

Problem Set 1 Solutions

Due: Monday, September 13

Problem 1. [24 points]

Translate the following sentences from English to predicate logic. The domain that you are working over is X , the set of people. You may use the functions $S(x)$, meaning that “ x has been a student of 6.042,” $A(x)$, meaning that “ x has gotten an ‘A’ in 6.042,” $T(x)$, meaning that “ x is a TA of 6.042,” and $E(x, y)$, meaning that “ x and y are the same person.”

(a) [6 pts] There are people who have taken 6.042 and have gotten A’s in 6.042

Solution. $\exists x \in X : S(x) \wedge A(x)$

Addendum: It was noted by a sharp student that in the wording of the problem, we used the word “people” rather than “at least person” - meaning that there must be at least two people who have gotten A’s. So, the correct answer for this problem would be

$$\exists x, y \in X : S(x) \wedge S(y) \wedge A(x) \wedge A(y) \wedge \neg E(x, y)$$

Because even we missed this, though, both answers are acceptable for this problem. ■

(b) [6 pts] All people who are 6.042 TA’s and have taken 6.042 got A’s in 6.042

Solution. $\forall x \in X : T(x) \wedge S(x) \Rightarrow A(x)$ ■

(c) [6 pts] There are no people who are 6.042 TA’s who did not get A’s in 6.042.

Solution. $\neg \exists x \in X : T(x) \wedge (\neg A(x))$ ■

(d) [6 pts] There are at least three people who are TA’s in 6.042 and have not taken 6.042

Solution.

$$\exists x, y, z \in X : (\neg E(x, y) \wedge \neg E(y, z) \wedge \neg E(x, z)) \wedge T(x) \wedge \neg S(x) \wedge T(y) \wedge \neg S(y) \wedge T(z) \wedge \neg S(z)$$

■

Problem 2. [24 points]

Use a truth table to prove or disprove the following statements:

(a) [12 pts]

$$\neg(P \vee (Q \wedge R)) = (\neg P) \wedge (\neg Q \vee \neg R)$$

Solution. We write out the truth table and check that the statements on the left and right always have the same outputs. Define $F(P, Q, R)$ to be the formula representing the left side of the equation above, and $G(P, Q, R)$ to be the right side.

P	Q	R	$F(P, Q, R)$	$G(P, Q, R)$
true	true	true	false	false
true	true	false	false	false
true	false	true	false	false
true	false	false	false	false
false	true	true	false	false
false	true	false	true	true
false	false	true	true	true
false	false	false	true	true

■

(b) [12 pts]

$$\neg(P \wedge (Q \vee R)) = \neg P \vee (\neg Q \vee \neg R)$$

Solution. Once again, write out the truth table for each side of the equation, letting $F(P, Q, R)$ be the left side and $G(P, Q, R)$ be the right side.

P	Q	R	$F(P, Q, R)$	$G(P, Q, R)$
true	true	true	false	false
true	true	false	false	true
true	false	true	false	true
true	false	false	true	true
false	true	true	true	true
false	true	false	true	true
false	false	true	true	true
false	false	false	true	true

The entries for F and G do not agree for the same inputs of P , Q , and R , so they are not equal.

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Problem 3. [24 points]

The binary logical connectives \wedge (*and*), \vee (*or*), and \Rightarrow (*implies*) appear often in not only computer programs, but also everyday speech. In computer chip designs, however, it is

considerably easier to construct these out of another operation, **nand**, which is simpler to represent in a circuit. Here is the truth table for **nand**:

P	Q	$P \text{ nand } Q$
true	true	false
true	false	true
false	true	true
false	false	true

(a) [12 pts] For each of the following expressions, find an equivalent expression using only **nand** and \neg (*not*), as well as grouping parentheses to specify the order in which the operations apply. You may use A , B , and the operators any number of times.

(i) $A \wedge B$

Solution. Observe from the truth table that $(A \text{ nand } B)$ is equivalent to $\neg(A \wedge B)$. We negate both expressions to produce

$$(A \wedge B) = \neg(A \text{ nand } B).$$

■

(ii) $A \vee B$

Solution. The negation of the negation of a statement is equivalent to the original statement. We derive the desired expression as follows, applying the identity $\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$:

$$\begin{aligned} (A \vee B) &= \neg(\neg(A \vee B)) \\ &= \neg((\neg A) \wedge (\neg B)) \\ &= (\neg A) \text{ nand } (\neg B). \end{aligned}$$

■

(iii) $A \Rightarrow B$

Solution. An implication is true whenever the antecedent is false or the consequent is true:

$$\begin{aligned} (A \Rightarrow B) &= ((\neg A) \vee B) \\ &= \neg(A \wedge (\neg B)) \\ &= A \text{ nand } (\neg B). \end{aligned}$$

■

(b) [4pts] It is actually possible to express each of the above using only **nand**, without needing to use \neg . Find an equivalent expression for $(\neg A)$ using only **nand** and grouping parentheses.

Solution. Observe from the truth table for **nand** that when P and Q are both **true**, $(P \text{ nand } Q)$ is **false**, and when P and Q are both **false**, $(P \text{ nand } Q)$ is **true**. Setting P and Q equal to each other, we see that the value of $(P \text{ nand } P)$ is the negation of the value of P . So an equivalent expression for $(\neg A)$ is

$$(A \text{ nand } A).$$

■

(c) [8pts] The constants **true** and **false** themselves may be expressed using only **nand**. Construct an expression using an arbitrary statement A and **nand** that evaluates to **true** regardless of whether A is **true** or **false**. Construct a second expression that always evaluates to **false**. Do not use the constants **true** and **false** themselves in your statements.

Solution. Observe from the truth table for **nand** that only three combinations of **true** and **false** produce **true**.

P	Q	$P \text{ nand } Q$
true	false	true
false	true	true
false	false	true

Furthermore, observe that of these three, two are interchangeable simply by negating both P and Q :

P	Q	$P \text{ nand } Q$
true	false	true
false	true	true

We therefore have a method by which we can produce a **true** value regardless of the value of P : make sure that Q is the negation of P . Then, as long as Q and P are different, the value of P has no bearing on the **true** value of $(P \text{ nand } Q)$.

So given an arbitrary proposition A , an expression that always evaluates to **true** is $(A \text{ nand } (\neg A))$. By part (b), this is equal to

$$(A \text{ nand } (A \text{ nand } A)).$$

The negation of **true** is **false**. So by substituting the previous expression for A in part (b), we can construct an expression that always evaluates to **false**:

$$(A \text{ nand } (A \text{ nand } A)) \text{ nand } (A \text{ nand } (A \text{ nand } A)).$$

■

Problem 4. [10 points] You have 12 coins and a balance scale, one of which is fake. All the real coins weigh the same, but the fake coin weighs less than the rest. All the coins visually appear the same, and the difference in weight is imperceptible to your senses. In at most 3 weighings, give a strategy that detects the fake coin. (Note: the scale in this problem is a scale with two dishes, which tips toward the side that is heavier. For clarification, do an image search for “balance scale”).

Solution. There are many ways to approach this problem. One way is this: separate the coins into three groups of four, and call them A, B, and C. First, weigh A against B. Because we know that the fake coin is lighter than the rest, this first weighing will tell us which group the fake coin is in. If A and B do not balance exactly, the fake coin is in the lighter group. Otherwise, it is in group C. Now, we have narrowed the fake coin to a group of four coins. First weigh two coins from that group against each other. If they do not balance, then the fake coin is the lighter of the two. Otherwise, the fake coin is one of the other two coins. In that case, we can just weigh those two coins against each other to determine the fake coin. In the worst case, this strategy takes 3 weighings. ■

Problem 5. [6 points] Prove the following statement by proving its contrapositive: if r is irrational, then $r^{1/5}$ is irrational. (Be sure to *state* the contrapositive explicitly.)

Solution. Proof. We prove the given statement by proving its contrapositive: if $r^{1/5}$ is rational, then r is rational.

By our assumption that $r^{1/5}$ is rational, there exists an integer a and a positive integer b such that:

$$r^{1/5} = \frac{a}{b}. \quad (1)$$

Since 5 is a positive integer, we may raise both sides of (1) to the power 5, which gives:

$$r = \frac{a^5}{b^5}$$

Since a^5 and b^5 are integers, this implies that r is rational. (Note that $b^5 \neq 0$, since b is positive.)

This proves the contrapositive, so the original statement is also true. □

Problem 6. [12 points] Suppose that $w^2 + x^2 + y^2 = z^2$, where w , x , y , and z always denote positive integers. (Hint: It may be helpful to represent even integers as $2i$ and odd integers as $2j + 1$, where i and j are integers)

Prove the proposition: z is even if and only if w , x , and y are even. Do this by considering all the cases of w, x, y being odd or even.

Solution. As the problem suggests, we will build a truth table to figure out what z is in each case of x, y, z being odd or even.

- w, x, y are all even. In this case, we can write w, x, y as $2i, 2j, 2k$, and the sum of squares is $4i^2 + 4j^2 + 4k^2 = 4(i^2 + j^2 + k^2)$. In this case, z can be an even integer when $i^2 + j^2 + k^2 = l^2$ for some integer l .
- w, x, y are all odd. In this case, each of their squares is odd, and the sum of three odd numbers is odd. However, if z is even, then z^2 is also even. So it cannot be that $z^2 = w^2 + x^2 + y^2$.

- One of w, x, y is odd. This case is the same as above. Assume x is odd. Then, x^2 is odd, but y^2 and w^2 are even. So the sum is once again, odd, which cannot equal the square of an even integer. So z cannot be even in that case either. The same argument can be repeated for when w and y are odd as well.
- Two of w, x, y are odd. Assume that w and x are odd and that y are even. Then, we can write the sum as $w^2 + x^2 + y^2 = (2i+1)^2 + (2j+1)^2 + (2k)^2$. This can be rewritten as $4i^2 + 4i + 1 + 4j^2 + 4j + 1 + 4k^2 = 4(i^2 + j^2 + k^2 + i + j) + 2$. Note that 4 does not divide this sum. However, if z was an even integer, then z^2 is a multiple of 4. So z cannot be an even integer in this case. The same argument is also used when x, y are odd, or when w, y are odd.

With this, we have covered all the cases for the parities of w, x, y , and obtain the following truth table:

w even	x even	y even	z even
true	true	true	true
true	true	false	false
true	false	true	false
true	false	false	false
false	true	true	false
false	true	false	false
false	false	true	false
false	false	false	false

Note: writing this truth table is not a required part of solving this problem. Rather, the truth table was supposed to suggest to the student the casework that needed to be done for this problem.

