Problem Set 10 Solutions

Due: Monday, November 15

Problem 1. [15 points] Suppose $\Pr\{\} : \mathcal{S} \to [0,1]$ is a *probability function* on a sample space, $\mathcal{S}$, and let B be an event such that $\Pr\{B\} > 0$. Define a function $\Pr_B\{\cdot\}$ on outcomes $w \in \mathcal{S}$ by the rule:

$$\Pr_B\{w\} = \begin{cases} \Pr\{w\} / \Pr\{B\} & \text{if } w \in B, \\ 0 & \text{if } w \notin B. \end{cases} \tag{1}$$

(a) [7 pts] Prove that $\Pr_B\{\cdot\}$ is also a probability function on $\mathcal{S}$ according to Definition 14.4.2.

Solution. We must show that $\Pr_B\{w\} \geq 0$ for all outcomes $w \in \mathcal{S}$, and

$$\sum_{w \in \mathcal{S}} \Pr_B\{w\} = 1. \tag{2}$$

But obviously $\Pr_B\{w\} \geq 0$ since both the numerator, $\Pr\{w\}$, and the denominator, $\Pr\{B\}$, in (1) are nonnegative. Also (2) holds because

$$\begin{aligned} \sum_{w \in \mathcal{S}} \Pr_B\{w\} &= \sum_{w \in B} \Pr_B\{w\} + \sum_{w \notin B} \Pr_B\{w\} \\ &= \sum_{w \in B} \frac{\Pr\{w\}}{\Pr\{B\}} + \sum_{w \notin B} 0 \\ &= \frac{\Pr\{B\}}{\Pr\{B\}} + 0 = 1. \end{aligned}$$

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(b) [8 pts] Prove that

$$\Pr_B\{A\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

for all $A \subseteq \mathcal{S}$.

Solution.

$$
\begin{aligned}
\Pr_B\{A\} &= \sum_{w \in A} \frac{\Pr\{w\}}{\Pr\{B\}} \\
&= \sum_{w \in A \cap B} \Pr_B\{w\} + \sum_{w \in A-B} \Pr_B\{w\} \\
&= \sum_{w \in A \cap B} \frac{\Pr\{w\}}{\Pr\{B\}} + \sum_{w \in A-B} \frac{\Pr\{w\}}{\Pr\{B\}} \\
&= \frac{\sum_{w \in A \cap B} \Pr\{w\}}{\Pr\{B\}} + \sum_{w \in A-B} 0 \\
&= \frac{\Pr\{A \cap B\}}{\Pr\{B\}}.
\end{aligned}
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Problem 2. **[20 points]**

(a) [10 pts] Here are some handy rules for reasoning about probabilities that all follow directly from the Disjoint Sum Rule. Use Venn Diagrams, or another method, to prove them.

$$\Pr\{A - B\} = \Pr\{A\} - \Pr\{A \cap B\} \qquad \text{(Difference Rule)}$$

Solution. Any set A is the disjoint union of $A - B$ and $A \cap B$, so

$$\Pr\{A\} = \Pr\{A - B\} + \Pr\{A \cap B\}$$

by the Disjoint Sum Rule.

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$$\Pr\left\{\bar{A}\right\} = 1 - \Pr\{A\} \qquad \text{(Complement Rule)}$$

Solution. $\bar{A} = \mathcal{S} - A$, so by the Difference Rule

$$\Pr\left\{\bar{A}\right\} = \Pr\{\mathcal{S}\} - \Pr\{A\} = 1 - \Pr\{A\}.$$

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$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} \qquad \text{(Inclusion-Exclusion)}$$

Solution. $A \cup B$ is the disjoint union of A and $B - A$ so

$$\begin{aligned}\Pr\{A \cup B\} &= \Pr\{A\} + \Pr\{B - A\} && \text{(Disjoint Sum Rule)}\\ &= \Pr\{A\} + (\Pr\{B\} - \Pr\{A \cap B\}) && \text{(Difference Rule)}\end{aligned}$$

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$$\Pr\{A \cup B\} \le \Pr\{A\} + \Pr\{B\}. \qquad \text{(2-event Union Bound)}$$

Solution. This follows immediately from Inclusion-Exclusion and the fact that $\Pr\{A \cap B\} \geq 0$. ■

$$\text{If } A \subseteq B, \text{ then } \Pr\{A\} \le \Pr\{B\}. \qquad \text{(Monotonicity)}$$

Solution.

$$\begin{aligned}\Pr\{A\} &= \Pr\{B\} - (\Pr\{B\} - \Pr\{A\})\\ &= \Pr\{B\} - (\Pr\{B\} - \Pr\{A \cap B\}) && \text{(since } A = A \cap B)\\ &= \Pr\{B\} - \Pr\{B - A\} && \text{(difference rule)}\\ &\le \Pr\{B\} && \text{(since } \Pr\{B - A\} \ge 0).\end{aligned}$$

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(b) [10 pts] Prove the following probabilistic identity, referred to as the **Union Bound**. You may assume the theorem that the probability of a union of *disjoint* sets is the sum of their probabilities.

Theorem. *Let $A_1, \ldots, A_n$ be a collection of events on some sample space. Then*

$$\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) \le \sum_{i=1}^{n} \Pr(A_i).$$

(*Hint: Induction*)

Solution. For all $n \geq 1$, let $P(n)$ be the proposition that the claim is true.

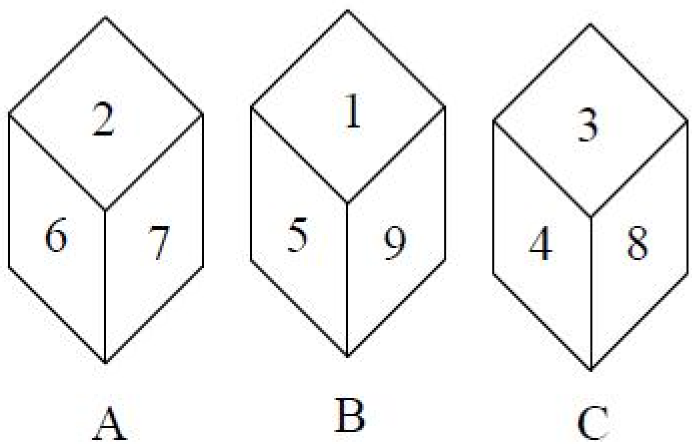
Base case: Trivially $\Pr(A_1) \le \Pr(A_1)$, so $P(1)$ is true.

Induction step: Assume that $P(n)$ is true and show $P(n+1)$ for $n \geq 1$.

$$\begin{aligned}\Pr(A_1 \cup A_2 \cup \cdots \cup A_{n+1}) &= \Pr((A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1})\\ &= \Pr(A_1 \cup A_2 \cup \cdots \cup A_n) + \Pr(A_{n+1})\\ &\quad - \Pr((A_1 \cup A_2 \cup \cdots \cup A_n) \cap A_{n+1}) && \text{(by Inclusion-Exclusion)}\\ &\le \Pr(A_1 \cup A_2 \cup \cdots \cup A_n) + \Pr(A_{n+1})\\ &\le \sum_{i=1}^{n} \Pr(A_i) + \Pr(A_{n+1}) && \text{(by Ind. Hyp.)}\\ &= \sum_{i=1}^{n+1} \Pr(A_i)\end{aligned}$$

Thus $P(n)$ is true and the result follows by induction. ■

Problem 3. [15 points] Recall the strange dice from lecture:

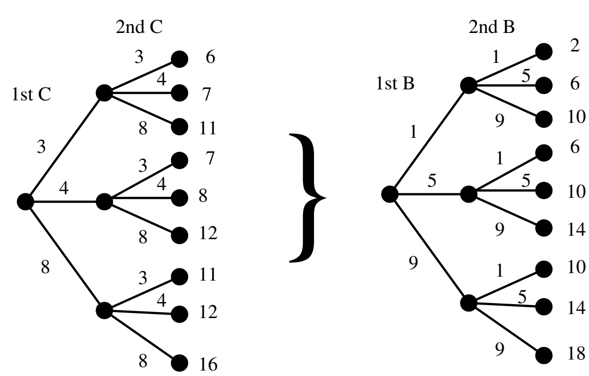


In the book we proved that if we roll each die once, then die A beats B more often, die B beats die C more often, and die C beats die A more often. Thus, contrary to our intuition, the "beats" relation $>$ is not transitive. That is, we have $A > B > C > A$.

We then looked at what happens if we roll each die twice, and add the result. In this problem we will show that the "beats" relation reverses in this game, that is, $A < B < C < A$, which is very counterintuitive!

(a) [5 pts] Show that rolling die C twice is more likely to win than rolling die B twice.

Solution. We draw the sample space. In the figure, it should be understood that the tree corresponding to B is connected to each leaf of the tree corresponding to C.



There are 81 leaves and the space is uniform, i.e., each outcome occurs with probability $(1/3)^4 = 1/81$. Let's work out the chances of winning. The sum of the two rolls of the B die is equally likely to be any element of the following multiset:

$$S_B = \{2, 6, 6, 10, 10, 10, 14, 14, 18\}.$$

The sum of the two rolls of the C die is equally likely to be any element of the following multiset:

$$S_C = \{6, 7, 7, 8, 11, 11, 12, 12, 16\}.$$

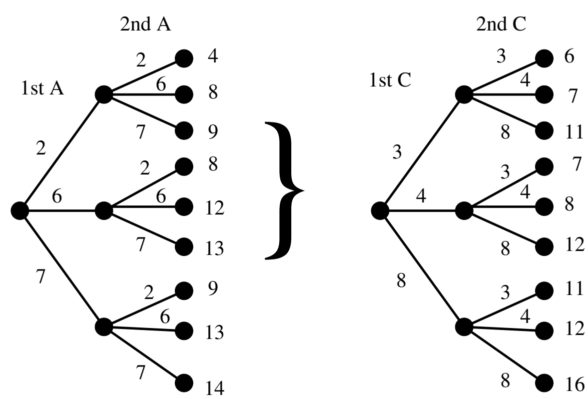
We can treat each outcome as a pair $(x, y) \in S_B \times S_C$, where C wins iff $y > x$. If $y = 6$, there is 1 value of x, namely $x = 2$, for which $y > x$. Continuing the count in this way, the number of pairs for which $y > x$ is

$$1 + 3 + 3 + 3 + 6 + 6 + 6 + 6 + 8 = 42,$$

while there are 2 ties and 37 cases where B wins. Thus, rolling die C twice is more likely to win than rolling die B twice. ■

(b) [5 pts] Show that rolling die A twice is more likely to win that rolling die C twice.

Solution. We draw the sample space. In the figure, it should be understood that the tree corresponding to C is connected to each leaf of the tree corresponding to A.



There are 81 leaves and the space is uniform, i.e., each outcome occurs with probability $(1/3)^4 = 1/81$. Let's work out the chances of winning. The sum of the two rolls of the C die is equally likely to be any element of the following multiset:

$$S_C = \{6, 7, 7, 8, 11, 11, 12, 12, 16\}.$$

The sum of the two rolls of the A die is equally likely to be any element of the following multiset:

$$S_A = \{4, 8, 8, 9, 9, 12, 13, 13, 14\}.$$

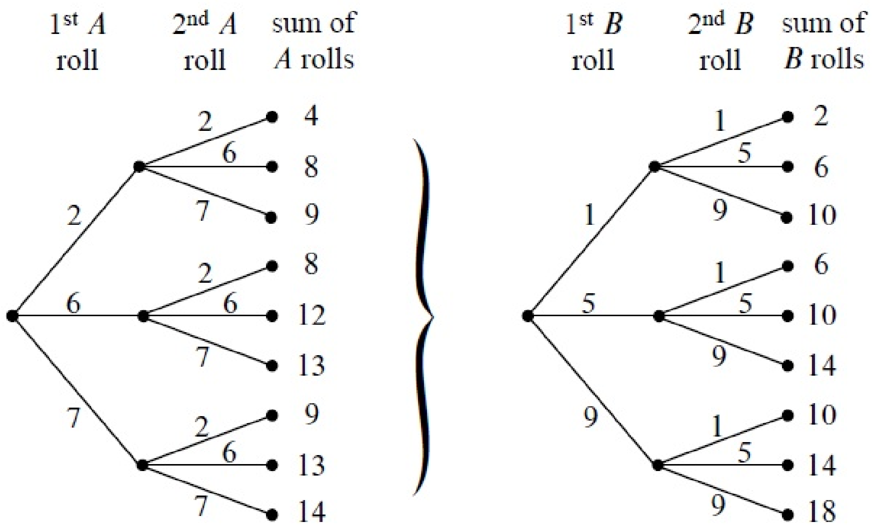
We can treat each outcome as a pair $(x, y) \in S_C \times S_A$, where A wins iff $y > x$. If $y = 4$, there is no x for which $y > x$. If $y = 8$, there are 3 values of x, namely $x = 6, 7, 7$, for which $y > x$. Continuing the count in this way, the number of pairs for which $y > x$ is

$$0 + 3 + 3 + 4 + 4 + 6 + 8 + 8 + 8 = 44,$$

while a similar count shows that there are only 33 pairs for which $x > y$, and there are 4 ties. Thus, rolling die A twice is more likely to win than rolling die C twice. ■

(c) [5 pts] Show that rolling die B twice is more likely to win than rolling die A twice.

Solution. We draw the sample space. In the figure, it should be understood that the tree corresponding to B is connected to each leaf of the tree corresponding to A.



There are 81 leaves and the space is uniform, i.e., each outcome occurs with probability $(1/3)^4 = 1/81$. Let's work out the chances of winning. The sum of the two rolls of the B die is equally likely to be any element of the following multiset:

$$S_B = \{2, 6, 6, 10, 10, 10, 14, 14, 18\}.$$

The sum of the two rolls of the A die is equally likely to be any element of the following multiset:

$$S_A = \{4, 8, 8, 9, 9, 12, 13, 13, 14\}.$$

We can treat each outcome as a pair $(x, y) \in S_B \times S_A$, where A wins iff $y > x$. If $y = 4$, there is no x for which $y > x$. If $y = 8$, there are 3 values of x, namely $x = 4, 8, 8$, for which $y > x$. Continuing the count in this way, the number of pairs for which $y > x$ is

$$1 + 3 + 3 + 3 + 3 + 6 + 6 + 6 + 6 = 37$$

while a similar count shows that there are 42 pairs for which $x > y$, and there are 2 ties. Thus, rolling die B twice is more likely to win than rolling die A twice. ■

Problem 4. [14 points] Let's play a game! We repeatedly flip a fair coin. You have the sequence HHT, and I have the sequence HTT. If your sequence comes up first, then you win. If my sequence comes up first, then I win. For example, if the sequence of tosses is:

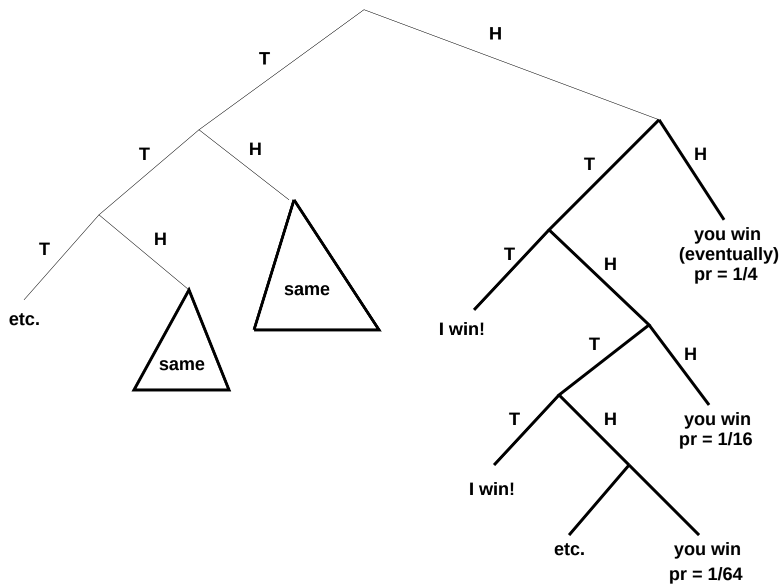
$$TTHTHT\underline{HHT}$$

then you win. What is the probability that you win? It may come as a surprise that the answer is very different from $1/2$.

This problem is tricky, because the game could go on for an arbitrarily long time. Draw enough of the tree diagram to see a pattern, and then sum up the probabilities of the (infinitely many) outcomes in which you win.

It turns out that for any sequence of three flips, there is another sequence that is likely to come up before it. So there is no sequence of three flips which turns up earliest! ... and given any sequence of three flips, knowing how to pick another sequence that comes up sooner more than half the time gives you a nice chance to fool people gambling at a bar :-)

Solution. A partial tree diagram is shown below. All edge probabilities are 1/2.



Let's first focus on the subtree shown in bold. Note that if two heads are flipped in a row, then you are guaranteed to win eventually. The sum of the probabilities of all your winning outcomes in this subtree is:

$$\begin{aligned}\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots &= \frac{1}{4}\cdot\frac{1}{1-1/4}\\ &= \frac{1}{3}\end{aligned}$$

The uppermost subtree marked **same** is identical to the one shown in bold, except that each outcome probability is reduced by 1/2, because it is one edge farther from the root. Thus, the sum of your winning outcomes in this subtree is 1/6. Similarly, the sum of your winning outcomes in the next subtree marked **same** is 1/12, and so forth. Overall, your probability of winning is:

$$\begin{aligned}\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\ldots &= \frac{1}{3}\cdot\frac{1}{1-1/2}\\ &= \frac{2}{3}\end{aligned}$$

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Problem 5. [15 points] We're interested in the probability that a randomly chosen poker hand (5 cards from a standard 52-card deck) contains cards from at most two suits.

(a) [7 pts] What is an appropriate sample space to use for this problem? What are the outcomes in the event, $\mathcal{E}$, we are interested in? What are the probabilities of the individual outcomes in this sample space?

Solution. The natural sample space to use consists of the $\binom{52}{5}$ possible poker hands. We define $\mathcal{E}$ to be the subset of outcomes in which the 5 cards on the outcome come from at most two suits. The sample space is *uniform*: Each hand is equally likely and comes up with probability $1/\binom{52}{5}$. ■

(b) [8 pts] What is $\Pr(\mathcal{E})$?

Solution. Since the sample space is uniform,

$$\Pr(\mathcal{E}) = \frac{|\mathcal{E}|}{\dbinom{52}{5}}.$$

We can count the size of $\mathcal{E}$ by cases. There are three cases: five cards of the same suit; four cards of one suit and one of another; and three cards of one suit and two of another.

For five of one suit, there are 4 ways to choose the suit and then $\binom{13}{5}$ ways to choose 5 cards from that suit.

For four of one suit and one of another, there are 4 ways to choose the larger suit and $\binom{13}{4}$ ways to choose 4 cards from that suit. Then there are 3 remaining suits from which to choose 1, and 13 choices for the 1 card of that suit.

Finally, for 3 of one suit and 2 of another, there are 4 ways to choose the suit of 3 and $\binom{13}{3}$ ways to choose cards for that suit, and there are 3 remaining suits to choose for the 2 cards, and $\binom{13}{2}$ choices for the 2 cards of that suit. So the total is

$$4 \cdot \binom{13}{5} + 4 \cdot \binom{13}{4} \cdot 3 \cdot 13 + 4 \cdot \binom{13}{3} \cdot 3 \cdot \binom{13}{2},$$

and the probability of at most two suits is

$$\frac{4 \cdot \dbinom{13}{5} + 4 \cdot \dbinom{13}{4} \cdot 3 \cdot 13 + 4 \cdot \dbinom{13}{3} \cdot 3 \cdot \dbinom{13}{2}}{\dbinom{52}{5}} = 88/595 \approx 0.15.$$

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Problem 6. [21 points]

I have a deck of 52 regular playing cards, 26 red, 26 black, randomly shuffled. They all lie face down in the deck so that you can't see them. I will draw a card off the top of the deck and turn it face up so that you can see it and then put it aside. I will continue to turn up cards like this but at some point while there are still cards left in the deck, you have to declare that you want the next card in the deck to be turned up. If that next card turns up black you win and otherwise you lose. Either way, the game is then over.

(a) [4 pts] Show that if you take the first card before you have seen any cards, you then have probability 1/2 of winning the game.

Solution. If we just record the sequence of black and red cards that will be drawn, there are $\binom{51}{25}$ sequences with first card black: 25 positions for the black cards chosen from the 51 remaining positions. Since there are $\binom{52}{26}$ sequences in all, the probability of winning on the first draw is $\binom{51}{25}/\binom{52}{26} = 26/52 = 1/2$. ■

(b) [4 pts] Suppose you don't take the first card and it turns up red. Show that you have then have a probability of winning the game that is greater than 1/2.

Solution. Suppose you take the next card after that. There are $\binom{50}{25}$ sequences that start with a red card and then a black and there are $\binom{51}{26}$ sequences that start with a red card. So then there is a $\binom{50}{25}/\binom{50}{26} = 26/51 > 1/2$ chance of winning. Any optimum strategy would have to guarantee a probability of winning as least as big as that. ■

(c) [4 pts] If there are r red cards left in the deck and b black cards, show that the probability of winning in you take the next card is $b/(r + b)$.

Solution. The probability is $\binom{b+r-1}{b-1}/\binom{b+r}{b} = b/(r + b)$. ■

(d) [9 pts] Either,

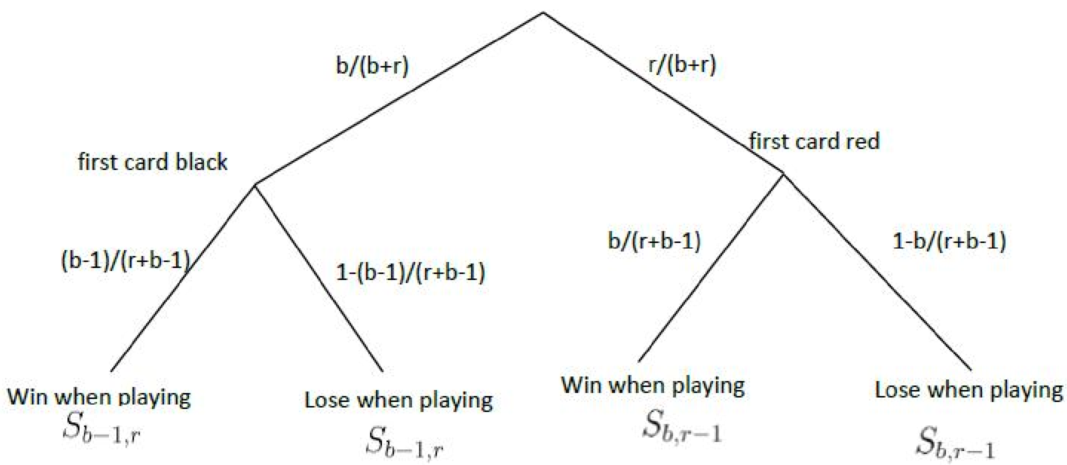
1. come up with a strategy for this game that gives you a probability of winning strictly greater than 1/2 and prove that the strategy works, or,
2. come up with a proof that no such strategy can exist.

Solution. There is no such strategy. Let $S_{b,r}$ be a strategy that acchieves the best probability of winning when starting with b black cards and r red cards. The claim is that $\Pr(\text{win by playing } S_{b,r}) = b/(r + b)$ for all b, r with at least $b > 0$ or $r > 0$.

Clearly $\Pr(\text{win by playing } S_{1,0}) = 1$ and $\Pr(\text{win by playing } S_{0,1}) = 0$. We prove the rest of the claim by induction on $r + b$. If the strategy $S_{b,r}$ is to take the next card, then $\Pr(\text{win by playing } S_{b,r}) = b/(r + b)$ as claimed. Suppose then that the strategy $S_{r,b}$ is to not take the first card, but to keep playing. Then the first card is going to be red with probability $r/(r + b)$ and it's going to be black with probability $b/(r + b)$. If the first card is black then we now have $b - 1$ black cards remaining and r red cards remaining and we should play $S_{b-1,r}$ which by our inductive hypothesis has a probability of $(b-1)/(b+r-1)$ of winning. On the other hand, if the first card is red, we now have $r-1$ red cards and b black cards remaining and we should play $S_{b,r-1}$ which by our inductive hypothesis has a probability of $b/(b+r-1)$ of winning. This gives us a total probability of

$$\Pr(\text{win by playing } S_{b,r}) = ((b-1)/(b-1+r))(b/(r+b)) + (b/(b+r-1))(r/(b+r)) = b/(b+r),$$

as claimed. This is summarized in the diagram below.



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