

Second-order cone programming

Computational Intelligence, Lecture 8

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- Norm
- Second-order cone programming
- SOCP to QCQP
- Friction cone as an SOCP

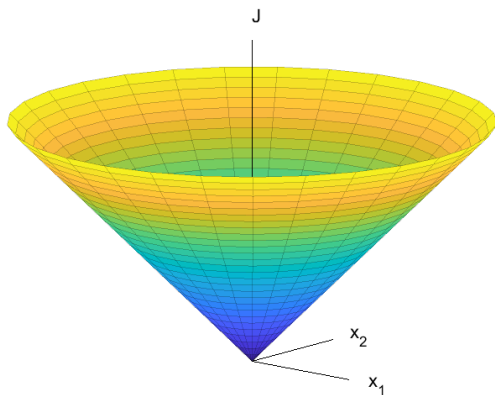
Let us consider a 2-norm as a function $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$f(\mathbf{x}) = \|\mathbf{x}\|_2 \tag{1}$$

$$f(\mathbf{x}) = \sqrt{\sum_{i=1}^n x_i^2} \tag{2}$$

We can describe 2-norm as a surface in the $\mathcal{S} \subset \mathbb{R}^{n+1}$ space:

$$\mathcal{S} = \{(J, \mathbf{x}) : J = \|\mathbf{x}\|_2\} \quad (3)$$



The shape of the surface $\mathcal{S} = \{(J, \mathbf{x}) : J = \|\mathbf{x}\|_2\}$ is a *cone*. We observe the following properties of a cone:

- There is a single tip point τ and a normal direction.
- Slicing cone with planes orthogonal to the normal direction, we produce ellipsoids (we can call it tangent sets).
- For any point p on the cone, the half-line from the tip point τ through p lies on the cone. The angle between this line and the normal is called *vertex angle*.

A second-order cone constraint has the following form:

$$||\mathbf{Ax} + \mathbf{b}|| \leq \mathbf{c}^\top \mathbf{x} + d \quad (4)$$

where $\mathbf{A} \in \mathbb{R}^{n,n}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $d \in \mathbb{R}$.

This constraint describes interior of a cone. The surface of the cone is an intersection of two surfaces:

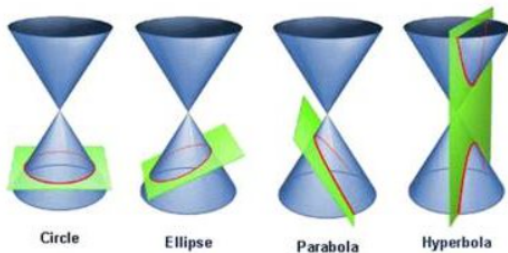
$$J = ||\mathbf{Ax} + \mathbf{b}|| \quad (5)$$

$$P = \mathbf{c}^\top \mathbf{x} + d \quad (6)$$

First is a cone and second is a plane. Their intersection is called a *conic section*.

THE ROLE OF THE FREE CONSTANT, 1

Typical conic sections are shown below:



As we can see, they represent ellipsoid and parabola. In order for them to represent a cone, the plane S needs to pass through the tip of the cone J . This can be achieved with the appropriate choice of constant d , which shifts S up or down.

The surface of a second-order cone (SOC) is:

$$||\mathbf{Ax} + \mathbf{b}|| = \mathbf{c}^\top \mathbf{x} + d \quad (7)$$

we can find a tip point; it corresponds to both right-hand side and left-hand side becoming zero:

$$\begin{cases} \mathbf{Ax} + \mathbf{b} = 0 \\ \mathbf{c}^\top \mathbf{x} + d = 0 \end{cases} \quad (8)$$

Given full rank matrix \mathbf{A} , the solution is $\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$. The system would hold if:

$$-\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b} + d = 0 \quad (9)$$

SECOND-ORDER CONE PROGRAMMING (SOCP)

General form

The general form of a Second-order cone program (SOCP) is:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \begin{cases} \|\mathbf{A}_i \mathbf{x} + \mathbf{b}_i\|_2 \leq \mathbf{c}_i^\top \mathbf{x} + d_i, \\ \mathbf{F} \mathbf{x} = \mathbf{g}. \end{cases} \end{array} \quad (10)$$

LP, QP and QCQP are subsets of SOCP.

SECOND-ORDER CONE PROGRAMMING

Special cases

We can write problem where our domain is a ball as SOCP:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \|\mathbf{x}\|_2 \leq d_i\end{array}\tag{11}$$

Same for ellipsoidal constraints:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \|\mathbf{A}_i \mathbf{x}\|_2 \leq d_i\end{array}\tag{12}$$

Set $\mathbf{c}_i = 0$ and recognize that $\|\mathbf{A}_i \mathbf{x} + \mathbf{b}_i\|_2 \leq d_i$ is the same as $(\mathbf{A}_i \mathbf{x} + \mathbf{b}_i)^\top (\mathbf{A}_i \mathbf{x} + \mathbf{b}_i) \leq d_i^2$ (since the first implies that d_i is non-negative).

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{array} \quad (13)$$

Now to make the cost quadratic:

$$\begin{aligned}
 & \underset{\mathbf{x}, t}{\text{minimize}} && t, \\
 & \text{subject to} && \begin{cases} \mathbf{x}^\top \mathbf{A}_0^\top \mathbf{A}_0 \mathbf{x} + 2\mathbf{b}_0^\top \mathbf{A}_0 \mathbf{x} + \mathbf{b}_0^\top \mathbf{b}_0 \leq t \\ \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases}
 \end{aligned} \tag{14}$$

Which is the same as:

$$\begin{aligned}
 & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{H}\mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\
 & \text{subject to} && \begin{cases} \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases}
 \end{aligned} \tag{15}$$

As long as $\mathbf{A}_0 = \sqrt{\mathbf{H}}$, and $\mathbf{b}_0 = 0.5\mathbf{A}_0^{-1}\mathbf{f}$.

Friction cone as an SOC

Friction force \mathbf{f}_τ together with normal reaction force \mathbf{f}_n together form contact reaction force \mathbf{f}_R :

$$\mathbf{f}_R = \mathbf{f}_n + \mathbf{f}_\tau \quad (16)$$

We can choose to represent the reaction force in a basis \mathbf{B} formed by concatenating normal direction \mathbf{n} and two tangent directions $\mathbf{t}_1, \mathbf{t}_2$.

$$\mathbf{f}_R = \mathbf{B} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = [\mathbf{n} \quad \mathbf{t}_1 \quad \mathbf{t}_2] \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} \quad (17)$$

We can prove that $f_n = \mathbf{n}^\top \mathbf{f}_R$:

$$\mathbf{n}^\top \mathbf{f}_R = \mathbf{n}^\top \begin{bmatrix} \mathbf{n} & \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = f_n \quad (18)$$

We can prove that $\begin{bmatrix} f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}^\top \mathbf{f}_R$:

$$\begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}^\top \mathbf{f}_R = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}^\top \begin{bmatrix} \mathbf{n} & \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \quad (19)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \begin{bmatrix} f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} \quad (20)$$

We can write friction cone constraint as follows:

$$\sqrt{f_{\tau,1}^2 + f_{\tau,2}^2} \leq \mu f_n \quad (21)$$

where μ is friction coefficient, f_{τ} is the magnitude of the friction force and f_n is the magnitude of the normal reaction force.

We can describe it as *element-wise description*. The simplicity of this description makes it quite attractive.

It is possible to re-write the same constraint as:

$$\left\| \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}^\top \mathbf{f}_R \right\| \leq \mu \mathbf{n}^\top \mathbf{f}_R \quad (22)$$

We can call it a *vector description*. The advantage of this description is the use of a single vector variable \mathbf{f}_R . It takes the form of a second-order cone (SOC) constraint.

Note that \mathbf{t}_1 and \mathbf{t}_2 are usually not given, and can be chosen arbitrarily, up to rotation. We can find them as a left null space of the normal vector: $\mathbf{T} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix} = \text{null}(\mathbf{n}^\top)$:

$$\|\mathbf{T}^\top \mathbf{f}_R\| \leq \mu \mathbf{n}^\top \mathbf{f}_R \quad (23)$$

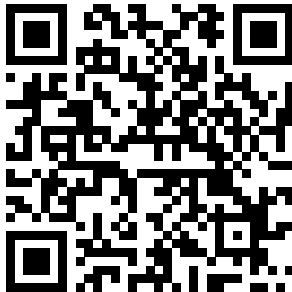
We can do the same with projectors:

$$\|(\mathbf{I} - \mathbf{n}\mathbf{n}^\top)\mathbf{f}_R\| \leq \mu\mathbf{n}^\top\mathbf{f}_R \quad (24)$$

Plot a cone from a given direction and a given vertex angle.

Lecture slides are available via Github, links are on Moodle:

github.com/SergeiSa/Computational-Intelligence-2024



Appendix A - canonical form

Consider the following SOC constraint:

$$\|\mathbf{Ax} + \mathbf{b}\|_2 \leq \mathbf{c}^\top \mathbf{x} + d \quad (25)$$

Let us consider a special case when $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{(n-1) \times n}$ and $\text{rank} \left(\begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix} \right) = n$. Then we can introduce the following substitution:

$$\xi = \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} \mathbf{E} \\ \mathbf{e}^\top \end{bmatrix} \quad (26)$$

where $\mathbf{I} \in \mathbb{R}^{n,n}$ is an identity matrix. Then constraint (25) becomes:

$$\|\mathbf{E}\xi\|_2 \leq \mathbf{e}^\top \xi \quad (27)$$

Notice that $\|\mathbf{E}\xi\|_2 \leq \mathbf{e}^\top \xi$ is equivalent to:

$$\sum_{i=1}^{n-1} \xi_i^2 \leq \xi_n^2 \quad (28)$$

which is a standard form of a cone. A map back from ξ to \mathbf{x} is given as:

$$\mathbf{x} = \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix}^{-1} \left(\xi - \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix} \right) \quad (29)$$

Appendix B - plotting cones

To plot a cone it is convenient to first use change of coordinates $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$, meaning $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b})$, giving us SOC:

$$\|\mathbf{y}\| = \mathbf{c}^\top \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b}) + d \quad (30)$$

Note that $d - \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b} = 0$ for a cone with a tip; so SOC becomes:

$$\|\mathbf{y}\| = \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{y} \quad (31)$$

To plot level sets of this cone we choose height of the level set h and pick point $\mathbf{y}_h = h \frac{\mathbf{A}^{-T}\mathbf{c}}{\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{A}^{-T}\mathbf{c}}$; we note that $\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{y}_h = h$. Then we consider points on the plane \mathcal{P} orthogonal to $\mathbf{c}^\top \mathbf{A}^{-1}$ and passing through \mathbf{y}_h :

$$\mathcal{P} = \mathbf{y}_h + \mathbf{T}\mathbf{z} : \forall \mathbf{z} \quad (32)$$

where $\mathbf{T} = \text{null}(\mathbf{c}^\top \mathbf{A}^{-1})$, so $\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{T} = 0$.

PLOTTING LEVEL SETS, 2

Since SOC becomes:

$$\|\mathbf{y}_h + \mathbf{T}\mathbf{z}\| = h \quad (33)$$

Since \mathbf{y}_h and $\mathbf{T}\mathbf{z}$ are orthogonal, it is equivalent to:

$$\|\mathbf{T}\mathbf{z}\| = g \quad (34)$$

where $g = \sqrt{h^2 - \mathbf{y}_h^\top \mathbf{y}_h}$. In the 3D case, this is a circle with radius g . We can find N consecutive evenly spaced points of this circle, resulting in the next sequence of \mathbf{y}_l :

$$\mathbf{y}_l = \mathbf{y}_h + \mathbf{T} \begin{bmatrix} g \cos(\varphi) \\ -g \sin(\varphi) \end{bmatrix}, \quad \varphi = 0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, \dots, 2\pi \quad (35)$$

$$\mathbf{x}_l = \mathbf{A}^{-1}(\mathbf{y}_l - \mathbf{b}) \quad (36)$$

The center of the ellipsoid representing this level set lies at the point $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{y}_h - \mathbf{b})$.

PLOTTING LEVEL SETS, 3

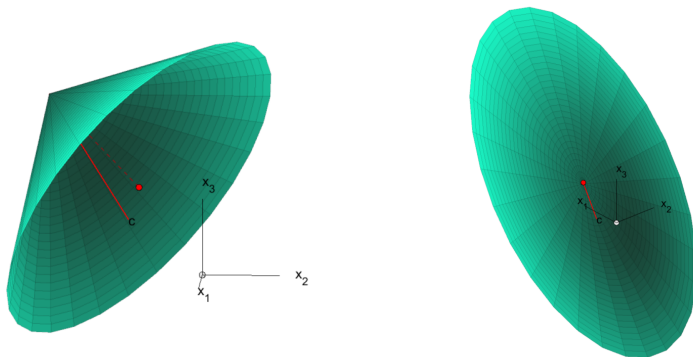


Figure 1: Cone. Dashed line - centers of level-sets.