Linear inequalities and H-polytopes

Computational Intelligence, Lecture 4

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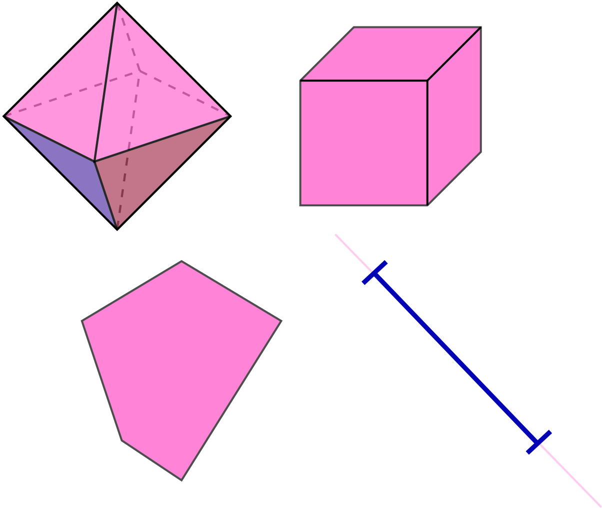
Before defining what a convex polytope is, let us look at examples:

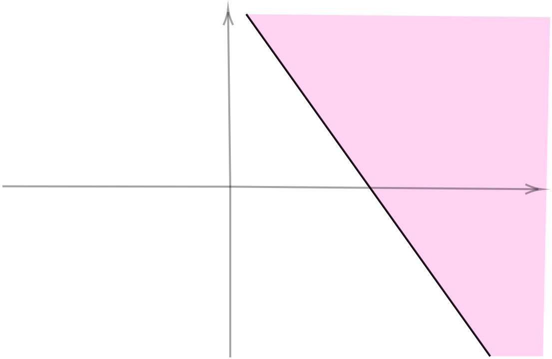


You can think of polytopes as geometric figures (or continuous sets of points) with linear edges, faces and higher-dimensional analogues.

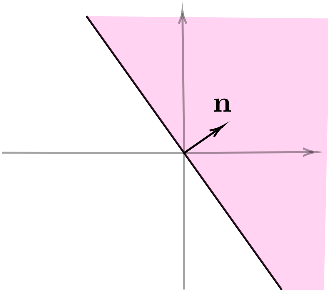
Definition

Convex polytopes are polytopes whose every two points can be connected with a line that would lie in the polytope. They can be bounded or unbounded.

We can define half-space as a set of all points $\mathbf{x}$, such that $\mathbf{a}^\top \mathbf{x} \leq b$. It has a very clear geometric interpretation. In the following image, the filled space is **not** in the half space.

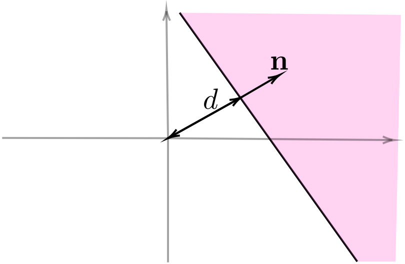


Consider half-space that passes through the origin, and defined by its normal vector $\mathbf{n}$:



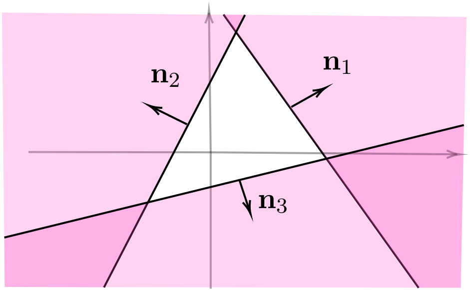
It is easy to see that this half-space can be defined as "all vectors $\mathbf{x}$, such that $\mathbf{n} \cdot \mathbf{x} \leq 0$", which is the same as using $\mathbf{n}$ instead of $\mathbf{a}$ in our original definition, setting $b = 0$.

In the general case there is some distance between the boundary of the half-space and the origin, let's say d.



Here the half space can be defined as "all vectors $\mathbf{x}$, such that $\mathbf{x}^\top \frac{\mathbf{n}}{||\mathbf{n}||} \leq d$". This is the same as making $\mathbf{a} = \mathbf{n}$ and $b = d||\mathbf{a}||$.

We can define a region of space as an *intersection* of half-spaces $\mathbf{a}_i^\top \mathbf{x} \leq b_i$:



Resulting region will be easily described as $\begin{bmatrix} \mathbf{a}_1^\top \\ \dots \\ \mathbf{a}_k^\top \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} b_1 \\ \dots \\ b_k \end{bmatrix}$

The last result allows us to write any convex polytope as a matrix inequality:

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{1}$$

And conversely, any matrix inequality (1) represents either an empty set or a convex polytope.

Definition

$\mathbf{A}\mathbf{x} \leq \mathbf{b}$ is called *H-representation* (half-space representation) of a polytope.

Convex polytopes have alternative representations, such as *V-representation*. In amounts to representing polytope as a set of its vertices.

Example

$V = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$ is a V-representation of a square.

Example

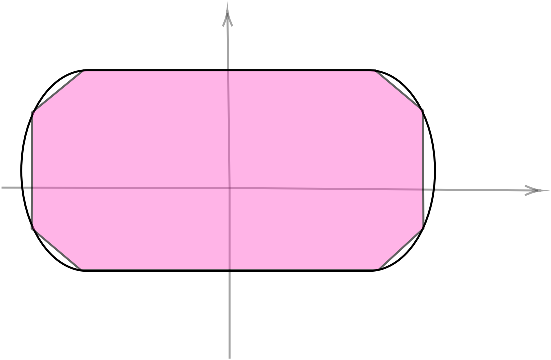
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an H-representation of the same square.

To transfer from H-representation to V-representation, you need to solve *vertex enumeration* problem, which is computationally expensive.

It is also possible to construct H-representation out of V-representation. Both algorithms are not convex.

Linear approximation of convex regions

Some convex regions can be easily approximated using polytopes.



Which allows to represent constraints on $\mathbf{x}$ to belong in such a region as a matrix inequality

Represent in matrix inequality form the following figures:

- Equilateral triangle
- A square
- Parallelepiped
- Trapezoid

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:

github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2023