An Example of a None-zero Walsh Series with Riesz-spaces' Coefficients and Vanishing Partial Sums S_{2^k}

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A none-zero Walsh series with coefficients from a Riesz-space and partial sums S_{2^k} (o) - converging to 0 is presented.

In this work we consider an example of a Walsh series [1] with coefficients belonging to a Riesz-space [2]. The partial sums S_{2^k} of this series (o) - converge to the element 0 of the Riesz-space [3, 4]. Nevertheless, the series is not the null series. Thus, we can not recover coefficients of the series using (o) - convergence. It is an additional argument for introduction of (u) - convergence (see [4]). There are some results of using it in [4, 5].

Theorem 1. There exist a series in the Walsh system with coefficients belonging to a Riesz-space R such that it (o) - converges to $0 \in R$ with respect to the subsequence of the natural numbers $n_k = 2^k,\ k = 0, 1, \dots$ but this series is not a null series.

Let f be a function $f\colon \mathbb{R} \to \mathbb{L}^0([0,1])$. By definition, put (see [3], p.300)

$$f(t) = 1_{]t-1/8,\, t+1/8[\cap]0,1[}, \quad t \in \mathbb{R}.$$

Notice that the space $\mathbb{L}^0([0,1])$ is a (super) Dedekind complete Riesz-space (see [3], p.300).

Lemma 1. Let $t, t_0 \in \mathbb{R}$ be points such that $0 \leqslant t_0 < t < t_0 + 1/8 \leqslant 1$, and $s \in \mathbb{R}$; then

$$\frac{f(t) - f(t_0)}{t - t_0}(s) = \begin{cases} -\frac{1}{t-t_0}, & \text{if } s \in]t_0 - 1/8,\, t - 1/8[, \\ \ \ \frac{1}{t-t_0}, & \text{if } s \in]t_0 + 1/8,\, t + 1/8[, \\ \ \ 0, & \text{otherwise.} \end{cases}$$

We now recall some definitions. Here we use the notions and conceptions of Walsh systems, a Riesz-space, (o) - convergence and ets. from [1, 2, 3, 4].

Definition 1. A function $F\colon [a,b] \to R$ is said to be differentiable at x_0 if

$$(o)\text{-}\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0}, \quad \text{exists in } R.$$

Half-bounded intervals $\Delta_m^{(k)} = \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right),\ 0 \leqslant m \leqslant 2^k - 1,\ k = 0, 1, \dots$ are called binary–intervals (or intervals) of the rank $k \geqslant 0$.

The set of all intervals of the rank k is called the binary-net $\mathfrak{M}_k$ of the rank k. Let $\Delta_x^{(k)}$ be a interval of the rank k such that the point x belongs to it. By $\{\Delta_x^{(k)}\} = \{[\alpha_k, \beta_k]_{k=0}^{\infty}\}$ denote a sequence of the intervals such that $x \in \Delta_x^{(k)}$ for any $k \geqslant 0$. There exists a nest of intervals $\{\Delta_x^{(k)}\}$ for any point $x \in [0,1]$.

Definition 2. Suppose that for a function $F\colon [a,b] \to R$, there exists a limit

$$(o)\text{-}\lim_{k \to \infty} \frac{F(\beta_k) - F(\alpha_k)}{\beta_k - \alpha_k}.$$

This limit is called the (o) - derivative with respect to the binary sequence of nets $\{\mathfrak{M}_k\}$ or $(o)\{\mathfrak{M}_k\}$ - derivative and is denoted by $(o)D_{\{\mathfrak{M}_k\}}F(x)$.

Lemma 2. The limit of the right of the function f at any point $t_0 \in [0,1]$ equals 0: $(o)f'_+(t_0) = 0 \in R$. (In fact, it means that $f'_+(t_0)(s) = 0$ is true almost everywhere on $[0,1]$.)

Likewise, we have $(o)f'_-(t_0) = 0 \in R$. It follows that $(o)f'(t_0) = 0 \in R$.

Hereafter we suppose that ϕ is a Riesz-space-valued function defined on the set of dyadic-rational points. We use the Henstock-Kurzweil type integral over $J = [0,1]$ with respect to the basis $\mathfrak{B}$ for Riesz-Space-valued functions (see [4]). In our case, the basis $\mathfrak{B}$ is a dyadic basis $\mathfrak{D}$. The elements of the set $\mathfrak{D}$ are pairs (I, x) such that $x \in I$ and I is any interval of the rank $k \in \mathbb{N}$.

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By definition, put

$$\phi_k(x) = \frac{\phi(\frac{m+1}{2^k} - \phi(\frac{m}{2^k})}{|\Delta_m^{(k)}|}, \text{ where } x \in \Delta_m^{(k)},\ 0 \leqslant m \leqslant 2^k - 1,\ k = 0, 1, \ldots.$$

Proposition 1.

$$\int_{\Delta_m^{(k)}} \phi_k(x)\, dx = \int_{\Delta_{2m}^{(k+1)}} \phi_k(x)\, dx + \int_{\Delta_{2m+1}^{(k+1)}} \phi_k(x)\, dx = \int_{\Delta_m^{(k+1)}} \phi_k(x)\, dx.$$

Let $S_n = \sum_{j=0}^{n-1} a_j w_j$ be the partial sum of a series in the Walsh system $\{w_j\}_j$, where coefficients a_j belong to a Riesz-space R.

The following theorem is needed for the sequel. This theorem can be proved similarly to proposition 3.1.1. from [1] for Riesz-space-valued functions.

Theorem 2. Let $\{k_j\}_j$be an increasing sequence of natural numbers and $\{\phi_k\}_j$ be a sequence of functions such that $\{\phi_k\}$ is a constant on any interval $\Delta_m^{(k_j)}$ and for any interval $\Delta_m^{(k_j)}$ the following condition holds

$$\int_{\Delta_m^{(k)}} \phi_k(x)\, dx = \int_{\Delta_m^{(k+1)}} \phi_k(x)\, dx.$$

Then, there exists a Walsh series such that sums $S_{2^{k_j}}(x)$ are equal to the functions $\{\phi_j\}$.

Take the function f as ϕ. The application of Theorem 2 yields that the sequence $\{f_k(x)\}$ defines a unique series with partial sums $S_{2^{k_j}}(x) = f_k(x)$ (where $k_j = j$), and for any $x \in \Delta_m^k$ the following condition holds

$$S_{2^k}(x) = \frac{f(\frac{m+1}{2^k}) - f(\frac{m}{2^k})}{|\Delta_m^{(k)}|}. \tag{I}$$

The limit as $k \to \infty$ for the fraction of the right side of equality (I) equals the $(o)\{\mathfrak{M}_k\}$ - derivative of the function f. Thus we have (o) - $\lim_{k\to\infty} S_{2^k}(x) = (o)D_{\{\mathfrak{M}_k\}} f(x)$.

To complete the example, we need the following proposition:

Proposition 2. If a function Y has the (o) - derivative $(o)Y'(x) = y(x)$ at some point x, then there exists a unique (o) - derivative with respect to the binary sequence of nets and $(o)D_{\{\mathfrak{M}_k\}} f(x) = y(x)$ at this point x.

Finally, we obtain (o) - $\lim_{k\to\infty} S_{2^k}(x) = (o)D_{\{\mathfrak{M}_k\}} f(x) = (o)f'(x) \equiv 0$ for any $x \in [0, 1]$.

We claim that our series is not a null series. Indeed,

$$S_1(x) = a_0 + w_1(x) = S_{2^0}(x) = \frac{f(1) - f(0)}{\Delta_0^{(0)}} = 1_{]7/8,1[} - 1_{]0,1/8[} \neq 0.$$

Hence it is clear that $a_0 \neq 0 \in R$. Thus Theorem 1 is proved.

'References'

[1] Golubov B., Efimov A., Skvortsov V. Walsh Series and Transforms: Theory and Applications (Mathematics and its Applications, Book 64), Springer, 1991, 368 p.

[2] Vulih B. Z. An Introduction to Functional Analysis [in Russian]. M.: Nauka, 1967. 415 p.

[3] Boccuto A. Differential and integral calculus in Riesz space. Tatra Mt. Math. Publ. 14. (1998), P. 419–438.

[4] Boccuto A., Skvortsov V. A. Henstock-Kurzweil type integration of Riesz-space-valued functions and applications to Walsh series//Real Anal. Exch. – 2003/04. – Vol. 29, N 1. – P. 419–438.

[5] Petrov S. V., Prostokishin V. M. On Representation of Riesz-space-valued Functions by Fourier Series on Multiplicative Systems. 2017 J. Phys.: Conf. Ser. 937 012040.