Robust convex programming

Computational Intelligence, Lecture 11

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Consider the following problem:

Example

Find smallest $x \in \mathbb{R}$, such that $x + y \geq 1$, where $|y| \leq 2$.

In that example we need to find optimal value of x subject to a constraint where another unknown variable is present; the solution we find has to satisfy the constraint for any allowed value of y. The solution here is $x = 3$

Consider the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \ \underset{\mathbf{y}}{\text{any}} \quad & ||\mathbf{x}||, \\ \text{subject to} \quad & \mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{1}$$

It is clear that worst-case scenario corresponds to the largest value of $\mathbf{d}^\top \mathbf{y}$. We note that $\max(\mathbf{d}^\top \mathbf{y}) = \max(||\mathbf{d}|| \cdot ||\mathbf{y}|| \cdot \cos(\mathbf{d}^\wedge \mathbf{y})) = ||\mathbf{d}|| p$; hence $\mathbf{y} = p \frac{\mathbf{d}}{||\mathbf{d}||}$.

Therefore $\mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \leq h$ becomes:

$$\mathbf{c}^\top \mathbf{x} + p||\mathbf{d}|| \leq h \tag{2}$$

Thus our problem becomes:

$$\begin{aligned} \min_{\mathbf{x}} \quad & ||\mathbf{x}||, \\ \text{subject to} \quad & \mathbf{c}^\top \mathbf{x} \leq h - p||\mathbf{d}|| \end{aligned} \tag{3}$$

Consider the following problem, where $\mathbf{x}^*$ is the desired value of $\mathbf{x}$:

$$\begin{aligned} \min_{\mathbf{x}} \ \text{any}_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & \mathbf{y}^\top \mathbf{D}\mathbf{x} \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{4}$$

This time worst-case scenario corresponds to $\mathbf{y}$ aligned with $\mathbf{D}\mathbf{x}$ and having its maximum possible length p. From that we conclude that $\mathbf{y} = p\frac{\mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||}$. Let us substitute it to $\mathbf{y}^\top \mathbf{D}\mathbf{x}$:

$$p\left(\frac{\mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||}\right)^\top \mathbf{D}\mathbf{x} = p\frac{\mathbf{x}^\top \mathbf{D}^\top \mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||} = p\frac{||\mathbf{D}\mathbf{x}||^2}{||\mathbf{D}\mathbf{x}||} = p||\mathbf{D}\mathbf{x}|| \tag{5}$$

Thus our problem becomes:

$$\begin{aligned} \min_{\mathbf{x}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & ||\mathbf{D}\mathbf{x}|| \leq \frac{h}{p} \end{aligned} \tag{6}$$

which is an SOCP.

Quadratic constraint, type 2

A more general case of the previous problem is:

$$\begin{aligned} \min_{\mathbf{x}} \ \text{any}_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{7}$$

We can rewrite $(\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h$ as:

$$\mathbf{y}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{8}$$

With that we see that the worse case scenario is $\mathbf{y}$ is aligned with $\mathbf{D}(\mathbf{x} - \mathbf{b})$ and has length p:

$$\mathbf{y} = p \frac{\mathbf{D}(\mathbf{x} - \mathbf{b})}{||\mathbf{D}(\mathbf{x} - \mathbf{b})||} \tag{9}$$

Quadratic constraint, type 2

Then $\mathbf{y}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) \leq h$ becomes:

$$p\frac{(\mathbf{x}-\mathbf{b})^\top \mathbf{D}^\top \mathbf{D}(\mathbf{x}-\mathbf{b})}{||\mathbf{D}(\mathbf{x}-\mathbf{b})||} - \mathbf{a}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) \leq h \tag{10}$$

which is the same as:

$$p||\mathbf{D}(\mathbf{x}-\mathbf{b})|| - \mathbf{a}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) \leq h \tag{11}$$

$$||\mathbf{D}(\mathbf{x}-\mathbf{b})|| \leq \frac{1}{p}\mathbf{a}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) + \frac{h}{p} \tag{12}$$

which is an SOCP constraint.

And thus we get:

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\
\text{subject to} \quad & ||\mathbf{D}(\mathbf{x} - \mathbf{b})|| \leq \frac{1}{p}\mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \frac{h}{p}
\end{aligned}
\tag{13}
$$

which is SOCP.

Quadratic constraint, type 3

A more general case of the previous problem is:

$$\begin{aligned} \min_{\mathbf{x}} \ \text{any}_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h, \\ & ||\mathbf{H}\mathbf{y} + \mathbf{f}|| \leq p \end{aligned} \tag{14}$$

where $\mathbf{H}$ is has an inverse. We start by making substitution:

$$\mathbf{v} = \mathbf{H}\mathbf{y} + \mathbf{f} \tag{15}$$

meaning $\mathbf{y} = \mathbf{H}^{-1}(\mathbf{v} - \mathbf{f})$:

$$(\mathbf{H}^{-1}(\mathbf{v} - \mathbf{f}) - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{16}$$

$$\mathbf{v}^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) - (\mathbf{H}^{-1}\mathbf{f} + \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{17}$$

$$\mathbf{v}^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) - (\mathbf{H}\mathbf{a} + \mathbf{f})^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{18}$$

Quadratic constraint, type 3

We can introduce notation:

$$\mathbf{M} = \mathbf{H}^{-\top}\mathbf{D} \tag{19}$$
$$\mathbf{g} = \mathbf{H}\mathbf{a} + \mathbf{f} \tag{20}$$

With that we can re-write our constraint:

$$\mathbf{v}^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) - \mathbf{g}^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h \tag{21}$$
$$(\mathbf{v} - \mathbf{g})^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h \tag{22}$$

And now we formulated type 3 problem as type 2:

$$\begin{aligned} \min_{\mathbf{x}} \operatorname*{any}_{\mathbf{v}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{v} - \mathbf{g})^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h, \\ & ||\mathbf{v}|| \leq p \end{aligned} \tag{23}$$

Try solving this problem on your own:

$$\begin{aligned}
\min_{\mathbf{x}} \ \operatorname*{any}_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\
\text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \mathbf{s}^\top \mathbf{y} + \mathbf{q}^\top \mathbf{x} \leq h, \\
& ||\mathbf{H}\mathbf{y} + \mathbf{f}|| \leq p
\end{aligned} \tag{24}$$

Consider the problem $\max_{\mathbf{x}}(||\mathbf{a}+\mathbf{x}||)$ and $||\mathbf{x}|| \leq p$. Let us open the norm:

$$||\mathbf{a}+\mathbf{x}|| = \sqrt{(\mathbf{a}+\mathbf{x})^\top(\mathbf{a}+\mathbf{x})} = \sqrt{\mathbf{a}^\top\mathbf{a} + 2\mathbf{a}^\top\mathbf{x} + \mathbf{x}^\top\mathbf{x}} \quad (25)$$

If $\mathbf{a}$ is a constant, then the expression $\mathbf{a}^\top\mathbf{a} + 2\mathbf{a}^\top\mathbf{x} + \mathbf{x}^\top\mathbf{x}$ attains a maximum when $\mathbf{x}^\top\mathbf{x} = p^2$ and $\mathbf{a}^\top\mathbf{x} = ||\mathbf{a}||p$. This implies that:

$$\mathbf{x} = p\frac{\mathbf{a}}{||\mathbf{a}||} \quad (26)$$

And thus:
$\max_{\mathbf{x}}(||\mathbf{a}+\mathbf{x}||) = ||\mathbf{a} + p\frac{\mathbf{a}}{||\mathbf{a}||}|| = ||\mathbf{a}||\left(1 + \frac{p}{||\mathbf{a}||}\right) = ||\mathbf{a}|| + p$

Robust CP: Conic constraint

Consider the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \ \operatorname*{any}_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & ||\mathbf{A}\mathbf{x} + \mathbf{b} + \mathbf{y}|| \leq \mathbf{c}^\top \mathbf{x} + d, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{27}$$

As we discussed in the last slide, expression $||\mathbf{A}\mathbf{x} + \mathbf{b} + \mathbf{y}||$ is biggest when $\mathbf{y} = p\frac{\mathbf{A}\mathbf{x}+\mathbf{b}}{||\mathbf{A}\mathbf{x}+\mathbf{b}||}$, and the conic constraint becomes:

$$||\mathbf{A}\mathbf{x} + \mathbf{b} + p\frac{\mathbf{A}\mathbf{x} + \mathbf{b}}{||\mathbf{A}\mathbf{x} + \mathbf{b}||}|| \leq \mathbf{c}^\top \mathbf{x} + d \tag{28}$$

$$||\mathbf{A}\mathbf{x} + \mathbf{b}|| \left(1 + \frac{p}{||\mathbf{A}\mathbf{x} + \mathbf{b}||}\right) \leq \mathbf{c}^\top \mathbf{x} + d \tag{29}$$

Robust CP: Conic constraint

Continue the derivation:

$$||\mathbf{A}\mathbf{x} + \mathbf{b}|| \left(1 + \frac{p}{||\mathbf{A}\mathbf{x} + \mathbf{b}||}\right) \leq \mathbf{c}^\top \mathbf{x} + d \tag{30}$$

$$||\mathbf{A}\mathbf{x} + \mathbf{b}|| + p \leq \mathbf{c}^\top \mathbf{x} + d \tag{31}$$

Finally the problem becomes:

$$\begin{aligned} \min_{\mathbf{x}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & ||\mathbf{A}\mathbf{x} + \mathbf{b}|| \leq \mathbf{c}^\top \mathbf{x} + d - p \end{aligned} \tag{32}$$

Lecture slides are available via Github, links are on Moodle:

github.com/SergeiSa/Computational-Intelligence-2024