$\boxed{N1}$

$\exists$ $x_1, \dots, x_n$ - выборка распр-я $Pois(\theta)$

$$p(k) = \frac{\theta^k}{k!} e^{-\theta}, \; k \in \mathbb{N}_0$$

① ~~ее~~ а.н.о. θ = ?

нужно найти $\hat{\theta}$: ⊛ $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow[n\to\infty]{d} N(0, \sigma^2(\theta))$

Попробуем найти ее, применив ЦПТ.

$$\underline{M x_1} = \sum_{k=0}^{\infty} k \cdot \frac{\theta^k}{k!} e^{-\theta} = \theta e^{-\theta} \sum_{k=1}^{\infty} \frac{\theta^{k-1}}{(k-1)!} =$$

$$= \theta e^{-\theta} \cdot \sum_{k=0}^{\infty} \frac{\theta^k}{k!} = \theta \cancel{e^{-\theta}} \cancel{e^{\theta}} = \underline{\theta}$$

$$M x_1^2 = M[x_1(x_1-1)] + M x_1$$

$$M[x_1(x_1-1)] = \sum_{k=0}^{\infty} k(k-1) \frac{\theta^k}{k!} e^{-\theta} = \theta^2 e^{-\theta} \sum_{k=2}^{\infty} \frac{\theta^{k-2}}{(k-2)!}$$

$$= \theta^2 \cancel{e^{-\theta}} \cancel{e^{\theta}} = \theta^2.$$

$$M x_1^2 = \theta^2 + \theta.$$

$$\Rightarrow \underline{D x_1} = M x_1^2 - (M x_1)^2 = \underline{\theta}.$$

Из ЦПТ: ~~$\sqrt{n}(\theta M x_1 - \theta$~~

$$\sqrt{n}(\bar{x} - M x_1) \xrightarrow[n\to\infty]{d} N(0, D x_1)$$

$$\Rightarrow \sqrt{n}(\bar{x} - \theta) \xrightarrow[n\to\infty]{d} N(0, \theta) \; ⊛⊛$$

т.е. $\underset{\text{а.н.о.}}{\hat{\theta}} = \bar{x}$ ~~и~~ с асимп. дисп. $\sigma^2(\theta) = \theta$.

(Сравнивая ⊛ и ⊛⊛ ~~пону~~ это очевидно)

(2) Асимп. дов. инт-л на θ = ?

1) $\sqrt{n} \cdot \frac{\hat{\theta} - \theta}{\sigma(\theta)} \xrightarrow[n\to\infty]{d} N(0,1)$

$\frac{1+\alpha}{2}$ - квантиль $N(0,1)$

2) $P\left(\left| \sqrt{n} \cdot \frac{\hat{\theta} - \theta}{\sigma(\theta)} \right| < z_{\frac{1+\alpha}{2}} \right) \xrightarrow[n\to\infty]{} \alpha$

равна $\frac{1-\alpha}{2}$ кван-тиль в силу четности e^{-x^2}

3) $-z_{\frac{1+\alpha}{2}} < \sqrt{n} \cdot \frac{\bar{X} - \theta}{\sqrt{\theta}} < z_{\frac{1+\alpha}{2}}$

$$-\frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}} < \underline{\frac{\bar{X}}{\sqrt{\theta}} - \sqrt{\theta}} < \frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}}$$

нелинейное выражение отн. θ,

(по теореме Слуцкого)?

и теореме о насл. сх-ти

$\sigma(\theta) \to \sigma(\hat{\theta})$

т.е. $-\frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}} < \frac{\bar{X} - \theta}{\sqrt{\bar{X}}} < \frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}}$

$$-\sqrt{\bar{X}} - \frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}} < -\frac{\theta}{\sqrt{\bar{X}}} < \frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}} - \sqrt{\bar{X}}$$

$$\bar{X} + \frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}}\sqrt{\bar{X}} > \theta > \bar{X} - \frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}}\sqrt{\bar{X}}$$

т.е. асимп. довер. инт-л для θ:

(уровня доверия α)

$$\underline{\left(\bar{X} - \sqrt{\bar{X}}\frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}} ; \bar{X} + \sqrt{\bar{X}}\frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}} \right)},$$

где z_β - β-квантиль $N(0,1)$

$\boxed{N2}$ $x_1, \dots, x_n$ – выборка распр-я Лапласа

$$p(x) = \cancel{\theta}\frac{1}{2}e^{-|x-\theta|}$$

$$\underline{\underline{Mx_1}} = \frac{1}{2}\int_{-\infty}^{\infty} x e^{-|x-\theta|}\,dx = \frac{1}{2}\int_{-\infty}^{\theta} x e^{x-\theta}\,dx +$$

$$+\frac{1}{2}\int_{\theta}^{\infty} x e^{-(x-\theta)}\,dx = \frac{1}{2}e^{-\theta}\int_{-\infty}^{\theta} x\,d(e^x) - \frac{1}{2}e^{\theta}\int_{\theta}^{\infty} x\,d(e^{-x})$$

$$= \frac{1}{2}e^{-\theta}\left(xe^x\Big|_{\cancel{-\infty}}^{\theta} - \int_{-\infty}^{\theta} e^x\,dx\right) -$$

$$-\frac{1}{2}e^{\theta}\left(xe^{-x}\Big|_{\theta}^{\infty} - \int_{\theta}^{\infty} e^{-x}\,dx\right) =$$

$$= \frac{1}{2}e^{-\theta}\left(\theta e^{+\theta} - e^{\theta}\right) + \frac{1}{2}e^{\theta}\left(\theta e^{-\theta} + e^{-\theta}\right) =$$

$$= \frac{1}{2}(\theta - \cancel{1}) + \frac{1}{2}(\theta + \cancel{1}) = \underline{\underline{\theta}}.$$

$$Mx_1^2 = \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x-\theta|}\,dx = \frac{1}{2}\int_{-\infty}^{\theta} x^2 e^{x-\theta}\,dx +$$

$$+\frac{1}{2}\int_{\theta}^{\infty} x^2 e^{-x+\theta}\,dx = \frac{e^{-\theta}}{2}\left(x^2e^x\Big|_{\cancel{-\infty}}^{\theta} - 2\int_{-\infty}^{\theta} xe^x\,dx\right) +$$

$$+ -\frac{e^{\theta}}{2}\left(x^2e^{-x}\Big|_{\theta}^{\cancel{\infty}} - 2\int_{\theta}^{\infty} xe^{-x}\,dx\right) =$$

$$= \frac{e^{-\theta}}{2}\left(\theta^2 e^{\theta} - 2(\theta e^{\theta} - e^{\theta})\right) +$$

$$+\frac{e^{\theta}}{2}\left(\theta^2 e^{-\theta} + 2(\theta e^{-\theta} + e^{-\theta})\right) =$$

$$= \frac{1}{2}\left(\theta^2 - \cancel{2\theta} + 2 + \theta^2 + \cancel{2\theta} + 2\right) = \theta^2 + 2.$$

$$\underline{\underline{Dx_1}} = Mx_1^2 - (Mx_1)^2 = \underline{\underline{2}}.$$

Из ЦПТ: $\sqrt{n}(\overline{x} - Mx_1) \xrightarrow[n\to\infty]{d} N(0, Dx_1)$

$$\sqrt{n}(\overline{x} - \theta) \xrightarrow[n\to\infty]{d} N(0, 2)$$

$\Rightarrow$ а.н.о. для θ — $\hat{\theta} = \overline{x}$ с асимп. дисп. $\sigma^2(\theta) = 2$

Асимп. дов. инт-л для θ:

$$-z_{\frac{1+\alpha}{2}} < \sqrt{n}\,\frac{\overline{x} - \theta}{\sqrt{2}} < z_{\frac{1+\alpha}{2}}$$

$$\overline{x} + \sqrt{2}\,\frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}} > \theta > \overline{x} - \sqrt{2}\,\frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}}$$

т.е. асимп. дов. инт-л для θ с уровнем доверия α:

$$\left(\overline{x} - \sqrt{2}\,\frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}};\ \overline{x} + \sqrt{2}\,\frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}}\right)$$