Problem Set 8 Solutions

Due: Tuesday, November 2 @ 7pm

Problem 1. [25 points] Find Θ bounds for the following divide-and-conquer recurrences. Assume $T(1) = 1$ in all cases. Show your work.

(a) [5 pts] $T(n) = 8T(\lfloor n/2 \rfloor) + n$

(b) [5 pts] $T(n) = 2T(\lfloor n/8 \rfloor + 1/n) + n$

(c) [5 pts] $T(n) = 7T(\lfloor n/20 \rfloor) + 2T(\lfloor n/8 \rfloor) + n$

(d) [5 pts] $T(n) = 2T(\lfloor n/4 \rfloor + 1) + n^{1/2}$

(e) [5 pts] $T(n) = 3T(\lfloor n/9 + n^{1/9} \rfloor) + 1$

Solution. We use the method of Akra-Bazzi for these problems.

(a) We see that $a = 8$, $b = 1/2$, $h = \lfloor n/2 \rfloor - n/2$ so $p = 3$ gives $ab^p = 1$.

$$T(n) = \Theta(n^3(1 + \int_1^n \frac{u}{u^4} du)) = \Theta(n^3(1 + \int_1^n u^{-3} du)) = \Theta(n^3).$$

(b) $a_1 = 2$, $b_1 = 1/8$, $h_1(n) = \lfloor n/8 \rfloor - n/8 + 1/n$, $g(n) = n$, $p = 1/3$,

$$\begin{aligned}
T(n) &= \Theta\left(n^p\left(1 + \int_1^n \frac{g(u)}{u^{p+1}} du\right)\right) \\
&= \Theta\left(n^{1/3}\left(1 + \int_1^n \frac{u}{u^{4/3}}\, du\right)\right) \\
&= \Theta\left(n^{1/3} + n^{1/3}\int_1^n u^{-1/3}\, du\right) \\
&= \Theta(n^{1/3} + n^{1/3}\frac{3}{2}(n^{2/3} - 1)) \\
&= \Theta(n).
\end{aligned}$$

(c) $a_1 = 7$, $b_1 = 1/20$,$a_2 = 2$, $b_2 = 1/8$, $h_1(n) = \lfloor n/20 \rfloor - n/20$, $h_2(n) = \lfloor n/8 \rfloor - n/8$, and $g(n) = n$. Finally, note that although we do not know what p is, we are guaranteed that $p < 1$.

$$\begin{aligned} T(n) = \Theta(n^p(1 + \int_1^n \frac{u}{u^{p+1}} du)) &= \Theta(n^p(1 + \int_1^n u^{-p} du)) \\ &= \Theta(n^p + n^p \frac{1}{1-p}(n^{1-p} - 1)) \\ &= \Theta(n). \end{aligned}$$

(d) $a_1 = 2$, $b_1 = 1/4$, $h_1(n) = \lfloor n/4 \rfloor - n/4 + 1$, $g(n) = n^{1/2}$, $p = 1/2$,

$$T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{u^{1/2}}{u^{3/2}} du)) = \Theta(n^{1/2} \log n).$$

(e) $a_1 = 3$, $b_1 = 1/9$, $h_1(n) = \lfloor n/9 + n^{1/9} \rfloor - n/9$, $g(n) = 1$, $p = 1/2$,

$$T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{1}{u^{3/2}} du)) = \Theta(n^{1/2}).$$

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Problem 2. [30 points] It is easy to misuse induction when working with asymptotic notation.

False Claim If

$$T(1) = 1 \text{ and}$$
$$T(n) = 4T(n/2) + n$$

Then T(n) = O(n).

False Proof We show this by induction. Let $P(n)$ be the proposition that $T(n) = O(n)$.

Base Case: $P(1)$ is true because $T(1) = 1 = O(1)$.

Inductive Case: For $n \geq 1$, assume that $P(n-1), \ldots, P(1)$ are true. We then have that

$$T(n) = 4T(n/2) + n = 4O(n/2) + n = O(n)$$

And we are done.

(a) [5 pts] Identify the flaw in the above proof.

(b) [10 pts] A simple attempt to prove $T(n) \neq O(n)$ via induction ultimately fails. We assume for sake of contradiction that $T(n) = O(n)$. Then there exists positive integer n_0 and positive real number c such that for all $n \geq n_0$, $T(n) \leq cn$. We then define $P(n)$ as the proposition that $T(n) \leq cn$.

We then proceed with strong induction.

Base Case, $n = n_0$: $P(n_0)$ is true, by assumption.

Inductive Step: We assume $P(n_0)$, $P(n_0 + 1)$, ..., $P(n - 1)$ true.

Fill in the rest of this proof attempt, and explain why it doesn't work.

Note: As this problem was updated so late, the graders will be instructed to be exceedingly lenient when grading this.

(c) [5 pts] Using Akra-Bazzi theorem, find the correct asymptotic behavior of this recurrence.

(d) [10 pts] We have now seen several recurrences of the form $T(n) = aT(\lfloor n/b \rfloor) + n$. Some of them give a runtime that is $O(n)$, and some don't. Find the relationship between a and b that yields $T(n) = O(n)$, and prove that this is sufficient.

Solution. (a) The flaw is that $P(n)$ is a predicate on n, whereas $O(n)$ is a statement not on n, but on the limit of n as n approaches infinity. $T(n) = O(n)$ does not depend on the value of n - it is either true or false.

(b) We first take some $n \geq 2n_0$. Then,

$$T(n) = 4T(n/2) + n$$

From the inductive hypothesis, $n/2 \geq n_0$, so $T(n/2) \leq cn/2$. So this means that

$$T(n) \leq 4cn/2 + n = 2cn + n = n(2c + 1)$$

Which is not less than cn. So the induction is simply not powerful enough.

(c) We have that $p = 2$, so $T = \Theta(n^2(1 + \int_1^n (u/u^3)du)) = \Theta(n^2)$.

(d) From analyzing the integral we can see that any case where $p < 1$ will give a linear solution, so having the condition $a < b$ is sufficient.

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Problem 3. [15 points] Define the sequence of numbers A_i by

$A_0 = 2$

$A_{n+1} = A_n/2 + 1/A_n$ (for $n \geq 1$)

Prove that $A_n \leq \sqrt{2} + 1/2^n$ for all $n \geq 0$.

Solution. *Proof.* The proof is by induction on n. Let $P(n)$ be the proposition that $A_n \leq \sqrt{2} + 1/2^n$.

Base case: $A_0 = 2 \leq \sqrt{2} + 1/2^0$ is true.

Inductive step: Let $n \geq 0$ and assume the inductive hypothesis $A_n \leq \sqrt{2}+1/2^n$. We need the following lemma.

Lemma. *For real numbers $x > 0$, $x/2 + 1/x \geq \sqrt{2}$.*

Proof. For real numbers $x > 0$,

$$\begin{aligned} & x/2 + 1/x \geq \sqrt{2} \\ \Leftrightarrow \quad & x^2 + 2 \geq 2\sqrt{2} \cdot x \\ \Leftrightarrow \quad & x^2 - 2\sqrt{2} \cdot x + 2 \geq 0 \\ \Leftrightarrow \quad & (x - \sqrt{2})^2 \geq 0, \end{aligned}$$

which is true. □

By using induction it is straightforward to prove that $A_n > 0$ for $n \geq 0$ (base case: $A_0 = 2 > 0$; inductive step: if $A_n > 0$, then $A_{n+1} = A_n/2 + 1/A_n > 0$). By the lemma, $A_n \geq \sqrt{2}$ for $n \geq 0$. Together with the inductive hypothesis this can be used in the following derivation:

$$\begin{aligned} A_{n+1} &= A_n/2 + 1/A_n \\ &\leq (\sqrt{2} + 1/2^n)/2 + 1/\sqrt{2} \\ &= \sqrt{2} + 1/2^{n+1}. \end{aligned}$$

This completes the proof. □

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Problem 4. [30 points] Find closed-form solutions to the following linear recurrences.

(a) [15 pts] $x_n = 4x_{n-1} - x_{n-2} - 6x_{n-3} \quad (x_0 = 3, x_1 = 4, x_2 = 14)$

Solution. The characteristic equation is $r^3 - 4r^2 + r + 6 = 0$.

Generally, solving a cubic equation is a difficult problem. However, we can find from inspection that the roots are:

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 2 \\ r_3 &= 3 \end{aligned}$$

Therefore a general form for a solution is

$$x_n = A(-1)^n + B(2)^n + C(3)^n.$$

Substituting the initial conditions into this general form gives a system of linear equations.

$$\begin{aligned}3 &= A + B + C\\ 4 &= -A + 2B + 3C\\ 14 &= A + 4B + 9C\end{aligned}$$

The solution to this linear system is $A = 1$, $B = 1$, and $C = 1$. The complete solution to the recurrence is therefore

$$x_n = (-1)^n + 2^n + 3^n.$$

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(b) [15 pts] $x_n = -x_{n-1} + 2x_{n-2} + n \quad (x_0 = 5, x_1 = -4/9)$

Solution. First, we find the general solution to the homogenous recurrence. The characteristic equation is $r^2 + r - 2 = 0$. The roots of this equation are $r_1 = 1$ and $r_2 = -2$. Therefore, the general solution to the homogenous recurrence is

$$x_n = A(-1)^n + B2^n.$$

Now we must find a particular solution to the recurrence, ignoring the boundary conditions. Since the inhomogenous term is linear, we guess there is a linear solution, that is, a solution of the form $an + b$. If the solution is of this form, we must have

$$an + b = -a(n-1) - b + 2a(n-2) + 2b + n$$

Gathering up like terms, this simplifies to

$$n(a + a - 2a - 1) + (b + a + b + 4a - 2b) = 0$$

which implies that

$$n = -5a$$

But a is a constant, so this cannot be so. So we make another guess, this time that there is a quadratic solution of the form $an^2 + bn + c$. If the solution is of this form, we must have

$$an^2 + bn + c = -[a(n-1)^2 + b(n-1) + c] + 2[a(n-2)^2 + b(n-2) + c] + n$$

which simplifies to

$$n^2(a + a - 2a) + n(b + b - 2a + 8a - 2b - 1) + (c + a - b + c - 8a + 4b - 2c) = 0$$

This simplifies to

$$n(6a - 1) + (-7a + 3b) = 0$$

This last equation is satisfied only if the coefficient of n and the constant term are both zero, which implies $a = 1/6$ and $b = 7/18$. Apparently, any value of c gives a valid particular solution. For simplicity, we choose $c = 0$ and obtain the particular solution:

$$x_n = \frac{1}{6}n^2 - \frac{7}{18}n.$$

The complete solution to the recurrence is the homogenous solution plus the particular solution:

$$x_n = A(-1)^n + B2^n + \frac{1}{6}n^2 - \frac{7}{18}n$$

Substituting the initial conditions gives a system of linear equations:

$$\begin{aligned} 5 &= A + B \\ -4/9 &= -A + 2B - +1/6 + 7/18 \end{aligned}$$

The solution to this linear system is $A = 3$ and $B = 2$. Therefore, the complete solution to the recurrence is

$$x_n = 3 + 2(-2)^n + \frac{1}{6}n^2 + \frac{7}{18}n$$

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