = 0$ with $\phi_n'(L) = \phi_n(0) = 0$. These are:

$\phi_n(x) = \sin\frac{(2n-1)\pi x}{2L}$ with $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \quad n = 1,2,3,\ldots$

$\Rightarrow$ assume $\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)}$. differentiate both sides with respect to t:

$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n'(t)\phi_n(t) = K\frac{\partial^2 u}{\partial x^2} + Q(x,t)$ multiplying by $\phi_n(x)$ and integrating by use of orthogonality

$$\Rightarrow b_n'(t) = \frac{\int_0^L \left(K\frac{\partial^2 u}{\partial x^2} + Q(x,t)\right)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$$

$$= \frac{\int_0^L \left(K\frac{\partial^2 u}{\partial x^2}\right)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} + \underbrace{\frac{\int_0^L Q(x,t)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}}_{q_n(t)}$$

Now use the Green's formula:

$$\int_0^L \left(u\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^2 u}{\partial x^2}\right)dx = \left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right)\Big|_0^L$$

$$\int_0^L v\frac{\partial^2 u}{\partial x^2}\,dx = \int_0^L u\frac{\partial^2 v}{\partial x^2}\,dx - \left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right)\Big|_0^L \qquad \text{we will let } v = \phi_n(x)$$

$$\Rightarrow \int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx = \int_0^L u\frac{d^2\phi_n}{dx^2}\,dx - \left(u\frac{d\phi_n}{dx} - \phi_n(x)\frac{\partial u}{\partial x}\right)\Big|_0^L$$

$$= \int_0^L u(-\lambda_n\phi_n(x))\,dx - u(L,t)\frac{d\phi_n}{dx}(L) + \phi_n(L)\frac{\partial u}{\partial x}(L,t) + u(0,t)\frac{d\phi_n}{dx}(0) - \phi_n(0)\frac{\partial u}{\partial x}(0,t)$$

$$= -\lambda_n\int_0^L u\phi_n(x)\,dx + B(t)\sin\frac{(2n-1)\pi\cdot\pi}{2L} + A(t)\left(\frac{(2n-1)\pi}{2L}\right)\cos\frac{(2n-1)\pi\cdot 0}{2L}$$

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx = -\lambda_n\int_0^L u\phi_n(x)\,dx + B(t)(-1)^{n+1} + A(t)\left(\frac{(2n-1)\pi}{2L}\right)$$ substituting u and using orthogonality

$$= -\lambda_n b_n(t)\int_0^L \phi_n^2(x)\,dx + (-1)^{n+1}B(t) + \left(\frac{(2n-1)\pi}{2L}\right)A(t)$$

So $b_n'(t) = q_n(t) + K\dfrac{(-1)^{n+1}B(t) + \frac{(2n-1)\pi}{2L}A(t)}{\int_0^L \phi_n^2(x)\,dx} - K\lambda_n b_n(t)$

$\int_0^L \phi_n^2(x)\,dx = L/2$ since $\phi_n = \sin(2n-1)\pi x/L$

$$\Rightarrow \boxed{b_n'(t) + K\lambda_n b_n(t) = q_n(t) + \frac{2K}{L}\left[(-1)^{n+1}B(t) + \left(\frac{(2n-1)\pi}{2L}\right)A(t)\right]}$$

Since $u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x) \longrightarrow b_n(0) = \frac{2}{L}\int_0^L f(x)\phi_n(x)\,dx$.

we can solve this for $b_n(t)$ using the integrating factor $w(t) = e^{K\lambda_n t}$ as usual.

giving according to (8.3.10):

$$\boxed{b_n(t) = b_n(0)\, e^{-k\lambda_n t} + e^{-k\lambda_n t}\int_0^t \frac{2k}{L}\left[(-1)^{n+1}B(\tau) + \left(\frac{(2n-1)\pi}{2L}\right)A(\tau)\right] e^{\lambda_n k \tau}\, d\tau}$$

$$b_n(0) = \frac{2}{L}\int_0^L f(x)\,\phi_n(x)\,dx$$

b) We need the eigenfunctions of the related homogeneous problem $\frac{d^2\phi_n}{dx^2} + \lambda_n \phi_n = 0$ with $\phi'_n(x) = \phi'(L) = 0$. These are:

$\boxed{\phi_n(x) = \cos(n\pi x/L)}$ $\Rightarrow$ assume $\boxed{u(x,t) = \sum_{n=0}^{\infty} b_n(t)\,\phi_n(x)}$

differentiating both sides respect to t:

$$\frac{\partial u}{\partial t} = \sum_{n=0}^{\infty} b'_n(t)\,\phi_n(x) = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

Multiplying by $\phi_n(x)$ and integrating by orthogonality

$$b'_n(t) = q_n(t) + \frac{\int_0^L k\frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$$

just as we did in part (a).

Again, we will use Green's formula:

$$\int_0^L v\frac{\partial^2 u}{\partial x^2}\,dx = \int_0^L u\frac{\partial^2 v}{\partial x^2}\,dx - \left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right)\Big|_0^L \quad \text{with } v = \phi_n(x)$$

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx = \int_0^L u\left(-\lambda_n\phi_n(x)\right)dx + \left(u(0,t)\frac{\partial\phi_n}{\partial x}(0) - \phi_n(0)\frac{\partial u}{\partial x}(0,t)\right) - \left(u(L,t)\frac{\partial\phi_n}{\partial x}(L) - \phi_n(L)\frac{\partial u}{\partial x}(L,t)\right)$$

$$= -\lambda_n\int_0^L u\,\phi_n(x)\,dx + \left(0 - (1)A(t)\right) - \left(0 - (-1)^n B(t)\right)$$

Substituting u and using orthogonality:

$$= -\lambda_n\left(\int_0^L \phi_n^2(x)\,dx\right)b_n(t) + (-1)^n B(t) - A(t)$$

So $b'_n(t) + \lambda_n k\, b_n(t) = q_n(t) + k\,\dfrac{(-1)^n B(t) - A(t)}{\int_0^L \phi_n^2(x)\,dx}$ $= \frac{L}{2}(1+\delta_{0n})\delta_n$

$$\Rightarrow \boxed{b'_n(t) + \lambda_n k\, b_n(t) = q_n(t) + \frac{k}{2(1+\delta_{0n})}\left[(-1)^n B(t) - A(t)\right]}$$

where $\lambda_n = (n\pi/L)^2$

From (8.3.10): $\boxed{b_n(t) = b_n(0)\,e^{-k\lambda_n t} + e^{-k\lambda_n t}\int_0^t \frac{k}{2(1+\delta_{0n})}\left[(-1)^n B(\tau) - A(\tau)\right] e^{\lambda k \tau}\, d\tau}$

with $b_n(0) = \frac{2}{L}\int_0^L f(x)\,\phi_n(x)\,dx$ just as we did in part (a).

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8.4.2 The eigenfunctions of the related homogeneous problem have the form $\boxed{\phi_n(x) = \sin(n\pi x/L)}$ with $\lambda_n = (n\pi/L)^2$

Assume $\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)} \Rightarrow \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = k\frac{\partial^2 u}{\partial x^2}$

Multiplying by $\phi_n(x)$ and integrating $\Rightarrow b_n'(t) = \frac{2k}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx$

Now use the Green's formula:

$$\int_0^L v\frac{\partial^2 u}{\partial x^2}\,dx = \int_0^L u\frac{\partial^2 v}{\partial x^2}\,dx - \left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right)\Big|_0^L \quad \text{let } v = \phi_n(x), \text{ then}$$

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx = -\lambda_n\int_0^L u\phi_n(x)\,dx + \left(u(0,t)\frac{d\phi_n}{dx}(0) - \phi_n(0)\frac{\partial u}{\partial t}(0,t)\right) - \left(u(L,t)\frac{d\phi_n}{dx}(L) - \phi_n(L)\frac{\partial u}{\partial t}(L,t)\right)$$

Substituting u and using orthogonality $\longrightarrow$

$$= -\lambda_n\left(\frac{L}{2}\right)b_n(t) + \left(A\left(\frac{n\pi}{L}\right) - 0\right) - \left(B\left(\frac{n\pi}{L}\right)(-1)^n - 0\right)$$

$$= -\frac{\lambda_n L}{2}b_n(t) + \left(\frac{n\pi}{L}\right)\left(A - (-1)^n B\right)$$

$$\Rightarrow b_n'(t) = \frac{2k}{L}\left[-\frac{\lambda_n L}{2}b_n(t) + \left(\frac{n\pi}{L}\right)\left(A - (-1)^n B\right)\right]$$

$$\boxed{b_n'(t) + \lambda_n k\, b_n(t) = \frac{2\pi n k}{L^2}\left(A - (-1)^n B\right)}$$ let $w(t) = e^{\lambda_n k t}$ be the integrating factor.

$$\Rightarrow \frac{d}{dt}\left(b_n(t)e^{\lambda_n k t}\right) = \frac{2k\pi n}{L^2}\left(A - (-1)^n B\right)e^{\lambda_n k t}$$ Now integrating:

$$b_n(t)e^{\lambda_n k t} - b_n(0) = \frac{1}{\lambda_n k}\cdot\frac{2kn\pi}{L^2}\left(A - (-1)^n B\right)e^{\lambda_n k t}\Big|_0^t$$

$$= \frac{L^2}{n^2\pi^2}\,\frac{2n\pi}{L^2}\left(A - (-1)^n B\right)\left(e^{\lambda_n k t} - 1\right)$$

Nothe that $b_n(0) = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$ just as we did in exercise (8.4.1).

$$\Rightarrow \boxed{b_n(t) = e^{-\lambda_n k t}b_n(0) + \frac{2\pi}{n\pi^2}\left(A - (-1)^n B\right)\left(1 - e^{-\lambda_n k t}\right)}$$

or substituting above:

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}e^{-(n\pi/L)^2 k t}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{n\pi}\left(A - (-1)^n B\right)\left(1 - e^{-(n\pi/L)^2 k t}\right)\right]\cdot \sin(n\pi x/L)$$

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8.4.3 The related homogeneous PDE has the form:

a) $$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + q(x)u \qquad u(0,t)=u(L,t)=0$$

we let $u(x,t) = X(x)T(t)$, then:

$$c(x)\rho(x)X(x)T'(t) = \frac{d}{dx}\left[K_0(x)X'(x)\right]T(t) + q(x)X(x)T(t)$$

$$\frac{T'(t)}{T(t)} = \frac{\frac{d}{dx}\left[K_0(x)X'(x)\right]}{c(x)\rho(x)X(x)} + \frac{q(x)}{c(x)\rho(x)} = -\lambda$$

we obtain the ODE for $X(x)$

$$\frac{\frac{d}{dx}\left[K_0(x)X'(x)\right]}{c(x)\rho(x)X(x)} + \frac{q(x)}{c(x)\rho(x)} = -\lambda \qquad \text{OR}$$

$$\frac{d}{dx}\left[K_0(x)X'(x)\right] + \left(q(x) + \lambda c(x)\rho(x)\right)X(x) = 0$$

we will assume that the eigenfunctions $\phi_n(x)$ of this equation are known. If $n \neq m$ then the eigenfunctions $\phi_m(x)$ and $\phi_n(x)$ are orthogonal in the sense that:

if $n \neq m$ $\int_0^L c(x)\rho(x)\phi_m(x)\phi_n(x)\,dx = 0$ since we have a Sturm-Liouville problem.

we will assume the solution of the nonhomogeneous PDE can be written as: $\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)}$ then:

$$c(x)\rho(x)\frac{\partial u}{\partial t} = c(x)\rho(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + q(x)u + f(x,t)$$

$$\Rightarrow \sum_{n=1}^{\infty} b_n'(t)\,c(x)\rho(x)\phi_n(x) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + q(x)u + f(x,t)$$

multiply both sides of this eq by $\phi_m(t)$ and integrating

$$\sum_{n=1}^{\infty} b_n'(t)\underbrace{\int_0^L c(x)\rho(x)\phi_n(x)\phi_m(x)\,dx}_{0 \text{ if } n\neq m \text{ by orthogonality}} = \int_0^L\left[\frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + q(x)u\right]\phi_m(x)\,dx + \int_0^L f(x,t)\phi_m(x)\,dx$$

$$\Rightarrow b_n'(t) = \frac{\int_0^L\left[\frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + q(x)u\right]\phi_n(x)\,dx}{\int_0^L c(x)\rho(x)\phi_n^2(x)\,dx} + \underbrace{\frac{\int_0^L f(x,t)\phi_n(x)\,dx}{\int_0^L c(x)\rho(x)\phi_n^2(x)\,dx}}_{f_n(t)}$$

since we expand $f(x,t) = \sum_{n=1}^{\infty} f_n(t)\phi_n(x)$

Now we must consider $\int_0^L \left[\frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right)+q(x)u\right]\phi_n(x)\,dx$

Recall the Green's formula:

$$\int_0^L [u\mathcal{L}(v)-v\mathcal{L}(u)]\,dx = p\left(u\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial x}\right)\Big|_0^L$$

wher $\mathcal{L}$ is any Sturm-Liouville operator. Here $p=K_0(x)$ and $q=q(x)$. Hence:

$$\int_0^L u\mathcal{L}(v)\,dx - p\left(u\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial x}\right)\Big|_0^L = \int_0^L v\mathcal{L}(u)\,dx$$

with $\mathcal{L}=\frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)+q$

we will let $v=\phi_n(x)$, thus:

$$\int_0^L\left[\frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right)+q(x)u\right]\phi_n(x)\,dx = \int_0^L u\mathcal{L}(\phi_n(x))\,dx - K_0(x)\left(u\phi_n'(x)-\phi_n(x)\frac{\partial u}{\partial x}\right)\Big|_0^L$$

$$=\int_0^L u\left(-\lambda_n c(x)\rho(x)\phi_n(x)\right)dx - K_0(L)\left(\beta(t)\phi_n'(L)\right)+K_0(0)\left(\alpha(t)\phi_n'(0)\right)$$

$$=-\lambda_n b_n(t)\underbrace{\int_0^L c(x)\rho(x)\phi_n^2(x)\,dx}_{\text{Here we used the expansion of } u=\sum_m b_m\phi_m(t) \text{ and orthogonality.}} + K_0(0)\alpha(t)\phi_n'(0)-\beta(t)K_0(L)\phi_n'(L)$$

Substituting above:

$$\Rightarrow \boxed{b_n'(t)+\lambda_n b_n(t) = \frac{\alpha(t)K_0(0)\phi_n'(0)-\beta(t)K_0(L)\phi_n'(L)}{\int_0^L c(x)\rho(x)\phi_n^2(x)\,dx}+f_n(t)}$$

Now using the initial condition

$$g(x)=u(x,0)=\sum_{n=1}^{\infty} b_n(0)\phi_n(x) \longrightarrow b_n(0)=\frac{\int_0^L c(x)\rho(x)g(x)\phi_n(x)\,dx}{\int_0^L c(x)\rho(x)\phi_n^2(x)\,dx}$$

we can solve the first-order linear ODE for $b_n(t)$ using (8.3.10):

$$\boxed{b_n(t)=b_n(0)e^{-\lambda_n K t}+e^{-\lambda_n K t}\int_0^t\left[\frac{\alpha(\tau)K_0(0)\phi_n'(0)-\beta(\tau)K_0(L)\phi_n'(L)}{\int_0^L c(x)\rho(x)\phi_n^2(x)\,dx}+f_n(\tau)\right]e^{\lambda_n K\tau}\,d\tau}$$

b) Let $r(x,t)=\beta(t)\frac{x}{L}+\left(1-\frac{x}{L}\right)\alpha(t)$ be the reference temperature, then suppose: $\boxed{u(x,t)=v(x,t)+r(x,t)}$

$$\Rightarrow c(x)\rho(x)\left(\frac{\partial v}{\partial t}-\frac{\partial r}{\partial t}\right)=\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial}{\partial x}(v+r)\right]+q(x)(v+r)+f(x,t)$$

$$c(x)\rho(x)\frac{\partial v}{\partial t}+c(x)\rho(x)\frac{\partial r}{\partial t}=\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial}{\partial x}v\right]+\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial r}{\partial x}\right]+q(x)v+q(x)r+f(x,t)$$

$$c(x)\rho(x)\frac{\partial v}{\partial t}+c(x)\rho(x)\left(\beta'(t)x+\left(1-\frac{x}{L}\right)\alpha'(t)\right)=\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial v}{\partial x}\right]+\frac{\partial}{\partial x}\left[K_0(x)\left(-\frac{\alpha(t)}{L}+\beta(t)\right)\right]$$
$$+q(x)v+q(x)r+f(x,t).$$

$$C(x)\rho(x)\frac{\partial v}{\partial t} + C(x)\rho(x)\left(\beta'(t)x + \left(1-\frac{x}{L}\right)\alpha'(t)\right) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial v}{\partial x}\right] + \frac{\partial K_0}{\partial x}\left(\beta(t) - \frac{\alpha(t)}{L}\right) + q(x)v + q(x)r + f(x,t)$$

$$C(x)\rho(x)\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial v}{\partial x}\right] + q(x)v + \left(\beta(t) - \frac{\alpha(t)}{L}\right)\frac{\partial K_0}{\partial x} + q(x)r(x,t) + f(x,t) - C(x)\rho(x)\left(\beta'(x)x + \left(1-\frac{x}{L}\right)\alpha'(t)\right) \equiv F(x,t)$$

$$C(x)\rho(x)\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial v}{\partial x}\right] + q(x)v + F(x,t)$$

Is now the form of the PDE. We note it is still nonhomogeneous, but the BC's are homogeneous $\left.\begin{array}{l} v(0,t)=0 \\ v(L,t)=0 \end{array}\right\}$ (You can check that).

The initial condition has now become:

$$v(x,0) = g(x) - r(x,0).$$

Now we will perform $\boxed{v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)}$

where the $\phi_n(x)$ are eigenfunctions of the related homogeneous PDE. For simplicity let write $v(x,0) = g(x) - r(x,0) \equiv G(x)$.

$\Rightarrow G(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x) \Rightarrow$ multiplying by $C(x)\rho(x)\phi_m(x)$ and integrating by use of orthogonality:

$$\boxed{a_n(0) = \frac{\int_0^L C(x)\rho(x)G(x)\phi_n(x)\,dx}{\int_0^L C(x)\rho(x)\phi_n^2(x)\,dx}}$$

Substituting the expansion for $v(x,t)$ in the PDE:

$$\sum_{n=1}^{\infty} a_n'(t)C(x)\rho(x)\phi_n(x) - \sum_{n=1}^{\infty} a_n(t)\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial \phi_n}{\partial x}\right] - \sum_{n=1}^{\infty} a_n(t)q(x)\phi_n(x) = F(x,t)$$

$$\sum_{n=1}^{\infty}\left[a_n'(t)C(x)\rho(x)\phi_n(x) - a_n(t)\underbrace{\left(\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial \phi_n}{\partial x}\right] + q(x)\phi_n(x)\right)}_{-\lambda_n\phi_n(x)C(x)\rho(x)}\right] = F(x,t)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(a_n'(t) + \lambda_n a_n(t)\right)C(x)\rho(x)\phi_n(x) = F(x,t)$$

multiply by $\phi_m(x)$ and integrate using orthogonality

$$\left(a_n'(t) + \lambda_n a_n(t)\right) = \boxed{\frac{\int_0^L F(x,t)\phi_n(x)\,dx}{\int_0^L C(x)\rho(x)\phi_n^2(x)\,dx} \equiv F_n(t)}$$

since we use an expansion for $F(x,t)$ $F(x,t) = \sum_n F_n(t)\phi_n(x)$

we can solve this first-order linear problem using (8.3.10):

$$\boxed{a_n(t) = a_n(0)e^{-\lambda_n t} + \int_0^t e^{-\lambda_n(t-s)}F_n(s)\,ds}$$

By: Herminso Villarraga Gómez

8.4.4 $u(x,t)=\sum_{n=0}^{\infty} C_n(t)\cos\left(\frac{n\pi x}{L}\right)$ is not a good solution because it don't satisfy the BC's $u(0,t)=u(L,t)=0$

except if $u(0,t)=\sum_{n=0}^{\infty} C_n(t)=0$

and $u(L,t)=\sum_{n=0}^{\infty} C_n(t)(-1)^n=0$

However if we try use $u(x,t)$ as was ~~defined~~ expanded in terms of $\cos\left(\frac{n\pi x}{L}\right)$ series, if du/dx is a piecewise smooth function then the Fourier cosine series of $u(x,t)$ can be differentiated term-by-term (It's no matter that don't satisfy BC's), see Haberman book pg 118. Then substituting in PDE:

$$\sum_{n=0}^{\infty} C_n'(t)\cos\frac{n\pi x}{L}=k\sum_{n=0}^{\infty}-\left(\frac{n\pi}{L}\right)^2 C_n(t)\cos\left(\frac{n\pi x}{L}\right)+Q(x,t)$$

since we expand $Q(x,t)$ in cosine series too.

$$\sum_{n=0}^{\infty}\left[C_n'(t)+k\left(\frac{n\pi}{L}\right)^2 C_n(t)\right]\cos\left(\frac{n\pi x}{L}\right)=\sum_{n=0}^{\infty} q_n(t)\cos\frac{n\pi x}{L}$$

$$\Rightarrow C_n'(t)+k\left(\frac{n\pi}{L}\right)^2=q_n(x)=\frac{2}{L(1+\delta_{0i})}\int_0^L Q(x,t)\cos\left(\frac{n\pi x}{L}\right)$$

this was found multiplying expansion of $Q(x,t)$ by $\cos(n\pi x/L)$ and integrating by use of the orthogonality.

but I don't sure!! since the expanded solution don't satisfy BC's.

By: Herminso Villarraga Gómez

8.5.1 Let $y(t)=\frac{1}{w_0}\int_0^t f(s)\sin w_0(t-s)\,ds$

$$=\frac{1}{w_0}\int_0^t f(s)\left(\sin w_0t\cos w_0s-\cos w_0t\sin w_0s\right)ds$$

$$=\frac{\sin w_0t}{w_0}\int_0^t f(s)\cos(w_0s)\,ds-\frac{\cos w_0t}{w_0}\int_0^t f(s)\sin w_0s\,ds$$

$$\frac{dy}{dt}=\cos w_0t\int_0^t f(s)\cos w_0s\,ds+\frac{f(t)\cos w_0t\sin w_0t}{w_0}$$
$$+\sin w_0t\int_0^t f(s)\sin w_0s\,ds-\frac{f(t)\cos w_0t\sin w_0t}{w_0}$$

$$\frac{dy}{dx}=\cos w_0t\int_0^t f(s)\cos w_0s\,ds+\sin w_0t\int_0^t f(s)\sin w_0s\,ds$$

$$\frac{d^2y}{dx^2}=-w_0\sin w_0t\int_0^t f(s)\cos w_0s\,ds+f(t)\cos^2 w_0t$$
$$+w_0\cos w_0t\int_0^t f(s)\sin w_0s\,ds+f(t)\sin^2 w_0t$$

$$\frac{d^2y}{dt^2}=-w_0\int_0^t f(s)\left(\sin w_0t\cos w_0s-\cos w_0t\sin w_0s\right)ds+f(t)$$

$$\frac{d^2y}{dt^2}+w_0^2\underbrace{\left(\frac{1}{w_0}\int_0^t f(s)\sin w_0(t-s)\,ds\right)}_{y}=f(t)\Rightarrow\boxed{\frac{d^2y}{dt^2}+w_0^2y=f(t)}$$

Assuming $w_0^2>0$, i.e $w_0>0\Rightarrow y(t)=A\cos w_0t+B\sin w_0t$

we can find a particular solution using variation of parametros method:

$$W=\begin{vmatrix}\cos w_0t & \sin w_0t\\ -w_0\sin w_0t & w_0\cos w_0t\end{vmatrix}=w_0(\cos^2 w_0t+\sin^2 w_0t)=w_0.$$

Let be $y(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$

$$\Rightarrow u_1'(t)=-\frac{y_2(t)f(t)}{W}=-\frac{\sin w_0t}{w_0}f(t)=-\frac{1}{w_0}\sin(w_0t)\cdot f(t)$$

$$u_1(t)=-\frac{1}{w_0}\int_0^t f(s)\sin w_0s\,ds$$

Similarly $u_2'(t)=\frac{y_1(t)f(t)}{W}=\frac{1}{w_0}f(t)\cos w_0t\rightarrow u_2(t)=\frac{1}{w_0}\int_0^t f(s)\cos w_0s\,ds$

$$\Rightarrow y(t)=-\frac{1}{w_0}\cos w_0t\int_0^t f(s)\sin w_0s\,ds+\frac{1}{w_0}\sin w_0t\int_0^t f(s)\cos w_0s\,ds$$

is a particular solution.

Hence the general solution has the form:

$$y(t) = c_1 \cos w_0 t + c_2 \sin w_0 t - \frac{\cos w_0 t}{w_0}\int_0^t f(s)\sin w_0 s\, ds + \frac{\sin w_0 t}{w_0}\int_0^t f(s)\cos w_0 s\, ds$$

$$y(0) = y_0 = c_1 - \frac{1}{w_0}\cdot 0 = 0 \rightarrow c_1 = y_0$$

$$\frac{dy}{dt} = -y_0 w_0 \sin w_0 t + c_2 w_0 \cos w_0 t + \sin w_0 t \int_0^t f(s)\sin w_0 s\, ds - \frac{\cos w_0 t}{w_0} f(t)\sin w_0 t + \cos w_0 t \int_0^t f(s)\cos w_0 s\, ds + \frac{\sin w_0 t}{w_0} f(t)\cos w_0 t$$

$$\frac{dy}{dt}(0) = v_0 = c_2 w_0 \Rightarrow c_2 = v_0/w_0$$

Hence the solution which satisfies the I.C. is:

$$y(t) = y_0 \cos w_0 t + \frac{v_0}{w_0}\sin w_0 t - \frac{\cos w_0 t}{w_0}\int_0^t f(s)\sin w_0 s\, ds + \frac{\sin w_0 t}{w_0}\int_0^t f(s)\cos w_0 s\, ds$$

$$= y_0 \cos w_0 t + \frac{v_0}{w_0}\sin w_0 t + \frac{1}{w_0}\int_0^t f(s)\left(\sin w_0 t \cos w_0 s - \cos w_0 t \sin w_0 s\right) ds$$

$$\boxed{y(t) = y_0 \cos w_0 t + \frac{v_0}{w_0}\sin w_0 t + \frac{1}{w_0}\int_0^t f(s)\sin w_0 (t-s)\, ds}$$

8.5.2 a) The eigenfunctions and eigenvalues of the associated homogeneous problem have the form:

$$\phi_n(x) = \sin(n\pi x/L) \qquad \text{with } \lambda_n = (n\pi/L)^2$$

Now we will assume $\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(t)}$ substituting in PDE:

$$\sum_{n=1}^{\infty} b_n''(t)\phi_n(t) = c^2 \sum_{n=1}^{\infty} -\lambda^2 b_n(t)\phi_n(t) + \underbrace{\sum_{n=1}^{\infty} q_n(t)\phi_n(t)}_{\text{where } Q(x,t) \text{ was expanded in eigenfunction series.}}$$

$$\Rightarrow \boxed{b_n''(t) + c^2\lambda_n^2 b_n(t) = q_n(t)} \qquad \lambda_n^2 = n\pi/L$$

The solution to the homogeneous part of the ODE is:

$$b_n(t) = C_{n,1}\cos\frac{cn\pi}{L}t + C_{n,2}\sin\frac{cn\pi}{L}t$$

the wronskian of this solution is just as we did in exercise (8.5.1) $W = cn\pi/L$, the using variation of parameters we have:

$$v_1'(t) = -\frac{L}{cn\pi} q_n(t)\sin\frac{cn\pi}{L}t \Rightarrow v_1(t) = \frac{-L}{cn\pi}\int_0^t q_n(s)\sin\frac{cn\pi}{L}s\, ds$$

$$v_2'(t) = \frac{L}{cn\pi} q_n(t)\cos\frac{cn\pi}{L}t \Rightarrow v_2(t) = \frac{L}{cn\pi}\int_0^t q_n(s)\cos\frac{cn\pi}{L}s\, ds$$

Thus after some algebra and trigonometry (OR using Mathematica to reduce the expresion), the particular solution has the form:

$$B_n(t) = \frac{L}{cn\pi}\int_0^t q_n(s)\sin\frac{n\pi}{L}(t-s)\,ds$$

thus $b_n(t) = C_{n,1}\cos\frac{cn\pi}{L}t + C_{n,2}\sin\frac{cn\pi}{L}t + \frac{L}{cn\pi}\int_0^t q_n(s)\sin\frac{cn\pi}{L}(t-s)\,ds$

Now we know:

$$u(x,t) = \sum_{n=1}^{\infty}\left(C_{n,1}\cos\frac{cn\pi}{L}t + C_{n,2}\sin\frac{nc\pi}{L}t + \frac{L}{cn\pi}\int_0^t q_n(s)\sin\frac{cn\pi}{L}(t-s)\,ds\right)\sin\frac{n\pi x}{L}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty}\left(C_{n,1} + 0 + 0\right)\sin\frac{n\pi x}{L}\,dx \rightarrow C_{n,1} = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$0 = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty}\left(\frac{C_{n,2}\,cn\pi}{L} + 0 + 0\right)\sin\frac{n\pi x}{L} = 0 \Rightarrow C_{n,2} = 0$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\cdot\cos\frac{cn\pi t}{L} + \frac{L}{cn\pi}\int_0^L q_n(s)\sin\frac{cn\pi}{L}(t-s)\,ds\right) \times \sin n\pi x/L}$$

b) Now suppose $Q(x,t) = g(x)\cos wt = \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x/L)$

$$\Rightarrow q_n(t) = \frac{2}{L}\int_0^L Q(x,t)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\cos wt\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

Let $g_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}$ for compact notation $\Rightarrow q_n(t) = g_n\cos wt$

Now:

$$\frac{L}{cn\pi}\int_0^t g_n\cos wt\sin\frac{cn\pi}{L}(t-s)\,ds$$

$$= \frac{L}{cn\pi}g_n\int_0^t \cos wt\sin\frac{cn\pi t}{L}\cos\frac{cn\pi s}{L} - \cos wt\cos\frac{cn\pi t}{L}\sin\frac{cn\pi s}{L}\,ds$$

$$= \frac{Lg_n}{cn\pi}\cos wt\sin\frac{cn\pi t}{L}\int_0^t\cos\frac{cn\pi s}{L}\,ds - \frac{Lg_n}{cn\pi}\cos wt\cos\frac{cn\pi t}{L}\int_0^t\sin\frac{cn\pi s}{L}\,ds$$

$$= \frac{Lg_n}{cn\pi}\cos wt\sin\frac{cn\pi t}{L}\cdot\frac{L}{cn\pi}\sin\frac{cn\pi s}{L}\Big|_0^t + \frac{Lg_n}{cn\pi}\cos wt\cos\frac{cn\pi t}{L}\frac{L}{cn\pi}\cos\frac{cn\pi s}{L}\Big|_0^t$$

$$= \frac{Lg_n}{cn\pi}\cos wt\sin\frac{cn\pi t}{L}\cdot\frac{L}{cn\pi}\sin\frac{cn\pi t}{L} + \frac{Lg_n}{cn\pi}\cos wt\cos\frac{cn\pi t}{L}\frac{L}{cn\pi}\left(\cos\frac{cn\pi t}{L} - 1\right)$$

$$= g_n\left(\frac{L}{cn\pi}\right)^2\cos wt\sin^2\frac{cn\pi t}{L} + g_n\left(\frac{L}{cn\pi}\right)^2\cos wt\cos^2\frac{cn\pi t}{L}$$

$$- g_n\left(\frac{L}{cn\pi}\right)^2\cos wt\cos\frac{cn\pi t}{L}$$

$$= g_n\left(\frac{L}{cn\pi}\right)^2 \cos \omega t - g_n\left(\frac{L}{cn\pi}\right)^2 \cos(\omega t) \cos \frac{cn\pi t}{L}$$

$$= g_n\left(\frac{L}{cn\pi}\right)^2 \cos \omega t \left(1 - \cos \frac{cn\pi t}{L}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left(C_{n,1} \cos \frac{cn\pi t}{L} + g_n\left(\frac{L}{cn\pi}\right)^2 \cos \omega t \left(1 - \cos \frac{cn\pi t}{L}\right)\right) \sin \frac{n\pi x}{L}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(\left(C_{n,1} - g_n\left(\frac{L}{cn\pi}\right)^2 \cos \omega t\right) \cos \frac{cn\pi t}{L} + g_n\left(\frac{L}{cn\pi}\right)^2 \cos \omega t\right) \sin \frac{n\pi x}{L}}$$

with $C_{n,1} = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$

$\rightarrow$ Resonance occurs if $\omega^2 = \left(\frac{cn\pi}{L}\right)^2$

(8.6.1) $\nabla^2 u = Q(x,y) \rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = Q(x,y)$

a) $u(x,0) = u(x,H) = 0 \rightarrow$ suppose $\boxed{u = \sum_{n=1}^{\infty} b_n(x) \sin \frac{n\pi y}{H}}$

$$\Rightarrow \nabla^2 u = \sum_{n=1}^{\infty} \frac{d^2 b_n}{dx^2} \sin \frac{n\pi y}{H} - \sum_{n=1}^{\infty} \left(\frac{n\pi}{H}\right)^2 b_n \sin\left(\frac{n\pi y}{H}\right) = Q(x,y)$$

OR $$\sum_{n=1}^{\infty} \left(\frac{d^2 b_n}{dx^2} - \left(\frac{n\pi}{H}\right)^2 b_n \right) \sin \frac{n\pi y}{H} = Q(x,y) = \sum_{n=1}^{\infty} q_n(x) \sin \frac{n\pi y}{H}$$

$$\Rightarrow \frac{d^2 b_n}{dx^2} - \left(\frac{n\pi}{H}\right)^2 b_n = q_n(x) \quad \text{with} \quad q_n(x) = \frac{2}{L}\int_0^L Q(x,y) \sin\left(\frac{n\pi}{H} y\right) dy$$

according to eq (8.6.9):

$$\boxed{b_n(x) = \sinh \frac{n\pi(L-x)}{H} \int_0^x q_n(\xi) \sinh \frac{n\pi \xi}{H} d\xi + \sinh \frac{n\pi x}{H} \int_x^L q_n(\xi) \sinh \frac{n\pi(L-\xi)}{H} d\xi}$$

* b) $u(0,y) = 0$; $u(L,y) = 1$
$u(x,0) = u(x,H) = 0$ $\Rightarrow$ The eigenfunctions of the related homogeneous problem have the form.

$$\varphi_{nm}(x,y) = \sin(n\pi x/L) \sin(m\pi y/H)$$

Let $\boxed{u(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}}$ Multiply by $\sin(n'\pi x/L) \sin(m'\pi y/H)$ and integrate

$$\int_0^H \int_0^L u(x,y) \sin \frac{n'\pi x}{L} \sin \frac{m'\pi y}{H} dx\,dy =$$

$$\sum_{n,m} b_{nm} \cdot \underbrace{\int_0^L \sin \frac{n\pi x}{L} \sin \frac{n'\pi x}{L} dx}_{(L/2)\delta_{nn'}} \cdot \underbrace{\int_0^H \sin \frac{m\pi y}{H} \sin \frac{m'\pi y}{H} dy}_{(H/2)\delta_{mm'}}$$

$$= b_{n'm'} \frac{LH}{4} \Rightarrow b_{nm} = \frac{4}{LH} \int_0^H \int_0^L u(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} dx\,dy$$

Since φ_{nm} are the eigenfunctions of the related homogeneous problem $\nabla^2 \varphi_{nm} = -\lambda_{nm} \varphi_{nm}$ with eigenvalues $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

$\Rightarrow \varphi_{nm} = -\frac{1}{\lambda_{nm}} \nabla^2 \varphi_{nm}$, thus:

$$b_{nm} = -\frac{1}{\lambda_{nm}} \frac{4}{LH} \int_0^H \int_0^L u(x,y) \nabla^2 \overbrace{\left[\sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\right]}^{\varphi_{nm}} dx\,dy$$

Now, we will need the Green's two-dimensional formula:

$$\int_0^H \int_0^L (u\nabla^2 v - v\nabla^2 u)\, dx\,dy = \oint (u\nabla v - v\nabla u) \cdot \hat{n}\, ds$$

$$\Rightarrow \int_0^H \int_0^L u \nabla^2 v\, dx\,dy = \int_0^H \int_0^L v \nabla^2 u\, dx\,dy + \oint (u\nabla v - v\nabla u) \cdot \hat{n}\, ds \Rightarrow$$

Since $u(x,0)=u(x,H)=0 \Rightarrow v(x,0)=u(x,0)-r(x)=-r(x)$
$v(x,H)=u(x,H)-r(x)=-r(x)$

thus $v(x,y)$ must satisfy: $v(x,0)=v(x,H)=-r(x)$ that lead to

$$\sum_{n=1}^{\infty} b(0)\sin\frac{n\pi x}{L}=\sum_{n=1}^{\infty} b(H)\sin\frac{n\pi x}{L}=-r(x)$$

multiplying by $\sin\frac{n'\pi x}{L}$ and integrating by use of orthogonalithy:

$$b(0)=b(H)=-\frac{2}{L}\int_0^L r(x)\sin\frac{n\pi x}{L}\,dx$$

$$=-\frac{2}{L}\int_0^L\left(-\frac{(L-x)}{L}+1\right)\sin\frac{n\pi x}{L}\,dx \qquad \text{Since } \int_0^L \sin\frac{n\pi x}{L}dx=0$$

hence

$$=-\frac{2}{L}\int_0^L \frac{x}{L}\sin\frac{n\pi x}{L}\,dx \qquad u=x/L \quad dv=\sin(n\pi x/L)dx \quad du=dx/L \quad v=-\frac{L}{n\pi}\cos(n\pi x/L)$$

$$=-\frac{2}{L}\left[-\frac{x}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L+\frac{1}{n\pi}\int_0^L\cos\frac{n\pi x}{L}\,dx\right] \to 0$$

$$\boxed{b(0)=b(H)=\frac{2}{n\pi}(-1)^n}$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

Now, substituting the expansion for $v(x,y)$ in $\nabla^2 v=Q(x,y)$ we have:

$$\sum_{n=1}^{\infty}-\left(\frac{n\pi}{L}\right)^2 b_n(y)\sin\frac{n\pi x}{L}+\sum_{n=1}^{\infty}\frac{d^2b_n(y)}{dy^2}\sin\left(\frac{n\pi x}{L}\right)=Q(x,y)$$

OR

$$\sum_{n=1}^{\infty}\left[\frac{d^2b_n}{dy^2}-\left(\frac{n\pi}{L}\right)^2 b_n\right]\sin\left(\frac{n\pi x}{L}\right)=Q(x,y)=\sum_{n=1}^{\infty}q_n(y)\sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{q_n(y)=\frac{2}{L}\int_0^L Q(x,y)\sin\frac{n\pi x}{L}\,dx}$$

$$\Rightarrow \quad \frac{d^2b_n}{dy^2}-\left(\frac{n\pi}{L}\right)^2 b_n=q_n(y) \quad \text{with} \quad b(0)=b(H)=\frac{2}{n\pi}(-1)^n$$

This is just the eq (8.6.6) but with nonhomogeneous BC's. To make the BC's homogeneous we need a reference function $\beta(y)$ such that $\gamma(0)=\gamma(H)=\frac{2}{n\pi}(-1)^n \Rightarrow \boxed{\beta_n(y)=\frac{2}{n\pi}\cos\frac{n\pi y}{H}\cos\frac{n\pi(y-H)}{H}}$

Let define $B_n(y)=b_n(y)-\beta_n(y) \Rightarrow b_n(y)=B_n(y)+\beta_n(y)$ and

$$\frac{d^2b_n}{dy^2}=\frac{d^2B_n}{dy^2}+\beta''(y)$$

Substituting in the PDE

$$\frac{d^2B_n}{dy^2}-\left(\frac{n\pi}{L}\right)^2 B_n=q_n(y)-\left[\beta''(y)-\left(\frac{n\pi}{L}\right)^2\beta(y)\right]\equiv\gamma_n(y) \quad \text{with} \quad B_n(0)=0,\ B_n(H)=0$$

this is just the eq (8.6.6) with solution given by eq (8.6.9):

$$B_n=\sinh\frac{n\pi(H-y)}{L}\int_0^y\gamma_n(\xi)\sinh\frac{n\pi\xi}{L}\,d\xi+\sinh\frac{n\pi y}{L}\int_y^H\gamma_n(\xi)\sinh\frac{n\pi(H-\xi)}{L}\,d\xi$$

with $\gamma_n(y)$ given above. Finally $\boxed{b_n(y)=B_n(y)+\beta_n(y)}$

d) $\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(L,y) = 0$ and $\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,H) = 0$

$\Rightarrow$ use the expansion $\boxed{u(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm} \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H}}$

By term-by-term differentiation (valid since u and the eigenfunctions satisfy the same ~~BC's~~ homogeneous BC's) we can substitute in

$$\nabla^2 u = Q(x,y) \quad \text{with } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\Rightarrow \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm} \nabla^2 \varphi_{nm} = Q(x,y) \quad \text{with } \varphi_{nm} = \cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}$$

the ~~eigenfuns~~ related eigenfunctions.

$$\nabla^2 \varphi_{nm} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi_{nm}(x,y) = -\underbrace{\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]}_{\lambda_{nm}}\varphi_{nm}(x,y) = -\lambda_{nm}\varphi_{nm}$$

$$\Rightarrow \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}(-\lambda_{nm}\varphi_{nm}) = Q(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_{nm}\varphi_{nm}(x,y)$$

$$\Rightarrow \boxed{A_{nm} = -\frac{q_{nm}}{\lambda_{nm}} = -\frac{4/LH(1+\delta_{n0})(1+\delta_{m0})}{(n\pi/L)^2 + (m\pi/H)^2}\int_0^H\int_0^L Q(x,y)\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}\,dx\,dy}$$

where we used an expansion in eigenfunctions $\varphi_{nm}(x,y)$ for $Q(x,y)$ with q_{nm} the Fourier coefficients. The factor $4/LH(1+\delta_{n0})(1+\delta_{m0})$ appears by the orthogonality of the cosine functions

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{n'\pi x}{L}\,dx = \frac{L}{2}(1+\delta_{0n})\cdot\delta_{nn'}$$

just as in equation (8.6.12).

e) $\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(L,y) = 0$ and $u(x,0) = \frac{\partial u}{\partial y}(x,H) = 0$

Now use the expansion $\boxed{u(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\frac{n\pi x}{L}\sin\frac{(2m-1)\pi y}{2H}}$

just as we did before in part (d) above $A_{nm} = -q_{nm}/\lambda_{nm}$ with $\lambda_{nm} = 1/[(n\pi/L)^2 + ((2n-1)\pi/2H)^2]$, where q_{nm} is the Fourier coefficient of the expansion:

$$Q(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} q_{nm}\cos\frac{n\pi x}{L}\sin\frac{(2m-1)\pi y}{2H}$$

Multiplying by $\varphi_{n'm'}$ and integrating:

$$\int_0^H\int_0^L Q(x,y)\cos\frac{n'\pi x}{L}\sin\frac{(2m'-1)\pi y}{2H}\,dy$$

$$= \sum_{n,m} q_{nm}\underbrace{\int_0^L \cos\frac{n'\pi x}{L}\cos\frac{n\pi x}{L}\,dx}_{\frac{L}{2}(1+\delta_{0n})\cdot\delta_{nn'}}\underbrace{\int_0^H \sin\frac{(2m-1)\pi y}{2H}\sin\frac{(2m'-1)\pi y}{2H}\,dy}_{(H/2)\delta_{mm'}}$$

$$\Rightarrow \boxed{q_{nm} = \frac{4}{HL}\frac{1}{(1+\delta_{0n})}\int_0^H\int_0^L Q(x,y)\cos(\ )\sin[\ ]\,dx\,dy}$$

(8.6.2) $\frac{d^2 b_n}{dy^2} - \left(\frac{n\pi}{L}\right)^2 b_n(y) = q_n(y) \qquad b_n(0) = b_n(H) = 0$

a) Suppose $\boxed{b_n(y) = \sum_{m=1}^{\infty} \alpha_m \sin\left(\frac{m\pi y}{H}\right)}$ substituting above:

$$-\sum_{m=1}^{\infty}\left[\left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right]\alpha_m \sin\frac{m\pi y}{H} = q_n(y) = \sum_{m=1}^{\infty} q_{nm}\sin\left(\frac{m\pi y}{H}\right)$$

where we used ~~differen~~ term-by-term differentiation (since $b_n(y)$ and $\sin(n\pi y/H)$ satisfy the same BC's) and an expansion for $q_n(y)$ in sine series.

$$\Rightarrow -\left[\left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right]\alpha_m = q_{nm} \Rightarrow \boxed{\alpha_m = \frac{-q_{nm}}{(n\pi/L)^2 + (m\pi/H)^2}}$$

where q_{nm} is the Fourier coefficient for the expansion of $q_n(y)$

$$q_{nm} = \frac{2}{H}\int_0^H q_n(y)\sin(m\pi y/H).$$

Hence (8.6.3) becomes: $u = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{q_{nm}}{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$

just like (8.6.11).

b) The solution (8.6.9) is:

$$b_n(y) = \sinh\frac{n\pi(H-y)}{L}\int_0^y q_n(\xi)\sinh\frac{n\pi\xi}{L}\,d\xi + \sinh\frac{n\pi y}{L}\int_y^H q_n(\xi)\sinh\frac{n\pi(H-\xi)}{L}\,d\xi$$

we can check that $b_n(0) = b_n(H) = 0$ reality. Also by differentiation and substitution we can se that $b_n(y)$ solves the differential equation $\frac{d^2\phi}{dy^2} - \left(\frac{n\pi}{L}\right)^2\phi = q_n(y)$ (See Haberman's book)

By othe hand also $\beta_n(y) = -\sum_{m=1}^{\infty}\frac{q_{nm}\sin(m\pi y/H)}{(n\pi/L)^2 + (m\pi/H)^2}$ found above satisfy $\beta_n(0) = \beta_n(H) = 0$.

And the differential equation. You can check that:

$$\frac{d^2\beta_n}{dy^2} - \left(\frac{n\pi}{L}\right)^2\beta_n = \sum_{m=1}^{\infty}\frac{+(m\pi/L)^2 q_{nm}\sin(m\pi y/H)}{(\ )^2 + (\ )^2} - \left(\frac{n\pi}{L}\right)^2\sum_{m=1}^{\infty}\frac{-q_{nm}\sin(m\pi y/H)}{(\ )^2 + (\ )^2}$$

$$= \sum_{m=1}^{\infty} q_{nm}\sin(m\pi y/H) = q_n(y) \checkmark$$

Thus $\beta_n(y)$ also solves the one-dimensional value problem. Since the solution is unique the $b_n(y) = \beta_n(y)$.

c) The expansion (8.6.11) is $u(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$

with b_{nm} given by (8.6.12) as:

$$b_{nm}=-\frac{1}{\lambda_{nm}}\underbrace{\frac{4}{LH}\int_0^H\int_0^L Q(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dx\,dy}_{q_{nm}}$$

since $Q(x,y)$ is expanded in $\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$

$$\boxed{b_{nm}=-\frac{q_{nm}}{(n\pi/L)^2+(m\pi/H)^2}}$$

So $$u(x,y)=-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q_{nm}}{(n\pi/L)^2+(m\pi/H)^2}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$$

just the same expression obtained in part (a) for $u(x,y)$.

8.6.3 $\nabla^2 u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=Q(r,\theta)$

a) $u(a,\theta)=0 \rightarrow$ the related two-dimensional eigenfunctions that satisfy the BC's including the periodic boundary conditions $u(r,-\pi)=u(r,\pi)$ and $\frac{\partial u}{\partial\theta}(r,-\pi)=\frac{\partial u}{\partial\theta}(r,-\pi)$

and $|u(0,\theta)|<\infty$, are:

$$\phi_{nm}\sim J_m(\sqrt{\lambda_{mn}}\,r)\begin{Bmatrix}\cos m\theta\\ \sin m\theta\end{Bmatrix}$$

See section 7.7 in Haberman's book.

where $J_m(\sqrt{\lambda_{mn}}\,a)=0$ for satisfy $u(a,\theta)=0$, thus $\boxed{\lambda_{mn}=\left(\frac{z_{mn}}{a}\right)^2}$

with $z_{mn}\rightarrow$ zeros of $J_m(z)$.

Thus let suppose:

$$\boxed{u(r,\theta)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}J_m(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos(m\theta)+B_{mn}\sin(m\theta)\right]}$$

Since $\nabla^2\phi_{nm}=-\lambda_{nm}\phi_{nm}$, because ϕ_{nm} satisfy the eigenvalue related problem.

$$\Rightarrow \nabla^2 u=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}-\lambda_{nm}J_m(\sqrt{\lambda_{mn}}\,r)\left[A_{mn}\cos m\theta+B_{mn}\sin m\theta\right]=Q(\ ,$$

From here, multiplying by $J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\begin{Bmatrix}\cos m'\theta\\ \sin m'\theta\end{Bmatrix}$ at both sides and using the orthogonality relations as usual:

$$\begin{Bmatrix}A_{nm}\\ B_{nm}\end{Bmatrix}=-\frac{1}{\lambda_{mn}}\left[\int_{-\pi}^{\pi}\int_0^a Q(r,\theta)\begin{Bmatrix}\cos m\theta\\ \sin m\theta\end{Bmatrix}J_m(\sqrt{\lambda_{mn}}\,r)\,r\,dr\,d\theta\Big/\int_{-\pi}^{\pi}\int_0^a\begin{Bmatrix}\cos^2 m\theta\\ \sin^2 m\theta\end{Bmatrix}J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr\,d\theta\right]$$

where $\int_{-\pi}^{\pi}\cos^2 m\theta\,d\theta=\pi(1+\delta_{0m})$ and $\int_0^a J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr=\frac{a^2}{2}J_{m+1}^2(\sqrt{\lambda_{mn}}\,a)$

$\int_{-\pi}^{\pi}\sin^2 m\theta\,d\theta=\pi(1-\delta_{0m})$

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b) $\frac{\partial u}{\partial r}(a,\theta)=0 \rightarrow$ The related eigenfunction are just the same used in part (a) but now we need $J'_m(\sqrt{\lambda_{mn}}\,a)=0$ to satisfy the BC.

Thus $\boxed{\lambda_{mn}=\left(\frac{z_{mn}}{a}\right)^2}$, but now $z_{mn} \rightarrow$ zeros of $J'_m(z)$.

Hence the expansion for $u(r,\theta)$ have the same form as we did in part (a) and the same expresion for the coefficients, only change the orthogonality relation for Bessel functions

$$\int_0^a J_m(\sqrt{\lambda_{mn}}\,r)J_m(\sqrt{\lambda_{mn'}}\,r)\,r\,dr=\frac{a^2}{2}\left(1-\frac{m^2}{\lambda_{mn}a^2}\right)J_m^2(\sqrt{\lambda_{mn}}\,a)\,\delta_{nn'}$$

$$\Rightarrow \int_0^a J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr=\frac{a^2}{2}\left(1-\frac{m^2}{\lambda_{mn}a^2}\right)J_m^2(\sqrt{\lambda_{mn}}\,a)$$

with $z_{mn}=\sqrt{\lambda_{mn}}\,a$ the n-th zero of $J'_m(z)$.

c) $u(a,\theta)=f(\theta) \rightarrow$ To make this BC homogeneous a reference function that solves the homogeneous problem $\nabla^2 u=0$ in polar coordinates inside a circle. That function is

$$\boxed{u_E(r,\theta)=\sum_{n=1}^{\infty}\left[A_n\cos n\theta+B_n\sin n\theta\right]r^n}$$ See Section (2.5.2) in book.

Since $u_E(a,\theta)=f(\theta)=\sum_{n=1}^{\infty}\left[A_n\cos(n\theta)+B_n\sin n\theta\right]a^n$, then multiplying by $\sin(n\theta)$ OR $\cos(n\theta)$ we can find the Fourier coefficients as in eqs (2.5.47):

$$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)d\theta \quad A_n=\frac{1}{\pi a^n}\int_{-\pi}^{\pi}f(\theta)\cos(n\theta)d\theta, \quad B_n=\frac{1}{\pi a^n}\int_{-\pi}^{\pi}f(\theta)\sin(n\theta)d\theta$$

Now let $v(r,\theta)=u(r,\theta)-u_E(r,\theta) \rightarrow \nabla^2 u=\nabla^2 v+\cancel{\nabla^2 u_E}^{\,0}$

thus substituting in original PDE: $\boxed{\nabla^2 v=Q(r,\theta)}$ with $v(a,\theta)=0$

just the problem that we did in part (a), hence the solution for $v(r,\theta)$ is the same wrote in part (a) and finally:

$$\boxed{u(r,\theta)=v(r,\theta)+u_E(r,\theta)}$$

d) That is similar to procedure in part (c), but now the reference function must satisfy $\frac{\partial u_E}{\partial r}(a,\theta)=f(\theta)$, hece in this case the Fourie coefficions of u_E are:

$$A_n=\frac{n}{\pi a^{n-1}}\int_{-\pi}^{\pi}f(\theta)\cos(n\theta)d\theta, \quad B_n=\frac{n}{\pi a^{n-1}}\int_{-\pi}^{\pi}f(\theta)\sin(n\theta)d\theta$$ and A_0 can be free.

The final solution is $\boxed{u(r,\theta)=v(r,\theta)+u_E(r,\theta)}$

with $v(r,\theta)$ the solution found in part (b).

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8.6.4 $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = Q(r,\theta)$ inside a circle of radious a.

The problem need satisfy $u(r,-\pi) = u(r,\pi)$; then the
$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial r}(r,\pi)$

related eigenfunctions are $\cos m\theta$ and $\sin m\theta$. Using an expansion in these eigenfunctions for $u(r,\theta)$:

$$\boxed{u(r,\theta) = \sum_{m=0}^{\infty} A_m(r)\cos(m\theta) + B_m(r)\sin(m\theta)}$$

and substituting in the PDE:

$$\sum_{n=0}^{\infty}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_m}{\partial r}\right) - \frac{m^2}{r^2}A_m\right)\cos m\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_m}{\partial r}\right) - \frac{m^2}{r^2}B_m\right)\sin m\theta = Q(r,\theta)$$

let be $Q(r,\theta) = \sum_{n=0}^{\infty} a_m(r)\cos(m\theta) + b_m(r)\sin(m\theta)$ an Fourier expansion for $Q(r,\theta)$:

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dA_m}{dr}\right) - \frac{m^2}{r^2}A_m = a(r) \qquad \frac{1}{r}\frac{d}{dr}\left(r\frac{dB_m}{dr}\right) - \frac{m^2}{r^2}B_m = b(r)$$

These are the radial equations to solve with the BC's (a), (b) (c) or (d) for $A_m(r)$ and $B_m(r)$. This way is hard to do.

Method 2: Let use the related eigenfunctions $J_m(\sqrt{\lambda_{mn}}\,r)$ with $\boxed{\lambda_{mn} = (z_{mn}/a)^2}$ and z_{mn} the zeros of $J_m(z)$ or $J_m'(z)$ dependind of which is the case.

$$\Rightarrow \boxed{u(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{mn}(\theta)\,J_m(\sqrt{\lambda_{mn}}\,r)}$$

where $J_m(\sqrt{\lambda_{mn}}\,r)$ satisfy Bessel's eq:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ_m}{dr}\right) + \left(\lambda_{mn} - \frac{m^2}{r^2}\right)J_m = 0$$

Thus sustituting the expansion for u in the PDE.

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \underbrace{\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ_m}{dr}\right)}_{\left(\frac{m^2}{r^2}-\lambda_{mn}\right)J_m \to \text{ by Bessel's eq.}} C_{mn} + \frac{1}{r^2}\frac{d^2C_{mn}}{d\theta^2}J_m = Q(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} q_{nm}(\theta)\,J_m(\sqrt{\lambda_{mn}}\,r)$$

we expand $Q(x,\theta)$ in Bessel's functions too.

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\left(\frac{m^2}{r^2} - \lambda_{mn}\right)C_{mn} + \frac{1}{r^2}\frac{d^2C_{mn}}{d\theta^2}\right]J_m(\sqrt{\lambda_{mn}}\,r) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} q_{mn}(\theta)\,J_m(\sqrt{\lambda_{mn}}\,r)$$

$$\Rightarrow \left(\frac{m^2}{r^2} - \lambda_{mn}\right)C_{mn} + \frac{1}{r^2}\frac{d^2C_{mn}}{d\theta^2} = q_{mn}(\theta)$$

OR $\frac{d^2C_{mn}}{d\theta^2} + m^2C_{mn} = \lambda_{mn}r^2C_{mn} + q_{mn}(\theta) \rightarrow$ Eq to solve.

Note that this have the quasi-form of (8.5.18) with its solution contain $C_{mn}(\theta) \sim C_{1mn}\cos(m\theta) + C_{2mn}\sin m\theta + \int \cdots$

8.6.5 $\nabla^2 u = Q(x,y) \qquad u|_s = 0$

Since the eigenfunctions of $\nabla^2\phi = -\lambda\phi$ satisfy $\phi|_s = 0$

$\Rightarrow$ let use $\boxed{u(x,y) = \sum_\lambda C_\lambda \varphi_\lambda(x,y)}$

Since φ_λ and $u(x,y)$ satisfy the same BC's we can ~~differentiate~~ term-by-term differentiate, then substitutin in the PDE:

$$\sum_\lambda C_\lambda \nabla^2\varphi_\lambda = Q(x,y) \qquad \text{but } \nabla^2\varphi_\lambda = -\lambda\varphi$$

$$\Rightarrow \sum_\lambda C_\lambda(-\lambda\varphi_\lambda) = Q(x,y)$$

If we can expand $Q(x,y)$ in functions φ_λ too: $Q(x,y) = \sum_\lambda q_\lambda \varphi_\lambda(x,y)$

$$\Rightarrow -\lambda\sum_\lambda C_\lambda\varphi_\lambda = \sum_\lambda q_\lambda \varphi_\lambda(x,y) \qquad \Rightarrow \boxed{C_\lambda = -\frac{q_\lambda}{\lambda}}$$

with q_λ the Fourier coefficient of $Q(x,y) \rightarrow q_\lambda = \dfrac{\iint Q(x,y)\varphi_\lambda\,dxdy}{\iint \varphi_\lambda^2\,dxdy}$

where we used the orthogonality of the eigenfunctions since $\nabla^2\phi = -\lambda\phi$ is a Sturm-Liouville problem.

8.6.6 $\nabla^2 u = e^{2y}\sin x$ with $u(0,y) = u(\pi,y) = 0$

$u(x,0) = 0 \quad u(x,L) = f(x)$

Since $u(0,y) = u(\pi,y) = 0 \rightarrow$ the related eigenfunctions are

$$\varphi_n(x) = \sin(n\pi x/\pi) = \sin(nx)$$

thus let $\boxed{u(x,y) = \sum_{n=1}^{\infty} a_n(y)\sin nx}$, substituting in the P.DE

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sum_{n=1}^{\infty}\left(-n^2 a_n + \frac{d^2 a_n}{dy^2}\right)\sin nx = e^{2y}\sin x$$

$$\Rightarrow \left(-n^2 a_n + \frac{d^2 a_n}{dy^2}\right) = e^{2y}\delta_{n1} \qquad \text{with} \qquad \delta_{n1} = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n\neq 1 \end{cases}$$

For $n\neq 1$: $\dfrac{d^2 a_n}{dy^2} - n^2 a_n(y) = 0 \rightarrow a_n(y) = \alpha_n \sinh(ny) + \beta_n\cosh(ny)$

For $n=1$ $\dfrac{d^2 a_1}{dy^2} - a_1(y) = e^{2y} \rightarrow a_1(y) = \alpha_1\sinh y + \beta_1\cosh y + \gamma e^{2y}$

Substituting a_1 in the differential equation:

$$0 + 4\gamma e^{2y} - \gamma e^{2y} = e^{2y} \rightarrow \boxed{\gamma = 1/3}$$

We can write $a_n(y) = \alpha_n\sinh(ny) + \beta_n\cosh(ny) + \frac{1}{3}e^{2y}\delta_{n1} \rightarrow$

Now, if $u(x,0)=0 \rightarrow a_n(0)=0$, thus

$$a_n(0)=0=\beta_n+\frac{1}{3}\delta_{n1} \rightarrow \beta_n=-\frac{1}{3}\delta_{n1}$$

$$\Rightarrow \boxed{a_n(y)=\alpha_n \sinh(ny)+\frac{1}{3}\left(-\cosh(ny)+e^{2y}\right)\delta_{n1}}$$

Now $a_n(L)=\alpha_n \sinh(nL)+\frac{1}{3}\left(e^{2L}-\cosh(nL)\right)\delta_{n1}$

$$\boxed{\alpha_n=\frac{1}{\sinh(nL)}\left[a_n(L)-\frac{1}{3}\left(e^{2L}-\cosh(nL)\right)\delta_{n1}\right]}$$

where $a_n(L)$ can be found of the condition $u(x,L)=f(x)$ if we use the expansion of $u(x,y)$:

$$f(x)=u(x,L)=\sum_{n=1}^{\infty} a_n(L)\sin(nx)$$ multiplying by $\sin(n'x)$, integrating and using orthogonality

$$\int_0^{\pi} f(x)\sin(n'x)\,dx=\sum_{n=1}^{\infty} a_n(L)\int_0^{\pi}\sin nx\,\sin n'x\,dx=\sum_{n=1}^{\infty} a_n(L)\frac{\pi}{2}\delta_{nn'}=a_{n'}(L)\frac{\pi}{2}$$

$$\Rightarrow \boxed{a_n(L)=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx}$$

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Solution to problems from Haberman Chapter 9.

9.2.1 $\dfrac{\partial u}{\partial t} = K\dfrac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad u(x,0) = g(x)$

a) $u(0,t) = u(L,t) = 0 \rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x/L)$

where $a_n(t) = \dfrac{2}{L}\int_0^L u(x,t)\sin\left(\dfrac{n\pi x}{L}\right)dx \rightarrow$ According to Fourier expansión.

Remember $\int_\alpha^b [u\mathcal{L}(v) - v\mathcal{L}(u)]\,dx = p\left(u\dfrac{dv}{dx} - v\dfrac{du}{dx}\right)\Big|_\alpha^b$ Green's Formula (5.5.8)

Let $v = \sin(n\pi x/L) \qquad \mathcal{L} = \partial^2/\partial x^2 \qquad p = 1$

$$\Rightarrow \int_0^L \left[u(x,t)(-1)\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi x}{L}\right) - \frac{\partial^2 u}{\partial x^2}\sin\left(\frac{n\pi x}{L}\right)\right]dx$$

$$= \cancel{\left[u(x,t)\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right) - \frac{\partial u}{\partial x}(x,t)\sin\left(\frac{n\pi x}{L}\right)\right]_0^L} \rightarrow 0$$

$u(0,t) = u(L,t) = 0$
$\sin(n\pi) = \sin 0 = 0.$

$$\Rightarrow \int_0^L \frac{\partial^2 u}{\partial x^2}\sin\left(\frac{n\pi x}{L}\right)dx = -\left(\frac{n\pi}{L}\right)^2\int_0^L u(x,t)\sin\left(\frac{n\pi x}{L}\right)dx \qquad (*)$$

Substituting the expansion for $u(x,t)$ in the differential equation, without differentiate term-by-term:

$\sum_{n=1} a_n'(t)\sin\left(\dfrac{n\pi x}{L}\right) = K\dfrac{\partial^2 u}{\partial x^2} + Q(x,t)$ multiplying by $\sin\left(\dfrac{n'\pi x}{L}\right)$ and integrating

$$\sum_{n=1}^{\infty} a_n'(t)\underbrace{\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n'\pi}{L}x\right)dx}_{(L/2)\delta_{nn'}} = K\int_0^L \frac{\partial^2 u}{\partial x^2}\sin\left(\frac{n'\pi x}{L}\right)dx + \int_0^L Q(x,t)\sin\frac{n\pi x}{L}dx \qquad (**)$$

Since we can expand $Q(x,t)$ as a Fourier series we have used

$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin\left(\dfrac{n\pi x}{L}\right)$ with $q_n(t) = \dfrac{2}{L}\int_0^L Q(x,t)\sin\left(\dfrac{n\pi x}{L}\right)dx$

with this and using (*) then:

$$a_{n'}'(t)\cdot\frac{L}{2} = -K\left(\frac{n'\pi}{L}\right)^2\int_0^L u(x,t)\sin\left(\frac{n'\pi x}{L}\right)dx + \int_0^L Q(x,t)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$a_n' + K\left(\frac{n\pi}{L}\right)^2\underbrace{\frac{2}{L}\int_0^L u(x,t)\sin\left(\frac{n\pi x}{L}\right)dx}_{a_n(t)} = \underbrace{\frac{2}{L}\int_0^L Q(x,t)\sin\left(\frac{n\pi x}{L}\right)dx}_{b_n(t)}$$

since $Q = \sum_{n=1}^{\infty} b_n(t)\sin\left(\dfrac{n\pi x}{L}\right)$ Fourier expansio for $Q(x,t)$

$$\boxed{a_n'(t) + K\left(\frac{n\pi}{L}\right)^2 a_n = q_n(t)} \rightarrow \text{Eq } (9.2.14)$$

Using the integrating factor $e^{k(n\pi/L)^2 t}$, multiplying by it

$$\underbrace{a_n'(t)\, e^{k(n\pi/L)^2 t} + k\left(\frac{n\pi}{L}\right)^2 e^{k(n\pi/L)^2} a_n(t)}_{\frac{d}{dt}\left[a_n(t)\, e^{k(n\pi/L)^2 t}\right]} = q_n(t)\, e^{k(n\pi/L)^2 t}$$

$$\frac{d}{dt}\left[a_n(t)\, e^{k(n\pi/L)^2 t}\right] = \quad ''$$

integrating $\quad a_n(t) e^{k(n\pi/L)^2 t} - a_n(0) = \int_0^t q_n(t_0)\, e^{k(n\pi/L)^2 t_0}\, dt_0$

$$a_n(t) = a_n(0)\, e^{-k(n\pi/L)^2 t} + e^{-k(n\pi/L)^2 t} \int_0^t q_n(t_0)\, e^{k(n\pi/L)^2 t_0}\, dt_0$$

where $a_n(0) = \frac{2}{L}\int_0^L u(x,0) \sin\left(\frac{n\pi x}{L}\right) = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)$ by the I.C: $u(x,0) = g(x)$

Subtituting $a_n(t)$ in the Fourier series expansion for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n(0)\, e^{-k(n\pi/L)^2 t} + e^{-k(n\pi/L)^2 t}\int_0^t q_n(t_0)\, e^{k(n\pi/L)^2 t_0}\, dt_0\right] \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty}\left[\left(\frac{2}{L}\int_0^L g(x_0) \sin\frac{n\pi x_0}{L}\, dx_0\right) e^{-k(n\pi/L)^2 t}\right.$$

$$\left. + e^{-k(n\pi/L)^2 t}\int_0^t \left(\frac{2}{L}\int_0^L Q(x_0,t_0) \sin\left(\frac{n\pi x_0}{L}\right) dx_0\right) e^{k(n\pi/L)^2 t_0}\, dt_0\right] \sin\left(\frac{n\pi x}{L}\right)$$

$$= \int_0^L g(x_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}\right) dx_0$$

$$+ \int_0^L\int_0^t Q(x_0,t_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 (t-t_0)}\right) dt_0\, dx_0$$

$$u(x,t) = \int_0^L g(x_0)\, G(x,t;x_0,0)\, dx_0 + \int_0^L\int_0^t Q(x_0,t_0)\, G(x,t;x_0,t_0)\, dt_0\, dx_0$$

With $\boxed{G(x,t;x_0,t_0) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 (t-t_0)}}$ Eq (9.2.19)

b) It's the same as in part a) but now the second member of Green's Formula is not zero. Since $u(0,t) = A(t)$, $u(L,t) = B(t)$ then:

$$\int_0^L \left[u(x,t)(-1)\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) - \frac{\partial^2 u}{\partial x^2}\sin\left(\frac{n\pi x}{L}\right)\right] dx$$

$$= \left[u(x,t)\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right) - \frac{\partial u}{\partial x}(x,t)\, \cancel{\sin\left(\frac{n\pi x}{L}\right)}\right]_0^L \to 0$$

$\sin(n\pi) = \sin 0 = 0$

But Now $u(0,t) = A(t)$, $u(L,t) = B(t)$

$$= \frac{n\pi}{L}\left[B(t) - A(t)\right]$$

$$\Rightarrow \int_0^L \frac{\partial^2 u}{\partial x^2}\sin\left(\frac{n\pi x}{L}\right) dx = -\left(\frac{n\pi}{L}\right)^2 \int_0^L u(x,t)\sin\left(\frac{n\pi x}{L}\right) dx - \left(\frac{n\pi}{L}\right)\left[B(t) - A(t)\right] \quad (*b)$$

From (**) in part (a) we have:

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$$a_n(t) = k\frac{2}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}\sin\left(\frac{n\pi x}{L}\right)dx + \overbrace{\frac{2}{L}\int_0^L Q(x,t)\sin(n\pi x/L)}^{q_n(x)}$$

using (*b) here, we have:

$$a_n(t) = -k\left(\frac{n\pi}{L}\right)^2 \underbrace{\frac{2}{L}\int_0^L u(x,t)\sin\left(\frac{n\pi x}{L}\right)dx}_{a_n(t)} - k\left(\frac{n\pi}{L}\right)\left[B(t)-A(t)\right]\frac{2}{L} + q_n(x)$$

$$\Rightarrow a_n'(t) + k\left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n(t) - \frac{2k}{L}\left(\frac{n\pi}{L}\right)\left[B(t)-A(t)\right]$$

using the same integrating factor from part a) $e^{k(n\pi/L)^2 t}$

$$\frac{d}{dt}\left[a_n(t)e^{k(n\pi/L)^2 t}\right] = \left\{q_n(t) - \frac{2}{L}k\left(\frac{n\pi}{L}\right)\left[B(t)-A(t)\right]\right\}e^{k(n\pi/L)^2 t} \quad \text{integrate}$$

$$a_n(t) = a_n(0)e^{-k(n\pi/L)^2 t} + e^{-k(n\pi/L)^2 t}\int_0^t \left\{q_n(t_0) - \frac{2k}{L}\left(\frac{n\pi}{L}\right)\left[B(t_0)-A(t_0)\right]e^{k\left(\frac{n\pi}{L}\right)^2 t_0}\right\}dt_0$$

then, subtituting in Fourier sine series for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n(0)e^{-k(n\pi/L)^2 t} + e^{-k(n\pi/L)^2 t}\int_0^t \left\{\quad\right\}dt_0\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty}\left[\left(\frac{2}{L}\int_0^L g(x_0)\sin\frac{n\pi x_0}{L}dx_0\right)e^{-k(n\pi/L)^2 t}\right.$$

$$+ e^{-k(n\pi/L)^2 t}\int_0^t\left(\frac{2}{L}\int_0^L Q(x_0,t_0)\sin\left(\frac{n\pi x_0}{L}\right)dx_0\right)e^{k(n\pi/L)^2 t_0}dt_0$$

$$\left. - e^{-k(n\pi/L)^2 t}\frac{2k}{L}\left(\frac{n\pi}{L}\right)\int_0^t\left[B(t_0)-A(t_0)\right]e^{k(n\pi/L)^2 t_0}dt_0\right]\sin\frac{n\pi x}{L}$$

$$= \int_0^L g(x_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\left(\frac{n\pi x}{L}\right)e^{-k(n\pi/L)^2 t}\right)dx$$

$$+ \int_0^L\int_0^t Q(x_0,t_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-k(n\pi/L)^2(t-t_0)}\right)dt_0\,dx_0$$

$$- \int_0^t k\left[B(t_0)-A(t_0)\right]\left(\sum_{n=1}^{\infty}\frac{2}{L}\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-k(n\pi/L)^2(t-t_0)}\right)dt_0$$

$$\boxed{\begin{aligned} u(x,t) = &\int_0^L g(x_0)G(x,t;x_0,0)\,dx_0 + \int_0^L\int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)\,dt_0\,dx_0 \\ &- \int_0^t k\,B(t_0)\frac{\partial G}{\partial x_0}(x,t;L,t_0)\,dt_0 + \int_0^t kA(t_0)\frac{\partial G}{\partial x_0}(x,t;0,t_0)\,dt_0 \end{aligned}}$$

with $G(x,t;x_0,t_0)$ the same given in part (a).

Note that here in this problem we cannot use differentiation term-by-term becase the boundary conditions for $u(x,t)$ are not homogeneous.

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c) $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \rightarrow u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos(n\pi x/L)$

with $a_n(t) = \frac{2}{L(1+\delta_{n0})} \int_0^L u(x,t) \cos\left(\frac{n\pi x}{L}\right) dx$ the Fourier coefficient of the expansion.

Now $\int_0^L \left[u(x,t)(-1)\left(\frac{n\pi}{L}\right)^2 \cos\left(\frac{n\pi x}{L}\right) - \frac{\partial^2 u}{\partial x^2} \cos\left(\frac{n\pi x}{L}\right)\right] dx = 0$ since $\frac{\partial u}{\partial x}(0,t)=0$ and $\frac{\partial u}{\partial x}(L,t)=0$.

just like in ~~problem~~ part (a).

$$\Rightarrow \int_0^L \frac{\partial^2 u}{\partial x^2} \cos\left(\frac{n\pi x}{L}\right) dx = -\left(\frac{n\pi}{L}\right)^2 \int_0^L u(x,t) \cos\left(\frac{n\pi x}{L}\right) dx \qquad (*c)$$

The differential equation with the series expansion for $u(x,t)$ is:

$$\sum_{n=0}^{\infty} a_n'(t) \cos\left(\frac{n\pi x}{L}\right) = K \frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

multiplying by $\cos(n'\pi x/L)$ and integrating:

$$\sum_{n=0}^{\infty} a_n'(t) \underbrace{\int_0^L \cos\frac{n\pi x}{L} \cos\frac{n'\pi x}{L} dx}_{\frac{L(1+\delta_{n0})}{2}\delta_{nn'}} = K \int_0^L \frac{\partial^2 u}{\partial x^2} \cos\left(\frac{n'\pi x}{L}\right) dx + \int_0^L Q(x,t) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$\Rightarrow a_{n'}'(t) = K \frac{2}{L(1+\delta_{n'0})} \int_0^L \frac{\partial^2 u}{\partial x^2} \cos\left(\frac{n'\pi x}{L}\right) dx + \underbrace{\frac{2}{L(1+\delta_{n'0})} \int_0^L Q(x,t) \cos\left(\frac{n\pi x}{L}\right) dx}$$

OR $a_n'(t) = -K\left(\frac{n\pi}{L}\right)^2 \underbrace{\frac{2}{L(1+\delta_{n0})} \int_0^L u(x,t) \cos\frac{n\pi x}{L} dx}_{a_n(t)} + q_n(t)$

where $(*c)$ was used and the Fourier Cosine expansion for $Q(x,t)$

$$\Rightarrow a_n'(t) + K\left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n(t) \qquad (**c)$$

Multiplying by the integrating factor $e^{K(n\pi/L)^2 t}$ and integrating we obtain $a_n(t)$ and substituting for $u(x,t)$ just like in part (a)

$$\boxed{u(x,t) = \int_0^L g(x_0) G(x,t;x_0,0) dx + \int_0^L \int_0^t Q(x_0,t_0) G(x,t;x_0,t_0) dt_0 dx_0.}$$

Note that for $t=0$, the first term correspond to the MEAN-VALUE distribution of temperature

with $\boxed{G(x,t;x_0,t_0) = \sum_{n=0}^{\infty} \frac{2}{L(1+\delta_{n0})} \cos\left(\frac{n\pi x_0}{L}\right) \cos\left(\frac{n\pi x}{L}\right) e^{-K(n\pi/L)^2(t-t_0)}}$

Method 2: Since the BC's are homogeneus we can differentiate term-by-term, then the differential equation with the expansion for $u(x,t)$ lead to:

$$\sum_{n=0}^{\infty} a_n'(t) \cos\left(\frac{n\pi x}{L}\right) = K(-1) \sum_{n=0}^{\infty} \left(\frac{n\pi}{L}\right)^2 a_n(t) \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=0}^{\infty} b_n(t) \cos\left(\frac{n\pi x}{L}\right)$$

where Fourier Cosine expansion for $Q(x,t)$ was used. $\longrightarrow$ $a_n'(t) = -K\left(\frac{n\pi}{L}\right)^2 a_n(t) + b_n(t)$ just the same $(**c)$. eq.

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d) Here we cannot use differentiation term-by-term since we don't have homogeneous BC's.

Now the right member in Green's Formula (8.4.12) with $v = \cos\frac{n\pi x}{L}$ is not zero since $\frac{\partial u}{\partial x}(0,t) = A(t)$ and $\frac{\partial u}{\partial x}(L,t) = B(t)$, then:

$$\int_0^L \left[u(x,t)(-1)\left(\frac{n\pi}{L}\right)^2 \cos\left(\frac{n\pi x}{L}\right) - \frac{\partial^2 u}{\partial x^2}\cos\left(\frac{n\pi x}{L}\right)\right]dx =$$

$\cos n\pi = (-1)^n$, $\cos 0 = 1$; $\frac{\partial u}{\partial x}(L,t) = B$, $\frac{\partial u}{\partial x}(0,t) = A$

$$\left[u(x,t)\left(-\frac{n\pi}{L}\right)\cancel{\sin\left(\frac{n\pi x}{L}\right)}^{\,0} - \frac{\partial u}{\partial x}(x,t)\cos\left(\frac{n\pi x}{L}\right)\right]_0^L = -(-1)^n B(t) - A(t)$$

$$\Rightarrow \int_0^L \frac{\partial^2 u}{\partial x^2}\cos(n\pi x/L)\,dx = -\left(\frac{n\pi}{L}\right)^2\int_0^L u(x,t)\cos\left(\frac{n\pi x}{L}\right)dx + (-1)^n B(t) - A(t) \qquad (*d)$$

Substituting the ~~series~~ cosine series expansion in $u(x,t)$ we have:

$$\sum_{n=0}^{\infty} a_n'(t)\cos\left(\frac{n\pi x}{L}\right) = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

multiplying by $\cos(n'\pi x/L)$ and integrating:

$$\sum_{n=0}^{\infty} a_n'(t)\underbrace{\int_0^L \cos\frac{n\pi x}{L}\cos\frac{n'\pi x}{L}\,dx}_{\frac{L(1+\delta_{n0})}{2}\delta_{nn'}} = k\int_0^L \frac{\partial^2 u}{\partial x^2}\cos\left(\frac{n'\pi x}{L}\right)dx + \int_0^L Q(x,t)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow a_n'(t) = -k\left(\frac{n\pi}{L}\right)^2 \underbrace{\frac{2}{L(1+\delta_{n0})}\int_0^L u(x,t)\cos\left(\frac{n\pi x}{L}\right)dx}_{a_n(t)} + k\frac{2}{L(1+\delta_{0n})}\left[(-1)^n B(t) - A(t)\right] + q_n(t)$$

where (*d) was used and the Fourier cosine series coefficient of $Q(x,t)$ identified:

$$\Rightarrow a_n'(t) + k\left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n(t) + \frac{2k}{L(1+\delta_{0n})}\left[(-1)^n B(t) - A(t)\right]$$

Multiplying by the integrating factor $e^{k(n\pi/L)^2 t}$ and integrating:

$$a_n(t) = a_n(0)e^{-k(n\pi/L)^2 t} + \int_0^t \left[q_n(t) + \frac{2k}{L(1+\delta_{0n})}\left((-1)^n B(t) - A(t)\right)\right]e^{-k(n\pi/L)^2(t-t_0)}dt_0$$

where $a_n(0) = \int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx \cdot \frac{2}{L(1+\delta_{0n})}$

Substituting in the ~~series~~ cosine series for $u(x,t)$:

$$u(x,t) = \int_0^L g(x_0)\sum_{n=0}^{\infty}\frac{2}{L(1+\delta_{0n})}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x_0}{x}\right)e^{-k(n\pi/L)^2}dx_0$$

$$+ \int_0^L\int_0^t Q(x_0,t_0)\sum_{n=0}^{\infty}\frac{2}{L(1+\delta_{0n})}\cos\frac{n\pi x}{L}\cos\frac{n\pi x_0}{L}e^{-k(n\pi/L)^2(t-t_0)}dt_0\,dx_0$$

$$+ \int_0^t k\sum_{n=0}^{\infty}\frac{2}{L(1+\delta_{0n})}\left[(-1)^n B(t) - A(t)\right]\cos\frac{n\pi x}{L}e^{-k(n\pi/L)^2(t-t_0)}dt_0$$

OR

$$\boxed{\begin{aligned} u(x,t) = &\int_0^L g(x_0)G(x,t;x_0,0)\,dx_0 + \int_0^L\int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)\,dt_0\,dx_0 \\ &+ \int_0^t kB(t_0)G(x,t;L,t_0)\,dt_0 - \int_0^t kA(t)G(x,t;0,t_0)\,dt_0 \end{aligned}}$$

with $G(x,t;x_0,t_0)$ the same given in part (b).

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9.2.2 $c\rho \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(K_0 \dfrac{\partial u}{\partial x}\right) + Q(x,t)$ $\qquad u(x,0) = g(x)$

$u(0,t) = u(L,t) = 0 \rightarrow$ ~~$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right)$~~

~~where $a_n(t) = \frac{2}{L}\int_0^L u(x,t)\sin\left(\frac{n\pi x}{L}\right)dx$ according to the sine Fourier expansion.~~

~~The differentiation term by term is useless since K_0 is function of x. Then we prefer use the Green's Formula just as in exercise (9.2.1) but now with $p = K_0$ and $\mathcal{L}(u) = \frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right)$, then~~

~~$\int_0^L [u\mathcal{L}(v) - v\mathcal{L}(u)]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_0^L$ Green's Formula (5.3.1).~~

~~Let $v = \sin(n\pi x/L)$, then since $u(0,t) = u(L,t) = 0$ and $v(0) = v(L) = 0$ the second member of the Green's Formula vanishes:~~

~~$\int_0^L \left[u(x,t)\frac{\partial}{\partial x}\left(K_0 \frac{n\pi}{L}\cos\frac{n\pi x}{L}\right) - \sin\frac{n\pi x}{L}\frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)\right]dx = 0$ (*)$$

~~$\Rightarrow \int_0^L \sin\frac{n\pi x}{L}\frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)dx = \int_0^L \left[u(x,t)K_0'(x)\frac{n\pi}{L}\cos\frac{n\pi x}{L} - \left(\frac{n\pi}{L}\right)^2 u(x,t)\sin\left(\frac{n\pi x}{L}\right)\right]dx$~~

~~Substituting the sine Fourier expansion for $u(x,t)$ in the PDE:~~

~~$c\rho\sum_{n=1}^{\infty} a_n'(t)\sin\left(\frac{n\pi x}{L}\right) = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q(x,t)$ multiplying by $\sin(n\pi x/L)$ and integrating.~~

~~$\sum_{n=1}^{\infty} a_n'(t)\int_0^L c\rho\sin\left(\frac{n\pi x}{L}\right)\sin\frac{n\pi x}{L}dx = \int_0^L \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)\sin\frac{n\pi x}{L}dx + \int_0^L Q(x,t)\sin\frac{n\pi x}{L}dx$~~

The homogeneous problem is $\dfrac{\partial u}{\partial t} = \dfrac{1}{c\rho}\dfrac{\partial}{\partial x}\left(K_0\dfrac{\partial u}{\partial x}\right)$, Let $u(x,t) = \phi(x)a(t)$

$\Rightarrow$ Separing variables and dividing by $\phi a \rightarrow \underbrace{\dfrac{1}{h}\dfrac{da}{dt}}_{-\lambda} = \underbrace{\dfrac{1}{c\rho\phi}\dfrac{d}{dx}\left(K_0\dfrac{d\phi}{dx}\right)}_{-\lambda}$

$\Rightarrow \dfrac{da}{dt} = -\lambda a(t) \rightarrow \boxed{a(t) = ce^{-\lambda t}}$

$\boxed{\dfrac{d}{dx}\left(K_0\dfrac{d\phi}{dx}\right) + \lambda c\rho\,\phi(x) = 0} \rightarrow$ Sturm-Liouville problem (5.3.2) with $p = K_0(x)$ $\quad q = 0$ and $\sigma(x) = c\rho(x)$

By Rayleigh quotient (5.3.8) we have $\lambda = \dfrac{\cancel{-K_0\phi\phi'(x)\big|_0^L}^{\,0} + \int_0^L K_0[\phi'(x)]^2dx}{\int_0^L \phi^2 c\rho\,dx} \geq 0$

Since $u(0,t) = u(L,t) = 0 \rightarrow \phi(0) = \phi(L) = 0$

$\lambda = 0 \rightarrow \phi'(x) = 0 \rightarrow \phi = cte$, but $\phi(0) = \phi(L) = 0 \Rightarrow cte = 0$

$\Rightarrow$ there are not solution for $\lambda = 0 \Rightarrow \boxed{\lambda > 0}$

Assuming that the Sturm-Liouville eigenvalue problem have known eigenfunctions $\phi_n(x)$ so that:

$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)}$ with $a_n(t) = c_n e^{-\lambda_n t}$

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Since $\phi_n(x)$ are solutions of a sturm-Liouville problem, they are orthogonals: $\int_0^L \phi_n(x)\phi_m(x)\sigma(x)dx = 0$ if $m \neq n$ eq (5.3.6)

then multiplying $u(x,t)$ by $\phi_m(x)\sigma(x)$ and integrating:

$$\int_0^L u(x,t)\phi_m(x)\, c\rho(x)dx = \sum_{n=1}^{\infty} a_n(t)\underbrace{\int_0^L \phi_n(x)\phi_m(x)\sigma(x)dx}_{0 \text{ if } m\neq n}$$

$$= a_m(t)\int_0^L \phi_m^2(x)\sigma\, dx$$

$$\Rightarrow a_n(t) = \int_0^L u(x,t)\phi_n(x)\, c\rho(x)dx \Big/ \int_0^L \phi_m^2(x)\sigma(x)dx \qquad \sigma \equiv c\rho$$

In particular since $u(x,0) = g(x)$ we have:

$$\boxed{a_n(0) = \int_0^L g(x)\phi_n(x)\, c\rho(x)dx \Big/ \int_0^L \phi_n^2(x)\sigma(x)dx}$$

Note that we can define $\mathcal{L}[\phi] = \frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right)$ such that the sturm-Liouville problem can be written as: $\mathcal{L}[\phi_n(x)] = -\lambda_n\sigma(x)\phi_n(x)$ with $\sigma \equiv c\rho(x)$.

Thus if we substute $u(x,t) = \sum a_n(t)\phi_n(x)$ in the PDE:

$$c\rho\sum_{n=1}^{\infty} a_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} a_n(t)\underbrace{\mathcal{L}[\phi_n(x)]}_{-\lambda_n c\rho\phi_n} + \sum_{n=1}^{\infty} q_n(t)\phi_n(x)$$

where we used expansión for $Q(x,t)$ as series of eigenfunction $\phi_n(x)$

$\Rightarrow a_n'(t) + \lambda_n a_n(x) = \frac{1}{c\rho}q_n(t)$ using the integrating factor $e^{\lambda_n t}$ multiplying by it and togethering

$$\frac{d}{dt}\left[a_n(t)e^{\lambda_n t}\right] = \frac{1}{c\rho}q_n(x)e^{\lambda_n t}$$ Now integrating:

$$a_n(t) = a_n(0)e^{-\lambda_n t} + e^{-\lambda_n t}\int_0^t q_n(t_0)e^{\lambda_n t_0}dt_0/\rho c$$ where $a_n(0)$ is given above.

Substituting $a_n(t)$ in series expansion for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n(0)e^{-\lambda_n t} + e^{-\lambda_n t}\int_0^t q_n(t_0)e^{\lambda_n t_0}e^{\lambda_n t_0}dt_0/c\rho\right]\phi_n(x)$$

$q_n(t)$

$$= \sum_{n=1}\left[\left(\frac{\int_0^L g(x_0)\phi_n(x_0)c\rho\, dx_0}{\int_0^L \phi_n^2(x)c\rho\, dx}\right)e^{-\lambda_n t} + e^{-\lambda_n t}\int_0^t\left(\frac{\int_0^L Q(x_0,t_0)\phi_n(x_0)c\rho\, dx_0}{\int_0^L \phi_n^2(x)c\rho\, dx}\right)e^{\lambda_n t_0}\frac{dt_0}{c\rho}\right] * \phi_n(x).$$

$$= \int_0^L g(x_0)\left(\sum_{n=1}^{\infty}\frac{1}{\int_0^L \phi_n^2(x)c\rho\, dx}\phi_n(x_0)\phi_n(x)e^{-\lambda_n t}\right)c\rho\, dx$$

$$+ \frac{1}{c\rho}\int_0^L\int_0^t Q(x_0,t_0)\underbrace{\left(\sum_{n=1}^{\infty}\frac{1}{\int_0^L \phi_n^2(x)c\rho\, dx}\phi_n(x_0)\phi_n(x)e^{-\lambda_n(t-t_0)}\right)}_{G(x,t;x_0,t_0)}c\rho\, dt_0 dx_0$$

$$\boxed{u(x,t) = \int_0^L g(x_0)G(x,t;x_0,0)c\rho\, dx_0 + \int_0^L\int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)c\rho\, dt_0 dx_0/c\rho}$$

with $\boxed{G(x,t;x_0,t_0) = \sum_{n=1}^{\infty} \frac{1}{\int_0^L \phi_n^2(x)\sigma\,dx}\phi_n(x_0)\phi_n(x)\,e^{-\lambda_n(t-t_0)}}$

Method 2: Using the Green's Formula (5.5.8)

$$\int_a^b \left[u\mathcal{L}(v) - v\mathcal{L}(u)\right]dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$$

Let $v = \phi_n(x)$, $p = K_0$ and $\mathcal{L}(u) = \frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right)$

$$\Rightarrow \int_0^L \left[u\mathcal{L}[\phi_n] - \phi_n\mathcal{L}(u)\right]\sigma\,dx = \cancel{K_0\left(u(x,t)\frac{d\phi}{dx} - \phi(x)\frac{du}{dx}\right)\Big|_0^L} \to 0$$

since $u(L,t) = u(0,t) = 0$, $\phi(L) = \phi(0) = 0$.

$$\int_0^L \phi_n\mathcal{L}(u)\,dx = \int_0^L u\mathcal{L}(\phi_n)\,dx = \int_0^L u\cdot(-\lambda_n\sigma\phi_n)\,dx = -\lambda_n\int_0^L u(x,t)\phi_n(x)\sigma\,dx \quad (*)$$

Now, substituing the expansion for $u(x,t)$ in the PDE, without differentiate term-by-term, then multiplying by $\phi_m(x)$ and integrating:

$$\sum_{n=1}^{\infty} a_n'(t)\underbrace{\int_0^L \phi_n(x)\phi_m(x)\sigma\,dx}_{0 \text{ if } n\neq m} = \underbrace{\int_0^L \mathcal{L}(u)\cdot\phi_m(x)\,dx}_{\text{using } (*)} + \int_0^L Q(x,t)\phi_m(x)\,dx$$

$$a_m'(t)\int_0^L \phi_m^2(x)\sigma\,dx = -\lambda_n\int_0^L u(x,t)\phi_m(x)\sigma\,dx + \int_0^L Q(x,t)\phi_m(x)\,dx$$

$$a_m'(t) = -\lambda_m\underbrace{\left(\frac{\int_0^L u(x,t)\phi_m(x)\sigma\,dx}{\int_0^L \phi_m^2(x)\sigma\,dx}\right)}_{a_m} + \underbrace{\left(\frac{\int_0^L Q(x,t)\phi_m(x)\,dx}{\int_0^L \phi_m^2(x)\sigma\,dx}\right)}_{q_n(t)/c\rho}$$

$\boxed{a_n'(t) + \lambda_n a_n(t) = q_n(t)/c\rho}$ Then continue as usual for find $u(x,t)$ and $G(x,t;x_0,t_0)$

9.2.3 $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + Q(x)$, $u(0,t) = u(L,t) = 0 \to u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow \sum_{n=1}^{\infty} a_n''(t)\sin\left(\frac{n\pi x}{L}\right) = c^2\sum_{n=1}^{\infty}(-1)\left(\frac{n\pi}{L}\right)^2 a_n(t)\sin\left(\frac{n\pi x}{L}\right) + \underbrace{\sum_{n=1}^{\infty} q_n(t)\sin\left(\frac{n\pi x}{L}\right)}_{\text{we used Fourier Sine expansion for } Q(x,t)}$$

$$a_n''(t) + c^2\left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n(t)$$

Note that we used differentiaton term-by-term because we had homogeneous BC's (Remember statement about - Pg 120 ch 3).

Using (8.5.19) the general solution is:

$$a_n(t) = c_1\cos\frac{n\pi c t}{L} + c_2\sin\frac{n\pi c t}{L} + \frac{L}{n\pi c}\int_0^t q_n(t_0)\sin\frac{n\pi c}{L}(t-t_0)\,dt_0 \quad (*)$$

Since $u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n(0)\sin\frac{n\pi x}{L} \to a_n(0) = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Similarly: $\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} a_n'(0) \sin \frac{n\pi x}{L} \rightarrow a_n'(0) = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)dx$

By other hand from (*):

$$a_n(0) = c_1 = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \; ; \; a_n'(0) = \frac{n\pi c}{\cancel{L}} c_2 = \frac{2}{\cancel{L}}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow a_n(t) = \left(\frac{2}{L}\int_0^L f(x_0) \sin \frac{n\pi x_0}{L} dx_0\right) \cos \frac{n\pi c t}{L} + \left(\frac{2}{n\pi c}\int_0^L g(x_0) \sin\left(\frac{n\pi x_0}{L}\right)dx_0\right) \sin \frac{n\pi c t}{L}$$

$$+ \frac{L}{n\pi c}\int_0^t \left(\frac{2}{L}\int_0^L Q(x_0,t) \sin \frac{n\pi x_0}{L} dx_0\right) \sin \frac{n\pi c}{L}(t-t_0)\,dt_0$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left[\quad \right] \sin \frac{n\pi x}{L}$$

$$= \int_0^L f(x_0) \sum_{n=1}^{\infty} \frac{2}{L} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L} dx_0$$

$$+ \int_0^L g(x_0) \sum_{n=1}^{\infty} \frac{2}{L} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} \frac{\sin n\pi c t/L}{n\pi c/L} dx_0$$

$$+ \int_0^L \int_0^t Q(x_0,t_0) \sum_{n=1}^{\infty} \frac{2}{L} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} \frac{\sin n\pi c(t-t_0)/L}{n\pi c/L} dt_0\, dx_0.$$

$$\boxed{u(x,t) = \int_0^L f(x_0) \frac{\partial G}{\partial t}(x,t;x_0,0)\,dx_0 + \int_0^L g(x_0) G(x,t;x_0,0)\,dx_0 + \int_0^L \int_0^t Q(x_0,t_0) G(x,t;x_0,t_0)\,dt_0\,dx_0}$$

with $\boxed{G(x,t;x_0,t_0) = \sum_{n=1}^{\infty} \frac{2}{L} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} \frac{\sin n\pi c(t-t_0)/L}{n\pi c/L}}$

Method 2: Using the Green's Formula (5.5.8) or (8.4.12):

$$\int_0^L \left(u \frac{\partial^2 v}{\partial x^2} - v \frac{\partial^2 u}{\partial x^2}\right) dx = \left(u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x}\right)\Big|_0^L$$

just as in the follow exercise but with homogeneous BC's. $(A(t)=0)$

9.2.4 (b) Let $v = \sin(n\pi x/L)$, then:

$$\int_0^L \left[u \cdot (-1)\left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L} - \sin \frac{n\pi x}{L} \frac{\partial^2 u}{\partial x^2}\right] dx = \left[u \cdot \left(\frac{n\pi}{L}\right) \cos \frac{n\pi x}{L} - \cancel{\sin \frac{n\pi x}{L} \frac{\partial u}{\partial x}}\right]_0^L$$

since $u(0,t) = A(t)$ and $u(L,t) = 0 \Rightarrow \; = -A(t) \cdot \frac{n\pi}{L}$

$$\Rightarrow \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cdot \frac{\partial^2 u}{\partial x^2} dx = -\left(\frac{n\pi}{L}\right)^2 \int_0^L u(x,t) \sin\left(\frac{n\pi x}{L}\right) dx + \left(\frac{n\pi}{L}\right) A(t). \quad (*)$$

By the method of eigenfunction expansion $u = \sum_{n=1}^{\infty} a_n(t) \sin(n\pi x/L)$

Substituting this in the PDE without differentiate term-by-term:

Expansion in Fourier Sine series for $Q(x,t)$.

$$\sum_{n=1}^{\infty} a_n''(t)\sin\left(\frac{n\pi x}{L}\right) = c^2\frac{\partial^2 u}{\partial x^2} + \sum_{n=1}^{\infty} q_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin\frac{n'\pi x}{L}$ and integrating!

$$\sum_{n=1}^{\infty} a_n''(t)\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}} = c^2\underbrace{\int_0^L \frac{\partial^2 u}{\partial x^2}\sin\frac{n'\pi x}{L}\,dx}_{\text{use }(*)} + \sum_{n=1}^{\infty} q_n \underbrace{\int_0^L \sin\frac{n\pi x}{L}\sin\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}}$$

$$\Rightarrow a_{n'}''(t)\frac{L}{2} = -c^2\left(\frac{n'\pi}{L}\right)^2\underbrace{\int_0^L u(x,t)\sin\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)a_n(t)} + c^2\left(\frac{n'\pi}{L}\right)A(t) + q_{n'}(t)\cdot\frac{L}{2}$$

$$\Rightarrow a_n''(t) + c^2\left(\frac{n\pi}{L}\right)^2 a_n(t) = c^2\frac{2}{L}\left(\frac{n\pi}{L}\right)A(t) + q_n(t)$$

→ This is identical to eq. obtained in exercise (9.2.3) when $A(t)=0$

By (8.5.19) the general solution for this is:

$$a_n(t) = C_1\cos\frac{n\pi c t}{L} + C_2\sin\frac{n\pi c t}{L} + \frac{L}{n\pi c}\int_0^t\left[\frac{2c^2}{L}\left(\frac{n\pi}{L}\right)A(t) + q_n(t)\right]\sin\frac{n\pi c}{L}(t-t_0)\,dt_0$$

then similar to excersice (9.2.3) we can obtain the solution for $u(x,t)$ as:

$$\boxed{\begin{aligned} u(x,t) &= \int_0^L f(x_0)\frac{\partial G}{\partial t}(x,t;x_0,0)\,dx_0 + \int_0^L g(x_0)G(x,t;x_0,0)\,dx_0 \\ &+ \int_0^L\int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)\,dt_0\,dx_0 + \int_0^t c^2 A(t_0)\frac{\partial G}{\partial x_0}(x,t;0,t_0)\,dt_0 \end{aligned}}$$

with $G(x,t;x_0,t_0)$ the same given in the excersice (9.2.3).

(9.2.4 (a)) $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \longrightarrow u(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos\left(\frac{n\pi x}{L}\right)$

differentiating term-by-term and substituting in the PDE:

$$\sum_{n=0}^{\infty} a_n''(t)\cos\left(\frac{n\pi x}{L}\right) = c^2\sum_{n=0}^{\infty}(-1)\left(\frac{n\pi}{L}\right)^2 a_n(t)\cos\frac{n\pi x}{L} + \underbrace{\sum_{n=0}^{\infty} q_n(t)\cos\left(\frac{n\pi x}{L}\right)}_{\text{We use Fourier Cosine series for } Q(x,t)}$$

$$\Rightarrow \boxed{a_n''(t) + c^2\left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n(t)},$$ by ~~(8.5.19)~~:

~~with $C_1\cos\frac{n\pi c t}{L} + C_2\sin\frac{n\pi c t}{L} + \frac{L}{n\pi c}\int_0^t q_n(t_0)\sin\frac{n\pi c}{L}(t-t_0)\,dt_0$,~~

~~Proceeding as in excersice (9.2.3) we obtain~~ Solution

~~but now with:~~

~~$G(x,t;x_0,t_0) = \frac{2}{L}\cos\frac{n\pi x_0}{L}\cos\left(\frac{n\pi x}{L}\right)\frac{\sin n\pi c(t-t_0)/L}{n\pi c/L}$~~

The solution will be obtained a few steps ahead.

<u>Method 2</u>: Using the Green's Formula

$$\int_0^L\left(u\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^2 u}{\partial x^2}\right)dx = \left(u\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}v\right)\Big|_0^L \quad \text{eq } (8.4.12)$$

just as ~~in the~~ follow way but with $B(t)=0$. However we keep $B(t)$ in the derivation of the solution doing finally $B(t)=0$.

(See next page for solution).

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9.2.4 c

Let $v = \cos(n\pi x/L)$ in the Greens Formula:

$$\int_0^L \left[u(x,t)\left(-(-1)\left(\frac{n\pi}{L}\right)^2 \cos\left(\frac{n\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L}\right)\frac{\partial^2 u}{\partial x^2}\right] dx = \left[u(x,t)\left(-\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L} - \cos\frac{n\pi x}{L}\frac{\partial u}{\partial x}\right]_0^L$$

Since $\frac{\partial u}{\partial x}(0,t)=0$ and $\frac{\partial u}{\partial x}(L,t)=0 \Rightarrow$ $\qquad = -(-1)^n B(t)$

$$\Rightarrow \int_0^L \cos\left(\frac{n\pi x}{L}\right)\frac{\partial^2 u}{\partial x^2}\,dx = -\left(\frac{n\pi}{L}\right)^2 \int_0^L u(x,t)\cos\left(\frac{n\pi x}{L}\right)dx + (-1)^n B(t) \qquad (*a)$$

By the method of eigenfunction expansion $u = \sum_{n=0}^{\infty} a_n(t)\cos(n\pi x/L)$

Substituing in the PDE:

$$\sum_{n=0}^{\infty} a_n''(t)\cos\left(\frac{n\pi x}{L}\right) = c^2\frac{\partial^2 u}{\partial x^2} + \sum_{n=0}^{\infty} q_n(t)\cos\left(\frac{n\pi x}{L}\right)$$

Multiply by $\cos n'\pi x/L$ and integrate.

$$\sum_{n=0}^{\infty} a_n''(t)\underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\frac{n'\pi x}{L}\,dx}_{\frac{L}{2}(1+\delta_{0n})\delta_{nn'}} = c^2\underbrace{\int_0^L \frac{\partial^2 u}{\partial x^2}\cos\frac{n'\pi x}{L}\,dx}_{\text{use } (*a)} + \sum_{n=0}^{\infty} q_n \underbrace{\int_0^L \cos\frac{n\pi x}{L}\cos\frac{n'\pi x}{L}\,dx}_{\frac{L}{2}(1+\delta_{0n})\delta_{nn'}}$$

$$\Rightarrow a_n''(t) = -c^2\left(\frac{n\pi}{L}\right)^2 \underbrace{\frac{2}{L(1+\delta_{0n})}\int_0^L u(x,t)\cos\left(\frac{n\pi x}{L}\right)dx}_{a_n(t)} + c^2\frac{2}{L(1+\delta_{0n})}(-1)^n B(t) + q_n(t)$$

$$a_n''(t) + c^2\left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n(t) + c^2\frac{2}{L(1+\delta_{0n})}(-1)^n B(t)$$

just the same equation for $a_n(t)$ when $B(t)=0$.

For $n\neq 0$ $\rightarrow$ $a_n''(t) + c^2\left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n(t) + c^2\frac{2}{L}(-1)^n B(t)$, and by (8.5.19)

$$a_n(t) = c_1\cos\frac{n\pi ct}{L} + c_2\sin\frac{n\pi ct}{L} + \frac{L}{n\pi c}\int_0^t \left[q_n + c^2\frac{2}{L}(-1)^n B(t)\right]\sin\frac{n\pi c(t-t_0)}{L}\,dt_0$$

Since $u(x,0) = f(x) = \sum_{n=0}^{\infty} a_n(0)\cos\frac{n\pi x}{L} \rightarrow a_n(0) = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$ (1)

$\frac{\partial u}{\partial x}(x,0) = g(x) = \sum_{n=0}^{\infty} a_n'(0)\cos\frac{n\pi x}{L} \rightarrow a_n'(0) = \frac{2}{L}\int_0^L g(x)\cos\frac{n\pi x}{L}\,dx$ (2)

By other hand: $a_n(0) = c_1 = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

$a_n'(0) = \frac{n\pi c}{L}c_2 = \frac{2}{L}\int_0^L g(x)\cos\frac{n\pi x}{L}\,dx$

$$\Rightarrow a_n(t) = \left(\frac{2}{L}\int_0^L f(x_0)\cos\frac{n\pi x_0}{L}dx_0\right)\cos\frac{n\pi ct}{L} + \left(\frac{2}{n\pi c}\int_0^L g(x_0)\cos\frac{n\pi x_0}{L}dx_0\right)\sin\frac{n\pi ct}{L}$$

$n\neq 0$

$$+ \frac{L}{n\pi c}\int_0^t \left(\frac{2}{L}\int_0^L Q(x_0,t_0)\cos\frac{n\pi x_0}{L}dx_0 + c^2\frac{2}{L}(-1)^n B\right)\cos(\ )\,dt_0$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\left[\qquad\qquad\right]\cos\frac{n\pi x}{L} + a_0(t)$$

with $a_0(t)$ the solution of the equation above when $n=0$.

Susbtituting here $a_{n\neq 0}(t)$ and exchange the order of integration and summ:

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$$u(x,t) = a_0(t) + \int_0^L f(x_0) \sum_{n=1}^{\infty} \frac{2}{L} \cos \frac{n\pi x_0}{L} dx_0 \cos\left(\frac{n\pi x}{L}\right) \cos \frac{n\pi c t}{L} dx_0$$
$$+ \int_0^L g(x_0) \sum_{n=1}^{\infty} \frac{2}{L} \cos \frac{n\pi x_0}{L} dx_0 \cos \frac{n\pi x}{L} \frac{\sin n\pi c t/L}{n\pi c/L} dx_0$$
$$+ \int_0^L \int_0^t Q(x_0,t_0) \sum_{n=1}^{\infty} \frac{2}{L} \cos \frac{n\pi x_0}{L} \cos \frac{n\pi x}{L} \frac{\sin n\pi c(t-t_0)/L}{n\pi c/L} dt_0 dx_0$$
$$+ \int_0^t c^2 B(t_0) \sum_{n=1}^{\infty} (-1)^n \frac{2}{L} \cos \frac{n\pi x}{L} \frac{\sin n\pi c(t-t_0)/L}{n\pi c/L} dt_0 .$$

For n=0: the equation for $a_n(t)$ with $n=0$ is:

$$a_0''(t) = q_0(t) + \frac{c^2}{L} B(t) \Rightarrow \boxed{a_0(t) = \int_0^t \int_0^{\tau} \left[q_0(t_0) + \frac{c^2}{L} B(t_0) \right] dt_0 d\tau + a_0'(0)\, t + a_0(0)}$$

with $a_0(0) = \frac{1}{L}\int_0^L f(x)dx$, $a'(0) = \frac{1}{L}\int_0^L g(x)dx$ according to (1) and (2)

and $q_0(t) = \frac{1}{L}\int_0^L Q(x,t)\,dx$ since $Q(x,t) = \sum_{n=0}^{\infty} q_n(t) \cos\left(\frac{n\pi x}{L}\right)$.

Then the final solution is:

$$\boxed{\begin{aligned} u(x,t) = a_0(t) &+ \int_0^L f(x_0) \frac{\partial G}{\partial t}(x,t;x_0,0)\,dx_0 + \int_0^L g(x_0) G(x,t;x_0,0)\,dx_0 \\ &+ \int_0^L \int_0^t Q(x_0,t_0) G(x,t;x_0,t_0)\,dt_0\,dx_0 + \int_0^t c^2 B(t_0) \frac{\partial G}{\partial x_0}(x,t;L,t_0)\,dt_0 \end{aligned}}$$

with $\boxed{G(x,t;x_0,t_0) = \sum_{n=1}^{\infty} \frac{2}{L} \cos \frac{n\pi x_0}{L} \cos \frac{n\pi x}{L} \frac{\sin n\pi c(t-t_0)/L}{n\pi c/L}}$

Do $B(t)=0$ for obtain answer for excersice 9.2.4(a).

Note that $a_0(0) = \frac{1}{L}\int_0^L f(x)dx \rightarrow$ Mean-value of the initial distribution $u(x,0) = f(x)$

$a_0'(0) = \frac{1}{L}\int_0^L g(x)\,dx \rightarrow$ Mean-value of the initial velocity $\frac{\partial u}{\partial t}(x,0) = g(x)$

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9.3.1 $G(x,x_0) = \begin{cases} -x(L-x_0)/L & x<x_0 \\ -x_0(L-x)/L & x>x_0 \end{cases}$

Let $G(x,x_0) = \sum_{j=1}^{\infty} G_j(x_0)\sin\left(\frac{j\pi x}{L}\right)$ → Fourier sine series.

where $G_j(x_0) = \frac{2}{L}\int_0^L G(x,x_0)\sin\left(\frac{j\pi x}{L}\right)dx$

$$= -\frac{2}{L}\int_0^{x_0}\frac{x(L-x_0)}{L}\sin\left(\frac{j\pi x}{L}\right)dx - \frac{2}{L}\int_{x_0}^{L}\frac{x_0(L-x)}{L}\sin\left(\frac{j\pi x}{L}\right)dx$$

$$\int \overset{u}{x}\,\overbrace{\sin\left(\frac{j\pi x}{L}\right)dx}^{dv} = -\frac{x\cos(j\pi x/L)}{(j\pi/L)} + \int\frac{\cos(j\pi x/L)}{(j\pi x/L)}dx$$

$$= -\frac{L}{j\pi}x\cos(j\pi x/L) + \frac{L^2}{(j\pi)^2}\sin(j\pi x/L)$$

$$\Rightarrow G_j(x_0) = -\frac{2}{L}\frac{(L-x_0)}{L}\frac{L}{j\pi}\left[-x_0\cos\left(\frac{j\pi x_0}{L}\right) + \frac{L}{j\pi}\sin\left(\frac{j\pi x_0}{L}\right)\right] + \frac{2}{L}x_0\frac{L}{j\pi}\left[\cos\left(\frac{j\pi L}{L}\right) - \cos\frac{j\pi x_0}{L}\right]$$

$$+\frac{2}{L}\frac{x_0}{L}\frac{L}{j\pi}\left[-L\cos j\pi + x_0\cos\frac{j\pi x_0}{L} + \frac{L}{j\pi}\left(0 - \sin j\pi x_0/L\right)\right]$$

$$= -\frac{2L}{j^2\pi^2}\sin\frac{j\pi x_0}{L} \Rightarrow \boxed{G(x,x_0) = -\sum_{n=1}^{\infty}\frac{2}{L}\frac{\sin(n\pi x_0/L)\sin(n\pi x/L)}{(n\pi/L)^2}}$$

9.3.2 a) $\frac{d^2u}{dx^2} = f(x)$ → Eq (9.3.1). By directly integration

$$\Rightarrow \frac{du}{dx} = \int_0^x f(\bar{x})d\bar{x} + C_1 \rightarrow u(x) = \int_0^x\left[\int_0^{x_0} f(\bar{x})d\bar{x}\right]dx_0 + C_1x + C_2$$

b) $u(0) = 0 = C_2$ $\quad u(L) = 0 = \int_0^L\left[\int_0^{x_0} f(\bar{x})d\bar{x}\right]dx_0 + C_1L \rightarrow C_1 = -\frac{1}{L}\int_0^L\left[\int_0^{x_0} f(\bar{x})d\bar{x}\right]dx_0$

$$\int_0^x\underbrace{\left[\int_0^{x_0} f(\bar{x})d\bar{x}\right]}_{u}\underbrace{dx_0}_{dv} = \left[x_0\int_0^{x_0} f(\bar{x})d\bar{x}\right]_{x_0=0}^{x_0=x} - \int_0^x x_0 f(x_0)dx_0$$ Using this in C_1 too,

$$\Rightarrow u(x) = x\int_0^x f(\bar{x})d\bar{x} - \int_0^x x_0 f(x_0)dx_0 - \underbrace{\frac{x}{L}\left(L\int_0^L f(\bar{x})d\bar{x} - \int_0^L x_0 f(x_0)dx_0\right)}$$

$$= \quad '' \quad - \quad '' \quad - \frac{x}{L}\left(L\int_0 f(\bar{x})d\bar{x} + L\int_x^L f(\bar{x})d\bar{x} - \int_0^x x_0 f(x_0)dx_0 - \int_0^L x_0 f(x_0)dx_0\right)$$

$$= -x\int_x^L\left(1-\frac{x_0}{L}\right)f(x_0)dx_0 - \int_0^L x_0\left(1-\frac{x}{L}\right)f(x_0)dx_0$$

OR $\boxed{u(x) = \int_0^L f(x_0)G(x,x_0)dx_0}$ with $\boxed{G(x,x_0) = \begin{cases} -x\left(1-\frac{x_0}{L}\right) & \text{if } x<x_0 \\ -x_0\left(1-\frac{x}{L}\right) & \text{if } x>x_0 \end{cases}}$

c)

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9.3.3 $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad u(0,L)=0,\ \frac{\partial u}{\partial x}(L,t)=0 \rightarrow u=\sum_{n=1} a_n(t)\sin\left(\frac{2n-1}{2L}\pi x\right)$

a)

$$\Rightarrow \sum_{n=1}\frac{da_n}{dt}\sin\left(\frac{2n-1}{2}\pi x\right) = -K\left(\frac{2n-1}{2L}\pi\right)^2\sum_{n=1}\left[a_n(t)+q_n(t)\right]\sin\left(\frac{2n-1}{2L}\pi x\right) \qquad Q=\sum q_n\sin(\ )$$

$$\Rightarrow \frac{da_n}{dt} + K\left(\frac{2n-1}{2L}\pi\right)^2 a_n(t) = q_n(t) \quad \text{Multiplying by } e^{K((2n-1)\pi/L)^2 t}$$

$$\frac{d}{dt}\left(a_n(t)e^{K[(2n-1)\pi/L]^2 t}\right) = q_n(t)e^{K[(2n-1)\pi/L]^2 t}$$

$$a_n(t)e^{K[(2n-1)\pi/L]^2 t} = \int_0^t q_n(t_0)e^{K[(2n-1)\pi/L]^2 t_0}dt_0 + a_n(0)$$

$$a_n(t) = a_n(0)e^{-K[(2n-1)\pi/L]^2 t} + e^{-K[(2n-1)\pi/L]^2 t}\int_0^t q_n(t_0)e^{K[(2n-1)\pi/L]^2 t_0}dt_0$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\left[\qquad\qquad\right]\sin\left(\frac{2n-1}{2L}\pi x\right) \qquad \text{with } a_n(0) = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{2n-1}{2L}\pi x\right)dx$$

$$= \int_0^L g(x_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{(2n-1)\pi x_0}{2L}\sin\frac{(2n-1)\pi x}{2L}e^{-K[(2n-1)\pi/L]^2 t}\right)dx_0$$

$$+\int_0^t\left(\sum_{n=1}^{\infty} q_n(t_0)\sin\frac{(2n-1)\pi x}{2L}e^{-K[(2n-1)\pi/L]^2(t-t_0)}\right)dt_0$$

$$q_n(t_0) = \frac{2}{L}\int_0^L Q(x,t)\sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

b)

$$\Rightarrow u(x,t) = \int_0^L g(x_0)G(x,t;x_0,0)\,dx_0 + \int_0^t\int_0^L Q(x_0,t_0)G(x,t;x_0,t_0)\,dx_0\,dt_0$$

$$\text{with } G(x,t;x_0,t_0) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{(2n-1)\pi x_0}{2L}\sin\frac{(2n-1)\pi x}{2L}e^{-K[(2n-1)\pi/L]^2(t-t_0)}$$

c) As $t\rightarrow\infty$, $G(x,t;x_0,0)\rightarrow 0$ such that the effect of the initial condition $u(x,0)=g(x)$ vanishes at $t\rightarrow\infty$. However, the steady source is still important as $t\rightarrow\infty$ since:

$$\int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)\,dt_0 \propto \int_0^t e^{-K[(2n-1)\pi/L]^2(t-t_0)}dt_0 = \left.\frac{e^{-K[\ ]^2(t-t_0)}}{K[\ ]^2}\right|_{t_0=0}^{t_0=t}$$

$$\frac{1-e^{-K[\ ]^2}}{K[\ ]^2} = \frac{1}{K[\ ]^2} \quad \text{for } t\rightarrow\infty$$

Thus, as $t\rightarrow\infty$ $\quad Q(x_0,t_0)=Q(x_0)$

$$u(x,t)\rightarrow u(x) = \int_0^L Q(x_0)\sum_{n=1}^{\infty}\frac{2\sin[(2n-1)\pi x_0/2L]\sin[(2n-1)\pi x/2L]}{LK[(2n-1)\pi/2L]^2}dx_0$$

$$u(x) = \int_0^L Q(x_0)G(x,x_0)\,dx_0$$

$$\text{with } \boxed{G(x,x_0) = \sum_{n=1}^{\infty}\frac{2}{L}\frac{\sin[(2n-1)\pi x_0/2L]\sin[(2n-1)\pi x/2L]}{K[(2n-1)\pi/2L]^2}dx_0}$$

Here we obtained the steady-state temperature distribution $u(x)$ by taking the limit as $t\rightarrow\infty$ of the time-dependent problem with a steady source $Q(x)$. $G(x,x_0)$ is the influence for the steady-state problem.

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9.3.4 a) $\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx = f(x_0)$

Let $f(x)=1 \Rightarrow \int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx = \int_{-\infty}^{\infty}\delta(x-x_0)\,dx = f(x_0)$

but $f(x)=1$ for all x including $x=x_0 \Rightarrow \boxed{\int_{-\infty}^{\infty}\delta(x-x_0)\,dx = 1}$

b) $\int_{-\infty}^{x}\delta(x_0-x_i)\,dx_0 \rightarrow$ We can analize two cases for $-\infty < x_0 \leq x$

- If $x<x_i \rightarrow x_0 \neq x_i$ for all $x_0 \in (-\infty, x) \Rightarrow \delta(x_0-x_i)=0$ by definition (9.3.26).

$$\Rightarrow \int_{-\infty}^{\infty}\delta(x_0-x_i)\,dx_0 = 0$$

- If $x>x_i \rightarrow \int_{-\infty}^{x}\delta(x_0-x_i)\,dx_0 = \int_{-\infty}^{x+\epsilon}\delta(x_0-x_i)\,dx + \int_{x_i+\epsilon}^{x}\underbrace{\delta(x_0-x_i)}_{0}\,dx_0$

with $x_i<\epsilon<x$. Now $x_0 \neq x_i$ for all $x_0 \in (x_i+\epsilon, x)$

Similarly $\int_{x_i+\epsilon}^{\infty}\delta(x_0-x_i)\,dx = 0 = \int_{x_i+\epsilon}^{x}\delta(x_0-x_i)\,dx_0$ (P.E.P)

$$\Rightarrow \int_{-\infty}^{x}\delta(x_0-x_i)\,dx_0 = \int_{-\infty}^{x+\epsilon}\delta(x_0-x_i)\,dx + \int_{x_i+\epsilon}^{\infty}\delta(x_0-x_i)\,dx_0$$

$$= \int_{-\infty}^{\infty}\delta(x_0-x_i)\,dx = 1 \text{ by } (9.3.29)$$

$$\Rightarrow \boxed{\int_{-\infty}^{x}\delta(x_0-x_i)\,dx_0 = \begin{cases} 0 & \text{if } x<x_i \\ 1 & \text{if } x>x_i \end{cases}}$$

c) $\int_{-\infty}^{\infty} f(x+s)\delta(s)\,ds = f(x) = \int_{-\infty}^{\infty} f(x-s)\delta(s)\,ds$

$x_0 = x+s \quad dx_0 = ds$ $\qquad$ $x_0 = x-s \quad dx_0 = -ds$

$s=\pm\infty \rightarrow x_0 = \pm\infty$ $\qquad$ $s=\pm\infty \rightarrow x_0 = \mp\infty$

$$\int_{-\infty}^{\infty} f(x_0)\delta(x_0-x)\,dx_0 = \int_{\infty}^{-\infty} f(x_0)\delta(x-x_0)(-dx_0)$$

OR $\int_{-\infty}^{\infty} f(x_0)\delta(x_0-x)\,dx_0 = \int_{-\infty}^{\infty} f(x_0)\delta(x-x_0)\,dx \Rightarrow \boxed{\delta(x_0-x)=\delta(x-x_0)}$

d) Let $y = c(x-x_0)$, $dy = c\,dx$ $\Rightarrow \int \delta[c(x-x_0)]\,dx = \int \delta(y)\frac{dy}{c}$

- If $c>0 \rightarrow x=\pm\infty \rightarrow y=\pm\infty \Rightarrow \int_{-\infty}^{\infty}\delta[c(x-x_0)]\,dx = \frac{1}{c}\int_{-\infty}^{\infty}\delta(y)\,dy = \frac{1}{c}$

- If $c<0 \rightarrow x=\pm\infty \rightarrow y=\mp\infty \Rightarrow \int_{-\infty}^{\infty}\delta[c(x-x_0)]\,dx = \frac{1}{c}\int_{+\infty}^{-\infty}\delta(y)\,dy$

$$= -\frac{1}{c}\int_{-\infty}^{\infty}\delta(y)\,dy = -\frac{1}{c}$$

$$\frac{1}{|c|} = \begin{cases} \frac{1}{c} & \text{if } c>0 \\ -\frac{1}{c} & \text{if } c<0 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta[c(x-x_0)]\,dx = \frac{1}{|c|} = \int_{-\infty}^{\infty} \frac{1}{|c|}\delta(x-x_0)\,dx$$

$$\Rightarrow \boxed{\delta[c(x-x_0)] = \frac{1}{|c|}\delta(x-x_0)}$$

9.3.5 a) $\frac{d^2u}{dx^2} = f(x)$ integrating between L and x by convenience:

$\frac{du}{dx} = \int_L^x f(\bar{x})\,d\bar{x} + C_1$ $\quad \frac{du}{dx}(L) = 0 = C_1$. Integrating again

$u(x) = \int_0^x \left(\int_L^{x_0} f(\bar{x})\,d\bar{x}\right) dx_0 + C_2$ $\quad u(0) = 0 = C_2$.

Now integrating by parts $\quad dw = dx_0 \quad v = \int_L^{x_0} f(\bar{x})\,d\bar{x}$

$w = x_0 \quad dv = f(x_0)\,dx_0$

$$\Rightarrow \boxed{u(x) = x\int_L^x f(\bar{x})\,d\bar{x} - \int_0^x x_0 f(x_0)\,dx_0}$$

b) Let u_1 be a homogeneous solution satisfying $u(0)=0 \rightarrow u_1 = x$, while let u_2 be an independent one satisfying $\frac{du}{dx}(L) = 0 \rightarrow u_2 = 1$.

According to (9.3.8), $p=1$, then $C = p\left(u_1\frac{du_2}{dx} - u_2\frac{du_1}{dx}\right) = -1$

$\Rightarrow$ from eqs (9.3.10-11) $\rightarrow \frac{dv_1}{dx} = +f$, $\frac{dv_2}{dx} = -xf$, ~~then from~~

$\Rightarrow v_1 = \int_0^x f(x_0)\,dx_0 + C_1$ and $v_2 = \int_0^x -x_0 f(x_0)\,dx_0 + C_2$ and from

(9.3.9): $u(x) = x\left(\int_0^x f(x_0)\,dx_0 + C_1\right) - \int_0^x x_0 f(x_0)\,dx_0 + C_2$

$u(0) = 0 = C_2$

(P.S.E.P) $\frac{du}{dx}(L) = \left(\int_0^L f(x_0)\,dx_0 + C_1\right) + \cancel{L f(L)} - \cancel{x f(x)}\Big|_{x=L} = 0 \rightarrow C_1 = -\int_0^L f(x_0)\,dx_0$

$$\Rightarrow u(x) = x\int_L^x f(x_0)\,dx_0 - \int_0^x x_0 f(x_0)\,dx_0.$$

c) Write

$$u(x) = \int_0^L f(x_0)G(x,x_0)\,dx_0 = -\int_0^x x_0 f(x_0)\,dx_0 + x\int_L^x f(x_0)\,dx_0$$

$$\Rightarrow \boxed{G(x,x_0) = \begin{cases} -x_0 & x_0 < x \\ -x & x_0 > x \end{cases}}$$

d) $u(x) = \sum_{n=1}^{\infty} a_n \sin\left[\frac{(2n-1)\pi x}{2L}\right] \Rightarrow -\sum_{n=1}^{\infty}\left(\frac{(2n-1)\pi}{2L}\right)^2 a_n \sin\left[\frac{(2n-1)\pi x}{2L}\right] = f(x)$

Multiplying by $\sin\left[\frac{(2n'-1)\pi x}{2L}\right]$ and integrating:

$$-\sum_{n=1}^{\infty}\left(\frac{(2n-1)\pi}{2L}\right)^2 a_n \underbrace{\int_0^L \sin\left[\frac{(2n-1)\pi x}{2L}\right]\sin\left[\frac{(2n'-1)\pi x}{2L}\right]dx}_{(L/2)\delta_{nn'}} = \int_0^L f(x)\sin\left[\frac{(2n'-1)\pi x}{2L}\right]dx$$

$$\Rightarrow -\left[\frac{(2n'-1)\pi}{2L}\right]^2 a_{n'}\frac{L}{2} = \int_0^L f(x)\sin\left[\frac{(2n'-1)\pi x}{2L}\right]dx \Rightarrow$$

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$$a_n = \frac{-2/L}{\left[\frac{(2n-1)\pi}{2L}\right]^2}\int_0^L f(x)\sin\left[\frac{(2n-1)\pi x}{2L}\right]dx$$, substituting in $u(x)$:

$$\Rightarrow u(x) = \sum_{n=1}^{\infty}\left(\frac{-2/L}{[\ \ ]^2}\int_0^L f(x_0)\sin\left[\frac{(2n-1)\pi x_0}{2L}\right]\right)\sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

$$= \int_0^L f(x_0)\left(-\sum_{n=1}^{\infty}\frac{2}{L}\,\frac{\sin[(2n-1)\pi x_0/2L]\sin[(2n-1)\pi x/2L]}{[(2n-1)\pi/2L]^2}\right)$$

OR $u(x) = \int_0^L f(x_0)\,G(x,x_0)\,dx_0$ ~~with~~

with $$\boxed{G(x,x_0) = -\frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi x_0/2L]\sin[(2n-1)\pi x/2L]}{[(2n-1)\pi/2L]^2}}$$

9.3.6 For $x \neq x_0 \rightarrow \frac{d^2G}{dx^2} = 0 \rightarrow G(x,x_0) = \begin{cases} a+bx & x<x_0 \\ c+dx & x>x_0 \end{cases}$

a)

Since $L > x_0 \rightarrow \frac{dG}{dx}(L,x_0) = 0 = d$

$0 < x_0 \rightarrow G(0,x_0) = 0 = a$ $\Rightarrow G(x,x_0) = \begin{cases} bx & x<x_0 \\ c & x>x_0 \end{cases}$

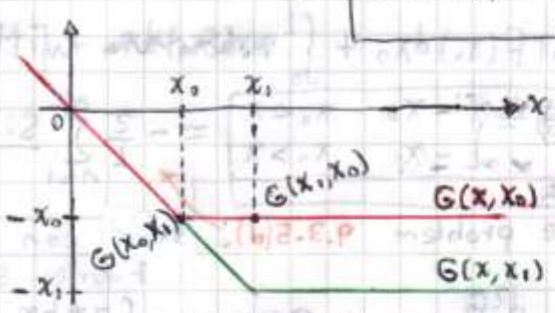
At $x = x_0$, by continuity $G(x_0^-,x_0) = G(x_0^+,x_0) \rightarrow bx_0 = c$

Now $\int_{x_0-\epsilon}^{x_0+\epsilon}\frac{d^2G}{dx^2}dx = \underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon}\delta(x-x_0)\,dx}$ $\rightarrow$ Discontinuity for the derivative

at $\lim \epsilon \rightarrow 0 \Rightarrow \frac{dG}{dx}\Big|_{x_0^-}^{x_0^+} = 1 \Rightarrow 0 - b = 1 \Rightarrow b = -1$, $c = x_0$

$$\Rightarrow \boxed{G(x,x_0) = \begin{cases} -x & x<x_0 \\ -x_0 & x>x_0 \end{cases}}$$

b)



c) That is the same answer as in Exercise 9.3.5 since $G(x,x_0) = G(x_0,x)$

9.3.7 a) $\frac{du}{dx} = \int_L^x f(\bar{x})\,d\bar{x} + C_1$ $\qquad \frac{du}{dx}(L) = C_1 = -h\,u(L)$

$u(0) = 0 \rightarrow C_2 = 0$

$u(x) = \int_0^x\left(\int_L^{x_0} f(\bar{x})\,d\bar{x}\right)dx_0 + C_1 x + C_2 = \int_0^x\left(\int_L^{x_0} f(\bar{x})\,d\bar{x}\right)dx_0 - h\,u(L)\cdot x + C_2$

$u(L) = \int_0^L\left(\int_L^{x_0} f(\bar{x})\,d\bar{x}\right)dx_0 + C_1 L = \int_0^L\left(\int_L^{x_0} f(\bar{x})\,d\bar{x}\right)dx_0 - h\,u(L)\cdot L$

$\Rightarrow (1+hL)\,u(L) = \int_0^L\left(\int_L^{x_0} f(\bar{x})\,d\bar{x}\right)dx_0 \rightarrow u(L) = \frac{1}{(1+hL)}\int_0^L(\qquad)\,dx_0$

$\Rightarrow u(x) = \int_0^x\left(\int_L^{x_0} f(\bar{x})\,d\bar{x}\right)dx_0 - \int_0^L\left(\int_L^{x_0} f(\bar{x})\,d\bar{x}\right)dx_0\,\frac{hx}{(1+hL)}$

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Integrating by parts $\quad dw = dx_0 \quad v = \int_L^{x_0} f(\bar{x})\,d\bar{x}$

$\quad w = x_0 \quad dv = f(x_0)\,dx_0$

From exercise 9.3.5(a): $\int_0^x \left(\int_L^{x_0} f(\bar{x})\,d\bar{x}\right) dx_0 = x\int_L^x f(\bar{x})\,d\bar{x} - \int_0^x x_0 f(x_0)\,dx_0$

$$\Rightarrow \boxed{u(x) = \left(x\int_L^x f(\bar{x})\,d\bar{x} - \int_0^x x_0 f(x_0)\,dx_0\right) - \frac{hx}{1+hL}\left(0\cancel{\int_L^L f(\bar{x})\,d\bar{x}}^{\,0} - \int_0^L x_0 f(x_0)\,dx_0\right)}$$

b) Now be $u_1(x) = 1$ and $u_2(x) = -hx$ such that $\frac{du_2}{dx}(L) + h u_1(L) = 0$

$$\Rightarrow C = p\left(u_1\frac{du_2}{dx} - u_2\frac{du_1}{dx}\right) = -h \Rightarrow \frac{dv_1}{dx} = -\frac{f u_2}{C} = -xf$$

$$\frac{dv_2}{dx} = \frac{f u_1}{C} = -\frac{f}{h}$$

$$\Rightarrow v_1 = \int_0^x -x_0 f(x_0)\,dx_0 + C_1 \quad \text{and} \quad v_2 = -\frac{1}{h}\int_0^x f(x_0)\,dx_0 + C_2$$

$$u = v_1\cdot u_1 + v_2\cdot u_2 = -\int_0^x x_0 f(x_0)\,dx_0 + C_1 + x\int_0^x f(x_0)\,dx_0 + C_2 x$$

just the solution of exercice 9.3.5(a)

$$\frac{du}{dL}(L) = -\cancel{Lf(L)} + \left(\int_0^L f(x_0)\,dx_0 + \cancel{Lf(L)}\right) + C_2 = -hu(L) \rightarrow C_2 = -hu(L) - \int_0^L f(x_0)\,dx_0$$

$$\Rightarrow u(x) = -\int_0^x x_0 f(x_0)\,dx_0 + C_1 + x\int_0^x f(x_0)\,dx_0 - xhu(L) - x\int_0^L f(x_0)\,dx_0$$

$$= -\int_0^x x_0 f(x_0)\,dx_0 + C_1 - x\int_x^L f(x_0)\,dx_0 - xhu(L) \qquad u(0) = 0 \rightarrow C_1 = 0$$

$$u(L) = -\int_0^L x_0 f(x_0)\,dx_0 + 0 + \cancel{L\int_L^L f(x_0)\,dx_0}^{\,0} - Lhu(L) \rightarrow \quad u(L) = \frac{-1}{1+hL}\int_0^L x_0 f(x_0)\,dx_0$$

$$\Rightarrow \boxed{u(x) = -\int_0^x x_0 f(x_0)\,dx_0 - x\int_x^L f(x_0)\,dx_0 + \frac{hx}{(1+hL)}\int_0^L x_0 f(x_0)\,dx_0}$$

↳ just the same answer of part (a)

c) $u(x) = \int_0^L G(x, x_0) f(x_0)\,dx_0$ with

$$\boxed{G(x,x_0) = \begin{cases} \frac{h}{1+hL}x_0 x - x_0 & x_0 < x \\ \frac{h}{1+hL}x x_0 - x & x_0 > x \end{cases}} = -\frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x_0/2L \,\sin(2n-1)\pi x/2L}{[(2n-1)\pi/2L]^2} + \frac{hxx_0}{(1+hL)}$$

d) According to the problem 9.3.5(d): ↳ You can show this finding the Fourier sine series of the left part.

9.3.8 For $x \neq x_0 \rightarrow \frac{d^2G}{dx^2} = 0 \rightarrow G(x,x_0) = \begin{cases} a + bx & x < x_0 \\ c + dx & x > x_0 \end{cases} \qquad G(0,x_0) = 0 \rightarrow a = 0$

Since $L > x_0 \rightarrow \frac{dG}{dx}(L) + hG(L) = 0 = d + h(c + dL) = d(1+hL) + hc$

$$\Rightarrow c = -\frac{d}{h}(1+hL)$$

At $x = x_0$, by continuity $G(x_0^-, x_0) = G(x_0^+, x_0) \rightarrow \cancel{a}^{\,0} + bx_0 = c + dx_0 \rightarrow c = (b-d)x_0$

Now $\int_{x_0-\epsilon}^{x_0+\epsilon}\frac{d^2G}{dx^2}\,dx = \int_{x_0-\epsilon}^{x_0+\epsilon}\delta(x-x_0)\,dx \xrightarrow[\epsilon\to 0]{} \left.\frac{dG}{dx}\right|_{x_0^-}^{x_0^+} = 1 \Rightarrow d - b = 1$

$c = -x_0$

$b = d - 1$

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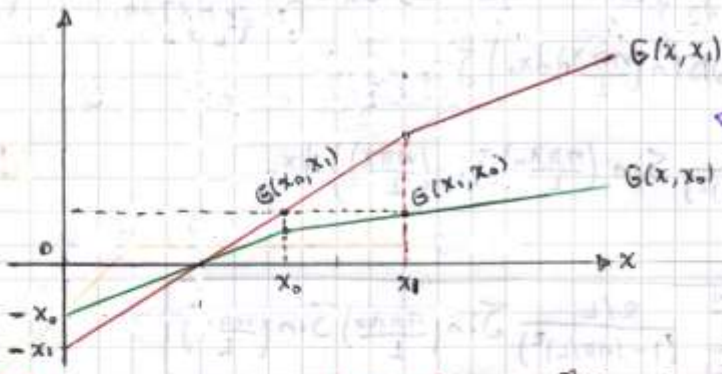
$$C = -x_0 = -\frac{d}{h}(1+hL) \rightarrow d = \frac{h x_0}{(1+hL)} \quad \text{and} \quad b = d-1 = \frac{h x_0}{1+hL} - 1$$

$$\boxed{G(x,x_0) = \begin{cases} \left(\frac{h x_0}{1+hL} - 1\right) x & x < x_0 \\ -x_0 + \frac{h x_0}{1+hL} x & x > x_0 \end{cases}}$$

Note that G satisfy the Simmetry $G(x,x_0) = G(x_0,x)$



Graph done with Mathematica choosing $\frac{h}{1+hL} = 1$, $x_0 = 1.5$, $x_1 = 2.5$

c) From excercise (9.3.6), we obtain the same answer.

$$\Rightarrow \boxed{G(x,x_0) = -\frac{2}{L}\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi x_0/2L]\,\sin[(2n-1)\pi x/2L]}{[(2n-1)\pi/2L]^2} + \frac{h}{1+hL} x_0 x}$$

9.3.9 Let u_1 be a homogeneous solution satisfying $u(0)=0 \rightarrow u_1 = \sin x$ while u_2 satisfies $u(L)=0 \rightarrow u_2 = \sin(x-L)$. Note thes are independent if $L \neq n\pi$.

$$C = \cancel{R}^{\,1}\left(u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx}\right) = \sin x \cos(x-L) - \sin(x-L)\cos x = \sin L \neq 0.$$

$$\frac{dv_1}{dx} = -\frac{f u_2}{C} = -\frac{f \sin(x-L)}{\sin L} \quad \text{and} \quad \frac{dv_2}{dx} = \frac{f u_1}{C} = \frac{f \sin x}{\sin L}$$

$$v_1 = -\frac{1}{\sin L}\int_0^x f(x_0)\sin(x_0-L)\,dx_0 + C_1 \qquad v_2 = \frac{1}{\sin L}\int_0^x f(x_0)\sin x_0\,dx_0 + C_2$$

$$u = v_1 u_1 + v_2 \cdot u_2 = -\frac{\sin x}{\sin L}\int_0^x f(x_0)\sin(x_0-L)\,dx_0 + C_1 \sin x + \frac{\sin(x-L)}{\sin L}\int_0^x f(x_0)\sin x_0\,dx_0 + C_2 \sin(x-L)$$

$$u(0) = 0 = C_2 \sin(-L) \rightarrow C_2 = 0$$

$$u(L) = 0 = -\int_0^L f(x_0)\sin(x_0-L)\,dx_0 + C_1 \sin L \rightarrow C_1 = \frac{1}{\sin L}\int_0^L f(x_0)\sin(x_0-L)\,dx_0$$

$$\rightarrow \boxed{u(x) = -\frac{\sin x}{\sin L}\int_L^x f(x_0)\sin(x_0-L)\,dx_0 + \frac{\sin(x-L)}{\sin L}\int_0^x f(x_0)\sin x_0\,dx_0}$$

b) $u(x) = \int_0^L f(x_0)\,G(x,x_0)\,dx_0$ with $\boxed{G(x,x_0) = \begin{cases} \sin x \sin(x_0-L)/\sin L & x < x_0 \\ \sin x_0 \sin(x-L)/\sin L & x > x_0 \end{cases}}$

9.3.10 $u(0) = u(L) = 0 \rightarrow u(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow \sum_{n=1}^{\infty}\left[-\left(\frac{n\pi}{L}\right)^2 + 1\right] a_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

Multiplying by $\sin(n\pi x/L)$ and integrating.

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$$\Rightarrow \sum_{n=1}^{\infty}\left[-\left(\frac{n\pi}{L}\right)^2+1\right]a_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}} = \int_0^L f(x)\sin\left(\frac{n'\pi x}{L}\right)dx$$

$$\left[-\left(\frac{n'\pi}{L}\right)^2+1\right]a_{n'}\,\frac{L}{2} = \int_0^L f(x)\sin\frac{n'\pi x}{L} \rightarrow a_n = \frac{2/L}{\left[1-\left(\frac{n\pi}{L}\right)^2\right]}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow u(x) = \sum_{n=1}^{\infty}\left(\frac{2/L}{(1-(n\pi/L)^2)}\int_0^L f(x_0)\sin\left(\frac{n\pi x_0}{L}\right)dx_0\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$= \int_0^L f(x_0)\left(\sum_{n=1}^{\infty}\frac{2/L}{(1-(n\pi/L)^2)}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right)dx_0$$

$$u(x) = \int_0^L f(x_0)\,G(x,x_0)\,dx_0$$

$$\boxed{G(x,x_0) = \sum_{n=1}^{\infty}\frac{2/L}{(1-(n\pi/L)^2)}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}$$

9.3.11 a) For $x \neq x_0 \rightarrow \frac{d^2G}{dx^2}+G=0 \rightarrow G(x,x_0) = a\sin(x+\phi)$

If $x<x_0 \rightarrow G(0)=0 \rightarrow \phi=0 \qquad G(x,x_0) = a\sin x$

If $x>x_0 \rightarrow G(L)=0 \rightarrow \phi=-L \qquad G(x,x_0) = b\sin(x-L)$

at $x=x_0 \rightarrow G(x_0^-,x_0) = G(x_0^+,x_0) \rightarrow a\sin x_0 = b\sin(x-L)$ (1)

$$\int_{x_0-\epsilon}^{x_0+\epsilon}\frac{d^2G}{dx^2}dx + \underbrace{\cancel{\int_{x_0-\epsilon}^{x_0+\epsilon}G(x,x_0)\,dx}}_{0 \text{ when } \epsilon\to 0} = \underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon}\delta(x-x_0)\,dx}_{1}$$

For $\epsilon\to 0 \Rightarrow \left.\frac{dG}{dx}\right|_{x_0^-}^{x_0^+} = 1 \rightarrow a\cos x_0 - b\cos(x_0-L) = 1$ (2)

Equations (1) - (2) may be ~~solved by~~ arranged as:

$$\begin{bmatrix}\sin x_0 & -\sin(x_0-L)\\ \cos x_0 & -\cos(x_0-L)\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}$$

In order for the matrix equation to be solvable, $\det[\,] \neq 0$

$$\Rightarrow \sin x_0\cos(x_0-L) - \cos x_0\sin(x_0-L) =$$

$$\sin x_0\left[\cancel{\cos x_0\cos(-L)} - \sin x_0\sin(-L)\right] - \cos x_0\left[\cancel{\sin x_0\cos(-L)} + \sin(-L)\cos x_0\right] =$$

$$-\sin^2 x_0\sin(-L) - \cos^2 x_0\sin(-L) = \sin L \neq 0$$

which is nonzero when $L \neq n\pi$.

From (1) $a = b\,\frac{\sin(x_0-L)}{\sin x_0} \Rightarrow$ (2): $b\left[\frac{\sin(x_0-L)}{\sin x_0}\cos x_0 - \cos(x_0-L)\right] = 1$

OR $b\left[\frac{\sin(x_0-L)\cos x_0 - \cos(x_0-L)\sin x_0}{\sin x_0}\right] = b\left[\frac{\sin(\cancel{x_0}-(\cancel{x_0}-L))}{\sin x_0}\right] = 1$

$$\Rightarrow b = \frac{\sin x_0}{\sin L} \quad \text{and} \quad a = \frac{\sin(x_0-L)}{\sin L} \Rightarrow \boxed{G(x,x_0) = \begin{cases}\frac{\sin(x_0-L)}{\sin L}\sin x & x<x_0\\ \frac{\sin x_0}{\sin L}\sin(x-L) & x>x_0\end{cases}}$$

Method 2: For $x \neq x_0 \rightarrow \frac{d^2G}{dx^2} + G = 0 \rightarrow G(x,x_0) = a\sin(x+\phi)$

- If $x=0 \Rightarrow x<x_0$, $G(0,x_0)=0 \rightarrow G \propto \sin x_<$
- If $x=L \Rightarrow x>x_0$, $G(L,x_0)=0 \rightarrow G \propto \sin(x_>-L)$

$\Rightarrow G = C\sin x_< \sin(x_>-L) \rightarrow$ This election garantee the simmetry and the continuity of $G(x,x_0)$ at $x=x_0$

OR
$$\boxed{G(x,x_0) = \begin{cases} C\sin x \sin(x_0-L) & x<x_0 \\ C\sin x_0 \sin(x-L) & x>x_0 \end{cases}}$$

Now, the jump condition on the derivative determines C:

$$\underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{d^2G}{dx^2}dx}_{\frac{dG}{dx}\Big|_{x_0^-}^{x_0^+} \text{ when } \epsilon\to 0} + \underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon} G(x,x_0)dx}_{0 \text{ by continuity}} = \underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x-x_0)dx}_{1}$$

$$\Rightarrow C\,(\underbrace{\cos x \sin(x_0-L) - \sin x_0 \cos(x-L)}_{\sin L}) = 1 \qquad \Rightarrow \boxed{C = \frac{1}{\sin L}}$$

b) $$G(x_0,x) = \begin{cases} C\sin x_0 \sin(x-L) & x_0<x \\ C\sin x \sin(x_0-L) & x_0>x \end{cases} = G(x,x_0)$$

9.3.12 a) We will use the same Green's function as we did previously in exercise (9.3.6):

$$\frac{d^2G}{dx^2} = \delta(x-x_0), \quad G(0,x_0)=0, \quad \frac{dG}{dx}(L,x_0)=0 \rightarrow \boxed{G(x,x_0) = \begin{cases} -x & x<x_0 \\ -x_0 & x>x_0 \end{cases}}$$

Note that the formula (9.3.51) does not hold due to the insulated left end condition that $G(x,x_0)$ satisfies. Recall the Green's identity

$$\int_0^L \left(u\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^2 u}{\partial x^2}\right)dx = \left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right)\Big|_0^L \qquad \text{eq (8.4.12)}$$

Let $v = G(x,x_0)$, then

$$\int_0^L \left(u\frac{d^2G}{dx^2} - G(x,x_0)\frac{d^2u}{dx^2}\right)dx = u(x)\frac{dG}{dx}\Big|_0^L - G(x,x_0)\frac{du}{dx}\Big|_0^L$$

Reversing the roles of x and x_0 and using the symmetry of $G(x,x_0)$.

$$\int_0^L \left(u(x_0)\delta(x-x_0) - G(x,x_0)f(x_0)\right)dx_0 = u(L)\cancel{\frac{dG}{dx_0}(x,L)}^{0} - u(0)\frac{dG}{dx_0}(x,0) - G(x,L)\frac{du}{dx_0}(L) + \cancel{G(x,0)}^{0}\frac{du}{dx_0}(0)$$

$$u(x) - \int_0^L G(x,x_0)f(x_0)dx_0 = -A(-1) - (-x)B$$

$$\boxed{u(x) = \int_0^L G(x,x_0)f(x_0)dx_0 + Bx + A}$$

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b) We will use the same Green's function as we did previously with the exercise (9.3.11):

$$\frac{d^2G}{dx^2}+G=\delta(x-x_0) \qquad G(0,x_0)=0,\ G(L,x_0)=0 \longrightarrow \boxed{G(x,x_0)=\begin{cases}\dfrac{\sin(x_0-L)}{\sin L}\sin x & x<x_0\\[2mm] \dfrac{\sin x_0}{\sin L}\sin(x-L) & x>x_0\end{cases}}$$

Again begin with Green's identity (8.4.12) with $v=G(x,x_0)$:

Doing $x\to x_0$ in the integral.

$$\int_0^L\left(u(x)\frac{d^2G}{dx^2}-G(x,x_0)\frac{d^2u}{dx^2}\right)dx=u(x)\frac{dG}{dx}\Big|_0^L-G(x,x_0)\frac{du}{dx}\Big|_0^L$$

$$\int_0^L\left[u(x_0)\left(\delta(x_0-x)-\cancel{G}\right)-G(x,x_0)\left(f(x_0)-\cancel{u(x_0)}\right)\right]dx_0=u(L)\frac{dG}{dx_0}(x,L)-u(0)\frac{dG}{dx_0}(x,0)-\cancelto{0}{G(x,L)}\frac{du}{dx_0}(L)+\cancelto{0}{G(x,0)}\frac{du}{dx_0}(0)$$

$$u(x)-\int_0^L G(x,x_0)f(x_0)\,dx_0=B\frac{\sin x}{\sin L}-A\frac{\sin(x-L)}{\sin L}$$

$$\boxed{u(x)=\int_0^L f(x_0)G(x,x_0)\,dx_0+\frac{B\sin x}{\sin L}-\frac{A\sin(x-L)}{\sin L}}$$

c) Using the Green's function of the PDE for $G(x,x_0)$ found in exercise (9.3.8):

$$\frac{d^2G}{dx^2}=\delta(x-x_0) \quad \text{with} \quad \frac{dG}{dx^2}(L,x_0)+hG(L,x_0)=0 \longrightarrow \boxed{G=\begin{cases}\dfrac{h}{1+hL}x_0x-x & x<x_0\\[2mm] \dfrac{h}{1+hL}xx_0-x_0 & x>x_0\end{cases}}$$

Begining with Green's identity (8.4.12) with $v=G(x,x_0)$:

$$\int_0^L\left(u(x_0)\frac{d^2G}{dx_0^2}-G(x,x_0)\frac{d^2u}{dx_0^2}\right)dx_0=u(x_0)\frac{dG(x,x_0)}{dx_0}\Big|_0^L-G(x,x_0)\frac{du}{dx_0}\Big|_0^L$$

$$\int_0^L\left(u(x_0)\delta(x_0-x)-G(x,x_0)f(x)\right)dx_0=u(L)\frac{dG(x,L)}{dx_0}-u(0)\frac{dG(x,0)}{dx_0}-G(x,L)\frac{du}{dx_0}(L)+\cancelto{0}{G(x,0)}\frac{du}{dx_0}(0)$$

$$u(x)-\int_0^L G(x,x_0)f(x)\,dx_0=\cancel{u(L)\left(-hG(x,L)\right)}-A\left(\frac{h}{1+hL}x\right)\cancel{G(x,L)\left(-hu(L)\right)}$$

$$\boxed{u(x)=\int_0^L f(x_0)G(x,x_0)\,dx_0+\frac{Ah}{1+hL}x}$$

9.3.13 a) $\dfrac{\partial^2\phi}{\partial t^2}=c^2\dfrac{\partial^2\phi}{\partial x^2}+g(x)e^{-i\omega t}$, let $\phi=u(x)e^{-i\omega t}$

$$\rightarrow -\omega^2u(x)e^{-i\omega t}=c^2\frac{d^2u}{dx^2}e^{-i\omega t}+g(x)e^{-i\omega t} \qquad \text{OR} \qquad \frac{d^2u}{dx^2}+k^2u(x)=f(x)$$

if we define $k\equiv\omega/c$ and $f(x)\equiv-g(x)/c^2$.

b) For $x\neq x_0\rightarrow \dfrac{d^2G}{dx^2}+k^2G=0 \longrightarrow G(x,x_0)=\begin{cases}Ae^{-ikx}+Be^{ikx} & x<x_0\\ Ce^{-ikx}+De^{ikx} & x>x_0\end{cases}$

e^{ikx} is a rightward propagating wave.
e^{-ikx} is a leftward propagating wave.

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The Green's function gives the response to a delta-function impulse at x_0. If this is an outward propagating wave (away from x_0), it must be that:

$$B = C = 0 \rightarrow G(x,x_0) = \begin{cases} A e^{-ikx} & x < x_0 \\ D e^{ikx} & x > x_0 \end{cases}$$

At $x = x_0$, $G(x_0^-, x_0) = G(x_0^+, x_0) \rightarrow A e^{-ikx_0} = D e^{ikx_0} \rightarrow D = A e^{-2ikx}$

$$\underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{d^2G}{dx^2}\,dx}_{\text{when } \epsilon \to 0} + k^2 \cancel{\int_{x_0-\epsilon}^{x_0+\epsilon} G(x,x_0)\,dx}_{\text{by continuity}} = \underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x-x_0)\,dx}_{1}$$

$$\frac{dG}{dx}\Big|_{x_0^-}^{x_0^+} = 1$$

$$ikDe^{ikx_0} + ikAe^{-ikx_0} = 1$$

$$ik(Ae^{-2ikx_0})e^{ikx_0} + ikAe^{-ikx_0} = 1 \rightarrow A = \frac{e^{ikx_0}}{2ik}, \quad D = \frac{e^{-ikx_0}}{2ik}.$$

$$\Rightarrow G(x,x_0) = \begin{cases} \frac{1}{2ik} e^{ik(x_0-x)} & x < x_0 \\ \frac{1}{2ik} e^{-ik(x_0-x)} & x > x_0 \end{cases} \quad \text{OR} \quad \boxed{G(x,x_0) = \frac{1}{2ik} e^{ik|x_0-x|}}$$

Method 2: For $x \neq x_0 \rightarrow \frac{d^2G}{dx^2} + k^2 G = 0 \rightarrow G(x,x_0) \propto e^{\pm ikx}$

~~If we normalize the amplitude of the wave function at x to be~~

by symmetry $G(x,x_0) = G(x_0,x)$, but $G(x_0,x) \propto e^{\pm ikx_0}$

then $G \propto e^{\pm ikx} e^{\pm ikx_0}$. For $x < x_0 \rightarrow e^{-ik(x-x_0)}$ outward wave propagating away x_0

$e^{ik(x-x_0)}$ outward wave propagating away from x_0.

$$\Rightarrow \boxed{G(x,x_0) = \begin{cases} C e^{-ik(x-x_0)} & x < x_0 \\ C e^{ik(x-x_0)} & x > x_0 \end{cases}}$$

This election preserve the symmetry and continuity of $G(x,x_0)$.

Now, the jump condition on the derivative ~~preser~~ determines C:

$$\underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{d^2G}{dx^2}\,dx}_{\epsilon \to 0} + \cancel{\int_{x_0-\epsilon}^{x_0+\epsilon} G(x,x_0)\,dx}_{\text{by continuity}} = \underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x-x_0)\,dx}_{1}$$

$$\frac{dG}{dx}\Big|_{x_0^-}^{x_0^+}$$

$$C\left[ik e^{+ik\cancel{(x_0^+ - x_0)}} - (-ik) e^{-ik\cancel{(x_0^- - x_0)}}\right] = 1 \rightarrow \boxed{C = \frac{1}{2ik}}$$

c) Starting with the Green's identity (8.4.12) with $v = G(x,x_0)$:

$$\int_{-\infty}^{\infty} \left(u(x)\frac{d^2G}{dx^2} - G(x,x_0)\frac{d^2u}{dx^2}\right)dx = \cancel{u(x)\frac{dG}{dx}\Big|_{-\infty}^{\infty}} - \cancel{G(x,x_0)\frac{du}{dx}\Big|_{-\infty}^{\infty}}$$

$$\int_{-\infty}^{\infty} \left[u(x)(\delta(x-x_0) - \cancel{G}) - G(f(x) - \cancel{u(x)})\right]dx =$$

we demand $u(\pm\infty), u'(\pm\infty) \rightarrow 0$ for physical wave functions.

$$u(x_0) = \int_{-\infty}^{\infty} f(x)\,G(x,x_0)\,dx$$

$$\text{OR} \quad \boxed{u(x) = \int_{-\infty}^{\infty} f(x_0)\,G(x_0,x)\,dx_0} \Rightarrow \boxed{\phi(x,t) = \int_{-\infty}^{\infty} f(x_0)\,\frac{e^{ik|x_0-x| - i\omega t}}{2ik}\,dx_0}$$

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9.3.14 ~~a) Using the Green's formula (5.5.12) with $v = G(x,x_0)$~~

Assuming that the appropriate Green's function is $G(x,x_0)$

$\Rightarrow \mathcal{L}(G) = \delta(x-x_0)$ and $\mathcal{L}(u) = f(x)$ $\quad \mathcal{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$

Using the Green's formula (5.5.8) with $v = G(x,x_0)$

$$\int_0^L \left[u(x)\mathcal{L}(G) - G\,\mathcal{L}(u)\right]dx = p\left(u\frac{dG}{dx} - G\frac{du}{dx}\right)\Big|_0^L$$

a) $$\int_0^L \left[u(x)\,\delta(x-x_0) - G(x,x_0)f(x)\right]dx = p(L)\,u(L)\frac{dG}{dx}(L,x_0) - p(0)\,u(0)\frac{dG}{dx}(0,x_0) - p(L)\,G(L,x_0)\frac{du}{dx}(L) + G(0,x_0)\frac{du}{dx}(0)\,p(0)$$

$$u(x_0) - \int_0^L G(x,x_0)f(x)\,dx = p(L)\,\beta\frac{dG}{dx}(L,x_0) - \alpha\frac{dG}{dx}(0,x_0)\,p(0) \qquad \underbrace{0 \qquad 0}$$

The appropriate G must satisfy the homogeneous BC's of $u(x)$.

$$\boxed{u(x) = \int_0^L G(x,x_0)f(x_0)\,dx_0 + \left(p(L)\beta\frac{dG}{dx_0}(x,L) - p(0)\alpha\frac{dG}{dx_0}(x,0)\right)}$$

where we used $G(x_0,x) = G(x,x_0)$ after interchage x_0 with x.

b) $$\int_0^L \left[u(x)\,\delta(x-x_0) - G(x,x_0)f(x)\right]dx = p(L)\cdot u(L)\frac{dG}{dx}(L,x_0) - p(0)\,u(0)\frac{dG}{dx}(0,x_0) - p(L)\,G(L,x_0)\frac{du}{dx}(L) + G(0,x_0)\frac{du}{dx}(0)\,p(0)$$

$$u(x_0) - \int_0^L G(x,x_0)f(x)\,dx = \quad 0 \quad + \quad 0 \quad - p(L)\,G(L,x_0)\,\beta + G(0,x_0)\,\alpha\,p(0)$$

$$\boxed{u(x) = \int_0^L G(x,x_0)f(x_0)\,dx_0 + \left(p(L)\alpha\,G(x,0) - p(0)\beta\,G(x,L)\right)}$$

we used $G(x,x_0) = G(x_0,x)$

c) $$\int_0^L \left[u(x)\,\delta(x-x_0) - G(x,x_0)f(x)\right]dx = p(L)\,u(L)\frac{dG}{dx}(L,x_0) - p(0)\,u(0)\frac{dG}{dx}(0,x_0) - p(L)\,G(L,x_0)\frac{du}{dx}(L) + p(0)\,G(0,x_0)\frac{du}{dx}(0)$$

$$u(x_0) - \int_0^L G(x,x_0)f(x)\,dx = \left(\quad 0 \quad - p(0)\alpha\frac{dG}{dx}(0,x_0) - G(L,x_0)\,\beta\,p(L) \quad 0 \quad\right)$$

$$\boxed{u(x) = \int_0^L G(x,x_0)f(x_0)\,dx_0 - \left(p(0)\alpha\frac{dG}{dx_0}(x,0) + p(L)\beta\frac{dG}{dx_0}(x,L)\right)}$$

d) In definitive, from last items we have that the solution for nonhomogeneous BC's is most general of (9.3.51) if we write:

$$\boxed{u(x) = \int_0^L f(x_0)\,G(x,x_0)\,dx_0 + p\left(u(x_0)\frac{dG}{dx_0}(x,x_0) - G(x,x_0)\frac{du}{dx_0}\right)\Big|_{x_0=0}^{x_0=L}}$$

the Green's function satisfies the related homogeneous BC's $G(0,x_0) = 0$ and $\frac{dG}{dx}(L,x_0) + hG(h,x_0) = 0$, while $u(x)$ satisfies the given conditions.

$$\Rightarrow u(x_0) = \int_0^L f(x)\,G(x_0,x)\,dx + p(L)\,u(L)\underbrace{\frac{dG}{dx}(x_0,L)}_{-hG(x_0,L)} - \underbrace{p(0)\,u(0)}_{A}\frac{dG}{dx}(x_0,0) - p(L)\,G(x_0,L)\underbrace{\frac{du}{dx}(L)}_{-h\,u(L)} + G(x_0,0)\frac{du}{dx}(0)\,p(0) \quad 0$$

$$\boxed{u(x) = \int_0^L f(x_0)\,G(x,x_0)\,dx - p(0)\,A\frac{dG}{dx_0}(0,x)}$$

$$L(G) = \delta(x-x_0) \qquad G(0,x_0) = G(L,x_0) = 0$$

9.3.15 a) For $x \neq x_0$, $L(G) = 0$, since $y_1(0) = 0$ and $y_2(L) = 0$ $\Rightarrow G(x,x_0) = \begin{cases} c_1 y_1(x) & x < x_0 \\ c_2 y_2(x) & x > x_0 \end{cases}$

At $x = x_0$, by continuity $c_1 y_1(x_0) = c_2 y_2(x_0)$ (1) and the jump condition for the derivative:

$$\underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{d}{dx}\left(p\frac{dG}{dx}\right)dx}_{\epsilon \to 0} + q\cancel{\int_{x_0-\epsilon}^{x_0+\epsilon} G(x,x_0)\,dx}_{\to 0} = \underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x-x_0)\,dx}_{1}$$

by continuity

$$p\frac{dG}{dx}\Big|_{x_0^-}^{x_0^+} = 1$$

$$p(x_0)\, c_2 \frac{dy_2}{dx}(x_0) - p(x_0)\, c_1 \frac{dy_1}{dx}(x_0) = 1 \qquad (2)$$

From (1) $c_2 = c_1 y_1(x_0)/y_2(x_0) \rightarrow p(x_0)\left[c_1 \frac{y_1(x_0)}{y_2(x_0)}\frac{dy_2}{dx}(x_0) - c_1 \frac{dy_1}{dx}(x_0)\right] = 1$

$$c_1 \underbrace{\left[y_1(x_0)\frac{dy_2}{dx}(x_0) - y_2(x_0)\frac{dy_1}{dx}(x_0)\right]}_{W(x_0)} = \frac{y_2(x_0)}{p(x_0)}$$

$$\Rightarrow c_2 = \frac{y_1(x_0)}{W(x_0)\,p(x_0)} \qquad \rightarrow c_1 = \frac{y_2(x_0)}{W(x_0)\,p(x_0)}$$

$$\Rightarrow \boxed{G(x,x_0) = \begin{cases} \frac{1}{W(x_0)\,p(x_0)} y_2(x_0) y_1(x) & x < x_0 \\ \frac{1}{W(x_0)\,p(x_0)} y_1(x_0) y_2(x) & x > x_0 \end{cases}}$$

Note that (1) - (2) would be arranged as $\begin{bmatrix} y_1(x_0) & -y_2(x_0) \\ \frac{dy_1}{dx}(x_0) & -\frac{dy_2}{dx}(x_0) \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-1}{p(x_0)} \end{pmatrix}$
which has solution if $W(x_0) \neq 0$.
The solution is just the same given above.

Method 2: For $x \neq x_0$, $L(G) = 0$. We have that $0 < x_0 < L$, then

Since $G(0,x_0) = 0$ and $y_1(0) = 0 \rightarrow G(x,x_0) \propto y_1(x_<)$
$G(L,x_0) = 0$ and $y_2(L) = 0 \rightarrow G(x,x_0) \propto y_2(x_>)$

$\Rightarrow G(x,x_0) = c\, y_1(x_<) y_2(x_>)$ $\longrightarrow$ This election guarantee the symmetry and the continuty of $G(x,x_0)$ at $x = x_0$

$$\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{d}{dx}\left(p\frac{dG}{dx}\right)dx + q\cancel{\int_{x_0-\epsilon}^{x_0+\epsilon} G(x,x_0)\,dx}_{\to 0} = \underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x-x_0)\,dx}_{1}$$

jump of the derivative ($\epsilon \to 0$)

$$p(x)\frac{dG}{dx}\Big|_{x_0^-}^{x_0^+} = 1$$

$$p(x)\left(c\, y_1(x_0)\frac{dy_2}{dx}\Big|_{x_0^+} - c\frac{dy_1}{dx}\Big|_{x_0^-} y_2(x_0)\right) = 1 \overset{\epsilon \to 0}{\Rightarrow} \boxed{c = \frac{1}{W(x_0)\,p(x_0)}}$$

Note that $x_< = \text{minor}\,(x,x_0)$, $x_> = \text{major}\,(x,x_0)$ $\Rightarrow \boxed{G(x,x_0) = \begin{cases} c\, y_1(x) y_2(x_0) & x < x_0 \\ c\, y_1(x_0) y_2(x) & x > x_0 \end{cases}}$

b) If $y_1(x) = c_0 y_2(x) \rightarrow W(x_0) = 0$, that is the wrong since we need $W(x_0) \neq 0$ for existence of an non-trivial solution.
Recall that we need $y_1(x)$ an $y_2(x)$ linearly independent.

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$\rightarrow p=1$ and $q=0 \Rightarrow \mathcal{L} = \frac{d^2}{dx^2} \Rightarrow \mathcal{L}(G) = \delta(x-x_0)$

9.3.16 $\frac{d^2G(x,x_0)}{dx^2} = \delta(x-x_0)$ Since $G(0,x_0)=0 \rightarrow G(0,x_0) \propto x$
$G(L,x_0)=0 \rightarrow G(L,x_0) \propto (L-x)$

Let $y_1(x)=x$, $y_2(x)=(L-x)$ $\Rightarrow G(x,x_0) = C\, y_1(x_<)\, y_2(x_>)$ $\rightarrow$ This election guarantee the symmetry and continuity

According to exercise (9.3.15), $\boxed{C = \frac{1}{p(x_0)W(x_0)} = -\frac{1}{L}}$

Recall $y_< = \text{minor}(x,x_0)$ and $y_> = \text{major}(x,x_0)$; then the answer is the same from eq (9.3.46):

$$\boxed{G(x,x_0) = \begin{cases} -\frac{1}{L}x(L-x_0) & x<x_0 \\ -\frac{1}{L}x_0(L-x) & x>x_0. \end{cases}}$$

9.3.17 a) Using the method of variation of parameters according to section 9.3.2, let $u_1(x) = y_1(x)$, $u_2(x) = y_2(x)$ $\rightarrow$ such that $y_1(0)=0$ and $y_2(L)=0$.

$\Rightarrow$ According to eqs (9.3.10) and (9.3.11):

$$\frac{dv_1}{dx} = -\frac{f\,y_2}{C}, \quad \frac{dv_2}{dx} = \frac{f\,y_1}{C}$$

with $\boxed{C = p(x)W(x)}$ according to (9.3.12)

$$v_1 = -\frac{1}{C}\int_0^x f(x_0)\,y_2(x_0)\,dx_0 + C_1, \qquad v_2 = +\frac{1}{C}\int_0^x f(x_0)\,y_1(x_0)\,dx_0 + C_2$$

The solution is (see 9.3.9):

$$u = v_1u_1 + v_2u_2 = y_1(x)\left(-\frac{1}{C}\int_0^x f(x_0)\,y_2(x_0)\,dx_0 + C_1\right) + y_2(x)\left(\frac{1}{C}\int_0^x f(x_0)\,y_1(x_0)\,dx_0 + C_2\right)$$

$$u(0) = 0 = \cancel{y_1(0)}^{\,0}\,C_1 + y_2(x)\cdot C_2 \rightarrow C_2 = 0$$

$$u(L) = 0 = y_1(L)\left(-\frac{1}{C}\int_0^L f(x_0)\,y_2(x_0)\,dx_0 + C_1\right) + \cancel{y_2(L)}^{\,0}\left(\qquad\right)$$

$$\Rightarrow C_1 = \frac{1}{C}\int_0^L f(x_0)\,y_2(x_0)\,dx_0$$

$$\Rightarrow u(x) = +\frac{y_1(x)}{C}\int_x^L y_2(x_0)\,f(x_0)\,dx_0 + \frac{y_2(x)}{C}\int_0^x y_1(x_0)\,f(x_0)\,dx_0$$

b) $\boxed{u(x) = \int_0^L G(x_0,x)\,f(x_0)\,dx_0}$ with $\boxed{G(x_0,x) = \begin{cases} \frac{1}{C}y_2(x)\,y_1(x_0) & x_0<x \\ \frac{1}{C}y_2(x_0)\,y_1(x) & x_0>x \end{cases}}$

c) That is the same answer of (9.3.15) since $G(x_0,x) = G(x,x_0)$

9.3.18 $\mathcal{L}(u) = f(x)$ with $u(0)=u(L)=0$ and $\mathcal{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q(x)$ $\rightarrow$ Sturm-Liouville operator

Let $\mathcal{L}(\phi) = -\lambda\sigma\phi$, with $\phi(x)$ subject to the same homogeneous BC's.

$\Rightarrow u(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)$. We can differentiate this twice term by since both $\phi_n(x)$ and $u(x)$ solve the same homogeneous BC's.

$$\mathcal{L}(u) = \sum_{n=1}^{\infty} a_n\,\mathcal{L}(\phi_n(x)) = \sum_{n=1}^{\infty} a_n\left[-\lambda_n\sigma\phi_n(x)\right] = f(x)$$

Multiplying by $\phi_{n'}(x)$ and integrating:

$$-\sum_{n=1}^{\infty} a_n \lambda_n \int_0^L \phi_n(x)\,\phi_{n'}(x)\,\sigma(x)\,dx = \int_0^L f(x)\,\phi_{n'}(x)\,dx$$

Since $\mathcal{L}(\phi) = -\lambda\sigma\phi$ is a sturm-Liouville problem, then the eigenfunctions are orthogonals (See Section 5.3.2 in Haberman's book)

$$\Rightarrow \int_0^L \phi_n(x)\,\phi_{n'}(x)\,\sigma(x)\,dx = 0 \quad \text{if} \quad n \neq n'$$

$$\Rightarrow -a_{n'}\lambda_{n'}\int_0^L \phi_{n'}^2(x)\,\sigma(x)\,dx = \int_0^L f(x)\,\phi_{n'}(x)\,dx \rightarrow a_n = -\frac{\int_0^L f(x)\,\phi_n(x)\,dx}{\lambda_n \int_0^L \phi_n^2\,\sigma(x)\,dx}$$

$$\Rightarrow u(x) = \sum_{n=1}^{\infty} a_n\,\phi_n(x) = \sum_{n=1}^{\infty}\left(\frac{-\int_0^L f(x_0)\,\phi_n(x_0)\,dx_0}{\lambda_n \int_0^L \phi_n^2(x)\,\sigma(x)\,dx}\right)\phi_n(x)$$

$$u(x) = \int_0^L f(x_0)\sum_{n=1}^{\infty}\frac{\phi_n(x_0)\,\phi_n(x)}{-\lambda_n\int_0^L \phi_n^2\,\sigma\,dx}\,dx_0$$

$$\boxed{u(x) = \int_0^L f(x_0)\,G(x_0,x)\,dx_0} \quad \text{with} \quad \boxed{G(x_0,x) = \sum_{n=1}^{\infty}\frac{\phi_n(x_0)\,\phi_n(x)}{-\lambda_n\int_0^L \phi_n^2\,\sigma\,dx}}$$

That is the same as (9.3.23) since $G(x_0,x) = G(x,x_0)$.

(9.3-19) a) $\mathcal{L}[G(x,x_s)] = \delta(x-x_s) \rightarrow G(x,x_s)$ is the response at x due to a concetrated source at x_s.

$$G(x,x_s) = \sum_{n=1}^{\infty}\frac{\phi_n(x)\,\phi_n(x_s)}{-\lambda_n\int_a^b \phi_n^2\,\sigma\,dx} \rightarrow \text{According to eq (9.3-23).}$$

Where $\phi_n(x)$ are the eigenfunction of the Sturm-Liouville problem $\mathcal{L}[\phi_n(x)] = -\lambda_n\sigma\phi_n(x)$.

Let place the concentrated source at a node x^* of the mode n^* and recall that a node x^* of the n^*th eigenfunction means $\phi(x^*) = 0$

$$\Rightarrow G(x,x^*) = \sum_{n\neq n^*}^{\infty}\frac{\phi_n(x)\,\phi_n(x^*)}{-\lambda_n\int_a^b \phi_n^2\,\sigma\,dx} \rightarrow$$ Thus, the amplitude of the response of the mode n^* no contribute in the expansion.

b) If $\phi_n(x) = \sin(n\pi x/L)$ and the concentrated source is placed at the node $x_s = L/2 \Rightarrow \phi_n(x_s) = \sin\left(\frac{n\pi}{L}\right) = 0$ if n is even

$$G(x, x_s = \tfrac{L}{2}) = \sum_{n\ \text{odd}}^{\infty}\frac{\sin(n\pi x/L)\,\sin(n\pi/L)}{-\lambda_n\int_a^b \sin^2(n\pi x/L)\,\sigma(x)\,dx} \rightarrow$$ We have no even harmonics

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$\mathcal{L} = \frac{d^2}{dx^2}$ for eq (9.3.41)

9.3.20 $\mathcal{L}(u) = f(x)$, $u(0) = u(L) = 0$

If we suppose $u(x) = \int_0^L f(x_0) G(x, x_0)\, dx_0$ $\rightarrow$ $u(0) = 0 \rightarrow G(0, x_0) = 0$, $u(L) = 0 \rightarrow G(L, x_0) = 0$

$$\Rightarrow \mathcal{L}(u) = \int_0^L f(x_0)\, \mathcal{L}[G(x, x_0)]\, dx_0 = f(x) = \int_0^L f(x_0)\, \delta(x - x_0)\, dx_0$$

$$\rightarrow \boxed{\mathcal{L}[G(x, x_0)] = \delta(x - x_0)} \quad \text{with} \quad \boxed{G(0, x_0) = G(L, x_0) = 0}$$

Now,

Suppose $\boxed{G(x, x_0) = \sum_{n=1}^{\infty} a_n \phi_n(x)}$ with $\phi_n(x)$ eigenfunctions of $\mathcal{L}$ such that $\mathcal{L}[\phi_n(x)] = -\lambda \sigma(x) \phi_n(x)$.

Assuming that term-by-term differentiation is permited since $\mathcal{L}$ is a sturm-Liouville operator and $\phi_n(x)$, $G(x, x_0)$ and $u(x)$ satisfying the same BC's:

$$\mathcal{L}[G(x, x_0)] = \sum_{n=1}^{\infty} a_n \mathcal{L}[\phi_n(x)] = \sum_{n=1}^{\infty} a_n [-\lambda_n \sigma \phi_n(x)] = \delta(x - x_0)$$

Multiplying by $\phi_{n'}(x)$ and integrating:

$$-\sum_{n=1}^{\infty} \lambda_n a_n \underbrace{\int_0^L \phi_n(x) \phi_{n'}(x) \sigma\, dx}_{0 \text{ if } n \neq n' \text{ by orthogonality of eigenfunctions of a Sturm-Liouville problem.}} = \int_0^L \delta(x - x_0) \phi_{n'}(x)\, dx = \phi_{n'}(x_0)$$

$$-\lambda_{n'} a_{n'} \int_0^L \phi_{n'}^2(x) \sigma(x)\, dx = \phi_{n'}(x_0) \longrightarrow \boxed{a_n = \frac{\phi_n(x_0)}{-\lambda_n \int_0^L \phi_n^2(x) \sigma(x)\, dx}}$$

Substituting this in $G(x, x_0)$ we obtain equation (9.3.23).

9.3.21 If $x \neq x_0 \Rightarrow \frac{dG}{dx} = 0 \rightarrow G(x, x_0) = \begin{cases} C_1 & x < x_0 \\ C_2 & x > x_0 \end{cases}$ $\quad G(0, x_0) = 0 = C_1$

If $\frac{dG}{dx} = \delta(x - x_0) \rightarrow$ the Green's function is not continuous at $x = x_0$.

$$\int_{x_0 - \epsilon}^{x_0 + \epsilon} \frac{dG}{dx}\, dx = \cancel{\int_{x_0 - \epsilon}^{x_0 + \epsilon} \delta(x - x_0)\, dx}^{\,1} \xrightarrow{\epsilon \to 0} G(x, x_0)\Big|_{x_0^-}^{x_0^+} = 1 \rightarrow C_2 - \cancel{C_1}^{\,0} = 1 \rightarrow C_2 = 1$$

$$\Rightarrow \boxed{G(x, x_0) = \begin{cases} 0 & x < x_0 \\ 1 & x > x_0 \end{cases}}$$ which is not symmetric

$$G(x_0, x) = \begin{cases} 0 & x_0 < x \\ 1 & x_0 > x \end{cases} \neq G(x, x_0) \rightarrow \text{No symmetric.}$$

9.3.22 If $x \neq x_0 \rightarrow \frac{dG}{dx} + G = 0 \rightarrow \frac{dG}{dx} = -G \rightarrow G = \begin{cases} C_1 e^{-x} & x < x_0 \\ C_2 e^{-x} & x > x_0 \end{cases}$

$G(0, x_0) = 0 = C_1$. The jump condition is obtained by integrating the defining differential equation.

$$\int_{x_0 - \epsilon}^{x_0 + \epsilon} \frac{dG}{dx}\, dx + \int_{x_0 - \epsilon}^{x_0 + \epsilon} G(x, x_0)\, dx = \underbrace{\int_{x_0 - \epsilon}^{x_0 + \epsilon} \delta(x - x_0)\, dx}_{1}$$

$$G(x, x_0)\Big|_{x_0^-}^{x_0^+} + \underbrace{\cancel{\int_{x_0^-}^{x_0} G\, dx}}_{0 \text{ if } x < x_0} + \underbrace{\cancel{\int_{x_0}^{x_0^+} G\, dx}}_{0 \text{ if } \epsilon \to 0} = 1 \Rightarrow C_2 e^{-x_0} - \cancel{C_1 e^{-x_0}}^{\,0} = 1 \rightarrow C_2 = e^{x_0}$$

$$\Rightarrow \boxed{G(x,x_0) = \begin{cases} 0 & x<x_0 \\ e^{-(x-x_0)} & x>x_0 \end{cases}}$$

which is not symmetric

$$G(x_0,x) = \begin{cases} 0 & x_0<x \\ e^{-(x_0-x)} & x_0>x \end{cases} \neq G(x,x_0)$$

9.3.23 For $x \neq x_0 \rightarrow \frac{d^4G}{dx^4} = 0 \rightarrow G(x,x_0) = \begin{cases} a+bx+cx^2+dx^3 & x<x_0 \\ e+fx+gx^2+hx^3 & x>x_0 \end{cases}$

$G(0,x_0) = \underline{0 = a}$, $\frac{dG}{dx}(0,x_0) = \underline{0 = b}$, $G(L,x_0) = 0 \rightarrow e = -(fL+gL^2+hL^3)$ (1)

$\frac{dG}{dx}(L,x_0) = 0 \rightarrow f = -(2gL+3hL^2$ (2)

Now integrating the differential equation and using the discontinuty in the 4th derivative:

$$\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{d^4G}{dx^4}dx = \underbrace{\int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x-x_0)dx}_{1}$$

$$\left.\frac{d^3G}{dx^3}\right|_{x_0^-}^{x_0^+} = 1 \rightarrow 6(\underline{h-d}) = 1 \rightarrow h = \frac{1}{6}+d \quad (3)$$

Now ~~integrating tw~~ by double, triple and four integrations we can see the continuity of G_{xx}, G_x an $G(x,x_0)$ at $x=x_0$

$$\Rightarrow 2c+6dx_0 = 2g+6hx_0 \quad (4)$$

$$0 + \cancel{b} + 2cx_0 + 3dx_0^2 = f+2gx_0+3hx_0^2 \quad (5)$$

$$0 + \cancel{a} + bx_0 + cx_0^2 + dx_0^3 = e+fx_0+gx_0^2+hx_0^3 \quad (6)$$

Solving (1) – (6), we obtain:

$$c = \frac{-2L^2x_0+3Lx_0^2-x_0^3}{4L^2}$$

$$d = \frac{2L^3-3Lx_0^2+x_0^3}{12L^3}$$

$$e = -x_0^3/6$$

$$f = x_0^2/2$$

$$g = (x_0^2-3Lx_0^2)/4L^2$$

$$h = (-x_0^3+3Lx_0^2)/12L^3, \quad a=b=0.$$

finally:

$$\Rightarrow \boxed{G(x,x_0) = \frac{1}{12L^3}\begin{cases} x^2(L-x_0)[2L^2(3x_0-x)+xx_0^2-Lx_0(2x+3x_0)] & x<x_0 \\ x_0^2(L-x)[2L^2(3x-x_0)+x_0^2x^2-Lx(2x_0+3x)] & x>x_0 \end{cases}}$$

which is clearly simmetric.

9.3.24 Let $L = \frac{d^4}{dx^4}$, from exercise 5.5.8 (b) $\int_0^L [uL(v)-vL(u)]dx = 0$

since u and v satisfy $\phi(0)=\phi(L)=0$ and $\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0$

jus the same conditions that we have for $u(x)$ and $G(x,x_0)$ if we take $G(x,x_0)$ from exercise (9.3.23).

then if $v = G(x, x_0)$:

exercise (9.3.23)

$$\int_0^L [u\mathcal{L}(G) - G\mathcal{L}(u)]\,dx = 0 \qquad \mathcal{L}(G) = \delta(x-x_0) \qquad \mathcal{L}(u) = f(x)$$

$$\Rightarrow \int_0^L [u(x)\delta(x-x_0) - G(x,x_0)f(x)]\,dx = 0$$

we used the symmetry of $G(x, x_0)$

$$u(x_0) - \int_0^L G(x,x_0)f(x)\,dx = 0 \rightarrow \boxed{u(x) = \int_0^L f(x_0)G(x,x_0)\,dx}$$

with $G(x, x_0)$ given ~~however~~ by the answer of exercise (9.3.23).

9.3.25 a) The simplest particular solution corresponds to the initial conditions $u(0) = u'(0) = 0$. Taking the Laplace transform of the differential equation yields

$$s^2\bar{u}(s) = F(s) \Rightarrow \bar{u}(s) = F(s)/s^2$$

Using the convolution theorem $u(x) = \int_0^x f(\bar{x})g(x-\bar{x})\,d\bar{x}$

where $g(x)$ is a function whose Laplace transform is $1/s^2$.

$\Rightarrow$ $g(x) = x$ which can be consulted in tables.

$$\Rightarrow \boxed{u(x) = \int_0^x f(\bar{x})(x-\bar{x})\,d\bar{x}} \rightarrow$$ Note that this is a convolution and relationate that with

$$u(x) = \int_0^L f(x_0)G(x,x_0)\,dx_0 \quad \text{with} \quad G(x,x_0) = \begin{cases} -x & x < x_0 \\ -x_0 & x > x_0 \end{cases}$$

b) Similarly as in part (a), the simplest particular solution corresponds to the BC's:

$$u(0) = u'(0) = u''(0) = u'''(0).$$

Taking the Laplace transform to the PDE, yields:

$$s^4\bar{u}(s) = F(s) \Rightarrow \bar{u}(s) = F(s)/s^4$$

$\Rightarrow$ $u(x) = \int_0^x f(\bar{x})g(x-\bar{x})\,d\bar{x} \rightarrow$ by convolution theorem

$g(x)$ is a function whose Laplace transform is $1/s^4$. From tables:

$$g(x) = x^3/3! \qquad \Rightarrow u(x) = \int_0^x f(\bar{x})(x-\bar{x})^3/3!\,d\bar{x}.$$

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9.3.26 For $x \neq x_0 \rightarrow \frac{d^2G}{dx^2} = G \rightarrow G(x,x_0) \sim \left\{ \begin{matrix} \sinh(x+\alpha) \\ \cosh(x+\beta) \end{matrix} \right\}$

a) $G(0,x_0)=0 \rightarrow G \propto \sinh x_<$
$G(L,x_0)=0 \rightarrow G \propto \sinh(x_>-L)$ $\Rightarrow \boxed{G(x,x_0) = C \sinh x_< \sinh(x_>-L)}$

$\boxed{\begin{matrix} x_< = \text{minor}(x,x_0) \\ x_> = \text{major}(x,x_0) \end{matrix}}$

$$\int_{x-\epsilon_0}^{x+\epsilon_0} \frac{d^2G}{dx^2}dx + \cancel{\int_{x-\epsilon_0}^{x+\epsilon_0} G(x,x_0)dx}^{\,0} = \underbrace{\int_{x-\epsilon_0}^{x+\epsilon_0} \delta(x-x_0)dx}$$

by continuity of $G(x,x_0)$

$\epsilon \rightarrow 0$ $\quad \left.\frac{dG}{dx}\right|_{x_0^-}^{x_0^+} + \quad = 1$

$$C\Big(\underbrace{\sinh x_0 \cosh(x_0-L) - \cosh x_0 \sinh(x_0-L)}_{W(x_0) \rightarrow \text{Wronskian}}\Big) = 1 \rightarrow \boxed{C = \frac{1}{W(x)}}$$

b) $G(0,x_0)=0 \rightarrow G \propto \sinh x_<$
$\frac{dG}{dx}(L,x_0)=0 \rightarrow G \propto \cosh(x_>-L)$ $\Rightarrow \boxed{G(x,x_0) = C \sinh x_< \cosh(x_>-L)}$

The jump condition for the derivative of $G(x,x_0)$ can be found integrating as in part (a)

$$\Rightarrow \left.\frac{dG}{dx}\right|_{x_0^-}^{x_0^+} = 1 \rightarrow C\big(\underbrace{\sinh x_0 \sinh(x_0-L) - \cosh x_0 \cosh(x_0-L)}_{W(x)}\big) = 1$$

$$\Rightarrow \boxed{C = 1/W(x)}$$

c) $\left.\begin{matrix} \frac{dG}{dx}(0,x_0)=0 \rightarrow G \propto \cosh x_< \\ \frac{dG}{dx}(L,x_0)=0 \rightarrow G \propto \cosh(x_>-L) \end{matrix}\right\} \Rightarrow \boxed{G(x,x_0) = C \cosh x_< \cosh(x_>-L)}$

$$\left.\frac{dG}{dx}\right|_{x_0^-}^{x_0^+} = 1 \rightarrow C\big(\underbrace{\cosh x_0 \sinh(x_0-L) - \sinh x_0 \cosh(x_0-L)}_{W(x)}\big) = 1 \rightarrow C = \frac{1}{W(x)}.$$

(d)–(e) For $x \neq x_0 \rightarrow \frac{d^2G}{dx^2} = G \rightarrow G \sim e^{\pm x}$, or combinations $\rightarrow \begin{matrix} \sinh(x+\alpha) \\ \cosh(x+\beta) \end{matrix}$

d) $\left.\begin{matrix} G(0,x_0)=0 \rightarrow G \propto \sinh x_< \\ G(\infty,x_0) \rightarrow 0 \rightarrow G \propto e^{-x_>} \end{matrix}\right\} \Rightarrow \boxed{G(x,x_0) = C e^{-x_>} \sinh x_<}$

$$\left.\frac{dG}{dx}\right|_{x_0^-}^{x_0^+} = 1 \rightarrow C\big(\underbrace{-e^{-x_0}\sinh x_0 - e^{-x_0}\cosh x_0}_{W(x)}\big) = 1 \rightarrow C = 1/W(x)$$

e) $\left.\begin{matrix} \frac{dG}{dx}(0,x_0)=0 \rightarrow G \propto \cosh x_< \\ G(\infty,x_0) \rightarrow 0 \rightarrow G \propto e^{-x_>} \end{matrix}\right\} \Rightarrow \boxed{G(x,x_0) = C e^{-x_>} \cosh x_<}$

$C = 1/W(x_0)$

f) $\left.\begin{matrix} G(-\infty,x_0) \rightarrow 0 \rightarrow G \propto e^{+x_<} \\ G(\infty,x_0) \rightarrow 0 \rightarrow G \propto e^{-x_>} \end{matrix}\right\} \Rightarrow G(x,x_0) = C e^{-(x_<+x_>)}$

$\boxed{G(x,x_0) = C e^{-(x_>-x_<)}}$ $\quad C = \frac{1}{W(x_0)}.$

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9.4.1 $\int_a^b [u\mathcal{L}(v) - v\mathcal{L}(u)]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$ Eq (9.3.54)

Let $v = \phi_h(x) \Rightarrow \int_a^b [\underbrace{u\mathcal{L}(\phi_h)}_{0} - \phi_h \underbrace{\mathcal{L}(u)}_{f(x)}] = \underbrace{p\left(u\frac{d\phi_h}{dx} - \phi_h\frac{du}{dx}\right)\Big|_a^b}_{0}$ since u and ϕ_h satisfy the same BC's.

$\Rightarrow \boxed{\int_a^b \phi_h f(x)\,dx = 0} \rightarrow$ Eq (9.4.9) $\rightarrow$ Solution exis only if hold this.

9.4.2 a) Now $0 < x < L$ and $u(0) = \alpha$, $u(L) = \beta$

$\Rightarrow \int_0^L [\underbrace{u\mathcal{L}(\phi_h)}_{0} - \phi_h \underbrace{\mathcal{L}(u)}_{f(x)}]\,dx = p\left(u\frac{d\phi_h}{dx} - \phi_h\frac{du}{dx}\right)\Big|_0^L$ since $\phi_h(L)=0$, $\phi_h(0)=0$

$= p(L)\underbrace{u(L)}_{\beta}\frac{d\phi_h(L)}{dx} - p(0)\underbrace{u(0)}_{\alpha}\frac{d\phi_h}{dx}(0) + 0$

$-\int_0^L \phi_h f(x)\,dx = p(L)\beta\frac{d\phi_h}{dx}(L) - p(0)\alpha\frac{d\phi_h}{dx}(0)$

b) By (9.4.3)a) $\Rightarrow \int_0^L \phi_h f(x)\,dx = 0$ for a solution of $\mathcal{L}(\phi_h) = 0$ to exist.

$\Rightarrow p(L)\beta\frac{d\phi_h}{dx}(L) - p(0)\alpha\frac{d\phi_h}{dx}(0) = 0 \rightarrow$ Condition for a solution to exist

b) If $\int_0^L \phi_h f(x)\,dx = 0 \rightarrow \lambda_n = 0$ according to eq (9.4.3)

Remember from eigenfunction expansions $\mathcal{L}(u) = f(x)$, $\mathcal{L}(\phi_n) = -\lambda_n \sigma \phi_n$

and $u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$.

If $\lambda_n = 0 \rightarrow \mathcal{L}(\phi_n) = 0 \rightarrow \phi_n$ are the ~~homogeneous~~ solutions of $\mathcal{L}(\phi_n) = 0 \rightarrow \phi_n = \phi_h \rightarrow$ homogeneous solutions of $\mathcal{L}(u) = f(x)$.

Since $u = \sum_{n=1}^{\infty} a_n \phi_n(x) \rightarrow \mathcal{L}(u) = \sum_{n=1}^{\infty} a_n \underbrace{\mathcal{L}(\phi_n)}_{0} = 0$ and satisfying the homogeneous equation, the a_n can be arbitrary and so there are an infinite number of solutions of $\mathcal{L}(u) = f(x)$

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9.4.3 $\dfrac{d^2u}{dx^2} + \gamma u = \sin x = f(x)$ (1)

a) $\gamma = 1$, $u(0) = u(\pi) = 0 \rightarrow \phi_h = \sin x$ is a nontrivial homogeneous sol. According to the Fredholm alternative, solutions to the nonhomogeneous problem (subject to homogeneous BC's) exist only if $f(x)$ is orthogonal to $\phi_h(x)$:

$\int_0^\pi f(x)\,\phi_h(x)\,dx = \int_0^\pi \sin x \sin x\,dx = \frac{\pi}{2} \neq 0 \rightarrow$ So there are not solutions to (1)

b) $\gamma = 1$, $u'(0) = u'(\pi) = 0 \rightarrow \phi_h = \cos x$

$\int_0^\pi f(x)\,\phi_h(x)\,dx = \int_0^\pi \sin x \cos x\,dx = 0 \rightarrow$ There are an infinite number of solutions.

c) $\gamma = -1$, $u(0) = u(\pi) = 0 \Rightarrow \dfrac{d^2\phi}{dx^2} - \phi = 0 \rightarrow$ homogeneous equation

$\phi_h(x) = c_1 e^{x} + c_2 e^{-x}$, but $\phi(0) = 0 = c_1 + c_2 \rightarrow c_2 = -c_1$

$\phi(\pi) = 0 = c_1(e^{\pi} - e^{-\pi}) \rightarrow c_2 = 0$

$\Rightarrow \phi_h(x) = 0 \rightarrow$ There are not non-trivial solutions of homogeneous equation and thus, there are not zero eigenvalues and so there is only a unique solution of (1) according to the Fredholm alternative.

d) $\gamma = 2$, $u(0) = u(\pi) = 0 \rightarrow \phi_h(x) = c_1 \sin(\sqrt{2}\,x) + c_2 \cos(\sqrt{2}\,x)$

$\phi_h(0) = 0 = c_2$

$\phi_h(\pi) = 0 = c_1 \sin \sqrt{2}\,\pi \rightarrow c_1 = 0$

There are not non-trivial solutions of homogeneous eq and thus, there are not zero eigenvalues and so there is only a unique solution of (1)

9.4.4 a) $\dfrac{d^2u}{dx^2} + u = e^{x} \rightarrow \phi_h = c_1 \cos x + c_2 \sin x$

$\phi_p = c_3 e^{x}$

$\Rightarrow u(x) = c_1 \cos x + c_2 \sin x + c_3 e^{x} \longrightarrow$

~~$\phi_p = \int_0^x e^{\tau} \sin(x-\tau)\,d\tau$~~

~~$u = \sin(x-\tau)$, $dv = e^{\tau}d\tau$~~

~~$du = \cos(x-\tau)\,d\tau$, $v = e^{\tau}$~~

~~$= e^{\tau}\sin(x-\tau)\Big|_0^x + \int_0^x e^{\tau}\cos(x-\tau)\,d\tau$~~

~~$u = \cos(x-\tau)$~~

~~$dv = e^{\tau}\sin(x-\tau)\,d\tau$~~

~~$\longrightarrow = \sin x + e^{\tau}\cos(x-\tau)\Big|_0^x - \int_0^x e^{\tau}\sin(x-\tau)\,d\tau$~~

~~$2\int_0^x e^{\tau}\sin(x-\tau) = \sin x + e^{x}(1 - \cos x)$~~

~~$\Rightarrow \phi_p = c e^{x}(1-\cos x) \Rightarrow \phi_p' = c e^{x}(1-\cos x) + c e^{x}\sin x$~~

~~$\frac{d^2}{dx^2}$~~

~~$\Rightarrow$ There are~~

~~$\int_0^\pi f(x)\,\phi_h(x)\,dx = 0$.~~

~~of solutions of $\mathcal{L}(u) = f(x)$ since~~

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$u(0)=0=c_1+c_3 \rightarrow c_3=-c_1$
$u(\pi)=0=-c_1+c_3e^{\pi} \rightarrow c_3=c_1e^{-\pi}$ } These are inconsistent, so that no solutions exist.

~~$\Rightarrow \phi_p'' = 2ce^x(1-\cos x)+ce^x\cos x+2ce^x\sin x$~~

~~$\phi_p''+\phi_p = 2ce^x(1-\cos x)+ce^x\cos x+2ce^x\sin x \neq e^x \Rightarrow c=0$~~

~~$\Rightarrow$ These are not solutions~~

That is according to the Fredholm Alternative, since because $\phi_h=\sin x\neq 0$ when $\mathcal{L}(u)=\frac{d^2u}{dx^2}+u$ and $\mathcal{L}(\phi_h)=-\lambda\sigma\phi_h$ for $\lambda=0$, and

$$\int_0^{\pi} f(x)\phi_h(x)\,dx=\int_0^{\pi} e^x\sin x\,dx=\frac{1}{2}(1+e^{\pi})\neq 0$$

Thus there are not solutions for the non-homogeneous equation $\mathcal{L}(u)=f(x)=e^x$ subject to the homogeneous BC's.

b) $\frac{d^2u}{dx^2}=e^x \rightarrow \left.\begin{array}{l}\phi_h=c_1+c_2x\\ \phi_p=c_3e^x\end{array}\right\} \Rightarrow \left.\begin{array}{l}u(x)=c_1+c_2x+c_3e^x\\ u'(0)=0=c_2+c_3 \rightarrow c_3=-c_2\\ u'(L)=0=c_2+c_3e \rightarrow c_3=-c_2e^{-L}\end{array}\right\}$

~~A particular solution is~~

$\Rightarrow$ There are not solutions. These are inconsistent so that no solutions exist.

That is the case according to Fredholm alternative when $\phi_h=c_1\neq 0$ for $\lambda=0$ ($\mathcal{L}(\phi)=-\lambda\sigma\phi$ with $\mathcal{L}=d^2/dx^2$), and

$$\int_0^L f(x)\phi_h(x)\,dx=c_1\int_0^L e^x\,dx=c_1e^L\neq 0 \rightarrow$$ Thus there are not solutions for $\mathcal{L}(u)=f(x)$ subject to homogeneous BC's.

c) $\frac{d^2u}{dx^2}+2u=e^x \longrightarrow \phi_h=c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x$
$\phi_p=c_3e^x$
$\phi_p''+2\phi_p=e^x \rightarrow c_3=\frac{1}{3}$ $\quad u(x)=\phi_h+\phi_p \Rightarrow \left.\begin{array}{l}\phi(0)=0=c_1+c_3 \rightarrow c_3=-c_1=1/3\\ \phi(\pi)=0=-\frac{1}{3}\cos\sqrt{2}\pi+c_2\sin\sqrt{2}\pi-\frac{1}{3}e^{\pi}\end{array}\right\} \Rightarrow$

This determine completely c_2, then the solution is unique!!

Method 2: Use eq (8.5.19): $\phi_p=\int_0^x e^{\tau}\frac{\sin[\sqrt{2}(x-\tau)]}{\sqrt{2}}\,d\tau$

Integrating by parts as in part (a)

$$=\frac{1}{6}\left(2e^x-2\cos(\sqrt{2}x)-\sqrt{2}\sin(\sqrt{2}x)\right)$$

$$\phi_p'=\frac{1}{6}\left(2e^x-2\cos(\sqrt{2}x)+2\sqrt{2}\sin(\sqrt{2}x)\right)$$

$$\phi_p''=\frac{1}{6}\left(2e^x+4\cos(\sqrt{2}x)+2\sqrt{2}\sin(\sqrt{2}x)\right)$$

$\Rightarrow \phi_p''(x)+2\phi_p(x)=e^x=f(x)$, then $\phi_p(x)$ is a particular solution that satisfy $\mathcal{L}(u)=f(x)$ if $\mathcal{L}=\frac{d^2}{dx^2}+2$ and $f(x)=e^x$, with homogeneous BC's.

$\Rightarrow \phi_p(x)=c_3e^x+c_1\cos(\sqrt{2}x)+c_2\sin(\sqrt{2}x)$

$\phi_p(0)=0 \rightarrow c_3=-c_1$, $\phi_p(\pi)=0=-c_1(e^{\pi}-\cos\sqrt{2}\pi)+c_2\sin\sqrt{2}\pi \Rightarrow$
$c_2=c_1(e^{\pi}-\cos\sqrt{2}\pi)/\sin\sqrt{2}\pi$

$$\Rightarrow C_2 = C_1\left[\frac{e^{\pi} - \cos\sqrt{2}\pi}{\sin\sqrt{2}\pi}\right]$$

$$\Rightarrow u(x) = C_1\left[-e^{x} + \cos\sqrt{2}x) + \frac{(e^{\pi} - \cos\sqrt{2}\pi)}{\sin\sqrt{2}\pi}\sin\sqrt{2}x\right]$$

Thus there is a unique solution. That is according to the Fredholm alternative because there are no solutions of the corresponding homogeneous problem i.e $\phi_h = 0$ when $\lambda = 0$ in the problem $\mathcal{L}(\phi) = -\lambda\sigma\phi$ with $\mathcal{L} = d^2/dx^2 + 2$.

d) $\dfrac{d^2u}{dx^2} + \left(\dfrac{\pi}{L}\right)^2 u = \beta + x \rightarrow \phi_h = C_1\cos(\pi x/L) + C_2\sin(\pi x/L)$

$\phi_p = Ax + B$

$\phi_p'' + \left(\frac{\pi}{L}\right)^2\phi_p = \beta + x \rightarrow A = \left(\frac{L}{\pi}\right)^2$, $B = \left(\frac{L}{\pi}\right)^2\beta$

$u(x) = \phi_h + \phi_p \rightarrow u(0) = 0 = C_1 + B \rightarrow B = -C_1$

$u(L) = 0 = -C_1 + AL + B$

$= 2B + AL \rightarrow B = -\frac{AL}{2}$

$\boxed{\beta = -L/2}$

$\cancel{\Rightarrow u(x) = C_1\left[\cos(\pi x/L) - \frac{2}{L}x - 1\right] + C_2\sin(\pi x/L)}$

$\cancel{u'(x) = C_1\left[-\left(\frac{\pi}{L}\right)^2\cos\left(\frac{\pi x}{L}\right)\right] - \left(\frac{\pi}{L}\right)^2 C_2\sin(\pi x/L)}$

$\cancel{u''(x) + \left(\frac{\pi}{L}\right)^2 u(x) = -\left(\frac{\pi}{L}\right)^2\left(C_1\cos\frac{\pi x}{L} + C_2\sin\frac{\pi x}{L}\right) + \left(\frac{\pi}{L}\right)^2\left[C_1\cos\pi x/L + C_2\sin\pi x/L\right]}$

$\cancel{\Rightarrow \left(\frac{2x}{L} - 1\right)C_1 = \left(\frac{x - L/2}{L/2}\right)C_1 = \frac{C_1 2}{L} \rightarrow C_1 = L/2 \text{ and } \boxed{\beta = -\frac{L}{2}}}$

C_2 is arbitrary, then there are infinite solutions of $u(x)$.

Method 2: Use eq (8.5.19): $\phi_p(x) = \displaystyle\int_0^x (\beta + \gamma)\frac{\sin[\pi(x-\gamma)/L]}{(\pi/L)}\,d\gamma$

$$= \frac{L^2[\pi(\beta + x) - \pi\beta\cos(\pi x/L) - L\sin(\pi x/L)]}{\pi^3}$$

$$\phi_p'(x) = \frac{L^2[\pi - \pi\cos(\pi x/L) + (\pi^2\beta/L)\sin(\pi x/L)]}{\pi^3}$$

$$\phi_p''(x) = \frac{L^2[(\pi^3/L^2)\beta\cos(\pi x/L) + (\pi^2/L)\sin(\pi x/L)]}{\pi^3}$$

$\Rightarrow$ You can verify that it satisfy $\phi_p''(x) + (\pi/L)^2\phi_p = \beta + x$.

$$\Rightarrow u(x) = C\sin(\pi x/L) + D\left[\frac{\pi(\beta + x) - \pi\beta\cos(\pi x/L) - L\sin(\pi x/L)}{\pi^3/L^2}\right]$$

$u(0) = 0$

$u(L) = 0 = 0 + D\dfrac{\pi(\beta + L) + \pi\beta}{\pi^3/L^2} = D\dfrac{(2\beta + L)}{\pi^2/L^3} \rightarrow D = 0$ OR $\beta = -L/2$

$\beta = -L/2 \rightarrow D$ is arbitrary, but C is arbitrary too, then there are infinite solution of $u(x)$.

That is according to the Fredholm alternative, because $\phi_h = C\sin(\pi x/L) \neq 0$ for $\lambda = 0$ in $\mathcal{L}(\phi) = -\lambda\sigma\phi$ and

$\displaystyle\int_0^L (\beta + x)\sin(\pi x/L) = 0$ when $\beta = -L/2$.

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9.4.5 $\frac{d^2u}{dx^2} + u = \beta + x \longrightarrow \phi_h(x) = C_1 \sin x + C_2 \cos x$

$\phi_h'(x) = C_1 \cos x - C_2 \sin x$

$\phi(-\pi) = \phi(\pi) \rightarrow C_2 \cancel{\cos(-\pi)} = C_2 \cancel{\cos \pi}$.

$\phi'(-\pi) = \phi'(\pi) \rightarrow C_1 \cancel{\cos(-\pi)} = C_1 \cancel{\cos \pi}$

$\Rightarrow$ C_1 and C_2 are arbitrary, and the general homogeneous solution holds for any C_1, C_2. Thus $\lambda_0 = 0$ is an eigenvalue with eigenfunction

$$\phi_0(x) = \sin x + \cos x \neq 0$$

There will be (infinitely) many solutions of $\mathcal{L}(u) = \beta + x$ if

$$\int_0^\pi f(x)\phi_h(x)\,dx = \int_0^\pi (\beta + x)(\sin x + \cos x)\,dx = 0$$

$$= \beta(-\cos x + \sin x)\Big|_0^\pi + \int_0^\pi x(\sin x + \cos x)\,dx$$

$u = x$, $du = dx$, $dv = \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}dx$, $v = \begin{Bmatrix}-\cos x\\ \sin x\end{Bmatrix}dx$

$$= 2\beta + x(-\cos x + \sin x)\Big|_0^\pi - \int_0^\pi (-\cos x + \sin x)\,dx$$

$$= 2\beta + \pi - (\sin x - \cos x)\Big|_0^\pi = 2\beta + \pi - 2 = 0$$

$$\boxed{\beta = \frac{2-\pi}{2}}$$

For $\beta \neq \frac{2-\pi}{2}$ there are not solutions.

9.4.6 a) $\frac{d^2u}{dx^2} + u = 1 \longrightarrow \phi_h = C_1 \cos x + C_2 \sin x$

$\phi_p = C_0 \rightarrow \phi_p'' + \phi_p = C_0 = 1$

$\Rightarrow u(x) = \phi_p + \phi_h = 1 + C_1 \cos x + C_2 \sin x$

$u(0) = 0 = 1 + C_1 \rightarrow C_1 = -1$
$u(\pi) = 0 = 1 - C_1 \rightarrow C_1 = 1$ } These are inconsistent so that <u>no solutions exist</u>.

To apply the Fredholm alternative, we note that ~~$\phi_h = \sin x$~~

$\phi_h = C_1 \cos x + C_2 \sin x$, $\phi(0) = 0 = C_1$

$\phi(\pi) = 0 = C_2 \cancel{\sin \pi}^{\,0} \rightarrow C_2$ arbitrary.

$\Rightarrow \phi_h = C_2 \sin x$.

Now, $\int_0^\pi f(x)\phi_h(x)\,dx = \int_0^\pi 1 \cdot C_2 \sin x\,dx = 2 \neq 0 \rightarrow$ <u>There are not solutions for</u> $\mathcal{L}[u] = f(x) = 1$

b) $\phi_h(x) = C_1 \cos x + C_2 \sin x$, $\phi_p(x) = C_0$, $u(x) = \phi_h + \phi_p \longrightarrow$ $u'(0) = 0 = C_2$, $u'(\pi) = 0 = -C_2$ $\Rightarrow C_1$ arbitrary

$\Rightarrow u(x) = C_0 + C_1 \cos x \longrightarrow$ There are <u>an infinite number of solutions</u> since C_0, C_1 are arbitrary.

- To apply the Fredholm alternative, use $\phi_h = C_1 \cos x + C_2 \sin x$.

$\phi_h'(0) = 0 = C_2$, $\phi_h'(\pi) = 0 = -C_2$ } $\Rightarrow C_1$ arbitrary

$\Rightarrow \phi_h(x) = C_1 \cos x$

Now, $\int_0^\pi f(x)\phi_h(x)\,dx = \int_0^\pi 1 \cdot C_1 \cos x\,dx = 0 \Rightarrow f(x) = 1$ is orthogonal to all homogeneous solutions $\phi_h = \cos x \Rightarrow$ There should be <u>an infinite number of solutions</u>.

c) $\phi_h = C_1 \cos x + C_2 \sin x, \quad \phi_p = 1$

$u(x) = \phi_h + \phi_p = 1 + C_1 \cos x + C_2 \sin x$

$$\left.\begin{array}{l} u(-\pi) = u(\pi) \rightarrow 1 - C_1 = 1 - C_1 \\ u'(-\pi) = u'(\pi) \rightarrow -C_2 = -C_2 \end{array}\right\} \Rightarrow$$ C_1 and C_2 arbitrary. There are an infinite number of solutions.

To apply the Fredholm alternative, we note that ~~both~~

$\phi_h = C_1 \cos x + C_2 \sin x \rightarrow \left.\begin{array}{l} \phi_h(-\pi) = \phi(\pi) \rightarrow -C_1 = -C_1 \\ \phi_h'(-\pi) = \phi_h'(\pi) \rightarrow -C_2 = -C_2 \end{array}\right\}$ C_1 and C_2 arbitrary

$\rightarrow \phi_h = \cos x + \sin x.$

Now $\int_{-\pi}^{\pi} f(x) \cdot \phi_h(x)\, dx = \int_{-\pi}^{\pi} 1 \cdot (\cos x + \sin x)\, dx = 0 \rightarrow$ Thus, there are an infinite number of solutions.

9.4.7 $\dfrac{d^2u}{dx^2} + 4u = \cos x$

a) $\left.\begin{array}{l} \phi_h = C_1 \cos 2x + C_2 \sin 2x \\ \phi_p = A \cos x \end{array}\right\} \rightarrow u(x) = A \cos x + C_1 \cos 2x + C_2 \sin 2x$

$\left.\begin{array}{l} u'(0) = 0 = 2C_2 \rightarrow C_2 = 0 \\ u'(\pi) = 0 = -2C_2 \rightarrow C_2 = 0 \end{array}\right\}$ C_1 arbitrary.

$$\left.\begin{array}{l} \Rightarrow u(x) = A \cos x + C_1 \cos 2x \\ u''(x) = -A \cos x - 4C_1 \cos 2x \end{array}\right\} \Rightarrow u''(x) + 4u(x) = 3A \cos x = \cos x$$

$A = 1/3$

$\Rightarrow \boxed{u(x) = \frac{1}{3} \cos x + C_1 \cos 2x} \rightarrow$ There are infinite number of solutions since C_1 can be arbitrary.

b) $u'(0) = u'(\pi) = 0 \rightarrow$ let $u(x) = \sum_{n=0}^{\infty} a_n \cos(nx)$

$u''(x) = \sum_{n=0}^{\infty} -n^2 a_n \cos(nx)$

$\Rightarrow u''(x) + 4u(x) = \sum_{n=0}^{\infty} (-n^2 + 4) a_n \cos nx = \cos x$

$(-n^2 + 4) a_n = \delta_{n1} \rightarrow \begin{cases} \text{For } n = 1 & 3a_1 = 1 \rightarrow a_1 = \frac{1}{3} \\ \text{For } n = 2 \rightarrow & a_2 \text{ arbitrary} \\ \text{For } n \neq 1, 2 \rightarrow & a_n = 0 \end{cases}$

$\Rightarrow \boxed{u(x) = \frac{1}{3} \cos x + a_2 \cos 2x} \rightarrow$ There are an infinite number of solutions since a_2 can be arbitrary.

c) To apply Fredholm alternative, we note that the homogeneous solution is: $\phi_h = C_1 \cos 2x$ since $\phi'(0) = \phi'(\pi) = 0$.

Now, $\int_0^{\pi} f(x) \phi_h(x)\, dx = \int_0^{\pi} \cos x \cdot C_1 \cos 2x\, dx = C_1 \frac{\pi}{2} \delta_{12} = 0 \Rightarrow$ There are an infinite number of solutions since $\phi_h \neq 0$.

That is consistent with the last answers.

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9.4.8 $\frac{d^2u}{dx^2} + u = \cos x$

a) $\phi_h = C_1 \cos x + C_2 \sin x \qquad \phi_p = Ax \sin x$

$u(x) = \phi_h + \phi_p = Ax \sin x + C_1 \cos x + C_2 \sin x \qquad \left.\begin{array}{l} u(0) = 0 = C_1 \\ u(\pi) = 0 = -C_1 \end{array}\right\}$ C_2 can be arbitrary

$\Rightarrow \left.\begin{array}{l} u(x) = (Ax + C_2)\sin x \\ u'(x) = (Ax + C_2)\cos x + A\sin x \\ u''(x) = -(Ax + C_2)\sin x + 2A\cos x \end{array}\right\} \Rightarrow \underbrace{u''(x) + u(x)}_{\cos x} = 2A\cos x \rightarrow A = 1/2$

$\Rightarrow \boxed{u(x) = \left(\frac{1}{2}x + C_2\right)\sin x}$ with C_2 arbitrary, which is an infinite number of solutions.

To apply the Fredholm alternative, we note that

$\phi_h = C_1 \cos x + C_2 \sin x, \qquad \left.\begin{array}{l} \phi(0) = 0 = C_1 \\ \phi(\pi) = 0 = -C_1 \end{array}\right\} \Rightarrow C_2$ arbitrary

$\phi_h = C_2 \sin x.$

Now $\int_0^\pi f(x)\phi_h(x)\,dx = \int_0^\pi \cos x \sin x\,dx = 0 \Rightarrow$ there are an infinite number of solutions.

b) $\phi_h = C_1 \cos x + C_2 \sin x \qquad \phi_p = Ax \sin x \rightarrow A = 1/2$ similar to part a)

$u(x) = \phi_h + \phi_p = \left(\frac{1}{2}x + C_2\right)\sin x + C_1 \cos x \qquad u(-\pi) = u(\pi) \rightarrow -C_1 = -C_1$

$u'(x) = (Ax + C_2)\cos x + \frac{1}{2}\sin x - C_1 \sin x$

$u'(-\pi) = u'(\pi) \rightarrow -\left(\frac{1}{2}(-\pi) + \cancel{C_2}\right) = -\left(\frac{1}{2}(\pi) + \cancel{C_2}\right) \rightarrow \frac{\pi}{2} = -\frac{\pi}{2} \rightarrow$ Inconsistent $\Rightarrow$ there are not solutions

$\Rightarrow u(x) = C_1 \cos x + C_2 \sin x$ with C_1, C_2 arbitrary, then there are an infinite number of solutions.

To apply the Fredholm alternative, we note that:

$\phi_h = C_1 \cos x + C_2 \sin x \qquad \phi(-\pi) = \phi(\pi) \rightarrow -C_1 = -C_1$

$\phi_h' = -C_1 \sin x + C_2 \cos x \qquad \phi'(-\pi) = \phi'(\pi) \rightarrow -C_2 = -C_2$

$\Rightarrow C_1, C_2$ can be arbitrary $\rightarrow \phi_h = C_1 \cos x + C_2 \sin x \neq 0.$

Now, $\int_{-\pi}^{\pi} f(x)\phi_h(x)\,dx = \int_{-\pi}^{\pi} \cos x\,(C_1 \cos x + C_2 \sin x)\,dx = \pi C_1 \rightarrow$ There are Not ~~infinite number~~ solutions

9.4.9 a) $G_m(x, x_0) = g_m(x, x_0) + \beta\,\phi_h(x_0)\,\phi_h(x) \rightarrow$ See after eq (9.4.16)

There are an infinite number of solutions if β can be any constant (independent of x and x_0). But now, we need that $G_m(x, x_0)$ be orthogonal to $\phi_h(x)$, that is:

$$0 = \int_a^b G_m(x, x_0)\,\phi_h(x)\,dx = \underbrace{\int_a^b g_m(x, x_0)\,\phi_h(x)\,dx}_{\neq 0} + \beta\,\phi_h(x_0)\int_a^b \phi_h^2(x)\,dx$$

$\neq 0$ Since g_m is not orthogonal to ϕ_h.

$\Rightarrow \beta = -\dfrac{\int_a^b g_m(x, x_0)\,\phi_h(x)\,dx}{\phi_h(x_0)\int_a^b \phi_h^2(x)\,dx} \Rightarrow$ Then β is unique and the generalized Green's function is then unique.

b) Apply Green's formula (9.3.54) with $u = G_m(x,x_1)$
$v = G_m(x,x_2)$.

$$\Rightarrow \int_a^b \Big[G_m(x,x_1)\, \mathcal{L}[G(x,x_2)] - G_m(x,x_2)\, \mathcal{L}[G(x,x_1)] \Big] dx = \rho \left(G_m(x,x_1) \frac{dG_m}{dx}(x,x_2) - G_m(x,x_2) \frac{dG}{dx}(x,x_1) \right) \Big|_a^b$$

$$\mathcal{L}[G(x,x_2)] = \left(\delta(x-x_2) - \frac{\phi_h(x)\phi_h(x_2)}{\int \phi_h^2(x)\,dx} \right) \qquad \mathcal{L}[G(x,x_1)] = \left(\delta(x-x_1) - \frac{\phi_h(x)\phi_h(x_1)}{\int \phi_h^2(x)\,dx} \right)$$

$\to 0$ since $G_m(x,x_1)$ and $G(x,x_2)$ satisfy the same BC's

But $\int_a^b G_m(x,x_i)\,\delta(x-x_j)\,dx = G_m(x_j,x_i)$

$$\Rightarrow \left(G_m(x_2,x_1) - \frac{\phi_h(x_2)\int \phi_h(x) G_m(x,x_1)\,dx}{\int \phi_h^2(x)\,dx} \right) - \left(G(x_1,x_2) - \frac{\phi_h(x_1)\int \phi_h(x) G_m(x,x_2)\,dx}{\int \phi_h^2(x)\,dx} \right) = 0$$

Now, $\int_a^b \phi_h(x)\, G_m(x,x_i)\,dx = 0$ since the generalized Green's function $G_m(x,x_i)$ is orthogonal to $\phi_h(x)$.

$$\Rightarrow G_m(x_2,x_1) - G_m(x_1,x_2) = 0 \quad \text{OR} \quad \boxed{G_m(x_2,x_1) = G_m(x_1,x_2)}$$

9.4-10 $\dfrac{d^2u}{dx^2} + u = f(x) \qquad u(0) = \alpha, \quad u(\pi) = \beta$

$$\Rightarrow \frac{d^2G}{dx^2} + G = \delta(x-x_0) \qquad G(0,x_0) = 0, \quad G(\pi,x_0) = 0$$

For $x \neq x_0 \to \dfrac{d^2G}{dx^2} + G = 0 \to G = C_1 \cos x + C_2 \sin x$

$$\left.\begin{array}{l} G(0,x_0) = 0 = C_1 \\ G(\pi,x_0) = 0 = -C_1 \end{array}\right\} \Rightarrow C_2 \text{ is arbitrary}$$

Thus $\phi_h = \sin x$ is a solution of the homogeneous eq.

A modified Green's function $G_m(x,x_0)$ must be introduced satisfying:

$$\mathcal{L}[G_m(x,x_0)] = \delta(x-x_0) + C\,\phi_h(x)$$

OR $\dfrac{d^2G_m}{dx^2} + G_m(x,x_0) = \delta(x-x_0) + C \sin x \qquad G_m(0,x_0) = G_m(\pi,x_0) = 0$

The constant C is chosen so that the right-side is orthogonal to $\phi_h \Rightarrow \int_0^\pi \sin x\,[\delta(x-x_0) + C \sin x]\,dx = 0$

$$\sin x_0 + C \underbrace{\int_0^\pi \sin^2 x\,dx}_{(\pi/2)\delta_{11}} = 0 \to \boxed{C = -\frac{2}{\pi} \sin x_0}$$

For $x \neq x_0 \to \dfrac{d^2G_m}{dx^2} + G_m = C \sin x \to G_h = C_1 \cos x + C_2 \sin x$

$$\phi_p = x(A\cos x + B \sin x)$$

$$\phi_p' = (A\cos x + B\sin x) + x(-A\sin x + B\cos x)$$

$$\phi_p'' = 2(-A\sin x + B\cos x) + x(-A\cos x - B\sin x)$$

$$\Rightarrow \phi_p'' + \phi_p = 2(-A\sin x + B\cos x) = C\sin x \to A = -\frac{C}{2}, \quad B = 0 \Rightarrow$$

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$$\rightarrow C = -\frac{2}{\pi}\sin x_0$$

$\Rightarrow \phi_p = -\frac{C}{2} x \cos x$. Now, $G_m = G_h + \phi_p$ where $G_h = C_1 \cos x + C_2 \sin x$

but **for $x \neq x_0$** we have two regions ($x < x_0$ and $x > x_0$), then:

$$G_m(x, x_0) = \frac{1}{\pi} x \sin x_0 \cos x + \begin{cases} C_1 \cos x + C_2 \sin x & x < x_0 \\ C_3 \cos x + C_4 \sin x & x > x_0. \end{cases}$$

$G_m(0, x_0) = 0 = C_1$

$G_m(\pi, x_0) = 0 = \sin x_0 (-1) - C_3 \rightarrow C_3 = -\sin x_0.$

Now, at $x = x_0$, by continuity $G_m(x_0^-, x_0) = G_m(x_0^+, x_0)$

$$\cancel{C_1} \cos x_0 + C_2 \cancel{\sin x_0} = (-\cancel{\sin x_0}) \cos x_0 + C_4 \cancel{\sin x_0}$$

$$C_2 = -\cos x_0 + C_4$$

$$\Rightarrow \quad G_m(x, x_0) = \frac{1}{\pi} x \sin x_0 \cos x + \begin{cases} (-\cos x_0 + C_4) \sin x & x < x_0 \\ -\sin x_0 \cos x + C_4 \sin x & x > x_0 \end{cases}$$

where C_4 is an arbitrary function of x_0.
We chose $G_m(x, x_0)$ to be symmetric, as is guaranteed by exercise (9.4.9), so that (9.4.17) may be derived. Then, imposing symmetry:

$$G_m(x, x_0) = G_m(x_0, x) = \frac{1}{\pi} x_0 \sin x \cos x_0 + \begin{cases} (-\cos x + C_4(x)) \sin x_0 & x_0 < x \\ -\sin x \cos x_0 + C_4(x) \sin x_0 & x_0 > x \end{cases}$$

$$\frac{1}{\pi} x \sin x_0 \cos x + (-\cos x_0 + C_4(x_0)) \sin x = \frac{1}{\pi} x_0 \sin x \cos x_0 - \sin x \cos x_0 + C_4(x) \sin x_0 \quad (1)$$

$$\frac{1}{\pi} x \sin x_0 \cos x - \sin x_0 \cos x + C_4(x_0) \sin x = \frac{1}{\pi} x_0 \sin x \cos x_0 + (-\cos x + C_4(x)) \sin x_0 \quad (2)$$

$$\frac{2}{\pi} x \sin x_0 \cos x - \left(\cos x_0 \sin x + \sin x_0 \cos x\right) + 2C_4(x_0) \sin x = \frac{2}{\pi} x_0 \sin x \cos x_0 - \left(\sin x \cos x_0 + \cos x \sin x_0\right) + 2C_4(x) \sin x_0$$

~~$2C_4(\sin x_0 - \sin x_0) = \frac{2}{\pi}(x_0 \sin x \cos x_0 - x \sin x_0 \cos x)$~~

~~$\Rightarrow (\sin x \cos x_0 + \sin x \cos x_0) - (\cos x \sin x_0 + \cos x \sin x_0)$~~

The more easy way to find $C_4(x_0)$ (or $C_4(x)$) is have in mind that we need $G_m(x, x_0)$ symetric, then comparing $G_m(x, x_0)$ and $G_m(x_0, x)$ we can suppose:

$$C_4(x_0) = a \sin x_0 + b x_0 \cos x_0$$

$$\Rightarrow \frac{1}{\pi} x \sin x_0 \cos x + (\cancel{-\cos x_0 + a \sin x_0 + b x_0 \cos x_0}) \sin x$$

$$= \frac{1}{\pi} x_0 \sin x \cos x_0 - \sin x \cos x_0 + (a \sin x + b x \cos x) \sin x_0.$$

$$= \left(\cancel{\frac{1}{\pi} x_0 \cos x_0 - \cos x_0 + a \sin x_0}\right) \sin x + b x \sin x_0 \cos x$$

$\Rightarrow b = 1/\pi$ and a an arbitrary constant.

$$\Rightarrow \boxed{G_m(x, x_0) = \frac{1}{\pi}\left(x \sin x_0 \cos x + x_0 \cos x_0 \sin x\right) + a \sin x_0 \sin x + \begin{cases} -\cos x_0 \sin x & x < x_0 \\ -\sin x_0 \cos x & x > x_0 \end{cases}}$$

Note that $G_m(x,x_0)$ automatically satisfy the jump condition $\left.\frac{dG_m}{dx}\right|_{x_0^-}^{x_0^+} = 1$, then integrate the differential equation around x_0 from $x_0-\epsilon$ to $x_0+\epsilon$ when $\epsilon \to 0$, don't gives additional information.

Thus $G_m(x,x_0)$ depend of the arbitrary constan α, yielding an infinite number of symmetric modified Green's functions.

To obtain a formula for $u(x)$, we use the Green's formula (9.3.54) with $v = G_m$:

$$\int_0^\pi \left[u\mathcal{L}(G_m) - G_m\mathcal{L}(u)\right]dx = p\left(u\frac{dG_m}{dx} - G_m\frac{du}{dx}\right)\Big|_0^\pi \quad \text{here } \mathcal{L} = \frac{d^2}{dx^2}+1 \Rightarrow p=1$$

$$\int_0^\pi \left[u(\delta(x-x_0)+c\sin x) - G_m f(x)\right]dx = \underbrace{u(\pi)}_{\beta}\frac{dG_m}{dx}(\pi) - \cancel{G_m(\pi)}_{0}\frac{du}{dx}(\pi) - \underbrace{u(0)}_{\alpha}\frac{dG_m}{dx}(0) + \cancel{G_m(0)}_{0}\frac{du}{dx}(0)$$

$$u(x_0) + \int_0^\pi \left[\left(\frac{-2}{\pi}\sin x_0\right)\overset{u(x)}{\sin x} - G_m f(x)\right]dx = \beta\frac{dG_m}{dx}(\pi) - \alpha\frac{dG_m}{dx}(0)$$

$$u(x_0) = \int_0^\pi f(x)G_m(x,x_0)\,dx + \frac{2}{\pi}\sin x_0\int_0^\pi u(x)\sin x\,dx + \beta\frac{dG_m}{dx}(\pi) - \alpha\frac{dG_m}{dx}(0)$$

We can use any modified Green's function. Choose $\alpha = 0$

$$\Rightarrow \frac{dG_m}{dx} = \frac{1}{\pi}\left(\sin x_0\cos x - x\sin x_0\sin x + x_0\cos x_0\cos x\right) + \begin{cases} -\cos x_0\cos x & x<x_0 \\ +\sin x_0\sin x & x>x_0\end{cases}$$

$$\frac{dG_m}{dx}(0) = \frac{1}{\pi}\left(\sin x_0 + x_0\cos x_0\right) - \cos x_0$$

$$\frac{dG_m}{dx}(\pi) = -\frac{1}{\pi}\left(\sin x_0 + x_0\cos x_0\right)$$

$$\Rightarrow u(x_0) = \int_0^\pi f(x)G_m(x,x_0)\,dx + \frac{2}{\pi}\sin x_0\int_0^\pi u(x)\sin x\,dx - \frac{\beta+\alpha}{\pi}\left(\sin x_0 + x_0\cos x_0\right) + \alpha\cos x_0$$

Switching $x \leftrightarrow x_0$ and using the symmetry $G_m(x_0,x) = G(x,x_0)$:

$$\boxed{u(x) = \int_0^\pi f(x_0)G_m(x,x_0)\,dx_0 + \left(\frac{2}{\pi}\int_0^\pi u(x_0)\sin x_0\right)\sin x - \frac{(\beta+\alpha)}{\pi}\left(\sin x + x\cos x\right) + \alpha\cos x}$$

This is an arbitrary constant multiple of the homogeneous solution $\phi_h \sim \sin x$.

~~This solution is consistent based on calculating $\int_0^\pi u(x)\sin x\,dx$~~

According to exercise (9.4.2) a solution ~~exists~~ only exists if

$$\int_0^\pi \phi_h(x)f(x)\,dx = p(0)\alpha\frac{d\phi_h}{dx}(0) - p(L)\beta\frac{d\phi_h}{dx}(L)$$

here $p(0) = p(L) = 1$

$\phi_h = \sin x \to \frac{d\phi_h}{dx} = \cos x \Rightarrow \boxed{\int_0^\pi f(x)\sin x\,dx = \alpha+\beta}$ Condition for solution to exist.

(9.4.11) a) $\int_0^L \frac{d^2 G_\alpha}{dx^2}\,dx = \cancelto{1}{\int_0^L \delta(x-x_0)\,dx} \rightarrow \left.\frac{dG_\alpha}{dx}\right|_0^L = 1$

$\underbrace{\frac{dG_\alpha}{dx}(L)}_{c} - \cancelto{0}{\frac{dG_\alpha}{dx}(0)} = 1 \rightarrow c = 1$

b) The modified Green's function represents a steady-state heat flow problem with heat source $-\delta(x-x_0)$ insulated at $x=0$ with heat flow to the right (out) at $x=L$ of $-c$. The total thermal energy generated inside per unit time is $\int_0^L -\delta(x-x_0)\,dx = -1$. For equilibrium, the heat generated inside must equal the heat flowing out. Thus

$$-1 = -c \rightarrow c = 1.$$

c) For $x \neq x_0 \rightarrow \frac{d^2 G_\alpha}{dx^2} = 0 \rightarrow G_\alpha(x,x_0) = \begin{cases} c_1 x + c_2 & x < x_0 \\ c_3 x + c_4 & x > x_0 \end{cases}$

$\frac{dG_\alpha}{dx}(0) = 0 = c_1, \quad \frac{dG_\alpha}{dx}(L) = c = 1 = c_3$

At $x = x_0$, $G(x_0^-, x_0) = G(x_0^+, x_0) \rightarrow \cancelto{0}{c_1} x_0 + c_2 = \overset{1}{c_3} x_0 + c_4 \Rightarrow G_\alpha(x,x_0) = \begin{cases} x_0 + c_4 & x < x_0 \\ x + c_4 & x > x_0 \end{cases}$

where c_4 can be function of x_0, $c_4 = c_4(x_0)$.
$G(x,x_0)$ automatically satisfies the jump condition $\left.\frac{dG_\alpha}{dx}\right|_{x_0^-}^{x_0^+} = 1$
Then integrate the differential equation around x_0 from $x_0-\epsilon$ to $x_0+\epsilon$ when $\epsilon \to 0$, don't gives additional information.

d) By symmetry $G_\alpha(x,x_0) = G_\alpha(x_0,x) = \begin{cases} x + c_4(x_0) & x_0 < x \\ x_0 + c_4(x_0) & x_0 > x \end{cases}$

$\Rightarrow c_4(x_0) = c_4(x) = \alpha \rightarrow$ an arbitrary constant independent of x_0.

$\Rightarrow \boxed{G(x,x_0) = \alpha + \begin{cases} x_0 & x < x_0 \\ x & x > x_0 \end{cases}}$ There is an infinite number of symmetric modified Green's function of this type, because if there is one any multiple of ϕ_n can be added.

e) Using the Green's formula (9.3.54) with $v = G_\alpha(x,x_0)$

$$\int_0^L \left(u\mathcal{L}(G_\alpha) - G_\alpha(x,x_0)\mathcal{L}(u)\right)dx = p\left(u\frac{dG_\alpha}{dx} - G_\alpha(x,x_0)\frac{du}{dx}\right)\Big|_0^L \quad \text{here } \mathcal{L} = d^2/dx^2,\ p = 1$$

$$\int_0^L \left(u\,\delta(x-x_0) - G_\alpha(x,x_0)f(x)\right)dx = u(L)\underbrace{\frac{dG_\alpha}{dx}(L)}_{c=1} - u(0)\cancelto{0}{\frac{dG_\alpha}{dx}(0)} - G_\alpha(L,x_0)\cancelto{0}{\frac{du}{dx}(L)} + G_\alpha(0,x_0)\cancelto{0}{\frac{du}{dx}(0)}$$

$$u(x_0) - \int_0^L G_\alpha(x,x_0)f(x)\,dx = u(L)$$

$\boxed{u(x) = \int_0^L G_\alpha(x,x_0)f(x_0)\,dx_0 + u(L)}$ where we used the symmetry of $G_\alpha(x,x_0)$.

$\cancel{u(L)} = \int_0^L \underbrace{G_\alpha(L,x_0)}_{\alpha+L} f(x_0)\,dx_0 + \cancel{u(L)} \rightarrow \boxed{\int_0^L f(x_0)\,dx_0 = 0} \rightarrow$ Condition for the solution to exist.

Thus $u(L)$ is arbitrary above (and may be replaced by an arbitrary constant), representing an arbitrary multiple of $\phi_n(x) = 1$.

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9.4.12 a) $\int_0^L \frac{d^2G_\alpha}{dx^2}\,dx = \cancel{\int_0^L \delta(x-x_0)\,dx} \to 1 \Rightarrow \overbrace{\frac{dG_\alpha}{dx}(L)}^{c} - \overbrace{\frac{dG_\alpha}{dx}(0)}^{-c} = 1 \Rightarrow \boxed{c = 1/2}$

For $x \neq x_0 \to \frac{d^2G_\alpha}{dx^2} = 0 \to G_\alpha(x,x_0) = \begin{cases} c_1 x + c_2 & x < x_0 \\ c_3 x + c_4 & x > x_0 \end{cases}$

$\frac{dG_\alpha}{dx}(0) = -c = c_1, \quad \frac{dG_\alpha}{dx}(L) = c = c_3$ where $\boxed{c = 1/2}$

At $x = x_0$, $G(x_0^-, x_0) = G(x_0^+, x_0) \to c_1 x_0 + c_2 = c_3 x_0 + c_4$

$-c x_0 + c_2 = c x_0 + c_4 \to c_2 = 2c x_0 + c_4$

$\Rightarrow G_\alpha(x,x_0) = \begin{cases} -cx + 2cx_0 + c_4 & x < x_0 \\ cx + c_4 & x > x_0 \end{cases}$ where $c_4 \equiv c_4(x_0)$, is a arbitrary constant function of x_0. and $c = 1/2$.

Since $c = 1/2$, $G(x,x_0)$ automatically satisfies the jump condition $\frac{dG_\alpha}{dx}\Big|_{x_0^-}^{x_0^+} = 1$. Then integrate the differential equation around x_0 from $x_0 - \epsilon$ to $x_0 + \epsilon$ when $\epsilon \to 0$, don't gives additional information.

b) By symmetry $G_\alpha(x,x_0) = G_\alpha(x_0,x) = \begin{cases} -cx_0 + 2cx + c_4(x) & x_0 < x \\ cx_0 + c_4(x) & x_0 > x \end{cases}$

$\Rightarrow \quad cx + c_4(x_0) = -cx_0 + 2cx + c_4(x)$

that implies $c_4(x_0) = -cx_0 + \alpha \qquad c_4(x) = -cx + \alpha$

$\Rightarrow \boxed{G_\alpha(x,x_0) = \alpha + \begin{cases} c(x_0 - x) & x < x_0 \\ c(x - x_0) & x > x_0 \end{cases}}$ There is an infinite number of symmetric modified Green's function of this type.

α can be an arbitrary constant.

That is not orthogonal to ϕ_h (you can show that). For find $G_\alpha(x,x_0)$ orthogonal to ϕ_h we need to proceed as in equation (9.4.15).

$\cancel{\int_0^L (u\mathcal{L}(G_\alpha) - G_\alpha(x,x_0)\mathcal{L}(u))\,dx = p\left(u\frac{dG_\alpha}{dx} - G_\alpha(x,x_0)\frac{du}{dx}\right)\Big|_0^L}$ here $\cancel{k=1\ d^2/dx^2}$ $\cancel{p=k}$.

$\cancel{\int_0^L (u\delta(x-x_0) - G_\alpha(x,x_0)f(x))\,dx = u(L)\frac{dG_\alpha}{dx}(L) - u(0)\frac{dG_\alpha}{dx}(0) - G_\alpha(L,x_0)\frac{du}{dx}(L) + G_\alpha(0,x_0)\frac{du}{dx}(0)}$

c) $\phi_h = Ax + B$

c) Option 1 Let $A = 1$, $B = 0 \Rightarrow$. A modified Green's function $G_m(x,x_0)$ must be introduced satisfying:

$$\mathcal{L}[G_m(x,x_0)] = \delta(x - x_0) + \beta\phi_h(x)$$

The constant β is chosen so that the right-side is orthogonal to $\phi_h \Rightarrow \int_0^L x\,[\delta(x-x_0) + \beta x]\,dx = 0$

$$x_0 + \beta\frac{L^3}{3} = 0 \to \beta = -\frac{3}{L^3}x_0$$

$\mathcal{L}[G_m(x,x_0)] = \delta(x - x_0) - \frac{3}{L^3}x_0 x \Rightarrow$

For $x \neq x_0$ $\rightarrow$ $\dfrac{d^2 G_m}{dx^2} = -\dfrac{3}{L^3} x_0 x$ $\rightarrow$ by direct integration

$$G_m = -\frac{x_0}{2L^3} x^3 + C_1 x + C_2$$

but we have two regions ($x < x_0$ and $x > x_0$)

$$\rightarrow \quad G_m(x, x_0) = -\frac{x_0}{2L^3} x^3 + \begin{cases} C_1 x + C_2 & x < x_0 \\ C_3 x + C_4 & x > x_0. \end{cases}$$

$$\frac{dG_m}{dx}(0) = -c = C_1 \qquad \frac{dG_m}{dx}(L) = -\frac{3x_0}{2L^3} L^2 + C_3 = c \rightarrow C_3 = c + \frac{3x_0}{2L}$$

where the constant c can be determined integrating the differential equation between 0 to L:

$$\int_0^L \frac{d^2 G_m}{dx^2} dx = \underbrace{\int_0^L \delta(x - x_0) dx}_{1} - \frac{3}{L^3} x_0 \int_0^L x \, dx$$

$$\underbrace{\frac{dG_m}{dx}(L)}_{c} - \underbrace{\frac{dG_m}{dx}(0)}_{-c} = 1 - \frac{3}{L^3} x_0 \frac{L^2}{2} \rightarrow c = \frac{1}{2} - \frac{3x_0}{4L} = -C_1$$

$$\Rightarrow C_3 = \frac{1}{2} + \frac{3}{4}\frac{x_0}{L}$$

At $x = x_0$, $G(x_0^-, x) = G(x_0^+, x) \rightarrow C_1 x_0 + C_2 = C_3 x_0 + C_4$

$$C_2 = (C_3 - C_1) x_0 + C_4 = x_0 + C_4$$

$$\Rightarrow G_m(x, x_0) = -\frac{x_0}{2L^3} x^3 + \begin{cases} -\left(\frac{1}{2} - \frac{3}{4}\frac{x_0}{L}\right) x + x_0 + C_4 & x < x_0 \\ \left(\frac{1}{2} + \frac{3}{4}\frac{x_0}{L}\right) x + C_4 & x > x_0 \end{cases}$$

where $C_4 \equiv C_4(x_0)$ is an arbitrary constant that can be function of x_0. The jump condition for the derivative ($dG_m/dx|_{x_0^-}^{x_0^+} = 1$) obtained by integrating the differential equation, is already satisfied by $G_m(x, x_0)$. Then apply that condition don't gives additional information.

By Symmetry $G_m(x, x_0) = G_m(x_0, x) = -\dfrac{x}{2L^3} x_0^3 + \begin{cases} -\left(\frac{1}{2} - \frac{3}{4}\frac{x}{L}\right) x_0 + x + C_4(x) & x_0 < x \\ \left(\frac{1}{2} + \frac{3}{4}\frac{x}{L}\right) x_0 + C_4(x) & x_0 > x. \end{cases}$

$$\Rightarrow -\frac{x_0}{2L^3} x^3 - \left(\frac{1}{2} - \cancel{\frac{3}{4}\frac{x_0}{L}}\right) x + x_0 + C_4(x_0) = -\frac{x}{2L^3} x_0^3 + \left(\frac{1}{2} + \cancel{\frac{3}{4}\frac{x}{L}}\right) x_0 + C_4(x)$$

$$C_4(x_0) = -\frac{x}{2L^3} x_0^3 - \frac{1}{2} x_0 \rightarrow$$ Contradiction since C_4 must not depend of x variable.

Option 2: $A = B = 1 \rightarrow \phi_h = x + 1 \rightarrow$ This election also lead to a contradiction, you can see that after a exhaustive calculation.

Option 3: Let $A = 0$ and $B = 1 \rightarrow \phi_h = 1 \Rightarrow \mathcal{L}[G_m(x, x_0)] = \delta(x - x_0) + \beta$

Since the rigth side must be orthogonal to ϕ_h:

$$\int_0^L 1 \cdot [\delta(x - x_0) + \beta] dx = 1 + \beta L = 0 \rightarrow \boxed{\beta = -1/L}$$

$$\Rightarrow \mathcal{L}[G_m(x, x_0)] = \delta(x - x_0) - \frac{1}{L} \quad \Rightarrow$$

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For $x \neq x_0$ $\rightarrow \dfrac{d^2G_m}{dx^2} = -\dfrac{1}{L} \Rightarrow G_m(x,x_0) = -\dfrac{1}{L}\dfrac{x^2}{2} + \begin{cases} c_1 x + c_2 & x < x_0 \\ c_3 x + c_4 & x > x_0 \end{cases}$

$\dfrac{dG_m}{dx}(0) = -c = c_1 \rightarrow c_1 = -c$

$\dfrac{dG_m}{dx}(L) = c = -1 + c_3 \rightarrow c_3 = 1 + c$

where the constant c can be determined integrating the differential equation between 0 to L:

$$\int_0^L \frac{d^2G_m}{dx^2}\,dx = \underbrace{\int_0^L \delta(x-x_0)\,dx}_{1} - \underbrace{\frac{1}{L}\int_0^L dx}_{1}$$

$$\underbrace{\frac{dG_m}{dx}(L)}_{c} - \underbrace{\frac{dG}{dx}(0)}_{-c} = 1 - 1 \Rightarrow c = 0 \Rightarrow c_1 = 0,\ c_3 = 1$$

$$G_m(x,x_0) = -\frac{1}{L}\frac{x^2}{2} + \begin{cases} c_2 & x < x_0 \\ x + c_4 & x > x_0 \end{cases}$$

At $x = x_0$, $G(x_0^-, x) = G(x_0^+, x) \rightarrow c_2 = x_0 + c_4$ where $c_4 \equiv c_4(x_0)$ is an arbitrary constant that can be function of x_0.

By Symmetry $G_m(x,x_0) = G_m(x_0,x) = -\dfrac{1}{L}\dfrac{x_0^2}{2} + \begin{cases} x + c_4(x) & x_0 < x \\ x_0 + c_4(x) & x_0 > x \end{cases}$

$$\Rightarrow -\frac{1}{L}\frac{x^2}{2} + \cancel{x_0} + c_4(x_0) = -\frac{1}{L}\frac{x_0^2}{2} + \cancel{x_0} + c_4(x) \rightarrow c_4(x_0) = -\frac{1}{L}\frac{x_0^2}{2} + \alpha$$

$$\Rightarrow \boxed{g_m(x,x_0) = \alpha - \frac{1}{2L}(x^2 + x_0^2) + \begin{cases} x_0 & x < x_0 \\ x & x > x_0 \end{cases}}$$

There is an infinite number of symmetric modified Green's functions of this type since α can be an arbitrary constant

Since $c = 0$, $G(x,x_0)$ atumatically satisfy the jump condition $\left.\dfrac{dG_m}{dx}\right|_{x_0^-}^{x_0^+} = 1$, then use this don't gives additional information. (21.4.P)

That is not orthogonal to $\phi_h(x) = 1$, but we can do that using

$$G_m(x,x_0) = g_m(x,x_0) + \gamma\,\phi_h(x_0)\,\phi_h(x)$$

and choosing γ such that G_m be orthogonal to ϕ_h:

$$\int_0^L \phi_h(x)G_m(x,x_0)\,dx = \int_0^{x_0} \phi_h(x)G_m(x,x_0)\,dx + \int_0^L \phi_h(x)G_m(x,x_0)\,dx \qquad \phi_h(x) = 1$$

$$= \cancel{\alpha x_0 - \frac{1}{2L}\left(\frac{x_0^3}{3} + x_0^2 x_0\right)} + \cancel{x_0 x_0} + \alpha L - \frac{1}{2L}\left(\frac{L^3}{3} + \cancel{x_0^2 L}\right) + \frac{L^2}{2}$$

$$-\left[\cancel{\alpha x_0 - \frac{1}{2L}\left(\frac{x_0^2}{3} + x_0^2 x_0\right)} + \cancel{\frac{x_0^2}{2}}\right] + \gamma L = 0$$

$$= \left(-\frac{L}{6} + \frac{L}{2} + \alpha - \gamma\right)L = 0 \rightarrow \gamma = -\left(\frac{L}{3} + \alpha\right)$$

$$\Rightarrow \boxed{G_m(x,x_0) = -\frac{L}{3} - \frac{1}{2L}(x^2 + x_0^2) + \begin{cases} x_0 & x < x_0 \\ x & x > x_0 \end{cases}}$$

This is a unique generalized Green's function, such as problem (9.4.9) says.

d) To obtain a formula for $u(x)$, we use the Green's formula (9.3.54) with $v = G_\alpha(x, x_0)$, the Green's function defined in part (b) if we want $u(x)$ in terms of the Neumann function

$$\int_0^L \left[u\mathcal{L}[G_\alpha] - G_\alpha(x,x_0)\mathcal{L}[u] \right] dx = p\left(u\frac{dG_\alpha}{dx} - G_\alpha \frac{du}{dx} \right)\Big|_0^L \quad \text{where } \mathcal{L} = d^2/dx^2, \; p = 1$$

$$\int_0^L \left[u(x)\delta(x-x_0) - G_\alpha(x,x_0)f(x) \right] dx = u(L)\underbrace{\frac{dG_\alpha}{dx}(L)}_{c} - u(0)\underbrace{\frac{dG_\alpha}{dx}(0)}_{-c} - G_\alpha(L,x_0)\frac{du}{dx}(L) + G_\alpha(0,x_0)\frac{du}{dx}(0)$$

$$u(x_0) - \int_0^L G_\alpha(x,x_0)f(x)dx = u(L)\cdot c - u(0)\cdot(-c) - c(L-x_0)\cdot\gamma + cx_0\cdot\beta$$

$$= c\left[u(0) + u(L) + (\gamma+\beta)x_0 - \gamma L\right] \quad \text{where we chose } \alpha = 0$$

Remember that $\boxed{c = 1/2}$ for this case.

Switching $x \leftrightarrow x_0$ and using the symmetry $G_\alpha(x_0, x) = G_\alpha(x, x_0)$:

$$\boxed{u(x) = \int_0^L G_\alpha(x,x_0)f(x_0)dx_0 + \frac{1}{2}\left[u(0) + u(L) + (\gamma+\beta)x - \gamma L\right]}$$

$$u(L) = \int_0^L G_\alpha(L,x_0)f(x_0)dx_0 + \frac{1}{2}\left[u(0) + u(L) + (\gamma+\beta)L - \gamma L\right]$$

$$\frac{1}{2}u(L) = \int_0^L c(L-x_0)f(x_0)dx_0 + \frac{1}{2}\left[u(0) + (\gamma+\beta)L - \gamma L\right] \quad \text{with } c = 1/2. \text{ Now,}$$

$$u(0) = \int_0^L G_\alpha(0,x_0)f(x_0)dx_0 + \frac{1}{2}\left[u(0) + u(L) - \gamma L\right]$$

$$\cancel{\frac{1}{2}u(0)} = \cancel{\int_0^L cx_0 f(x_0)dx_0} + \left(\int_0^L c(L-\cancel{x_0})f(x_0)dx_0 + \frac{1}{2}\left[\cancel{u(0)} + (\gamma+\beta)L - \gamma L\right]\right) - \frac{1}{2}\gamma L$$

Since $c = 1/2$ $\Rightarrow$ $\boxed{\int_0^L f(x_0)dx_0 = (2L-1)\gamma + \beta}$ $\rightarrow$ Condition for the existence of the solution.

9.4.13 Just as in problem (9.4.10): $\dfrac{d^2u}{dx^2} + u = f(x)$ $\quad u(0) = \alpha = 1$, $u(\pi) = \beta = 4$

$$\Rightarrow \quad \frac{d^2G}{dx^2} + G = \delta(x-x_0) \quad \text{with } G(0,x_0) = G(\pi,x_0) = 0$$

For $x \neq x_0 \Rightarrow \dfrac{d^2G}{dx^2} + G = 0 \rightarrow G = C_1\cos x + C_2\sin x$

$$\left.\begin{aligned} G(0,x_0) &= 0 = C_1 \\ G(\pi,x_0) &= 0 = -C_1 \end{aligned}\right\} \Rightarrow C_2 \text{ is arbitrary}$$

Thus $\phi_h = \sin x$ is a solution of the homogeneous equation. A modified Green's function $G_m(x,x_0)$ must be introduced satisfying

$$\mathcal{L}[G_m(x,x_0)] = \delta(x-x_0) + c\,\phi_h(x)$$

Since the right-hand side of this equation is orthogonal to $\phi_h(x)$ there are an infinite number of solutions.

According to exercise (9.4.2) a solution exists if

$$\int_0^\pi \phi_h(x)f(x)dx = p(0)\alpha\frac{d\phi_h}{dx}(0) - p(L)\beta\frac{d\phi_h}{dx}(L) \quad \text{here } p(L) = p(0) = 1 \text{ and } \phi_h(x) = \sin x$$

$$\Rightarrow \boxed{\int_0^\pi f(x)\sin x\,dx = \alpha + \beta = 5} \rightarrow \text{Condition on } f(x) \text{ for the solution to exist.}$$

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$$\mathcal{L} = \frac{d^2}{dx^2} + 1$$

Periodic BC's

9.4.14 $\frac{d^2u}{dx^2} + u = f(x)$ with $u(-L) = u(L)$ $\quad \frac{du}{dx}(-L) = \frac{du}{dx}(L)$

a) Let be $\phi_h(x)$ a homogeneous solution satisfying the same BC's. Then use the Green's formula (9.3.54) with $v = \phi_h$

$$\Rightarrow \int_{-L}^{L} [u\underbrace{\mathcal{L}(\phi_h)}_{0} - \phi_h \underbrace{\mathcal{L}(u)}_{f(x)}]\,dx = p\left(u\frac{d\phi_h}{dx} - \phi_h\frac{du}{dx}\right)\Big|_{-L}^{L} \quad \text{here } p = 1$$

$$= \cancel{u(L)\frac{d\phi_h}{dx}(L)} - \cancel{u(-L)\frac{d\phi_h}{dx}(-L)} = 0$$

Since both satisfy periodic BC's

$$\Rightarrow \boxed{\int_{-L}^{L} \phi_h(x) f(x)\,dx = 0} \rightarrow \text{Condition for the existence of the solution.}$$

b) $\frac{d^2G_m}{dx^2} + G_m = \delta(x - x_0) + C_1\cos\left(\frac{\pi x}{L}\right) + C_2\sin\left(\frac{\pi x}{L}\right)$

Since $\phi_h(x) = \cos(\pi x/L)$ OR $\sin\left(\frac{\pi x}{L}\right)$, we need that this function be orthogonal to the right-hand side of the equation above:

$$\int_{-L}^{L} \cos\left(\frac{\pi x}{L}\right)\left[\delta(x - x_0) + C_1\cos\left(\frac{\pi x}{L}\right)\right]x = 0$$

$$\underbrace{\cos(\pi x_0/L)}_{} + C_1\underbrace{\int_{-L}^{L}\cos^2\frac{\pi x}{L}\,dx}_{L} = 0 \rightarrow \boxed{C_1(x_0) = -\frac{1}{L}\cos(\pi x_0/L)}$$

Similarly:

$$\int_{-L}^{L}\sin\left(\frac{\pi x}{L}\right)\left[\delta(x - x_0) + C_2\sin\left(\frac{\pi x}{L}\right)\right]dx = 0 \rightarrow \boxed{C_2(x_0) = -\frac{1}{L}\sin(\pi x_0/L)}$$

c) To obtain a formula for $u(x)$ use the Green's formula (9.3.54) with $v = G_m(x, x_0)$

$$\int_{-L}^{L}\left[u\underbrace{\mathcal{L}[G_m]}_{\delta(x-x_0)} - G_m(x, x_0)\mathcal{L}(u)\right]dx = p\left(u\frac{dG_m}{dx} - G_m\frac{du}{dx}\right)\Big|_{0}^{L} \quad \text{where } \mathcal{L} = \frac{d^2}{dx^2} + 1, \; p = 1$$

$$\int_{-L}^{L}\left[u(x)\underbrace{\left(\delta(x - x_0) + C_1\cos\frac{\pi x}{L} + C_2\sin\frac{\pi x}{L}\right)}_{u(x_0)} - G_m(x, x_0)f(x)\right]dx$$

$$= u(L)\frac{dG_m}{dx}(L) - u(-L)\frac{dG_m}{dx}(-L) - G_m(L, x_0)\frac{du}{dx}(L) + G_m(-L, x_0)\frac{du}{dx}(-L)$$

Since u satisfy the periodic BC's

$$= u(L)\left[\frac{dG_m}{dx}(L) - \frac{dG_m}{dx}(-L)\right] - \frac{du}{dx}(L)\left[G(L, x_0) - G_m(-L, x_0)\right]$$

$$\Rightarrow \boxed{\begin{aligned} u(x) = {} & \int_{-L}^{L} G_m(x, x_0)f(x)\,dx + C_1(x)\int_{-L}^{L} u(x_0)\cos\left(\frac{\pi x_0}{L}\right)dx_0 + C_2(x)\int_{-L}^{L} u(x_0)\sin\left(\frac{\pi x_0}{L}\right)dx_0 \\ & + u(L)\left[\frac{dG_m}{dx_0}(L) - \frac{dG_m}{dx_0}(-L)\right] - \frac{du}{dx}(L)\left[G(x, L) - G_m(x, -L)\right] \end{aligned}}$$

We obtained $u(x)$ interchanging $x \leftrightarrow x_0$ and using the symmetry of $G_m(x, x_0)$. The second and third ~~equations~~ integrals are constants and represent arbitrary multiples of the homogeneous solutions. Since $C_1(x) \sim \cos(\pi x/L)$ and $C_2(x) \sim \sin(\pi x/L)$.

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9.5.1 Let suppose $G(\vec{x},\vec{x}_0)=\sum_\lambda a_\lambda \phi_\lambda(\vec{x})$ and $\nabla^2 G=\sum_\lambda b_\lambda \phi_\lambda(\vec{x})$ where $\phi_\lambda(\vec{x})$ are the eigenfunctions of $\nabla^2\phi_\lambda=-\lambda\phi_\lambda$, subject to the same BC's of $u(\vec{x})$ in (9.5.9).

Use the Green's formula with $u=\phi_\lambda(\vec{x})$ and $v=G(\vec{x},\vec{x}_0)$:

$$\int_V(\phi_\lambda\nabla^2 G-G\nabla^2\phi_\lambda)\,dV=\cancel{\oint_S(\phi_\lambda\nabla G-G\nabla\phi_\lambda)\cdot\hat{n}\,ds}\to 0$$

Since $\phi_\lambda(\vec{x})$ and $G(\vec{x},\vec{x}_0)$ satisfy the same homogeneous BC's.

$$\sum_\lambda\int_V\left[\phi_\lambda(b_\lambda\phi_\lambda)-(a_\lambda\phi_\lambda)\nabla^2\phi_\lambda\right]dV=0$$

$$\Rightarrow \phi_\lambda[b_\lambda\phi_\lambda-a_\lambda\nabla^2\phi_\lambda]=0 \to \phi_\lambda=0 \quad \text{or} \quad b_\lambda\phi_\lambda-a_\lambda\nabla^2\phi_\lambda=0$$

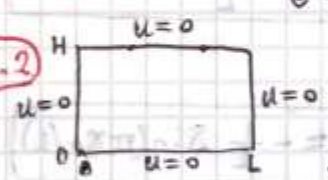
Sice we don't want trivial solutions $\phi_\lambda\neq 0$, then holds the last equality. Now

$$\nabla^2\phi_\lambda=-\lambda\phi_\lambda \Rightarrow b_\lambda\phi_\lambda-a_\lambda(-\lambda\phi_\lambda)=0 \to \boxed{b_\lambda=-\lambda a_\lambda}$$

then $\nabla^2 G=\sum_\lambda(-\lambda a_\lambda)\phi_\lambda=\sum_\lambda a_\lambda\nabla^2\phi_\lambda=\nabla^2\left(\sum_\lambda a_\lambda\phi_\lambda\right)$

and thus $\nabla^2 G$ may be obtained by term-by-term differentiation of (9.5.10) OR $G=\sum_\lambda a_\lambda\phi_\lambda$.

9.5.2



$u(0,y)=u(L,y)=0 \to u(x,y)\sim \sin(n\pi x/L)$

$u(x,0)=u(x,H)=0 \to u(x,y)\sim \sin(m\pi y/H)$

a)

$$\Rightarrow \boxed{u(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{nm}\sin(n\pi x/L)\sin(m\pi y/H)}$$

$$\underset{f(\vec{x})}{\nabla^2 u}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}-a_{nm}\left[\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\sin(n\pi x/L)\sin(m\pi y/H)$$

Multiplying both sides by $\sin(n'\pi x/L)\sin(m'\pi y/H)$ and integrating:

$$\int_0^L\int_0^H f(\vec{x})\sin\left(\frac{n'\pi x}{L}\right)\sin\left(\frac{m'\pi y}{H}\right)dy\,dx$$

$$=-\sum_{n,m}a_{nm}\left[\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\underbrace{\int_0^L\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}}\underbrace{\int_0^H\sin\left(\frac{m\pi y}{L}\right)\sin\left(\frac{m'\pi y}{H}\right)dy}_{(H/2)\delta_{mm'}}$$

$$=-a_{n'm'}\left[\left(\frac{n'\pi}{L}\right)^2+\left(\frac{m'\pi}{H}\right)^2\right]\frac{LH}{4}$$

$\vec{x}_0=(x_0,y_0)$

$$\boxed{a_{nm}=-\frac{4}{LH}\frac{1}{[(n\pi/L)^2+(m\pi/H)^2]}\int_0^L\int_0^H f(\vec{x}_0)\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{m\pi y_0}{H}\right)dy_0\,dx_0}$$

b) Substituting a_{nm} in the expansión for $u(\vec{x})$: and intechanging the summ with the integration:

$$u(\vec{x})=\int_0^L\int_0^H f(\vec{x}_0)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{-4}{LH}\frac{\sin(n\pi x_0/L)\sin(m\pi y_0/H)\sin(n\pi x/L)\sin(m\pi y/H)}{[(n\pi/L)^2+(m\pi/H)^2]}dy_0\,dx_0$$

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$$\mathcal{L} = \frac{d^2}{dx^2} + 1$$

Periodic BC's

9.4.14 $\frac{d^2u}{dx^2} + u = f(x)$ with $u(-L) = u(L)$ $\quad \frac{du}{dx}(-L) = \frac{du}{dx}(L)$

a) Let be $\phi_h(x)$ a homogeneous solution satisfying the same BC's. Then use the Green's formula (9.3.54) with $v = \phi_h$

$$\Rightarrow \int_{-L}^{L} [u\underbrace{\mathcal{L}(\phi_h)}_{0} - \phi_h \underbrace{\mathcal{L}(u)}_{f(x)}]\,dx = p\left(u\frac{d\phi_h}{dx} - \phi_h\frac{du}{dx}\right)\Big|_{-L}^{L} \quad \text{here } p = 1$$

$$= u(L)\frac{d\phi_h}{dx}(L) - u(-L)\frac{d\phi_h}{dx}(-L) = 0$$ Since both satisfy periodic BC's

$$\Rightarrow \boxed{\int_{-L}^{L} \phi_h(x) f(x)\,dx = 0} \rightarrow \text{Condition for the existence of the solution.}$$

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$$= u(L)\left[\frac{dG_m}{dx}(L) - \frac{dG_m}{dx}(-L)\right] - \frac{du}{dx}(L)\left[G(L, x_0) - G_m(-L, x_0)\right]$$

$$\Rightarrow \boxed{\begin{aligned} u(x) = {} & \int_{-L}^{L} G_m(x, x_0)f(x)\,dx + C_1(x)\int_{-L}^{L} u(x_0)\cos\left(\frac{\pi x_0}{L}\right)dx_0 + C_2(x)\int_{-L}^{L} u(x_0)\sin\left(\frac{\pi x_0}{L}\right)dx_0 \\ & + u(L)\left[\frac{dG_m}{dx_0}(L) - \frac{dG_m}{dx_0}(-L)\right] - \frac{du}{dx}(L)\left[G(x, L) - G_m(x, -L)\right] \end{aligned}}$$

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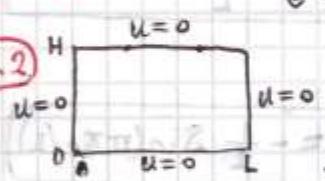
then $\nabla^2 G=\sum_\lambda(-\lambda a_\lambda)\phi_\lambda=\sum_\lambda a_\lambda\nabla^2\phi_\lambda=\nabla^2\left(\sum_\lambda a_\lambda\phi_\lambda\right)$

and thus $\nabla^2 G$ may be obtained by term-by-term differentiation of (9.5.10) OR $G=\sum_\lambda a_\lambda\phi_\lambda$.

9.5.2



$u(0,y)=u(L,y)=0 \to u(x,y)\sim \sin(n\pi x/L)$

$u(x,0)=u(x,H)=0 \to u(x,y)\sim \sin(m\pi y/H)$

a) $$\Rightarrow \boxed{u(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{nm}\sin(n\pi x/L)\sin(m\pi y/H)}$$

$$\underset{\underset{f(\vec{x})}{\parallel}}{\nabla^2 u}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}-a_{nm}\left[\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\sin(n\pi x/L)\sin(m\pi y/H)$$

Multiplying both sides by $\sin(n'\pi x/L)\sin(m'\pi y/H)$ and integrating:

$$\int_0^L\int_0^H f(\vec{x})\sin\left(\frac{n'\pi x}{L}\right)\sin\left(\frac{m'\pi y}{H}\right)dy\,dx$$

$$=-\sum_{n,m}a_{nm}\left[\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\underbrace{\int_0^L\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}}\underbrace{\int_0^H\sin\left(\frac{m\pi y}{L}\right)\sin\left(\frac{m'\pi y}{H}\right)dy}_{(H/2)\delta_{mm'}}$$

$$=-a_{n'm'}\left[\left(\frac{n'\pi}{L}\right)^2+\left(\frac{m'\pi}{H}\right)^2\right]\frac{LH}{4}$$

$\vec{x}_0=(x_0,y_0)$

$$\boxed{a_{nm}=-\frac{4}{LH}\frac{1}{\left[(n\pi/L)^2+(m\pi/H)^2\right]}\int_0^L\int_0^H f(\vec{x}_0)\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{m\pi y_0}{H}\right)dy_0\,dx_0}$$

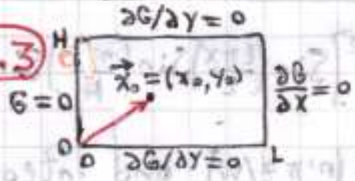
b) Substituting a_{nm} in the expansión for $u(\vec{x})$: and intechanging the summ with the integration:

$$u(\vec{x})=\int_0^L\int_0^H f(\vec{x}_0)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{-4}{LH}\frac{\sin(n\pi x_0/L)\sin(m\pi y_0/H)\sin(n\pi x/L)\sin(m\pi y/H)}{\left[(n\pi/L)^2+(m\pi/H)^2\right]}dy_0\,dx_0$$

OR $\quad u(\vec{x}) = \int_0^L \int_0^H f(\vec{x}_0)\, G(\vec{x},\vec{x}_0)\, dx_0\, dy_0$

with
$$\boxed{G(\vec{x},\vec{x}_0) = -\frac{4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin(n\pi x_0/L)\sin(m\pi y_0/H)\sin(n\pi x/L)\sin(m\pi y/H)}{[(n\pi/L)^2+(m\pi/H)^2]}}$$

9.5.3 a)



$G(0,y)=0,\ \frac{\partial G}{\partial x}(L,0)=0 \rightarrow G \sim \sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi x}{L}\right]$

$\frac{\partial G}{\partial y}(x,0) = \frac{\partial G}{\partial y}(x,H) = 0 \rightarrow G \sim \cos(m\pi y/H)$

$$\boxed{G(\vec{x},\vec{x}_0) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm}(\vec{x}_0)\sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi x}{L}\right]\cos\left(\frac{m\pi y}{H}\right)}$$

$$\nabla^2 G = \frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} -a_{nm}\left[\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi x}{L}\right]\cos\left(\frac{m\pi}{H}y\right)$$

$\overset{\shortparallel}{\delta(\vec{x}-\vec{x}_0)}$ multiplying by $\sin\left(\frac{2n'-1}{2}\right)\frac{\pi x}{L}\cos\left(\frac{m'\pi y}{H}\right)$ and integrating:

$$\int_0^L\int_0^H \delta(\vec{x}-\vec{x}_0)\sin\left[\left(\frac{2n'-1}{2}\right)\frac{\pi x}{L}\right]\cos\left(\frac{m'\pi y}{H}\right)dy\,dx = \sin\left[\left(\frac{2n'-1}{2}\right)\frac{\pi x_0}{L}\right]\cos\left(\frac{m'\pi y_0}{H}\right)$$

$$= -\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm}\left[\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\underbrace{\int_0^L \sin\left(\frac{2n-1}{2}\frac{\pi x}{L}\right)\sin\left(\frac{2n'-1}{2}\frac{\pi x}{L}\right)dx}_{(L/2)\delta_{nn'}}\underbrace{\int_0^H \cos\frac{m\pi y}{H}\cos\frac{m'\pi y}{H}dy}_{(H/2)\delta_{mm'}}$$

$$= -a_{n'm'}\left[\left(\frac{2n'-1}{2}\frac{\pi}{L}\right)^2+\left(\frac{m'\pi}{H}\right)^2\right]\frac{LH}{4}$$

$$\boxed{a_{nm} = -\frac{4}{LH}\left\{\sin\left(\frac{2n-1}{2}\frac{\pi x_0}{L}\right)\cos\left(\frac{m\pi y_0}{H}\right)\Big/\left[\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\right\}}$$

Note that $G(\vec{x},\vec{x})$ is symmetric

Method 2: let $x_< \equiv$ smaller of x and x_0 and similarly for y.
$x_> \equiv$ larger of x and x_0

$G(0,y)=0 \rightarrow G \sim \sin K_x x_<$ $\qquad G(x,0)=0 \rightarrow G \sim \cos K_y y_<$

$\frac{\partial G}{\partial x}(L,y)=0 \rightarrow G \sim \sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x_>\right]$ $\qquad G(x,H)=0 \rightarrow G \sim \cos\left(\frac{m\pi}{H}y_>\right)$

The symmetry demand that: $G(\vec{x},\vec{x}_0) \sim \sin\left(\frac{2n-1}{2}\frac{\pi}{L}x_<\right)\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x_>\right)\cos\left(\frac{m\pi}{H}y_<\right)\cos\left(\frac{m\pi}{H}y_<\right)$

$$\Rightarrow \boxed{G(\vec{x},\vec{x}_0) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm}\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x\right)\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x_0\right)\cos\left(\frac{m\pi}{H}y\right)\cos\left(\frac{m\pi}{H}y_0\right)}$$

where we used that is indifferent have $x>x_0$ or $x<x_0$.

By a similar way as above integrating the equation $\nabla^2 G = \delta(\vec{x}-\vec{x}_0)$ previously multiplyied by $\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x\right)\cos(m\pi y/H)$ we find:

$$\boxed{C_{nm} = -\frac{4}{LH}\Big/\left[\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]}$$

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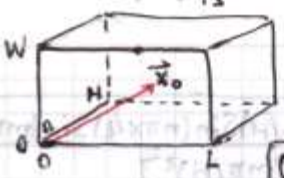
b) $G|_s = 0$



$G(0,y,z) = G(L,y,z) = 0 \rightarrow G \sim \sin(\ell\pi x/L)$

$G(x,0,z) = G(x,H,z) = 0 \rightarrow G \sim \sin(m\pi y/H)$

$G(x,y,0) = G(x,y,W) = 0 \rightarrow G \sim \sin(n\pi z/W)$

$$\boxed{G(x,y,z) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{\ell mn} \sin\left(\frac{\ell\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{n\pi z}{W}\right)}$$

$$\underset{\delta(\vec{x}-\vec{x}_0)}{\nabla^2 G} = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} = -\sum_{\ell,m,n} a_{\ell mn}\left[\left(\frac{\ell\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{W}\right)^2\right]\sin\left(\frac{\ell\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{n\pi z}{W}\right)$$

multiplying by $\sin(\ell'\pi x/L)\sin(m'\pi y/H)\sin(n'\pi z/W)$ and integrating:

$$\int_0^W\int_0^H\int_0^L \delta(\vec{x}-\vec{x}_0)\sin\left(\frac{\ell'\pi x}{L}\right)\sin\left(\frac{m'\pi y}{H}\right)\sin\left(\frac{n'\pi z}{W}\right)dx\,dy\,dz = \sin\left(\frac{\ell'\pi x_0}{L}\right)\sin\left(\frac{m'\pi y_0}{H}\right)\sin\left(\frac{n'\pi z_0}{W}\right)$$

$$= -\sum_{\ell,m,n} a_{\ell mn}\left[\ \right]\underbrace{\int_0^L \sin\frac{\ell\pi x}{L}\sin\frac{\ell'\pi x}{L}\,dx}_{(L/2)\delta_{\ell\ell'}}\ \underbrace{\int_0^H \sin\frac{m\pi y}{H}\sin\frac{m'\pi y}{H}\,dy}_{(H/2)\delta_{mm'}}\ \underbrace{\int_0^W \sin\frac{n\pi z}{W}\sin\frac{n'\pi z}{W}\,dz}_{(W/2)\delta_{nn'}}$$

$$= -a_{\ell'm'n'}\left[\left(\frac{\ell'\pi}{L}\right)^2 + \left(\frac{m'\pi}{H}\right)^2 + \left(\frac{n'\pi}{W}\right)^2\right]\frac{LHW}{8}$$

$$\boxed{a_{\ell mn} = -\frac{8}{LHW}\sin\left(\frac{\ell\pi x_0}{L}\right)\sin\left(\frac{m\pi y_0}{L}\right)\sin\left(\frac{n\pi z_0}{W}\right)\Big/\left[\left(\frac{\ell\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{W}\right)^2\right]}$$

Note that $G(\vec{x},\vec{x}_0) = G(\vec{x}_0,\vec{x})$ thus G is symmetric.

c) $G|_s = 0$

$G(r,0) = G(r,\pi) = 0 \rightarrow G \sim \sin(m\theta)$

$G(0,\theta) = G(a,\theta) = 0 \rightarrow G \sim J_m(\sqrt{\lambda_{mn}}\,r)$ with $\lambda_{mn} = (z_{mn}/a)^2$

$z_{mn} \rightarrow$ zeros of $J_m(z)$.

Since $J_0(0) = 1 \rightarrow m \neq 0$.

see exercise (7.7.9)

$$\Rightarrow \boxed{G(r,\theta;r_0,\theta_0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} J_m(\sqrt{\lambda_{mn}}\,r)\sin m\theta} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\phi_{mn}(r,\theta)$$

where $\phi_{mn} = J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)$ are the eigenfunctions of $\nabla^2\phi_{mn} = -\lambda_{mn}\phi_{mn}$

$$\Rightarrow \underset{\delta(\vec{r}-\vec{r}_0)}{\nabla^2 G} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\nabla^2\phi_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}(-\lambda_{mn}\phi_{mn})$$

$$\delta(\vec{r}-\vec{r}_0) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}a_{mn}J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)$$

Multiplying by $J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\sin(m'\theta)$ and integrating:

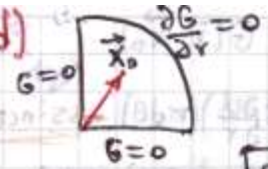
$$\int_0^\pi\int_0^a \delta(\vec{r}-\vec{r}_0)J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\sin(m'\theta)\,(r\,dr\,d\theta) = J_{m'}(\sqrt{\lambda_{m'n'}}\,r_0)\sin(m'\theta_0)$$

$$= -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}a_{mn}\int_0^a J_m(\sqrt{\lambda_{mn}}\,r)J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr\ \underbrace{\int_0^\pi \sin(m\theta)\sin(m'\theta)\,d\theta}_{(\pi/2)\delta_{mm'}}$$

$$= -\sum_{n=1}^{\infty}\lambda_{m'n}a_{m'n}\underbrace{\int_0^a J_{m'}(\sqrt{\lambda_{m'n}}\,r)J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr}_{0 \text{ if } m\neq m'}\cdot\frac{\pi}{2}$$

$$= \frac{-\pi}{2}\lambda_{m'n'}a_{m'n'}\int_0^a J_{m'}^2(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr \rightarrow \boxed{a_{mn} = \frac{-2}{\pi}\frac{J_m(\sqrt{\lambda_{mn}}\,r_0)\sin(m\theta_0)}{\lambda_{mn}\int_0^a J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr}}$$

$\frac{a^2}{2}J_{m+1}^2(\sqrt{\lambda_{mn}}\,a)$

d) 

$G(r,0)=G(r,\pi/2)=0 \rightarrow G\sim \sin(2m\theta)$

$\frac{\partial G}{\partial r}(a,\theta)=0 \rightarrow G\sim J_m(\sqrt{\lambda_{mn}}\,r)$ with $\lambda_{mn}=(z_{mn}/a)^2$

$z_{mn}\rightarrow$ zeros of $J_m'(z)$

$$\boxed{G(r,\theta;r_0,\theta_0)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} J_m(\sqrt{\lambda_{mn}}\,r)\sin(2m\theta)}$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\phi_{mn}$$

where $\phi_{mn}=J_m(\sqrt{\lambda_{mn}}\,r)\sin(2m\theta)$ are the eigenfunctions of $\nabla^2\phi_{mn}=-\lambda_{mn}\phi_{mn}$, thus:

$$\nabla^2 G=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\nabla^2\phi_{mn}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}(-\lambda_{mn}\phi_{mn})$$

$$\delta(\vec{r}-\vec{r}_0)=-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}a_{mn}J_m(\sqrt{\lambda_{mn}}\,r)\sin(2m\theta)$$

Multiplying by $J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\sin(2m'\theta)$ and integrating:

$$\int_0^{\pi/2}\int_0^a \delta(\vec{r}-\vec{r}_0)J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\sin(2m'\theta)\,(r\,dr\,d\theta)=J_{m'}(\sqrt{\lambda_{m'n'}}\,r_0)\sin(2m'\theta_0)$$

$$=-\sum_{m,n}\lambda_{mn}a_{mn}\int_0^a J_m(\sqrt{\lambda_{mn}}\,r)J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr\underbrace{\int_0^{\pi/2}\sin 2m\theta\,\sin 2m'\theta\,d\theta}_{(\pi/4)\delta_{mm'}}$$

$$=-\sum_{n=1}^{\infty}\lambda_{m'n}a_{m'n}\underbrace{\int_0^a J_{m'}(\sqrt{\lambda_{m'n}}\,r)J_{m'}(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr}_{0\text{ if }m\neq m'}\cdot\frac{\pi}{4}$$

$$=-\frac{\pi}{4}\lambda_{m'n'}a_{m'n'}\int_0^a J_{m'}^2(\sqrt{\lambda_{m'n'}}\,r)\,r\,dr$$

$$\boxed{a_{mn}=-\frac{4}{\pi}\frac{J_m(\sqrt{\lambda_{mn}}\,r_0)\sin(2m\theta_0)}{\lambda_{mn}\int_0^a J_m^2(\sqrt{\lambda_{mn}}\,r)\,r\,dr}}$$

$$\rightarrow \frac{a^2}{2}\left[1-\left(\frac{m}{\sqrt{\lambda_{mn}}\,a}\right)^2\right]J_m^2(\sqrt{\lambda_{mn}}\,a)$$

9.5.4 Use the Green's formula with $v=G(\vec{x},x_0)$:

$$\int_V(u\nabla^2 G-G\nabla^2 u)\,dV=\oint_S(u\nabla G-\cancel{G\nabla u})\cdot\hat{n}\,dS$$

$G|_S=0 \rightarrow G=0$ on the boundary.

$$\int_V(u\,\delta(\vec{x}-\vec{x}_0)-G\,f(x))\,dV=\oint_S h(\vec{x})\nabla G\cdot\hat{n}\,dS$$

$u=h(\vec{x})$ on the boundary

$$u(\vec{x}_0)-\int_V G\,f(x)\,dV=\oint_S h(\vec{x})\nabla G\cdot\hat{n}\,dS$$

By switching $\vec{x}\leftrightarrow\vec{x}_0$ and using the symmetry $G(\vec{x}_0,\vec{x})=G(\vec{x},\vec{x}_0)$

$$\Rightarrow \boxed{u(\vec{x})=\int_V G(\vec{x},\vec{x}_0)f(\vec{x}_0)\,d\vec{x}_0+\oint_S h(\vec{x}_0)\nabla_{\vec{x}_0}G\cdot\hat{n}\,dS_0}$$

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9.5.5 Again, using the Green's formula but in 2-D with $v = G(\vec{x}, \vec{x}_0)$

$\nabla u = \frac{\partial u}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\theta}$

$$\int_S (u\nabla^2 G - G\nabla^2 u)\,d\alpha = \oint (u\nabla G - G\nabla u)\cdot\hat{n}\,dl = \int_{-\pi}^{\pi}\left(u\frac{\partial G}{\partial r} - G\frac{\partial u}{\partial r}\right)(a\,d\theta) \rightarrow \text{since } \hat{n}=\hat{r}$$

$$\int_S (u\delta(\vec{r}-\vec{r}_0) - Gf(\vec{r}))\,d\alpha = \int_{-\pi}^{0}\left(u\underbrace{\frac{\partial G}{\partial r}}_{0} - G\underbrace{\frac{\partial u}{\partial r}}_{h_2(\theta)}\right)(a\,d\theta) + \int_0^{\pi}\left(\underbrace{u\frac{\partial G}{\partial r}}_{h_1(\theta)} - \underbrace{G}_{G=0}\frac{\partial u}{\partial r}\right)(a\,d\theta)$$

$$u(\vec{r}_0) - \int G f(\vec{r})\,d\alpha =$$

on the Boundary $-\pi<\theta<0$, $r=a$. $G=0$ on the Boundary $0<\theta<\pi$, $r=a$

$$\Rightarrow u(\vec{r}_0) = \int G(r,\theta;r_0,\theta_0)f(r,\theta)\,d\alpha - \int_{-\pi}^{0} G(\theta,r;\theta_0,r_0)h_2(\theta)(a\,d\theta) + \int_0^{\pi} h_1(\theta)\frac{\partial G}{\partial r}(a\,d\theta)$$

By switching $\vec{r} \longleftrightarrow \vec{r}_0$ and using $G(\vec{r}_0;\vec{r}) = G(\vec{r},\vec{r}_0)$ we have:

$$\boxed{u(\vec{r}) = \int_S f(\vec{r}_0)G(\vec{r},\vec{r}_0)\,d\alpha_0 + \int_0^{\pi} h_1(\theta_0)\frac{\partial G}{\partial r_0}\cdot a\,d\theta_0 - \int_{-\pi}^{0} h_2(\theta_0)G(\vec{r},\vec{r}_0)\,a\,d\theta_0}$$

9.5.6 a) Use the Green's formula with $v = \phi_n$

$$\int_V (u\nabla^2\phi_n - \phi_n\nabla^2 u)\,dv = \oint_S (u\nabla\phi_n - \phi_n\nabla u)\cdot\hat{n}\,ds \rightarrow 0$$

since both u and ϕ_n satify the same BC's

$$\int_V (u\nabla^2\phi_n - \phi_n f(\vec{x}))\,dv = 0$$

Let suppose for an instant that $\nabla^2\phi_n = -\lambda\phi_n$, obviously with $\lambda = 0$ for homogeneous solutions ϕ_n to exist.

$$\int_V u(-\lambda\phi_n)\,dv = \int_V \phi_n f(\vec{x})\,dv$$

$$\lambda = -\frac{\int_V \phi_n(\vec{x})f(\vec{x})\,dv}{\int u(\vec{x})\phi_n(\vec{x})\,dv}$$

If $\int_V \phi_n(\vec{x})f(\vec{x})\,dv \neq 0 \rightarrow \lambda \neq 0$ and there are not solutions for $\nabla^2\phi_n = 0$, thus $u(x)$ cannot be expanded in eigenfunctions of homogeneous problem as in (9.5.10).

b) If $\int_V \phi_n(\vec{x})f(\vec{x})\,dv = 0 \rightarrow \lambda = 0$ and there are homogeneous solutions ϕ_n for the problem $\nabla^2\phi = -\lambda\phi$ $(\lambda=0)$, then if we suppose an ~~eigen~~ expansión for $u(\vec{x})$ in eigenfunctions $\phi_n(\vec{x})$

$$u(\vec{x}) = \sum_i a_i\phi_{n_i} \rightarrow \int u(\vec{x})\phi_{n_j}(\vec{x})\,dv = \sum_i a_i \underbrace{\int_V \phi_{n_i}\phi_{n_j}\,dv}_{0 \text{ if } i\neq j \text{ by orthogonality}}$$

$$= a_j\int \phi_{n_j}^2\,dv$$

Substituting above $\lambda a_j = -\frac{\int_V \phi_n(\vec{x})f(\vec{x})\,dv}{\int \phi_{n_j}^2\,dv} = 0$, then a_j can be arbitrary and ther are infinite number of solutions for $u(\vec{x})$.

Method 2: Assume directly $u(\vec{x}) = \sum_i a_i\phi_{n_i}$ for start. Like in problem (9.5.1) we can prove that is able differentiate term-by-term, then

$f(\vec{x}) = \nabla^2 u = \sum_i a_i\nabla^2\phi_{n_i} = -\sum_i a_i(\lambda_i\phi_{n_i})$ where for an instant we supposed $\nabla^2\phi_n = -\lambda\phi_n$, with $\lambda=0$ for the ϕ_n to exist.

Multiplying by ϕ_{n_j} and integrating we obtain:

$$\lambda_j a_j = -\frac{\int_V \phi_{n_j}(\vec{x})f(\vec{x})\,dv}{\int_V \phi_{n_j}^2\,dv}$$

and the arguments are the same that we did above.

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9.5.7 a) Since $\phi_h = 1 \rightarrow \nabla^2\phi_h = 0$ and $\nabla\phi_h = 0 \rightarrow \nabla\phi_h\cdot\hat{n} = 0$.
Thus $\phi_h = 1$ is a homogeneous solution satisfying the BC's.

b) The solution exist only if $\int_V \phi_h(\vec{x})\cdot f(\vec{x})\,dv = 0$ according to (9.5.14), and there an infinite number of solutions in that case.

c) $\mathcal{L}[G_m(\vec{x},\vec{x}_0)] = \delta(\vec{x}-\vec{x}_0) + c\phi_h(\vec{x})$ where c is determined ~~deman~~ demanding that ϕ_h is orthogonal to the right-hand side.

$\mathcal{L} = \nabla^2$

$$0 = \int_V \phi_h(\vec{x})[\delta(\vec{x}-\vec{x}_0) + c\,\phi_h(\vec{x})]\,dv = \phi_h(x_0) + c\int_V \phi_h^2(\vec{x})\,dv$$

$$\Rightarrow c = -\phi_h(\vec{x}_0)/\int_V \phi_h^2(\vec{x})\,dv$$

d) Use the Green's formula with $u = G_m(\vec{x},\vec{x}_0)$

$$\int_V (u\nabla^2 G_m - G_m\nabla^2 u)\,dv = \oint_S (u\nabla G_m - G_m\nabla u)\cdot\hat{n}\,ds \rightarrow 0 \text{ since } \nabla u\cdot\hat{n}|_S = 0,\ \nabla G_m\cdot\hat{n}|_S = 0$$

Both u and G_m satisfy the same BC's.

$$\int_V \left(u[\delta(\vec{x}-\vec{x}_0) + c\phi_h(\vec{x})] - G_m f(\vec{x})\right)dv = 0$$

$$u(\vec{x}_0) + c\int_V u(\vec{x})\phi_h(\vec{x})\,dv - \int_V G_m(\vec{x},\vec{x}_0)f(\vec{x})\,dv = 0$$

Switching $\vec{x} \longleftrightarrow \vec{x}_0$, using $G_m(\vec{x}_0,\vec{x}) = G_m(\vec{x},\vec{x}_0)$ and substituting the value of c from part (c):

$$\boxed{u(\vec{x}) = \int_V G_m(\vec{x},\vec{x}_0)f(\vec{x}_0)\,dV_0 + \frac{\phi_h(\vec{x})}{\int_V \phi_h^2(\vec{x})\,dv}\int_V u(\vec{x}_0)\phi_h(\vec{x}_0)\,dV_0}$$

where $\phi_h(\vec{x}) = \phi_h(\vec{x}_0) = 1$. Note that the second term is a multiple of the homogeneous solution, and thus a simple particular solution of $u(\vec{x})$ is:

$$u(\vec{x}) = \int_V G_m(\vec{x},\vec{x}_0)f(\vec{x}_0)\,dV_0.$$

9.5.8 All is the same as in exercise (9.5.7) that we did above, except that in part (d) now

$$\nabla u\cdot\hat{n}|_S = h(\vec{x}),$$

however for G_m the BC's is $\nabla G_m\cdot\hat{n}|_S = 0$, just as before

$$\Rightarrow u(\vec{x}_0) + c\int_V u(\vec{x})\phi_h(\vec{x})\,dv - \int_V G_m(\vec{x},\vec{x}_0)f(\vec{x}) = \oint_S (\underbrace{u\nabla G_m}_{0} - G_m\underbrace{\nabla u\cdot\hat{n}}_{h(\vec{x})})\,ds$$

Again switching $\vec{x} \longleftrightarrow \vec{x}_0$, using the symmetry of $G_m(\vec{x},\vec{x}_0)$ and substituting c:

$$\boxed{u(\vec{x}) = \int_V G_m(\vec{x},\vec{x}_0)f(\vec{x}_0)\,dV_0 + \frac{\phi_h(\vec{x})}{\int_V \phi_h^2(\vec{x})\,dv}\int_V u(\vec{x}_0)\phi_h(\vec{x}_0)\,dV_0 - \oint_S G_m(\vec{x},\vec{x}_0)h(\vec{x}_0)\,ds.}$$

where $\phi_h(\vec{x}) = \phi_h(\vec{x}_0) = 1$. Since the second term is a multiple of ~~the~~ the homogeneous solution, then a simple particular solution of $u(\vec{x})$ is:

$$u(\vec{x}) = \int_V G_m(\vec{x},\vec{x}_0)f(\vec{x}_0)\,dV_0 - \oint_S G_m(\vec{x},\vec{x}_0)h(\vec{x}_0)\,ds.$$

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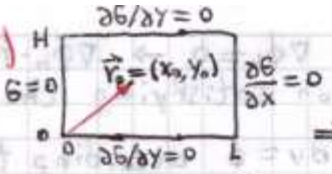
9.5.9 a)



$$G(0,x_0)=0,\ \frac{\partial G}{\partial x}(L,x_0)=0 \rightarrow G \sim \sin\left(\frac{2n-1}{2}\frac{\pi}{L}x\right)$$

$$\Rightarrow \boxed{G(x,y,x_0,y_0)=\sum_{n=1}^{\infty} a_n(y)\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x\right)} \rightarrow \begin{cases} a_n'(0)=0 \\ a_n'(H)=0 \end{cases}$$

$$\nabla^2 G = \frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2} = \sum_{n=1}^{\infty}\left[-\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 a_n + \frac{d^2 a_n}{dy^2}\right]\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x\right) = \delta(\vec{r}-\vec{r}_0) = \delta(x-x_0)\,\delta(y-y_0)$$

Multiplying by $\sin\left(\frac{2n'-1}{2}\frac{\pi}{L}x\right)$ at both sides and integrating:

$$\sum_{n=1}^{\infty}\left[-\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 a_n + \frac{d^2 a_n}{dy^2}\right]\underbrace{\int_0^L \sin\left(\frac{2n-1}{2}\frac{\pi}{L}x\right)\sin\left(\frac{2n'-1}{2}\frac{\pi}{L}x\right)dx}_{(L/2)\delta_{nn'}} = \delta(y-y_0)\int_0^L \delta(x-x_0)\sin\left(\frac{2n'-1}{2}\frac{\pi}{L}x\right)dx$$

$$\left[-\left(\frac{2n'-1}{2}\frac{\pi}{L}\right)^2 a_{n'} + \frac{d^2 a_{n'}}{dy^2}\right]\cdot\frac{L}{2} = \delta(y-y_0)\sin\left(\frac{2n'-1}{2}\frac{\pi}{L}x_0\right)$$

OR $$\frac{d^2 a_n}{dy^2} - \left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 a_n = \frac{2}{L}\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x_0\right)\delta(y-y_0) \quad \text{with} \quad \begin{matrix} a_n'(0)=0 \\ a_n'(H)=0 \end{matrix}$$

For $x \neq x_0$ $$\frac{d^2 a_n}{dy^2} - \underbrace{\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2}_{\alpha_n^2} a_n = 0 \rightarrow a_n(y) = \begin{cases} C_1\cosh(\alpha_n y+\gamma_1) & x<x_0 \\ C_2\cosh(\alpha_n y+\gamma_2) & x>x_0 \end{cases}$$

Since $0<x_0<H \rightarrow$ $a_n'(0)=0=C_1\alpha_n\sinh\gamma_1 \rightarrow \gamma_1=0$
$a_n'(H)=0=C_2\alpha_n\sinh(\alpha_n H+\gamma_2) \rightarrow \gamma_2=-\alpha_n H$

At $x=x_0$, by continuity of $G(x,x_0) \Rightarrow a_n(y_0^-)=a_n(y_0^+)$

$$C_1\cosh\alpha_n y_0 = C_2\cosh\alpha_n(y_0-H)$$

$$\Rightarrow C_2 = C_1\frac{\cosh\alpha_n y_0}{\cosh\alpha_n(y_0-H)}$$

Now integrating the differential equation for $a_n(y)$ about x_0:

$$\underbrace{\int_{y_0-\epsilon}^{y_0+\epsilon}\frac{d^2 a_n}{dy^2}dy}_{da_n/dy\,\big|_{y_0^-}^{y_0^+}} - \left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2\underbrace{\int_{y_0-\epsilon}^{y_0+\epsilon} a_n(y)\,dy}_{\to 0 \text{ by continuity of } a_n(y)} = \frac{2}{L}\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x_0\right)\underbrace{\int\delta(y-y_0)\,dy}_{1}$$

$$\lim_{\epsilon\to 0}\left(C_1\frac{\cosh\alpha_n y_0}{\cosh\alpha_n(y_0-H)}\right)\alpha_n\sinh\alpha_n(y_0-H) - C_1\alpha_n\sinh\alpha_n y_0 = \frac{2}{L}\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x_0\right)$$

$$\alpha_n C_1\underbrace{\left[\cosh\alpha_n y_0\sinh\alpha_n(y_0-H) - \sinh\alpha_n y_0\cosh\alpha_n(y_0-H)\right]}_{\sinh\alpha_n H} = \frac{2}{L}\sin(\text{"})\cdot\cosh\alpha_n(y_0-H)$$

$$\Rightarrow C_1 = \frac{2}{\alpha_n L}\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x_0\right)\frac{\cosh\alpha_n(y_0-H)}{\sinh\alpha_n H}$$

$$\Rightarrow \boxed{a_n(y) = \frac{2}{\alpha_n L}\frac{\sin\alpha_n x_0}{\sinh\alpha_n H}\begin{cases}\cosh\alpha_n(y_0-H)\cosh\alpha_n y & y<y_0 \\ \cosh\alpha_n y_0\cosh\alpha_n(y-H) & y>y_0\end{cases}}$$

With $a_n(y)$ substituted in $G(x,y;x_0,y_0)$ the symmetry is exhibited explicitly. In exercise 9.5.3(a) this same Green's function was represented as a double Fourier ~~sine~~ series in both x and y. A third representation of this Green's function is also posible; instead of using a Fourier sine series in x, we could have used a Fourier sine series in y.

Method 2: If we demand the symmetry early it's more fast way to find the solution.

Let $x_<$ ($x_>$) the smaller (larger) betwen x_0 and x, an similarly for y_0 and y.

the symmetry demand this.

$$\left.\begin{array}{l} G(0,x_0;y,y_0)=0 \rightarrow G\sim \sin\alpha_n x_< \\ \frac{\partial G}{\partial x}(L,x_0;y,y_0)=0 \rightarrow G\sim \sin\left(\frac{2n-1}{2}\frac{\pi}{L}x_>\right)\end{array}\right\} \Rightarrow G\sim \sin\alpha_n x_< \sin\alpha_n x_>$$

with. $\boxed{\alpha_n=\frac{2n-1}{2}\frac{\pi}{L}}$

$$\Rightarrow \boxed{G(x,y;x_0,y_0)=\sum_{n=1}^{\infty} a_n(y)\sin\alpha_n x_0 \sin\alpha_n x}$$

where we used that is indiferent have $x>x_0$ or $x<x_0$ becase is a product of sines functions.

$$\frac{\partial G}{\partial x}(x,x_0,0,y_0)=\frac{\partial}{\partial x}(x,x_0,H,y_0)=0 \rightarrow a_n'(0)=a_n'(H)=0.$$

$$\nabla^2 G=\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=\sum_{n=1}^{\infty}\left[-\alpha_n^2 a_n+\frac{d^2 a_n}{dy^2}\right]\sin\alpha_n x_0 \sin\alpha_n x=\delta(x-x_0)\delta(y-y_0)$$

Multiplying both sides by $\sin(\alpha_{n'}x)$ and integrating:

$$\sum_{n=1}^{\infty}\left[-\alpha_n^2 a_n+\frac{d^2 a_n}{dy^2}\right]\sin\alpha_n x_0 \underbrace{\int_0^L \sin\alpha_n x \sin\alpha_{n'} x\,dx}_{(L/2)\delta_{nn'}}=\delta(y-y_0)\underbrace{\int \delta(x-x_0)\sin\alpha_{n'}x\,dx}$$

$$\left[-\alpha_{n'}^2 a_{n'}+\frac{d^2 a_{n'}}{dy^2}\right]\cancel{\sin(\alpha_{n'}x_0)}\cdot\frac{L}{2}=\delta(y-y_0)\cancel{\sin(\alpha_{n'}x_0)}$$

$$\Rightarrow \frac{d^2 a_n}{dy^2}-\alpha_n^2 a_n(y)=\frac{2}{L}\delta(y-y_0) \quad \text{with } a_n'(0)=0,\ a_n'(H)=0$$

For $x\neq x_0$ $\Rightarrow$ $\frac{d^2 a_n}{dy^2}-\alpha_n^2 a_n(y)=0 \rightarrow a_n\sim \cosh(\alpha_n y+\gamma)$

$$\left.\begin{array}{l} a_n'(0)=0 \rightarrow a_n\sim \cosh\alpha_n y_< \\ a_n'(H)=0 \rightarrow a_n\sim \cosh\alpha_n(y_>-H)\end{array}\right\} \Rightarrow a_n=C_n\cosh\alpha_n y_< \cosh\alpha_n(y_>-H)$$

The jump condition for the derivative of $a_n(y)$, $\left.\frac{da_n}{dy}\right|_{y_0^-}^{y_0^+}=\frac{2}{L}$ gives:

$$\alpha_n C_n\Big[\underbrace{\cosh\alpha_n y_0 \sinh\alpha_n(y_0-H)-\sinh\alpha_n y_0\cosh\alpha_n(y_0-H)}_{\sinh(\alpha_n H)}\Big]=\frac{2}{L}$$

$$\Rightarrow C_n=\frac{2}{\alpha_n L}\Big/\sinh\alpha_n H$$

$$\Rightarrow \boxed{a_n(y)=\frac{2}{\alpha_n L}\frac{\cosh\alpha_n y_< \cosh\alpha_n(y_>-H)}{\sinh\alpha_n H}}$$

$y_<\equiv$ Smaller of y and y_0
$y_>\equiv$ Larger of y an y_0.

Using Fourier sine series in y (y-dependent eigenfunctions):

$$\left.\begin{array}{l} \frac{\partial G}{\partial y}(y=0)=0 \rightarrow G\sim \cos(\beta_m y_<) \\ \frac{\partial G}{\partial y}(y=H)=0 \rightarrow G\sim \cos(m\pi y_>/H)\end{array}\right\} \Rightarrow G\sim \cos(\beta_m y_<)\cos(\beta_m y_>)$$

$\boxed{\beta_m=m\pi/H}$

$$\Rightarrow \boxed{G(\vec{x},\vec{x}_0)=\sum_{m=0}^{\infty} b_m(x)\cos\beta_m y_0 \cos\beta_m y}$$

$\rightarrow$ where we used that is indiferent $x>x_0$ or $x<x_0$ in the product of both functions.

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$$\nabla^2 G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \sum_{m=0}^{\infty}\left[\frac{d^2 b_m}{dx^2} - \beta_m^2 b_m\right]\cos\beta_m y_0 \cos\beta_m y = \delta(x-x_0)\delta(y-y_0)$$

Multiplying by $\cos\beta_{m'} y$ at both sides and integrating:

$$\sum_{m=0}^{\infty}\left[\frac{d^2 b_m}{dx^2} - \beta_m^2 b_m\right]\cos\beta_m y_0 \underbrace{\int_0^H \cos\beta_m y \cos(\beta_{m'} y)\,dy}_{\frac{H}{2}(1+\delta_{m0})\delta_{mm'}} = \delta(x-x_0)\underbrace{\int_0^H \delta(y-y_0)\cos(\beta_{m'} y)\,dy}$$

$$\left[\frac{d^2 b_{m'}}{dx^2} - \beta_{m'}^2 b_{m'}\right]\cancel{\cos\beta_{m'} y_0}\cdot\frac{H}{2}(1+\delta_{m'0}) = \delta(x-x_0)\,\cancel{\cos(\beta_{m'} y_0)}$$

OR $\quad \dfrac{d^2 b_m}{dx^2} - \beta_m^2 b_m = \dfrac{2}{H(1+\delta_{m0})}\delta(x-x_0)$ with $b_m(0)=0$, $b_m'(L)=0$ since $G(x=0)=0$, $\frac{\partial G}{\partial x}(x=L)=0$.

For $x \neq x_0 \Rightarrow \dfrac{d^2 b_m}{dx^2} - \beta_m^2 b_m = 0$, $\quad b_m(0)=0 \to b_m \sim \sinh\beta_m x_<$, $\quad b_m'(0)=0 \to b_m \sim \cosh\beta_m(x_>-L)$

$$\Rightarrow b_m(x) = C_n \sinh\beta_m x_< \cosh\beta_m(x_>-L)$$

The jump condition for the derivative of $b_m(x)$ at $x=x_0$ gives:

$$\left.\frac{db_m}{dx}\right|_{x_0^-}^{x_0^+} = \beta_m C_n\Big[\underbrace{\sinh\beta_m x_0 \sinh\beta_m(x_0^+-L) - \cosh\beta_m x_0^- \cosh\beta_m(x_0-L)}_{\cosh(\beta_m L)}\Big] = \frac{2}{H(1+\delta_{m0})}$$

$$\Rightarrow C_m = \frac{2/\cosh(\beta_m L)}{\beta_m H(1+\delta_{m0})}$$

$$\Rightarrow \boxed{b_m(x) = \frac{2/\cosh(\beta_m L)}{\beta_m H(1+\delta_{m0})}\sinh\beta_m x_< \cosh\beta_m(x_>-L)}$$

We can write this as: $\quad b_m(x) = \dfrac{2/\cosh\beta_m L}{\beta_m(1+\delta_{m0})}\begin{cases}\sinh\beta_m x_0 \cosh\beta_m(x-L) & x<x_0\\ \sinh\beta_m x \cosh\beta_m(x_0-L) & x>x_0\end{cases}$

If we expand $b_m(x)$ in a sine Fourier series we obtain the same form for the Green's function like in exercise 9.5.3(a).

b)

$G(\theta=0)=G(\theta=\pi)=0 \Rightarrow G\sim\sin(m\theta)$

$0<\theta<\pi \to \left.\begin{matrix}G(\theta=0)=0\to G\sim\sin m\theta_<\\ G(\theta=\pi)=0\to G\sim\sin m\theta_>\end{matrix}\right\} \Rightarrow G\sim\sin m\theta_<\sin m\theta_>$ Election that preserve symmetry

$$\Rightarrow \boxed{G(\vec{x},\vec{x}_0) = \sum_{m=1}^{\infty} A_m(r)\sin m\theta_0 \sin m\theta}$$

where we used that is indifferent have $\theta>\theta_0$ or $\theta<\theta_0$ because it is a product of both functions.

$$\nabla^2 G = \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + \frac{1}{r^2}\frac{d^2 G}{d\theta^2} = \sum_{m=1}^{\infty}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_m}{dr}\right) - \frac{m^2}{r^2}A_m\right]\sin m\theta_0 \sin m\theta = \delta(\vec{x}-\vec{x}_0)$$

In polar coordinates $\delta(\vec{x}-\vec{x}_0) = K\delta(r-r_0)\delta(\theta-\theta_0)$ such that

$$1 = \int_0^{2\pi}\int_0^{\infty} K\delta(r-r_0)\delta(\theta-\theta_0)(r\,dr\,d\theta) = K r_0 \to K = \frac{1}{r_0} \to \delta(x-x_0) = \frac{1}{r_0}\delta(r-r_0)\delta(\theta-\theta_0)$$

Multiplying by $\sin m'\theta$ and integrating in $d\theta \Rightarrow$

$$\Rightarrow \sum_{m=1}^{\infty}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_m}{dr}\right)-\frac{m^2}{r^2}A_m\right]\sin m\theta_0 \underbrace{\int_0^{\pi}\sin m\theta \sin m'\theta\, d\theta}_{(\pi/2)\delta_{mm'}} = \frac{1}{r_0}\delta(r-r_0)\underbrace{\int_0^{\pi}\delta(\theta-\theta_0)\sin m'\theta}$$

$$\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_{m'}}{dr}\right)-\frac{m'^2}{r^2}A_{m'}\right]\cancel{\sin m'\theta}\cdot\frac{\pi}{2} = \frac{1}{r_0}\delta(r-r_0)\cancel{\sin m'\theta_0}.$$

OR $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_m}{dr}\right)-\frac{m^2}{r^2}A_m = \frac{2}{\pi r_0}\delta(r-r_0)$$

For $x\neq x_0$ $\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_m}{dr}\right)-\frac{m^2}{r^2}A_m=0 \rightarrow A_m\sim r^{\pm m}$ let $r_< \equiv$ smaller $\{r, r_0\}$, $r_> \equiv$ larger $\{r, r_0\}$

$A_m(r=0)=$ finite $\Rightarrow A_m\sim r_<^m$

The domain don't include infinite $(0<r<a) \Rightarrow G\sim C_{1m}r_>^m + C_{2m}r_>^{-m}$

$G(r=0)=0 \rightarrow A_m(r=a)=0 \rightarrow C_{1m}a^m + C_{2m}a^{-m}=0 \rightarrow C_{2m}=-C_{1m}a^{2m}$

$\Rightarrow A_m = C r_<^m\left[r_>^m - a^{2m}r_>^{-m}\right] \rightarrow$ This election preserve the symmetry.

The jump condition at $r=r_0$ is obtained integrating above $r=r_0$:

$$\underbrace{\int_{r_0-\epsilon}^{r_0+\epsilon}\frac{d}{dr}\left(r\frac{dA_m}{dr}\right)dr}_{r\frac{dA_m}{dr}\Big|_{r_0^-}^{r_0^+}} - m^2\int_{r_0-\epsilon}^{r+\epsilon}\frac{A_m}{r}dr = \frac{2}{\pi r_0}\underbrace{\int_{r_0-\epsilon}^{r_0+\epsilon}\delta(r-r_0)\,r\,dr}_{r_0}$$

$\rightarrow 0$ By continuity of A_m since G is continous.

$$\Rightarrow \cancel{r_0}\,C\left\{r_0^m\left(m r_0^{m-\cancel{1}}+m a^{2m}r_0^{-m-\cancel{1}}\right)-m r_0^{m-\cancel{1}}\left(r_0^m-a^{2m}r_0^{-m}\right)\right\}=\frac{2}{\pi}$$

$$Cm\left\{\left(r_0^{2m}+a^{2m}\right)-\left(r_0^{2m}-a^{2m}\right)\right\}=2mCa^{2m}=\frac{2}{\pi}$$

$$C=a^{-2m}/m\pi$$

$$\Rightarrow \boxed{A_m(r,r_0)=\frac{1}{m\pi}\left(\frac{r_<}{a}\right)^m\left[\left(\frac{r_>}{a}\right)^m-\left(\frac{a}{r_>}\right)^m\right]}$$

Method 2: $G(\theta=0)=G(\theta=\pi)=0 \rightarrow G\sim\sin m\theta \Rightarrow G=\sum_{m=1}^{\infty}A_m(r)\sin m\theta$

$$\nabla^2 G=\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)-\frac{1}{r^2}\frac{d^2G}{d\theta}=\sum_{m=1}^{\infty}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_m}{dr}\right)-\frac{m^2}{r^2}A_m\right]\sin\theta=\frac{1}{r_0}\delta(r-r_0)\delta(\theta-\theta_0)$$

multiplying by $\sin m'\theta$ and integrating as before:

$$\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_m}{dr}\right)-\frac{m^2}{r^2}A_m\right]=\frac{2}{\pi r_0}\sin m\theta_0\,\delta(r-r_0)$$

For $r\neq r_0 \Rightarrow A_m=\begin{cases}C_{1m}r^m+C_{2m}r^{-m} & r<r_0\\ C_{3m}r^m+C_{4m}r^{-m} & r>r_0\end{cases}$ $C_{2m}=0$ since A_m is finite for $r\rightarrow 0$.

$G(r=a)=0 \rightarrow A(r=a)=C_{3m}a^m+C_{4m}a^{-m}=0 \rightarrow C_{4m}=-C_{3m}a^{2m}$

The continuity of $G(\vec{r},\vec{r}_0)$ at $\vec{r}=\vec{r}_0$ demand that $A_m(r)$ must be continuous at $r=r_0 \Rightarrow C_{1m}r_0^m=C_{3m}\left[r_0^m-a^{2m}r_0^{-m}\right] \Rightarrow$

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$$C_{1m} = C_{3m}\left[1-\left(\frac{a}{r_0}\right)^{2m}\right] \Rightarrow A_m(r,r_0) = \begin{cases} C_{3m}\left[1-\left(\frac{a}{r_0}\right)^{2m}\right] r^m & r<r_0 \\ C_{3m}\left[1-\left(\frac{a}{r}\right)^{2m}\right] r^m & r>r_0 \end{cases}$$

The jump condition for $A_m(r,r_0)$ obtained before is $r\frac{dA_m}{dr}\Big|_{r_0^-}^{r_0^+} = \frac{2}{\pi}\sin m\theta_0$

$$\Rightarrow \cancel{r_0}\, C_{3m}\left\{\left(m r_0^{m-\cancel{1}} + m a^{2m} r_0^{-m-1}\right) - \left(m r_0^{m-\cancel{1}} - \left(\frac{a}{r_0}\right)^{2m} m r^{m-\cancel{1}}\right)\right\} = \frac{2}{\pi}\sin m\theta_0$$

$$2mC_{3m}\frac{a^{2m}}{r_0^m} = \frac{2}{\pi}\sin m\theta_0$$

$$\Rightarrow C_{3m} = \frac{1}{m\pi}\left(\frac{r_0}{a^2}\right)^m \sin(m\theta_0)$$

$$\hookrightarrow \boxed{A_m(r,r_0) = \begin{cases} \frac{1}{m\pi a^{2m}}\left[1-\left(\frac{a}{r_0}\right)^{2m}\right] r^m r_0^m \sin m\theta_0 & r<r_0 \\ \frac{1}{m\pi a^{2m}}\left[1-\left(\frac{a}{r}\right)^{2m}\right] r^m r_0^m \sin m\theta_0 & r>r_0 \end{cases}}$$

This is the same answer written before using $r_<$ and $r_>$.

9.5.10 a) $\int_V [u\mathcal{L}(v) - v\mathcal{L}(u)]\,dV = \int_V [u(\nabla^2 v + \cancel{k^2 v}) - v(\nabla^2 u + \cancel{k^2 u})]\,dV$

$$= \oint_S (u\nabla v - v\nabla u)\cdot\hat{n}\,dS$$

Problem spherical symmetric

b) $\nabla^2 G + k^2 G = \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dG}{d\rho}\right) + k^2 G = \delta(\vec{x}-\vec{x}_0)$ with $\rho \equiv |\vec{x}-\vec{x}_0|$

Spherical coordinates.

For $x \neq x_0$ $\rightarrow \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dG}{d\rho}\right) + k^2 G = 0$ let $G = Y/\rho \rightarrow \frac{dG}{d\rho} = \frac{1}{\rho}\frac{dY}{d\rho} - \frac{Y}{\rho^2}$

$$\rightarrow \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho\frac{dY}{d\rho} - Y\right) + k^2\frac{Y}{\rho} = \frac{1}{\rho^2}\left(\cancel{\frac{dY}{d\rho}} + \rho\frac{dY^2}{d\rho^2} - \cancel{\frac{dY}{d\rho}}\right) + k^2\frac{Y}{\rho} = 0$$

$$\frac{d^2Y}{d\rho^2} + k^2 Y = 0 \rightarrow Y = C_1 e^{ik\rho} + C_2 e^{-ik\rho}$$

$$G = \frac{Y}{\rho} \rightarrow \boxed{G = \frac{C_1 e^{ik\rho} + C_2 e^{-ik\rho}}{\rho}}$$

Since the wave-equation is $\frac{\partial^2\phi}{\partial t^2} = c^2\nabla^2\phi + g(\vec{x})e^{-i\omega t}$

$$\Rightarrow \phi = u(\vec{x})e^{-i\omega t} \rightarrow -\omega^2 u\cdot e^{-i\omega t} = c^2\nabla^2 u\cdot e^{-i\omega t} + g(\vec{x})e^{-i\omega t}$$

$$\hookrightarrow \boxed{\nabla^2 u + k^2 u = f(\vec{x})} \qquad k = \omega/c, \quad f = g/c^2$$

Thus $\frac{C_1 e^{i(k\rho-\omega t)} + C_2 e^{-i(k\rho+\omega t)}}{\rho}$

$e^{i(k\rho-\omega t)} \longrightarrow$ outgoing wave

$e^{-i(k\rho+\omega t)} \longrightarrow$ incoming wave $(\omega/k>0)$

If we are interesting in an outward-propagating wave $\rightarrow C_2 = 0$

$\Rightarrow G = C_1 e^{ik\rho}/\rho$ To determine C_1 integrate the differential equation over a small sphere centered at $\vec{x}=\vec{x}_0$.

$$\Rightarrow \int_{V_\epsilon}\nabla^2 G\,dV_\epsilon + \cancel{\int_{V_\epsilon} G\,dV_\epsilon} = \underbrace{\int_{V_\epsilon}\delta(\vec{x}-\vec{x}_0)}_{1} \Longrightarrow$$

$V_\epsilon \rightarrow 0$ $\quad$ 0 by continuity of G at $\vec{x}=\vec{x}_0$

Note that if $\rho \rightarrow 0$ $\Rightarrow \vec{x} \rightarrow \vec{x}_0$.

with ϵ the radious of the small sphere centered at $\vec{x}=\vec{x}_0$.

$$\lim_{\epsilon\to 0}\int_{V_\epsilon} \nabla^2 G\, dV_\epsilon = 1$$ OR using the divergence theorem

$$\lim_{\epsilon\to 0}\oint_{S_\epsilon} \nabla G\cdot\hat{n}\, dS_\epsilon = 1 \qquad \nabla G = \frac{\partial G}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial G}{\partial \theta}\hat{\theta} + \frac{1}{\rho\sin\theta}\frac{\partial G}{\partial \phi}.$$

but $\hat{n}=\hat{\rho} \Rightarrow \vec{\nabla} G\cdot\hat{n} = \frac{\partial G}{\partial \rho}$ in spherical coordinates.

On the surface of the sphere $\nabla G\cdot\hat{n} = \frac{\partial G}{\partial \rho} = \text{constant}$

$$\Rightarrow \lim_{\epsilon\to 0}\frac{\partial G}{\partial \rho}\oint_{S_\epsilon} dS_\epsilon = \lim_{\rho\to 0}\frac{\partial G}{\partial \rho}4\pi\rho^2 = 1$$ where we used that $\rho=\epsilon$ since ϵ is the radious of the small sphere.

$$\lim_{\rho\to 0}\left[\left(\frac{-C_1}{\rho^2}e^{ik\rho} + ikC_1\frac{e^{ik\rho}}{\rho}\right)4\pi\rho^2\right] = -C_1(4\pi) = 1 \rightarrow C_1 = -1/4\pi$$

Substituting that and $\rho = |\vec{x}-\vec{x}_0| \Rightarrow$ $$\boxed{G(\vec{x},\vec{x}_0) = -\frac{e^{ik|\vec{x}-\vec{x}_0|}}{4\pi|\vec{x}-\vec{x}_0|}}$$

Green's function for an outward-propagating wave.

c) Let $r=|\vec{x}-\vec{x}_0|$ in 2-D $\Rightarrow$ problem circularly symmetric

$$\Rightarrow \nabla^2 G + k^2 G = \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + k^2 G = \delta(\vec{r}-\vec{r}_0)$$ → cylindrical coordinates.

Bessel eq of zeroth order

If $x\neq x_0 \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + k^2 G = 0 \rightarrow \frac{d}{dr}\left(r\frac{dG}{dr}\right) + \left(k^2 r - \frac{0^2}{r}\right)G = 0$

$$\Rightarrow G = C_1 J_0(kr) + C_2 Y_0(kr)$$

From (7.8.3): $J_m(z\to\infty) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)$

$Y_m(z\to\infty) \sim \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)$

$$\Rightarrow G \sim \left(\frac{2}{\pi k r}\right)^{1/2}\left[C_1\cos(kr-\pi/4) + C_2\sin(kr-\pi/4)\right]$$ for r large.

$$= \left(\frac{2}{\pi k r}\right)^{1/2}\left[C_1\frac{e^{i(kr-\pi/4)}+e^{-i(kr-\pi/4)}}{2} + C_2\frac{e^{i(kr-\pi/4)}-e^{-i(kr-\pi/4)}}{2i}\right]$$

$$= (2\pi k r)^{-1/2}\left[(C_1 - iC_2)e^{i(kr-\pi/4)} + (C_1+iC_2)e^{-i(kr-\pi/4)}\right]$$

The time-dependent problem is obtained by multiplying by $e^{-i\omega t}$. $\phi = u(\vec{x})e^{-i\omega t}$. We note that $e^{i(kr-\omega t)}$ is outgoing if $\omega/k>0$, while $e^{i(kr+\omega t)}$ is incoming.

Thus, to be outgoing $\rightarrow C_1 + iC_2 = 0 \rightarrow C_1 = -iC_2$

Integrating the differential equation over a circle of radius r centered at $\vec{x}=\vec{x}_0$, we obtain:

by continuity of G at $\vec{x}=\vec{x}_0$ when $r\to 0$. Note that if $r\to 0 \Rightarrow \vec{x}\to\vec{x}_0$

$$\underbrace{\int_A \nabla^2 G\, dA}_{\oint \nabla G\cdot\hat{n}\,dl \text{ by theore of divergence}} + K^2 \cancel{\int_A G\, dA}^{\,0} = \underbrace{\int_A \delta(\vec{x}-\vec{x}_0)\,dA}_{1} \Rightarrow \lim_{r\to 0}\oint \nabla G\cdot\hat{n}\,dl = 1$$

Since G is only function of $r \Rightarrow \nabla G\cdot\hat{n} = \frac{dG}{dr}$ = constant over the contorn.

$$\Rightarrow \lim_{r\to 0}\oint \nabla G\cdot\hat{n}\,dl = \lim_{r\to 0}\frac{dG}{dr}(2\pi r) = 1$$

Using the asymptotic expansions for Bessel functions for small arguments (7.7.33):

$$J_m(z\to 0)\sim 1 \quad \text{for } m=0$$
$$Y_m(z\to 0)\sim \frac{2}{\pi}\ln z \quad \text{for } m=0$$

$$\Rightarrow G(r\to 0)\sim C_1 + \frac{2C_2}{\pi}\ln(Kr) \to \frac{dG}{dr} = \frac{2C_2}{\pi r} \Rightarrow \lim_{r\to 0}\frac{dG}{dr}(2\pi r) = 4C_2 = 1$$
$$C_2 = 1/4$$

Since $C_1 = -iC_2 \Rightarrow \boxed{G = \frac{1}{4}\left[Y_0(Kr) - iJ_0(Kr)\right]}$

9.5.11 a) $\delta(\vec{x}-\vec{x}_0) \longrightarrow$ positive source at $\vec{x}=\vec{x}_0$
$\delta(\vec{x}-\vec{x}^*) \longrightarrow$ positive image at $\vec{x}=\vec{x}^*$

Since the BC's for G is the insulated kind. $(\partial G/\partial y|_{y=0}=0.)$

By symmetry no heat will flow across $y=0$, as desired.
Since $\vec{x}^*$ is an reflection of $\vec{x}_0=(x_0,y_0)$ across $y=0 \to \vec{x}^*=(x_0,-y_0)$

$$\Rightarrow \nabla^2 G(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0)+\delta(\vec{x}-\vec{x}^*) = \delta(x-x_0)\delta(y-y_0)+\delta(x-x_0)\delta(y+y_0)$$

This problem will give us a function G that satisfy $\nabla^2 G = \delta(\vec{x}-\vec{x}_0)$ when restricted to the upper half plane satisfying the BC's.

For $\vec{x}\neq\vec{x}_0^* \Rightarrow \nabla^2 G(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0)$ let $r=|\vec{x}-\vec{x}_0| \to$ Cylindrical coordinates in 2D

If $\vec{x}\neq\vec{x}_0 \to \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)=0 \to G = C_1\ln r + C_2$

Now, $\int \nabla^2 G\, dA = \int \delta(\vec{x}-\vec{x}_0)\,dA = 1$, $\nabla G\cdot\hat{n}|_s = \frac{\partial G}{\partial r}|_s$ = constant

$$\oint \nabla G\cdot\hat{n}\,dl = \frac{\partial G}{\partial r}\oint dl = \frac{\partial G}{\partial r}(2\pi r) = 1$$

Derivative normal to a circle in 2D

$$\lim_{r\to 0}\oint \nabla G\cdot\hat{n}\,dl = \lim_{r\to 0}\frac{C_1}{r}(2\pi r) = 1 \to C_1 = 1/2\pi$$

C_2 is an arbitrary additive constant. For convenience we let $C_2=0$ ignoring the contribution at ∞

$$\Rightarrow G = \frac{1}{2\pi}\ln r \longrightarrow G = \frac{1}{2\pi}\ln\sqrt{(x-x_0)^2+(y-y_0)^2}$$

For $\vec{x}\neq\vec{x}_0 \Rightarrow \nabla^2 G(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0^*) \to G^* = \frac{1}{2\pi}\ln\sqrt{(x-x_0)^2+(y+y_0)^2}$

Thus by superposition the solution to this problem is just:

$$\boxed{G(\vec{x},\vec{x}_0) = \frac{1}{4\pi}\ln\left[(x-x_0)^2+(y-y_0)^2\right] + \frac{1}{4\pi}\ln\left[(x-x_0)^2+(y+y_0)^2\right]}$$

So all that remains is to check $\frac{\partial G}{\partial y}\Big|_{y=0} = 0$. Indeed.

$$\frac{\partial G}{\partial y}\Big|_{y=0} = \frac{1}{4\pi}\frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2}\Big|_{y=0} + \frac{1}{4\pi}\frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2}\Big|_{y=0}$$

$$= \frac{1}{4\pi}\frac{-2y_0}{(x-x_0)^2+y_0^2} + \frac{1}{4\pi}\frac{2y_0}{(x-x_0)^2+y_0^2} = 0 \quad \checkmark$$

b) Applying the Green's formula with $v = G$:

$$\int_R (u\nabla^2 G - G\nabla^2 u)\,dA = \oint (u\nabla G - G\nabla u)\cdot\hat{n}\,dl \qquad \nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}$$

outward unit normal vector

where R denotes the upper half plane and $\hat{n} = (0,-1)$ is the unit ~~vector~~ outward normal vector. Since $\frac{\partial G}{\partial y}\Big|_{y=0} = 0 \rightarrow \nabla G\cdot\hat{n}\Big|_s = 0$ and using $\nabla^2 G = \delta(\vec{x}-\vec{x}_0)$, $\nabla^2 u = f(\vec{x})$:

$$u(\vec{x}_0) - \int_R G(\vec{x},\vec{x}_0)f(\vec{x})\,dA = -\oint G(\vec{x},\vec{x}_0)\left(\frac{\partial u}{\partial x}\hat{\imath} + \frac{\partial u}{\partial y}\hat{\jmath}\right)\cdot(0,-1)\,dl$$

$$u(\vec{x}_0) - \int_{-\infty}^{\infty}\int_0^{\infty} G(\vec{x},\vec{x}_0)f(\vec{x})\,dy\,dx = \int_{-\infty}^{\infty} G(y=0)\,h(x)\,dx$$

$$\Rightarrow \boxed{u(\vec{x}) = \int_0^{\infty}\int_{-\infty}^{\infty} G(\vec{x},\vec{x}_0)f(\vec{x}_0)\,dx_0\,dy_0 + \int_{-\infty}^{\infty} G(y_0=0)\,h(x_0)\,dx_0}$$

You can actually check that

$$\frac{\partial}{\partial y}\left(\int_{-\infty}^{\infty} G(y_0=0)h(x_0)\,dx_0\right) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{h(x_0)}{(x-x_0)^2+y^2}\,dx_0 \quad \text{just } (9.5.49).$$

9.5.12 a) $\delta(\vec{x}-\vec{x}_0) \rightarrow$ positive source at $\vec{x} = \vec{x}_0 = (x_0,y_0)$ with $y_0 < 0$

$\delta(\vec{x}-\vec{x}^*) \rightarrow$ positive image at $\vec{x} = \vec{x}^* = (x_0,-y_0)$

↳ Reflection of $\vec{x} = \vec{x}_0$ across the plane $y=0$

$$\Rightarrow \nabla^2 G(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0) + \delta(\vec{x}-\vec{x}_0^*)$$

↳ This is the same problem in exercise 9.5.11(a) and the solution is the same for Green's function.

b) Now $\hat{n} = (0,1)$ since the physical region of interest is $y<0$, the lower half plane. Recall that $\vec{n}$ is a outward unit normal vector. The problem is similar to exercise 9.5.11(b), only change a sign:

$$\Rightarrow \boxed{u(\vec{x}) = \int_0^{\infty}\int_{-\infty}^{\infty} G(\vec{x},\vec{x}_0)f(\vec{x}_0)\,dx_0\,dy_0 - \int_{-\infty}^{\infty} G(y_0=0)\,h(x_0)\,dx_0}$$

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9.5.13 a) Similarly to exercise 9.5.11(a), to satisfy the BC that $\partial G/\partial y|_{y=0} = 0$, we introduce a positive image source but now at $\vec{x}^* = (x_0, -y_0, z_0)$, that is the reflection of the true source at $\vec{x}_0 = (x_0, y_0, z_0)$.

$$\Rightarrow \nabla^2 G(\vec{x}, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0)\,\delta(\vec{x} - \vec{x}_0^*)$$

For $\vec{x} \neq \vec{x}_0^*$ $\Rightarrow \nabla^2 G = \delta(\vec{x} - \vec{x}_0)$ let $\rho = |\vec{x} - \vec{x}_0| \rightarrow$ spherical coordinates

If $\vec{x} \neq \vec{x}_0 \rightarrow \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2 \frac{dG}{d\rho}\right) = 0 \rightarrow G = \frac{C_1}{\rho} + C_4$

Now, $\underbrace{\int \nabla^2 G\, dv}_{} = \int \delta(\vec{x} - \vec{x}_0)\, dv = 1$ $\quad \nabla G \cdot \hat{n}|_s = \frac{\partial G}{\partial \rho}\Big|_s = cte$

$\oint_s \nabla G \cdot \hat{n}\, ds = \frac{\partial G}{\partial \rho}\Big|_s \oint ds = \frac{\partial G}{\partial \rho}\Big|_s (4\pi\rho^2) = 1$ $\rightarrow$ Derivative normal to a sphere in 3D.

$$\lim_{\rho \to 0} \oint \nabla G \cdot \hat{n}\, ds = \lim_{r \to 0} -\frac{C_1}{\rho^2}(4\pi\rho^2) = 1 \rightarrow C_1 = -1/4\pi.$$

C_2 is an arbitrary additive constant. For convenience we let $C_2 = 0$ ignoring the contribution at ∞.

$$G = -\frac{1}{4\pi}\frac{1}{\rho} \rightarrow G = -\frac{1}{4\pi}\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{1/2}$$

For $\vec{x} \neq \vec{x}_0 \rightarrow \nabla^2 G = \delta(\vec{x} - \vec{x}_0^*) \rightarrow G = -\frac{1}{4\pi}\left[(x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2\right]^{1/2}$

By superposition:

$$\boxed{G = -\frac{1}{4\pi}\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{-1/2} - \frac{1}{4\pi}\left[(x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2\right]^{1/2}}$$

It can be verified that $\partial G/\partial y|_{y=0} = 0$.

b) From the Green's formula with $v = G$:

$$\int_V (u\nabla^2 G - G\nabla^2 u)\, dv = \oint_s (u\nabla G - G\nabla u)\cdot \hat{n}\, ds \qquad \hat{n} = -\hat{j} \quad \nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

outward unit normal vector

$\rightarrow 0$ since $\frac{\partial G}{\partial y}\Big|_{y=0} = 0 \rightarrow \nabla G \cdot \hat{n}|_s = 0$

$$\int_V \left(u\,\delta(\vec{x} - \vec{x}_0) - G f(\vec{x})\right) dv = \oint G \frac{\partial u}{\partial y}\Big|_s ds$$

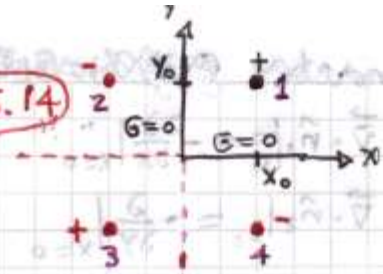
$$u(\vec{x}_0) - \int_V G f(\vec{x})\, dv = \oint_s G \frac{\partial u}{\partial y}\Big|_{y=0} \cdot ds \qquad \text{since } G(\vec{x}_0, \vec{x}) = G(\vec{x}, \vec{x}_0)$$

$$\Rightarrow u(\vec{x}) = \int_V G(\vec{x}, \vec{x}_0) f(\vec{x}_0)\, dv_0 + \oint_s G(\vec{x}, \vec{x}_0)\Big|_{y_0=0} \cdot h(x_0, z_0)\, ds_0$$

$$\boxed{u(\vec{x}) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} G(\vec{x}, \vec{x}_0) f(\vec{x}_0)\, dx_0\, dy_0\, dz_0 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(y_0 = 0)\, h(x_0, z_0)\, dx_0\, dz_0}$$

By: Herminso Villarraga Gómez

9.5.14



To satisfy the boundary conditions that $G=0$ on the boundaries of the first quadrant $(x=0, y\geq 0)$ and $(x\geq 0, y=0)$ we introduce three image sources, negative ones at $(-x_0, y_0)$ and $(x_0, -y_0)$ and a positive one at $(-x_0, -y_0)$.

The Green's function in the first quadrant is the superposition of the four infinite-space two-dimensional Green's functions each satisfying

$$\nabla^2 G_i = \delta(\vec{x}-\vec{x}_i) \qquad i=1,2,3,4 \text{ sources}$$

The solution is $G_i = \frac{1}{2\pi}\ln|\vec{x}-\vec{x}_i|$ according to exercise 9.5.11(a) or equation (9.5.31)

Then, by superposition:

$$G = \frac{1}{2\pi}\left\{\ln|\vec{x}-\vec{x}_1| - \ln|\vec{x}-\vec{x}_2| + \ln|\vec{x}-\vec{x}_3| - \ln|\vec{x}-\vec{x}_4|\right\}$$

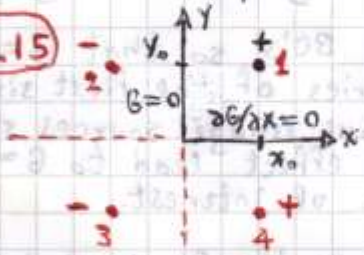
$$= \frac{1}{2\pi}\ln\left[\frac{|\vec{x}-\vec{x}_1||\vec{x}-\vec{x}_3|}{|\vec{x}-\vec{x}_2||\vec{x}-\vec{x}_4|}\right] \qquad \text{with } |\vec{x}-\vec{x}_i| = \left[(x-x_i)^2+(y-y_i)^2\right]^{1/2}$$

$$\boxed{G = \frac{1}{4\pi}\ln\left\{\frac{[(x-x_0)^2+(y-y_0)^2][(x+x_0)^2+(y+y_0)^2]}{[(x+x_0)^2+(y-y_0)^2][(x-x_0)^2+(y+y_0)^2]}\right\}}$$

We can verify that $G=0$ along $x=0$ and $y=0$.

9.5.15

a)



A first reflection across $x=0$ show the existence of a negative image to satisfy $G=0$ on the boundary $x=0$. Now, a reflection across $y=0$ show the existence of two images with the same signs of the generating sources to do $\partial G/\partial x|_{y=0}=0$.

In a similar way as we did in exercise (9.5.14) by superposition:

$$G = \frac{1}{2\pi}\left\{\ln|\vec{x}-\vec{x}_1| - \ln|\vec{x}-\vec{x}_2| - \ln|\vec{x}-\vec{x}_3| + \ln|\vec{x}-\vec{x}_4|\right\}$$

$$= \frac{1}{2\pi}\ln\left[\frac{|\vec{x}-\vec{x}_1||\vec{x}-\vec{x}_4|}{|\vec{x}-\vec{x}_2||\vec{x}-\vec{x}_3|}\right]$$

$$\boxed{G = \frac{1}{4\pi}\ln\left\{\frac{[(x-x_0)^2+(y-y_0)^2][(x-x_0)^2+(y+y_0)^2]}{[(x+x_0)^2+(y-y_0)^2][(x+x_0)^2+(y+y_0)^2]}\right\}} \rightarrow \text{Verify } G|_{x=0}=0 \text{ and } \partial G/\partial x|_{y=0}=0$$

b) Using the Green's formula with $v=G$

$$\int_R (u\nabla^2 G - G\nabla^2 u)\,dA = \oint (u\nabla G - G\nabla u)\cdot\hat{n}\,dl \qquad \nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$$

$$\int_R (u\,\delta(\vec{x}-\vec{x}_0) - G f(\vec{x}))\,dA = \oint (u\nabla G - G\nabla u)\cdot\hat{n}\,dl$$

$$u(\vec{x}_0) = \int_R G(\vec{x},\vec{x}_0) f(\vec{x})\,dA + \oint \left(u(\vec{x})\nabla G\cdot\hat{n}|_s - G(\vec{x},\vec{x}_0)\nabla u\cdot\hat{n}|_s\right)dl$$

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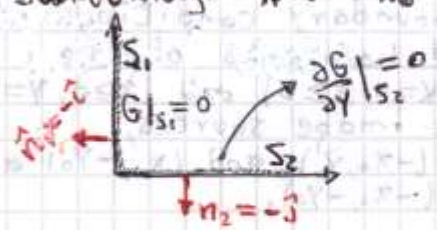
Switching $\vec{x}$ and $\vec{x}_0$ and using the symmetry $G(\vec{x},\vec{x}_0)=G(\vec{x}_0,\vec{x})$



$$\nabla=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j} \rightarrow \vec{\nabla}\cdot\hat{n}_1\big|_{S_1}=-\frac{\partial}{\partial x}\Big|_{x=0}$$

$$\vec{\nabla}\cdot\hat{n}_2\big|_{S_2}=-\frac{\partial}{\partial y}\Big|_{y=0}$$

$$S=S_1+S_2$$

$$u(\vec{x}_0)=\int_R G(\vec{x},\vec{x}_0)f(\vec{x})\,dA+\oint_{S_1}\Big(u(\vec{x})\vec{\nabla}G\cdot\hat{n}_1\big|_{S_1}-\cancelto{0}{G\big|_{S_1}}\vec{\nabla}u\cdot\hat{n}_1\big|_{S_1}\Big)dl_1+\oint_{S_2}\Big(u(\vec{x})\cancelto{0}{\vec{\nabla}G\cdot\hat{n}_2\big|_{S_2}}-G\big|_{S_2}\cancelto{0}{\vec{\nabla}u\cdot\hat{n}_2\big|_{S_2}}\Big)dl_2$$

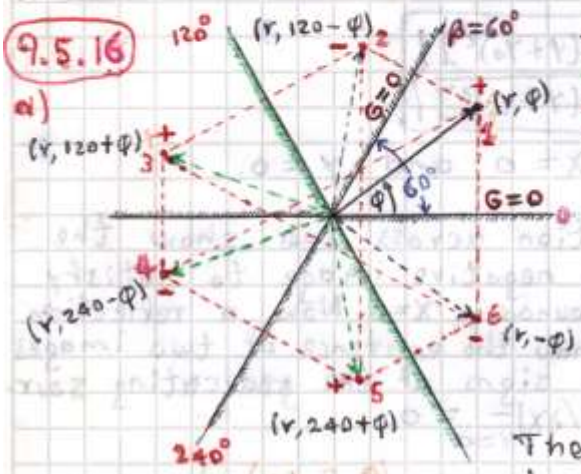
$$u(\vec{x}_0)=\int_0^\infty\int_0^\infty G(\vec{x},\vec{x}_0)f(\vec{x})\,dx\,dy-\int_0^\infty \underbrace{u(0,y)}_{g(y)}\frac{\partial G}{\partial x}\Big|_{x=0}dy+\int_0^\infty G\big|_{y=0}\underbrace{\frac{\partial u}{\partial y}\Big|_{y=0}}_{h(x)}dx$$

Switching $\vec{x}\longleftrightarrow\vec{x}_0$ and usin the symmetry $G(\vec{x}_0,\vec{x})=G(\vec{x},\vec{x}_0)$.

$$\boxed{u(\vec{x})=\int_0^\infty\int_0^\infty G(\vec{x},\vec{x}_0)f(\vec{x}_0)\,dx_0\,dy_0+\int_0^\infty G\big|_{y_0=0}h(x_0)\,dx_0-\int_0^\infty g(y_0)\frac{\partial G}{\partial x_0}\Big|_{x_0=0}dy_0}$$

9.5.16

a)



Boundaries as mirrors.
"mirror Images" of the boundaries.
All posibles reflections across the principal boundaries and across the virtual mirrors (or mirror images).

To satisfy the BC's so that $G=0$ on the boundaries of the first sixtant we introduce five image sources such that the total effect lead to $G=0$ on the boundarie of interest.

The Green's function in the first sixtant is the superposition of the six infinite-space two-dimensional Green's functions each satisfying:

$$\nabla^2 G_i=\delta(\vec{x}-\vec{x}_i)\qquad i=1,2,3,4,5,6 \text{ sources}$$

The solution for G_i as in exercise 9.5.11(a) or equation (9.5.31) is $\longrightarrow$ $G_i=\frac{1}{2\pi}\ln|\vec{x}-\vec{x}_i|$

Then, by superposition: $G=\frac{1}{2\pi}\sum_{i=1}^{\infty}(-1)^{i+1}\ln|\vec{x}-\vec{x}_i|$

$$\boxed{G=\frac{1}{2\pi}\ln\left[\frac{|\vec{x}-\vec{x}_1||\vec{x}-\vec{x}_3||\vec{x}-\vec{x}_5|}{|\vec{x}-\vec{x}_2||\vec{x}-\vec{x}_4||\vec{x}-\vec{x}_6|}\right]}$$

where $|\vec{x}-\vec{x}_i|=\left[(x-r\cos(2(i-1)60+\varphi))^2+(y-r\sin(2(i-1)60+\varphi))^2\right]^{1/2}$ For $i=1,3,5$

$|\vec{x}-\vec{x}_i|=\left[(x-r\cos(2(i-1)60-\varphi))^2+(y-r\sin(2(i-1)60-\varphi))^2\right]^{1/2}$ for $i=2,4,6$.

b) From the exercises 9.5.11, 9.5.14 and 9.5.15 we can see that the method of images extendable for other angles wedges boundaries such that the angle between the planes boundaries is

$\beta = 180°/n$ with n an integer, and are necessary $2n-1$ images to supply the BC's $\rightarrow G=0$ on the boundaries

In general by superposition: $\boxed{G = \frac{1}{2\pi}\sum_{i=1}^{2n}(-1)^{i+1}\ln|\vec{x}-\vec{x}_i|}$

where $|\vec{x}-\vec{x}_i| = \left[(x-r\cos(2(i-1)\beta+\phi))^2 + (y-r\sin(2(i-1)\beta+\phi))^2\right]^{1/2}$ for i odd

$|\vec{x}-\vec{x}_i| = \left[(x-r\cos(2(i-1)\beta-\phi))^2 + (y-r\sin(2(i-1)\beta-\phi))^2\right]^{1/2}$ for i even

being $\vec{x}_1 = (r\cos\phi, r\sin\phi)$ the position of the original source and $\vec{x}_i = (r\cos(2(i-1)\beta\pm\phi), r\sin(2(i-1)\beta\pm\phi))$ the position of the image sources.

The method is also aplicable if the BC's is $\partial G/\partial n|_s = 0$, but now the term $(-1)^{i+1}$ is onecessary since the image sources are all positive.

9.5.17 a) $\mathcal{L}[G_m] = \delta(\vec{x}-\vec{x}_0) + c\,\phi_h(\vec{x})$ Eq (9.4.15)

Let $\mathcal{L} = \nabla^2$ and $\phi_h = 1 \Rightarrow \nabla^2 G_m = \delta(\vec{x}-\vec{x}_0) + c$

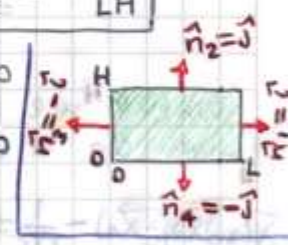
According to the equation (9.4.15) or Fredholm alternative the solution exist since the right-hand side of the equation above is orthogonal to ϕ_h:

$$0 = \int \phi_h[\delta(\vec{x}-\vec{x}_0)+c]\,dv = \int_0^L\int_0^H[\delta(\vec{x}-\vec{x}_0)+c]\,dy\,dx = 1 + cHL$$

$$\Rightarrow \boxed{c = -\frac{1}{LH}}$$

$\Rightarrow \nabla^2 G_m(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0) - \frac{1}{LH}$ with $\vec{\nabla}G_m\cdot\hat{n}|_s = 0$

or $\left(\frac{\partial G_m}{\partial x}\hat{i} + \frac{\partial G_m}{\partial y}\hat{j}\right)\cdot\hat{n}\Big|_s = 0$



$\Rightarrow \frac{\partial G}{\partial x}\Big|_{x=0} = 0 = \frac{\partial G}{\partial x}\Big|_{x=L} \Rightarrow G \sim \cos\left(\frac{n\pi x}{L}\right)$

$\frac{\partial G}{\partial y}\Big|_{y=0} = 0 = \frac{\partial G}{\partial y}\Big|_{y=H} \Rightarrow G \sim \cos\left(\frac{m\pi y}{H}\right)$

$$G(\vec{x},\vec{x}_0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{nm}(\vec{x}_0)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$$

Since we can differentiate term-by-term according to exercise (9.5.1) $\Rightarrow$

$$\nabla^2 G = \sum_{m,n} -a_{nm}\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$$

$$\parallel$$

$$\delta(\vec{x}-\vec{x}_0) - \frac{1}{LH}$$

where $\delta(\vec{x}-\vec{x}_0) = \delta(x-x_0)\delta(y-y_0)$.

Multiplying by $\cos\left(\frac{n'\pi x}{L}\right)\cos\left(\frac{m'\pi y}{H}\right)$ and integrating:

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$$\int_0^H\int_0^L \delta(x-x_0)\delta(y-y_0)\cos\left(\frac{n'\pi x}{L}\right)\cos\left(\frac{m'\pi y}{H}\right)dx\,dy - \frac{1}{LH}\overbrace{\int_0^L \cos\frac{n'\pi x}{L}dx}^{L\delta_{n'0}}\overbrace{\int_0^H \cos\frac{m'\pi y}{H}dy}^{H\delta_{m'0}}$$

$$= \sum_{n,m} -a_{nm}\left[\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\underbrace{\int_0^L \cos\frac{n\pi x}{L}\cos\frac{n'\pi x}{L}dx}_{(L/2)\delta_{nn'}}\underbrace{\int_0^H \cos\frac{m\pi y}{H}\cos\frac{m'\pi y}{H}dy}_{(H/2)\delta_{mm'}}$$

$$\cos\frac{n'\pi x_0}{L}\cos\frac{m'\pi y_0}{H} - \delta_{0n'}\delta_{0m'} = -a_{n'm'}\left[\left(\frac{n'\pi}{L}\right)^2+\left(\frac{m'\pi}{H}\right)^2\right]\frac{LH}{4}$$

$\Rightarrow$ For $n\neq 0, m\neq 0 \rightarrow$ $\boxed{a_{nm} = -\frac{4}{LH}\frac{\cos(n\pi x_0/L)\cos(m\pi y_0/H)}{(n\pi/L)^2+(m\pi/H)^2}}$

For $n=m=0 \rightarrow \underbrace{1-\delta_{00}\delta_{00}}_{0} = -a_0\underbrace{[0^2+0^2]}_{0}\frac{LH}{4} \rightarrow a_{00}$ can be arbitrary

$$\Rightarrow \boxed{G_m = a_{00} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4/LH}{(n\pi/L)^2+(m\pi/H)^2}\cos\left(\frac{n\pi x_0}{L}\right)\cos\left(\frac{m\pi y_0}{H}\right)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)}$$

which is symmetric. Now, since a_{00} can be arbitrary there are an infinite number of solutions.

b) Like in part (a) we can see that $c = -1/HL$

$\Rightarrow \nabla^2 G_m(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0) - \frac{1}{LH}$ with $\nabla G_m\cdot\vec{n}|_s = 0$, this imply (just as before in (a))

$$\left.\begin{array}{l}\frac{\partial G}{\partial x}\Big|_{x=0} = 0 \rightarrow G\sim\cos\alpha_n x_< \\ \frac{\partial G}{\partial x}\Big|_{x=L} = 0 \rightarrow G\sim\cos\left(\frac{n\pi x_>}{L}\right)\end{array}\right\} \Rightarrow G\sim\cos\alpha_n x_<\cos\alpha_n x_>$$

this election preserve the symmetry

$\boxed{\alpha_n = (n\pi/L)}$

$$\Rightarrow \boxed{G_m = \sum_{n=0}^{\infty}a_n(y)\cos\alpha_n x_0\cos\alpha_n x}$$

where we used that is indifferent to have $x>x_0$ or $x<x_0$ since we do a product of the two functions.

$$\delta(\vec{x}-\vec{x}_0) - \frac{1}{LH} = \nabla^2 G = \sum_{n=0}^{\infty}\left[\frac{d^2a_n}{dy^2} - \alpha_n^2 a_n\right]\cos\alpha_n x_0\cos\alpha_n x$$

Multiplying by $\cos\alpha_{n'}x$ and integrating between $0\leq x\leq L$:

$$\int_0^L \delta(x-x_0)\delta(y-y_0)\cos\alpha_{n'}x\,dx - \frac{1}{LH}\underbrace{\int_0^L\cos\alpha_{n'}x\,dx}_{L\delta_{n'0}}$$

$$= \sum_{n=0}^{\infty}\left[\frac{d^2a_n}{dy^2} - \alpha_n^2 a_n\right]\cos\alpha_n x_0\underbrace{\int_0^L\cos\alpha_n x\cos\alpha_{n'}x\,dx}_{(L/2)\delta_{nn'}}$$

$$\delta(y-y_0)\cos\alpha_{n'}x_0 - \frac{1}{H}\delta_{n'0} = \left[\frac{d^2a_{n'}}{dy^2} - \alpha_n^2 a_n\right]\cos(\alpha_{n'}x_0)\cdot\frac{L}{2} \Rightarrow$$

For $n \neq 0$ $\rightarrow$ $\delta(y-y_0)\cancel{\cos\alpha_n x_0} = \left[\frac{d^2 a_n}{dy^2} - \alpha_n^2 a_n\right]\cancel{\cos(\alpha_n x_0)} \cdot \frac{L}{2}$

$$\frac{2}{L}\delta(y-y_0) = \left[\frac{d^2 a_n}{dy^2} - \alpha_n^2 a_n\right]$$

- **For $y \neq y_0$** $\rightarrow$ $\frac{d^2 a_n}{dy^2} - \alpha_n^2 a_n(y) = 0 \rightarrow a_n \sim \cosh(\alpha_n y + \gamma)$

$\vec{\nabla} G_m \cdot \hat{n}\big|_S = 0 \rightarrow \frac{\partial G}{\partial y}\Big|_{y=0} = 0 \rightarrow a_n'(0) = 0 \rightarrow a_n \sim \cosh \alpha_n y_<$

$\frac{\partial G}{\partial y}\Big|_{y=H} = 0 \rightarrow a_n'(H) = 0 \rightarrow a_n \sim \cosh \alpha_n (y_> - H)$

$\Rightarrow a_n(y) = C_n \cosh \alpha_n y_< \cosh \alpha_n (y_> - H)$

The jump condition for the derivative of $a_n(y)$: $\frac{da_n}{dy}\Big|_{y_0^-}^{y_0^+} = \frac{2}{L}$ obtained integrating in a sphere of radious ϵ about $y = y_0$ when $\epsilon \to 0$. This condition gives:

$$\alpha_n C_n \Big[\underbrace{\cosh \alpha_n y_0 \sinh \alpha_n (y_0 - H) - \sinh \alpha_n y_0 \cosh \alpha_n (y_0 - H)}_{\sinh(\alpha_n H)}\Big] = \frac{2}{L}$$

$\Rightarrow C_n = \frac{2}{\alpha_n L} \Big/ \sinh \alpha_n H$

$$\Rightarrow \boxed{\underset{n\neq 0}{a_n(y)} = \frac{2}{\alpha_n L}\,\frac{\cosh \alpha_n y_< \cosh \alpha_n (y_> - H)}{\sinh \alpha_n H}}$$

$y_< \equiv$ Smaller of $\{y, y_0\}$
$y_> \equiv$ Larger of $\{y, y_0\}$

For $n = 0$ $\rightarrow$ $\delta(y-y_0) - \frac{1}{H} = \frac{d^2 a_0}{dy^2} \cdot \frac{L}{2}$ OR $\frac{d^2 a_0}{dy^2} = \frac{2}{L}\delta(y-y_0) - \frac{2}{LH}$

- **For $y \neq y_0$** $\rightarrow$ $\frac{d^2 a_0}{dy^2} = -\frac{2}{LH} \rightarrow a_0 = -\frac{y^2}{LH} + \begin{cases} c_1 y + c_2 & y < y_0 \\ c_3 y + c_4 & y > y_0 \end{cases}$

$\vec{\nabla} G_m \cdot \hat{n}\big|_S = 0 \rightarrow \frac{\partial G}{\partial y}\Big|_{y=0} = 0 \rightarrow a_0'(0) = 0 = c_1$

$\frac{\partial G}{\partial y}\Big|_{y=L} = 0 \rightarrow a_0'(H) = 0 = -\frac{2}{L} + c_3 \rightarrow c_3 = \frac{2}{L}$

By continuity of G at $x = x_0 \rightarrow a(x_0^-) = a(x_0^+) \rightarrow c_2 = \frac{2}{L}y_0 + c_4$

$$\Rightarrow a_0(y, y_0) = -\frac{y^2}{LH} + \begin{cases} \frac{2}{L}y_0 + c_4(y_0) & y < y_0 \\ \frac{2}{L}y + c_4(y_0) & y > y_0 \end{cases}$$

where we used that c_4 can be function of x_0. Now, by Symmetry

$G_m(\vec{x}, \vec{x}_0) = G(\vec{x}_0, \vec{x}) \rightarrow a_0(y, y_0) = a(y_0, y) = -\frac{y_0^2}{LH} + \begin{cases} \frac{2}{L}y + c_4(y) & y_0 < y \\ \frac{2}{L}y_0 + c_4(y) & y_0 > y \end{cases}$

$\Rightarrow -\frac{y^2}{LH} + \cancel{\frac{2}{L}y_0} + c_4(y_0) = -\frac{y_0^2}{LH} + \cancel{\frac{2}{L}y_0} + c_4(y) \rightarrow c_4(y_0) = -\frac{y_0^2}{LH} + \beta$

$$\Rightarrow \boxed{a_0(y,y_0) = \beta - \frac{1}{LH}(y^2+y_0^2) + \begin{cases} 2y_0/L & \text{if } y<y_0 \\ 2y/L & \text{if } y>y_0 \end{cases}}$$

Both a_n with $n\neq 0$ and a_0 are symmetric, the the associated Green's function also is symmetric defined as before.
Note that β can be **an** arbitrary constant, thus there are an infinite number of solutions.

9.5.18 $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ with $u(a,\theta) = h(\theta)$

We look for special product solutions $u(r,\theta) = \phi(\theta)g(r)$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dg}{dr}\right)\phi(\theta) + \frac{1}{r^2}g(r)\frac{d^2\phi}{d\theta^2} = 0 \quad \text{divide by } \frac{1}{r^2}g(r)\phi(\theta).$$

$$\Rightarrow \underbrace{\frac{r}{g}\frac{d}{dr}\left(r\frac{dg}{dr}\right)}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}}_{-\lambda} = 0 \Rightarrow \frac{d^2\phi}{d\theta^2} = -\lambda\phi \quad r\frac{d}{dr}\left(r\frac{dg}{dr}\right) = \lambda g$$

By peridic BC's on the circle $\Rightarrow \begin{matrix}\phi(-\pi)=\phi(\pi) \\ \phi'(-\pi)=\phi'(\pi)\end{matrix} \Rightarrow \phi \sim \left\{\begin{matrix}\sin n\theta \\ \cos n\theta\end{matrix}\right\} \quad \lambda = n^2$

$$r\frac{d}{dr}\left(r\frac{dg}{dr}\right) = n^2 g \longrightarrow \begin{matrix} g = C_1 + C_2 \ln r & \text{for } n=0 \\ g \sim r^{\pm n} & \text{for } n\neq 0. \end{matrix}$$ (See section 2.5.2)

But $|g(\theta)|<\infty \rightarrow C_2 = 0$ and $g\sim r^{+n}$ only for $n\neq 0$.

Finally: $u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n\cos n\theta + B_n \sin n\theta\right] r^n$

Since $h(\theta) = u(a,\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n\cos n\theta + B_n\sin n\theta\right] a^n$

we can find A_0, A_n and B_n multiplying ~~and~~ conveniently by $\cos n'\theta$ or $\sin n'\theta$ and integrating by use the orthogonality relations for the trigonometric functions, leading to the Fourier coefficients which can be written directly from equations (3.2.2) as:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta, \quad A_n = \frac{a^{-n}}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos n\theta\, d\theta \quad B_n = \frac{a^{-n}}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin n\theta\, d\theta$$

$$\Rightarrow u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\bar\theta)\,d\bar\theta + \sum_{n=1}^{\infty}\left[\frac{a^{-n}}{\pi}\int_{-\pi}^{\pi} h(\bar\theta)\cos n\bar\theta\, d\bar\theta \cos n\theta + \frac{a^{-n}}{\pi}\int_{-\pi}^{\pi} h(\bar\theta)\sin n\bar\theta\, d\bar\theta \sin n\theta\right] r^n$$

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$$\Rightarrow u(r,\theta)=\int_{-\pi}^{\pi}\left[\frac{1}{2\pi}h(\bar\theta)+\sum_{n=1}^{\infty}\frac{1}{\pi}h(\bar\theta)\left(\underbrace{\cos n\bar\theta\cos n\theta+\sin n\bar\theta\sin n\theta}\right)\left(\frac{r}{a}\right)^n\right]d\bar\theta$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}h(\bar\theta)\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\bar\theta)\right]d\bar\theta$$

Now for $n=0 \rightarrow \left(\frac{r}{a}\right)^n\cos n(\theta-\bar\theta)=1 \rightarrow$ thus we can start the sum from $n=0$ if substract 1, that's gives:

$$\boxed{u(r,\theta)=\frac{1}{\pi}\int_{0}^{2\pi}h(\bar\theta)\underbrace{\left[-\frac{1}{2}+\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\bar\theta)\right]}_{I(r,\theta,\theta_0)}d\bar\theta}$$

Where integrate in $-\pi\le\theta\le\pi$ is equivalent to integrate from 0 to 2π.

Since $\cos n(\theta-\bar\theta)=\mathrm{Re}\, e^{in(\theta-\bar\theta)}$

$$\Rightarrow u(r,\theta)=\frac{1}{\pi}\int_{0}^{2\pi}h(\bar\theta)\left[-\frac{1}{2}+\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\mathrm{Re}\, e^{in(\theta-\bar\theta)}\right]d\bar\theta$$

$$=\frac{1}{\pi}\int_{0}^{2\pi}h(\bar\theta)\left[-\frac{1}{2}+\mathrm{Re}\left\{\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\bar\theta)}\right\}\right]d\bar\theta$$

$$=\frac{1}{\pi}\int_{0}^{2\pi}h(\bar\theta)\left[-\frac{1}{2}+\mathrm{Re}\left\{\underbrace{\sum_{n=0}^{\infty}\left(\frac{r}{a}e^{i(\theta-\bar\theta)}\right)^n}\right\}\right]d\bar\theta$$

This is the geometric series which converges to:

$$\sum_{n=0}^{\infty}\left(\frac{r}{a}e^{i(\theta-\bar\theta)}\right)^n=\frac{1}{1-\frac{r}{a}e^{i(\theta-\bar\theta)}}=\frac{a}{a-re^{i(\theta-\bar\theta)}}\left(\frac{a-re^{-i(\theta-\bar\theta)}}{a-re^{-i(\theta-\bar\theta)}}\right)$$

$$=\frac{a^2-are^{-i(\theta-\bar\theta)}}{a^2-ar\left(e^{i(\theta-\bar\theta)}+e^{-i(\theta-\bar\theta)}\right)+r^2}=\frac{a^2-are^{-i(\theta-\bar\theta)}}{a^2+r^2-2ar\cos(\theta-\bar\theta)}$$

$$\Rightarrow \mathrm{Re}\left\{\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\bar\theta)}\right\}=\frac{a^2-ar\cos(\theta-\bar\theta)}{a^2+r^2-2ar\cos(\theta-\bar\theta)}$$ and thus:

$$-\frac{1}{2}+\mathrm{Re}\left\{\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\bar\theta)}\right\}=\frac{-[a^2+r^2-\cancel{2ar\cos(\theta-\bar\theta)}]+2(a^2-\cancel{2ar\cos(\theta-\bar\theta)})}{2[a^2+r^2-2ar\cos(\theta-\bar\theta)]}$$

$$=\frac{a^2-r^2}{2[a^2+r^2-2ar\cos(\theta-\bar\theta)]}$$

$$\boxed{u(r,\theta)=\frac{1}{2\pi}\int_{0}^{2\pi}h(\bar\theta)\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\bar\theta)}d\bar\theta}$$

just the Poisson's integral formula (9.5.61)

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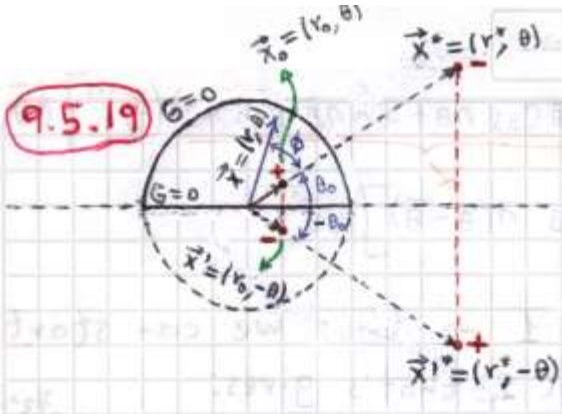
9.5.19



From the example worked in section 9.5.9 for a circle, the negative image source located at $\vec{x}^* = \frac{a^2}{r_0^2}\vec{x}_0$ will do $G|_{r=a}=0$.

Now for satysfy $G|_{\theta=0}=G_{\theta=\pi}=0$ (or along the line $y=0$) we need introduce two more image source located at $\vec{x}'=(r_0,-\theta)$ and $\vec{x}'^*=(r^*,-\theta)=\frac{a^2}{r_0^2}\vec{x}'^*$

The source image located at $\vec{x}'$ can be seen as a reflection of the source through the x-axis and need bee negative to satisfy $G|_{y=0}=0$. Similarly the source image located at $\vec{x}'^*$ can be seen as a reflection of the source image located at $\vec{x}^*$ through the x-axis, but also can be seen as the image of the $\vec{x}'$ though the dashed lower semi-circle.

From the eq (9.5.57) the Green's function ~~inside a semi-circle~~ due to the source located at $\vec{x}_0=(r_0,\theta_0)$ and the image source located at $\vec{x}^*=(r^*,\theta)=(a^2/r_0,\theta)$ is:

$$G_1(\vec{x},\vec{x}_0)=\frac{1}{4\pi}\ln\left(a^2\frac{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}{r^2r_0^2+a^4-2rr_0a^2\cos(\theta-\theta_0)}\right) \qquad \text{here } \phi=\theta-\theta_0$$

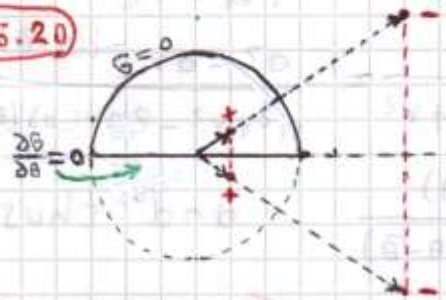
In a similar way the Green's function due to the image sources located at $\vec{x}'=(r_0,-\theta)$ and $\vec{x}'^*=(r^*,-\theta)=(a^2/r_0,-\theta)$ is

$$G_2(\vec{x},\vec{x}_0)=-\frac{1}{4\pi}\ln\left(a^2\frac{r^2+r_0^2-2rr_0\cos(\theta+\theta_0)}{r^2r_0^2+a^4-2rr_0a^2\cos(\theta+\theta_0)}\right) \qquad \text{here } \phi=\theta+\theta_0.$$

Then by superposition $\boxed{G(\vec{r},\vec{r}_0)=G_1(\vec{x},\vec{x}_0)+G_2(\vec{x},\vec{x})}$

You can verify that $G|_{y=0}=0$ just when $\theta=0$ or $\theta=\pi$.

9.5.20



Now for do $\left.\frac{\partial G}{\partial\theta}\right|_{y=0}=0$ we need that the image sources resulting from the reflection ~~through a certain axis~~) ~~have the same sign~~ of the other sources through the x-axis (or $y=0$) have the same sign of the reflected sources. See exercise 9.5.19, then now

$$G_2(\vec{x},\vec{x}_0)=+\frac{1}{4\pi}\ln\left(a^2\frac{r^2+r_0^2-2rr_0\cos(\theta+\theta_0)}{r^2r_0^2+a^4-2rr_0a^2\cos(\theta+\theta_0)}\right)$$

Thus now:

$$\boxed{G(\vec{x},\vec{x}_0)=G_1+G_2=\frac{1}{4\pi}\left\{\ln\left(a^2\frac{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}{r^2r_0^2+a^4-2rr_0a^2\cos(\theta-\theta_0)}\right)+\ln\left(a^2\frac{r^2+r_0^2-2rr_0\cos(\theta+\theta_0)}{r^2r_0^2+a^4-2rr_0a^2\cos(\theta+\theta_0)}\right)\right\}}$$

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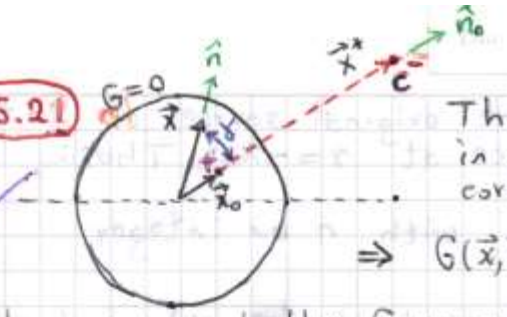
9.5.21



That is similar to problem worked in section 9.5.9, but now in spherical cordinates

$$\Rightarrow \quad G(\vec{x},\vec{x}_0) = -\frac{1}{4\pi|\vec{x}-\vec{x}_0|} + \frac{c}{4\pi|\vec{x}-\vec{x}^*|}$$

where we used the Green's function (9.5.31) adecuated for the infinite space in spherical coordinates.

~~We will show that there exists a point $\vec{x}^*$ such that $G(\vec{x},\vec{x}_0)$ vanishes on the surface of the sphere $|\vec{x}|=a$. In order for this to occur~~

$$\cancel{|\vec{x}-\vec{x}_0| = k|\vec{x}-\vec{x}^*| \quad \text{when } |\vec{x}|=a}$$

Note that by convenience we introduced a ~~[illegible]~~ magnified image by a factor c located at $\vec{x}^*$. Then the corresponding Negative Green's problem is.

$$\nabla^2 G(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0) - c\,\delta(\vec{x}-\vec{x}^*)$$

Since we define as $\hat{n}$ the unitary vector directed along the $\vec{x}$ and $\hat{n}_0$ the unitary vector directed along $\vec{x}_0$:

$$G(|\vec{x}|=a) = -\frac{1}{4\pi|a\hat{n}-r_0\hat{n}_0|} + \frac{c}{4\pi|a\hat{n}-r^*\hat{n}_0|} = 0 \quad \text{for satisfy the BC's}$$

$$\Rightarrow \quad \frac{1}{\sqrt{a^2+r_0^2-2ar_0\cos\gamma}} = \frac{c}{\sqrt{a^2+r^{*2}-2ar^*\cos\gamma}}$$

$$\frac{1}{a\sqrt{1+\left(\frac{r_0}{a}\right)^2-2\frac{r_0}{a}\cos\gamma}} = \frac{c}{r^*\sqrt{\left(\frac{a}{r^*}\right)^2+1-2\left(\frac{a}{r^*}\right)\cos\gamma}}$$

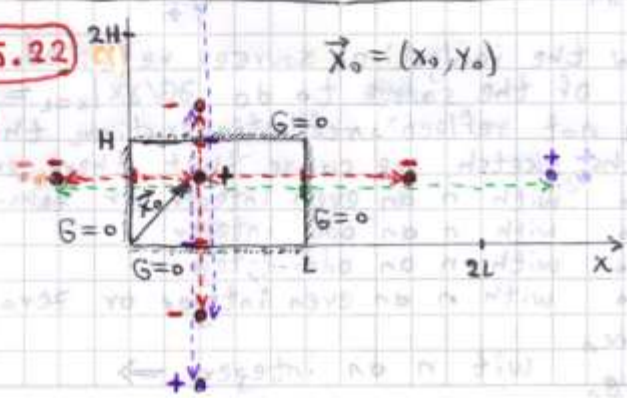
For satisfy that $\Rightarrow \frac{1}{a} = \frac{c}{r^*}$ and $\frac{r_0}{a} = \frac{a}{r^*} \rightarrow \boxed{r^* = a^2/r_0}$

and $\boxed{c = \frac{r^*}{a} = \frac{a}{r_0}}$

$$\Rightarrow \boxed{G(\vec{x},\vec{x}_0) = -\frac{1}{4\pi\sqrt{r^2+r_0^2-2ar_0\cos\gamma}} + \frac{a/r_0}{4\pi\sqrt{r^2+(a^2/r_0)^2-2a(a^2/r_0)\cos\gamma}}}$$

9.5.22

a)



To satisfy $G=0$ at $x=0$, a negative source must be introduced at $(-x_0, y_0)$. Now, to satisfy $G=0$ at $x=L$, a negative image source at $(-x_0+2L, y_0)$ and a positive source at (x_0+2L, y_0). This process must continue indefinitely There will be positive sources at x_0+2nL and negative sources at $-x_0+2nL$ where n is an arbitrary integer positive or negative

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we can include $n=0$ to give the original source at $x=x_0$ and the respective first image source at $x=-x_0$.
Thus if we define

$$\boxed{\begin{aligned}\alpha_n &= x_0+2nL\\ \beta_n &= -x_0+2nL\end{aligned}} \quad \text{with } n \text{ an integer}$$

the negative image sources are located at (β_n, y_0) and the positive sources are located at (α_n, y_0), the last including the position of the original source at (x_0, y_0) when $n=0$.

In a similar way to satisfy $G=0$ at $y=0$ and $y=L$ we find negative image source located at (x_0, δ_n) and the positive sources at (x_0, γ_n) with

$$\boxed{\begin{aligned}\gamma_n &= y_0+2nH\\ \delta_n &= -y_0+2nH\end{aligned}} \quad \text{with } n \text{ an integer.}$$

From eq (9.5.31) the response to a Dirac delta function in 2-D in $\vec{x}=(x,y)$ due to a positive source located at (α_n, y_0) is

$$G_1 = \frac{1}{2\pi}\ln\sqrt{(x-\alpha_n)^2+(y-y_0)^2} = \frac{1}{4\pi}\ln\left[(x-\alpha_n)^2+(y-y_0)^2\right]$$

and for a negative source located at (β_n, y_0)

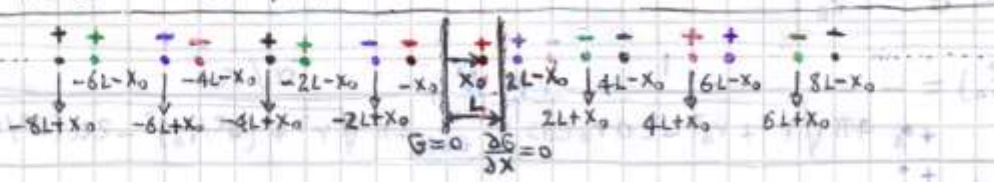
$$G_2 = -\frac{1}{4\pi}\ln\left[(x-\beta_n)^2+(y-y_0)^2\right]$$

Now by superposition $\Rightarrow$ $G = G_1+G_2 = \frac{1}{4\pi}\ln\left[\frac{(x-\alpha_n)^2+(y-y_0)^2}{(x-\beta_n)^2+(y-y_0)^2}\right]$

Additionating the others Green's functions due to the all others sources including the sources located at (x_0, δ_n) and (x_0, γ_n), then by superposition:

$$\boxed{G = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\ln\left[\frac{(x-\alpha_n)^2+(y-y_0)^2}{(x-\beta_n)^2+(y-y_0)^2}\right] + \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\ln\left[\frac{(x-x_0)^2+(y-\gamma_n)^2}{(x-x_0)^2+(y-\delta_n)^2}\right]}$$

b)



Thats similar to part (a) but now the images of sources reflected through $x=L$ preserve the sign of the source to do $\partial G/\partial x|_{x=L}=0$ a insolated boundary. There are not vertical reflections since there are not boundaries in y. From the sketch we canse that there are:

- positive sources at $x=2n+x_0$ with n an even integer or zero.
- negative sources at $x=2n+x_0$ with n an odd integer
- positive sources at $x=2n-x_0$ with n an odd integer
- negative sources at $x=2n-x_0$ with n an even integer or zero

$(-1)^n$ sources at $x=2n+x_0 \equiv \alpha_n$ wit n an integer $\Rightarrow$

$-(-1)^n$ sources at $x=2n-x_0 \equiv \beta_n$

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Similar as in part (a)

$$\boxed{G = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}(-1)^n \ln\left[\frac{(x-\alpha_n)^2+(y-y_0)^2}{(x-\beta_n)^2+(y-y_0)^2}\right]}$$

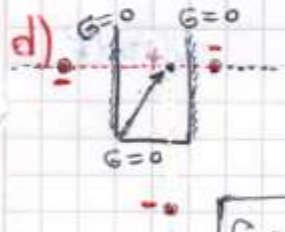
c) Just like in parte (a) we can see that for satisfy $G=0$ at $x=0$ and $x=L$ we need negative image sources located at (β_n, y_0, z_0) and positive sources at (α_n, y_0, z_0) with $\alpha_n = x_0 + 2nL$ and $\beta_n = -x_0 + 2nL$. But now from eq (9.5.31) the response to a Dirac delta function in 3-D at $\vec{x} = (x,y,z)$ due to a positive source located at (x', y', z') is:

$$G = -\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|} = -\frac{1}{4\pi}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

Using this then we have here:

$$\boxed{G = -\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\left(\frac{1}{\sqrt{(x-\alpha_n)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{1}{\sqrt{(x-\beta_n)^2+(y-y_0)^2+(z-z_0)^2}}\right)}$$

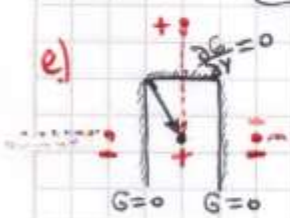
d)



That is the same problem as in part (a) but now there is only one image source for $y < y_0$ since there ~~are~~ is only ~~boundaries~~ one boundary for y. That cutt the second sum in the answer of part (a):

$$\boxed{G = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\ln\left[\frac{(x-\alpha_n)^2+(y-y_0)^2}{(x-\beta_n)^2+(y-y_0)^2}\right] + \frac{1}{4\pi}\ln\left[\frac{(x-x_0)^2+(y-\gamma_0)^2}{(x-x_0)^2+(y-\delta_0)^2}\right]}$$

e)



Now there are only one image source for $y > y_0$ since there is not lower boundary in y. That cut off the terms for $n \neq 0$ in the second term of the answer of part (a), but now since the image is positive at $y = -y_0$ for satisfy $\partial G/\partial y|_{y=0} = 0$:

$$\boxed{G = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\ln\left[\frac{(x-\alpha_n)^2+(y-y_0)^2}{(x-\beta_n)^2+(y-y_0)^2}\right] + \frac{1}{4\pi}\left[\ln\left[(x-x_0)^2+(y-\gamma_0)^2\right] + \ln\left[(x-x_0)^2+(y-\delta_0)^2\right]\right]}$$