

Problem Set 9 Solutions

Due: Monday, November 8

Problem 1. [10 points]

(a) [5 pts] Show that of any $n + 1$ distinct numbers chosen from the set $\{1, 2, \dots, 2n\}$, at least 2 must be relatively prime. (*Hint:* $\gcd(k, k + 1) = 1$.)

Solution. Treat the $n + 1$ numbers as the pigeons and the n disjoint subsets of the form $\{2j - 1, 2j\}$ as the pigeonholes. The pigeonhole principle implies that there must exist a pair of consecutive integers among the $n + 1$ chosen which, as suggested in the hint, must be relatively prime. ■

(b) [5 pts] Show that any finite connected undirected graph with $n \geq 2$ vertices must have 2 vertices with the same degree.

Solution. In a finite connected graph with $n \geq 2$ vertices, the domain for the vertex degrees is the set $\{1, 2, \dots, n - 1\}$ since each vertex can be adjacent to at most all of the remaining $n - 1$ vertices and the existence of a degree 0 vertex would violate the assumption that the graph be connected. Therefore, treating the n vertices as the pigeons and the $n - 1$ possible degrees as the pigeonholes, the pigeonhole principle implies that there must exist a pair of vertices with the same degree. ■

Problem 2. [10 points] Under Siege!

Fearing retribution for the many long hours his students spent completing problem sets, Prof. Leighton decides to convert his office into a reinforced bunker. His only remaining task is to set the 10-digit numeric password on his door. Knowing the students are a clever bunch, he is not going to pick any passwords containing the forbidden consecutive sequences "18062", "6042" or "35876" (his MIT extension).

How many 10-digit passwords can he pick that don't contain forbidden sequences if each number $0, 1, \dots, 9$ can only be chosen once (i.e. without replacement)?

Solution. The number of passwords he can choose is the number of permutations of the 10 digits minus the number of passwords containing one or more of the forbidden words, which we will find using inclusion-exclusion.

There are 6 positions 18062 could appear and the remaining digits could be any permutation of the remaining 5 digits. Therefore, there are $6 \cdot 5!$ passwords containing 18062. Similarly, there are $7 \cdot 6!$ passwords containing 6042 and $6 \cdot 5!$ passwords containing 35876.

Each of the forbidden words contain the digit 6 and since he must choose each number exactly once, the only way two forbidden words can appear in the same password is if they overlap at 6. The only case where this can happen is if the password contains 35876042 and there are $3 \cdot 2!$ such passwords.

By inclusion-exclusion the total number of passwords not containing any of the forbidden words is

$$10! - (6 \cdot 5! + 7 \cdot 6! + 6 \cdot 5!) + 3 \cdot 2! = 3622326$$

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Problem 3. [50 points] Be sure to show your work to receive full credit. In this problem we assume a standard card deck of 52 cards.

(a) [4 pts] How many 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

Solution. There is a bijection with sequence of the form:

(value of pair, suits of pair, value of other three cards, suits of other three cards)

Thus, the number of hands with a single pair is:

$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1,098,240$$

Alternatively, there is also a 3!-to-1 mapping to the tuple:

(value of pair, suits of pair,
value 3rd card, suit 3rd card, value 4th card, suit 4th card, value 5th card, suit 5th card)

Thus, the number of hands with a single pair is:

$$\frac{13 \cdot \binom{4}{2} \cdot 12 \cdot 4 \cdot 11 \cdot 4 \cdot 10 \cdot 4}{3!} = 1,098,240$$

■

(b) [4 pts] For fixed positive integers n and k , how many nonnegative integer solutions x_0, x_1, \dots, x_k are there to the following equation?

$$\sum_{i=0}^k x_i = n$$

Solution. There is a bijection from the solutions of the equation to the binary strings containing n zeros and k ones where x_0 is the number of 0s preceding the first 1, x_k is the number of 0s following the last 1 and x_i is the number of 0s between the i^{th} and $(i+1)^{th}$ 1 for $0 < i < k$.

$$\binom{n+k}{k}$$

■

(c) [4pts] For fixed positive integers n and k , how many nonnegative integer solutions x_0, x_1, \dots, x_k are there to the following equation?

$$\sum_{i=0}^k x_i \leq n$$

Solution. There is a bijection from the solutions of

$$\begin{aligned} \sum_{i=0}^k x_i &\leq n \\ &= n - x_{k+1} \end{aligned} \quad (\text{for some } x_{k+1} \geq 0)$$

and the solutions of

$$\begin{aligned} \sum_{i=0}^{k+1} x_i &= n. \\ \binom{n+k+1}{k+1} \end{aligned}$$

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(d) [4pts] How many simple undirected graphs are there with n vertices?

Solution. There are $\binom{n}{2}$ potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:

$$2^{\binom{n}{2}}$$

■

(e) [4pts] How many directed graphs are there with n vertices (self loops allowed)?

Solution. There are n^2 potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:

$$2^{n^2}$$

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(f) [4 pts] How many tournament graphs are there with n vertices?

Solution. There are no self-loops in a tournament graph and for each of the $\binom{n}{2}$ pairs of distinct vertices a and b , either $a \rightarrow b$ or $b \rightarrow a$ but not both. Therefore, the number of tournament graphs is:

$$2^{\binom{n}{2}}$$

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(g) [4 pts] How many acyclic tournament graphs are there with n vertices?

Solution. For any path from x to y in a tournament graph, an edge $y \rightarrow x$ would create a cycle. Therefore in any acyclic tournament graph, the existence of a path between distinct vertices x and y would require the edge $x \rightarrow y$ also be in the graph. That is, the "beats" relation for such a graph would be transitive. Since each pair of distinct players are comparable (either $x \rightarrow y$ or $y \rightarrow x$) we can uniquely rank the players $x_1 < x_2 < \dots < x_n$. There are $n!$ such rankings. ■

(h) [4 pts] How many numbers are there that are in the range $[1..700]$ which are divisible by 2, 5 or 7?

Solution. Let S be the set of all numbers in range $[1..700]$. Let S_2 be the numbers in this range divisible by 2, S_5 be the numbers in this range divisible by 5 and S_7 be the numbers in this range divisible by 7. By inclusion-exclusion, the number of elements in S divisible by 2, 5 or 7 is

$$\begin{aligned} n &= |S_2| + |S_5| + |S_7| - |S_2S_5| - |S_2S_7| - |S_5S_7| + |S_2S_5S_7| \\ &= \frac{700}{2} + \frac{700}{5} + \frac{700}{7} - \frac{700}{2 \cdot 5} - \frac{700}{2 \cdot 7} - \frac{700}{5 \cdot 7} + \frac{700}{2 \cdot 5 \cdot 7} \\ &= 350 + 140 + 100 - 70 - 50 - 20 + 10 \\ &= 460. \end{aligned}$$

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(i) [9 pts] In how many ways can you arrange n books on k bookshelf (assuming the order of books on a shelf matters?)

Solution.

$$n! \cdot \binom{n+k-1}{k-1}$$

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(j) [9 pts] How about if there has to be at least 1 book at each bookshelf?

Solution.

$$k! \cdot \binom{n}{k} \cdot (n-k)! \cdot \binom{n-1}{k-1}$$

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Problem 4. [15 points] Give a combinatorial proof of the following theorem:

$$n2^{n-1} = \sum_{k=1}^n k \binom{n}{k}$$

(Hint: Consider the set of all length- n sequences of 0's, 1's and a single *.)

Solution. Let $P = \{0, \dots, n-1\} \times \{0, 1\}^{n-1}$. On the one hand, there is a bijection from P to S by mapping (k, x) to the word obtained by inserting a * just after the k th bit in the length- $n-1$ binary word, x . So

$$|S| = |P| = n2^{n-1} \tag{1}$$

by the Product Rule.

On the other hand, every sequence in S contains between 1 and n nonzero entries since the *, at least, is nonzero. The mapping from a sequence in S with exactly k nonzero entries to a pair consisting of the set of positions of the nonzero entries and the position of the * among these entries is a bijection, and the number of such pairs is $\binom{n}{k}k$ by the Generalized Product Rule. Thus, by the Sum Rule:

$$|S| = \sum_{k=1}^n k \binom{n}{k}$$

Equating this expression and the expression (1) for $|S|$ proves the theorem. ■

Problem 5. [15 points] At a congressional hearing, there are $2n$ members present. Exactly n of them are Democrats and n of them are Republicans. The members want to select a smaller subcommittee of size n from within those present at the hearing. However, since the Democrats currently hold majority, they want there to be more Democrats than Republicans in the committee. In how many ways can you select such a committee? (Hint: Consider two cases: n odd and n even.)

Solution. First, look at the case when n is odd. There are

$$\binom{2n}{n}$$

ways to choose any subcommittee of size n . n is odd, so the number of Democrats is always different than the number of Republicans in a committee. Each of these committees will therefore either have more Democrats or more Republicans. However, there is an equal number of Democrats and Republicans present at the hearing, so the number of committees with more Republicans in them should by symmetry equal to the number of committees with more Democrats in them. Consequently, the number of committees with more Democrats than Republicans is

$$\frac{1}{2} \binom{2n}{n}.$$

■

If, however, n is even, then $\binom{n}{\frac{n}{2}}\binom{n}{\frac{n}{2}}$ committees will have an equal number of Democrats and Republicans (select $\frac{n}{2}$ out of n Democrats and $\frac{n}{2}$ out of n Republicans). The number of committees where one party has a majority is therefore $\binom{2n}{n} - \binom{n}{\frac{n}{2}}\binom{n}{\frac{n}{2}}$. Again by symmetry, there must be

$$\frac{1}{2} \left(\binom{2n}{n} - \binom{n}{\frac{n}{2}} \binom{n}{\frac{n}{2}} \right)$$

committees with more Democrats than Republicans.