On Representation of Riesz-space-valued Functions by Fourier Series on Multiplicative Systems

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The result that was established in [1] for Walsh series with coefficients belonging to a Riesz-space is generalized for multiplicative systems.

In this work the result that was established in [1] for Walsh series [3] with coefficients belonging to a Riesz-space [2] is generalized for multiplicative systems. These systems are going to be very popular for solving applied tasks nowadays [5],[6]. In particular, we consider conditions, for which such series are certain generalized Fourier series, where in order to write down Fourier's formulae we use the Henstock-Kurzweil's type integral [4]. This integral is defined with respect to the basis $\mathcal{B}_{\mathbb{P}}$ generated by intervals $\{\delta_r^{(k)}\}$ of the special form [3]:

$$\delta_r^{(k)} = \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right), \quad k = 0, 1, 2, \dots, \quad 0 \leqslant r \leqslant m_k - 1.$$

Now we recall the definition of the multiplicative systems using the $\mathbb{P}$-adic expansion for the rational numbers; by definition, put

$$m_0 = 1, \quad m_j = \prod_{s=1}^{j} p_s,$$

where p_s is a member of a sequence of the set of natural numbers

$$\mathbb{P} = \{p_1, p_2, \dots, p_j, \dots\}, \quad p_j \geqslant 2, \ j \geqslant 1,$$

and any integer $n \geqslant 0$ has the form

$$n = \sum_{i=1}^{k} \alpha_j m_{j-1}, \quad 0 \leqslant \alpha_j \leqslant p_j, \ j = 1, 2, ..., k.$$

Take the point $x \in [0,1)$ and consider the series $x = \sum\limits_{j=1}^{\infty} \frac{x_j}{m_j}$, where $0 \leqslant x_j \leqslant p_{j-1}, \ j \geqslant 1$. Here for $\mathbb{P}$-adic rational x we use only the finite expansions. The point x is called the $\mathbb{P}$-adic rational point if there exists a finite representation $x = \sum\limits_{j=1}^{K} \frac{x_j}{m_j}, \ K \in \mathbb{N}$. Using the introduced notations, by definition, put

$$\chi_n(x) = \exp\left(2\pi i \sum_{j=1}^{k} \frac{\alpha_j x_j}{p_j}\right).$$

With pointwise (o)-converges (order converges) we use (u)-converges on a set (see [1]), according to the definition:

Let Λ be any nonempty set, R be Dedekind complete Riesz space and $D = N^{\Lambda}$. The sequence of R-valued functions $(S_n \colon \Lambda \to R)_n$ (u)-*converges* to the function $S \colon \Lambda \to R$ (with respect to order convergence) if there exists an (o)-net $(a_\nu)_{\nu \in D}$ such that for any $\nu \in D$ the following condition holds:

$$\sup\{|S_n(x) - S(x)| : x \in \Lambda, \, n \leqslant \nu(x)\} \leqslant a_\nu.$$

Let $S_n = \sum_{j=0}^{n-1} a_j \chi_j$ be the partial sum of a series in the multiplicative system $\{\chi_j\}_j$, where coefficients a_j belong to a Riesz-space R.

Our main result is as follows.

Theorem 1. If R is a regular Riesz-space and the series

$$\sum_{j}^{\infty} a_j \chi_j, \quad a_j \in R \tag{*}$$

(u) - converges to a function f on a set $[0,1] \setminus E$, where E is a countable subset of the interval $[0,1]$, then f is $H_{\mathcal{B}}$ - integrable on $[0,1]$, and the series (*) is the Fourier series of f in the sense of the $H_{\mathcal{B}}$ - integral.

For proving this theorem the function $\psi(\delta_i^{(k)}) = \int_{\delta_i^{(k)}} S_{m_k}$ of the intervals $\delta_i^{(k)}$ is used, where S_{m_k} is the partial sum of the series (*). This function is additive with respect to the measure algebra, generated by the intervals $\{\delta_r^{(k)}\}$. It follows from the equality

$$\psi(\delta_r^{(k)}) = \psi(\delta_{rp_{k+1}}^{(k+1)}) + \psi(\delta_{rp_{k+1}+1}^{(k+1)}) + \cdots + \psi(\delta_{rp_{k+1}+p_{k+1}-1}^{(k+1)}) = \sum_{s=rp_{k+1}}^{(r+1)p_{k+1}-1} \psi(\delta_s^{(k+1)}),$$

where $\delta_r^{(k)} = \bigcup_{s=rp_{k+1}}^{(r+1)p_{k+1}-1} \delta_s^{(k+1)}$.

Theorem 1 follows as a result of applying the statement (see [2]) to the function ψ.

Theorem 2. Let R be a regular Riesz-space, $\mathcal{B}$ a fixed basis, $f\colon [a,b] \to R$ and let τ be the R - value $\mathcal{B}$ - interval function, such that for some countable set $Q \in [a,b]$ the function f is the (u) - derivative of τ with respect to the basis $\mathcal{B}$ on the set $[a,b] \setminus Q$ and τ is (o) - continuous with respect to the basis $\mathcal{B}$ on Q. Then f is $H_{\mathcal{B}}$ - integrable over $[a,b]$, and

$$(H_{\mathcal{B}}) \int_a^b f = \tau([a,b]).$$

Note. The basis $\mathcal{B}$ is the basis $\mathcal{B}_{\mathbb{P}}$ consisting of all pairs (I,x), where $x \in I = \left(\frac{i}{m_k}, \frac{i+1}{m_k}\right)$ and $k \in N \cup \{0\}$, and $i = 0, 1, \ldots, m_{k-1}$.

'References'

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