

Linear inequalities and polytopes

Computational Intelligence, Lecture 5

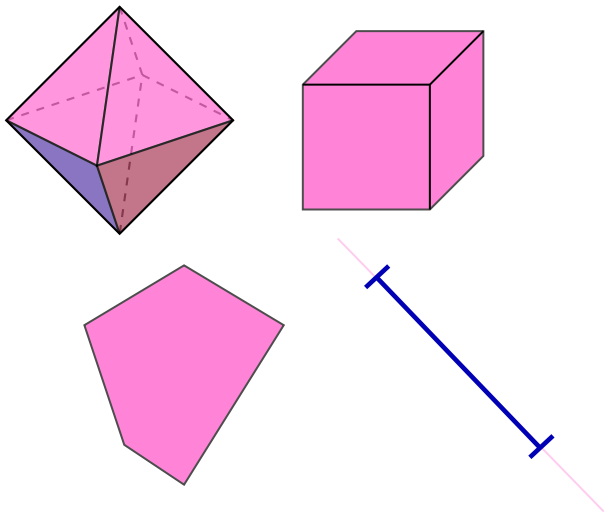
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- Convex polytopes
- Half-spaces
- H-representation
- V-representation
- G-representation (Zonotopes)
- Linear approximation of convex regions

CONVEX POLYTOPES

Before defining what a convex polytope is, let us look at examples:



You can think of polytopes as geometric figures (or continuous sets of points) with linear edges, faces and higher-dimensional analogues.

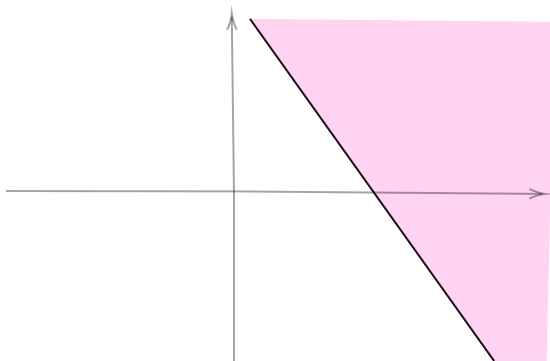
Definition

Convex polytopes are polytopes whose every two points can be connected with a line that would lie in the polytope. They can be bounded or unbounded.

HALF-SPACES

Definition

We can define half-space as a set of all points \mathbf{x} , such that $\mathbf{a}^\top \mathbf{x} \leq b$. It has a very clear geometric interpretation. In the following image, the filled space is **not** in the half space.



HALF-SPACES

Construction. Simple case

Consider half-space that passes through the origin, and defined by its normal vector \mathbf{n} :



It is easy to see that this half-space can be defined as "all vectors \mathbf{x} , such that $\mathbf{n} \cdot \mathbf{x} \leq 0$ ", which is the same as using \mathbf{n} instead of \mathbf{a} in our original definition, setting $b = 0$.

HALF-SPACES

Construction. General case

In the general case there is some distance between the boundary of the half-space and the origin, let's say d .



Here the half space can be defined as "all vectors \mathbf{x} , such that $\mathbf{x}^\top \frac{\mathbf{n}}{\|\mathbf{n}\|} \leq d$ ". This is the same as making $\mathbf{a} = \mathbf{n}$ and $b = d\|\mathbf{a}\|$.

HALF-SPACES

Combination

We can define a region of space as an *intersection* of half-spaces
 $\mathbf{a}_i^\top \mathbf{x} \leq b_i$:



Resulting region will be easily described as
$$\begin{bmatrix} \mathbf{a}_1^\top \\ \dots \\ \mathbf{a}_k^\top \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} b_1 \\ \dots \\ b_k \end{bmatrix}$$

The last result allows us to write any convex polytope as a matrix inequality:

$$\mathbf{Ax} \leq \mathbf{b} \tag{1}$$

And conversely, any matrix inequality (1) represents either an empty set or a convex polytope.

Definition

$\mathbf{Ax} \leq \mathbf{b}$ is called *H-representation* (half-space representation) of a polytope.

We can use containment in an H-polytope as a part of convex optimization problem. For example, the following QP includes such constraint:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \mathbf{A} \mathbf{x} \leq \mathbf{b}. \end{array} \tag{2}$$

Convex polytopes have alternative representations, such as *V-representation*. It amounts to representing polytope as a set of its vertices.

Example

$V = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$ is a V-representation of a square.

Example

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an H-representation of the same square.

Given points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$ their convex hull is represented as:

$$\mathcal{P} = \left\{ \mathbf{x} = \sum_{i=1}^N \alpha_i \mathbf{x}_i : \sum_{i=1}^N \alpha_i = 1, \alpha_i \in [0, 1] \right\} \quad (3)$$

See Appendix for an illustration of this formula.

We can use containment in an V-polytope as a part of convex optimization problem. For example, the following QP includes such constraint:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \left\{ \begin{array}{l} \mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{v}_i, \\ \sum_{i=1}^n \alpha_i = 1, \\ \alpha_i \geq 0. \end{array} \right. \end{array} \quad (4)$$

Notice that the constraint amounts to equating \mathbf{x} to a convex combination of the vertices of the V-polytope.

To transfer from H-representation to V-representation, you need to solve *vertex enumeration* problem, which is computationally expensive.

It is also possible to construct H-representation out of V-representation. Both algorithms are not convex.

ZONOTOPES: G-REPRESENTATION

A zonotope \mathcal{Z} is a symmetric polytope defined by its *center* \mathbf{c} and *generator* \mathbf{G} :

$$\mathcal{Z} = \{\mathbf{x} : \mathbf{x} = \mathbf{G}\boldsymbol{\beta} + \mathbf{c}, \|\boldsymbol{\beta}\|_{\infty} \leq 1\} \quad (5)$$

The set $\{\boldsymbol{\beta} : \|\boldsymbol{\beta}\|_{\infty} \leq 1\}$ is a hypercube and zonotope \mathcal{Z} is a projection (shadow) of this hypercube onto a lower-dimensional space; the projection is defined by the matrix \mathbf{G} .

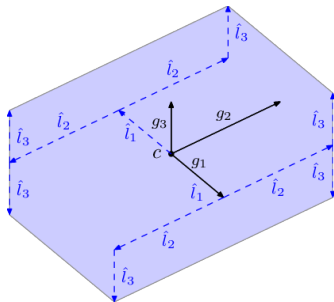


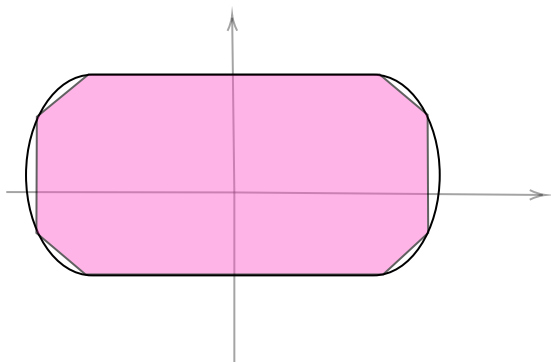
Figure 1: Zonotope ([Source](#))

We can use containment in an G-polytope as a part of convex optimization problem. For example, the following QP includes such constraint:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{x} = \mathbf{G} \boldsymbol{\beta} + \mathbf{c}, \\ -1 \geq \beta_i \geq 1. \end{cases} \end{array} \quad (6)$$

LINEAR APPROXIMATION OF CONVEX REGIONS

Some convex regions can be easily approximated using polytopes.



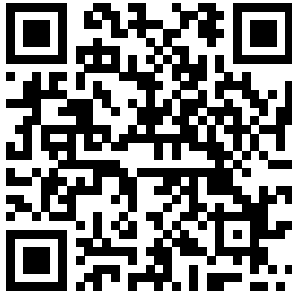
Which allows to represent constraints on \mathbf{x} to belong in such a region as a matrix inequality

Write H-representation of the following polytopes:

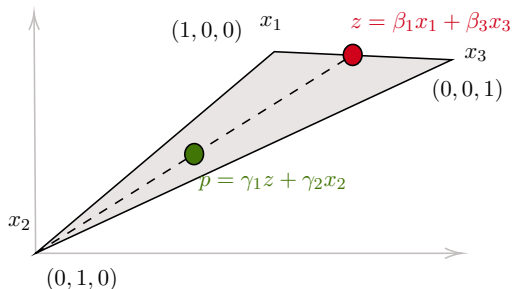
- Equilateral triangle
- Square
- Parallelepiped
- Trapezoid

Lecture slides are available via Github, links are on Moodle:

github.com/SergeiSa/Computational-Intelligence-2024



APPENDIX A - CONVEX HULL, 1

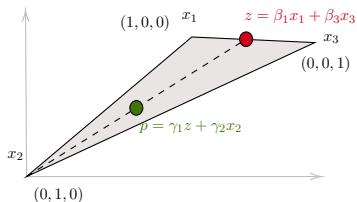


Let us illustrate the convex combination formula. Let \mathcal{P} be convex hull of points \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{x}_3 :

$$\mathcal{P} = \left\{ \mathbf{x} = \sum_{i=1}^3 \alpha_i \mathbf{x}_i : \sum_{i=1}^3 \alpha_i = 1, \alpha_i \in [0, 1] \right\} \quad (7)$$

Let \mathbf{z} be a convex combination of \mathbf{x}_1 and \mathbf{x}_3 : $\mathbf{z} = \beta_1 \mathbf{x}_1 + \beta_3 \mathbf{x}_3$. Then any $\mathbf{p} \in \mathcal{P}$ is expressed as a convex combination of \mathbf{z} and \mathbf{x}_2 : $\mathbf{p} = \gamma_1 \mathbf{z} + \gamma_2 \mathbf{x}_2$.

APPENDIX A - CONVEX HULL, 2



We can express \mathbf{p} as:

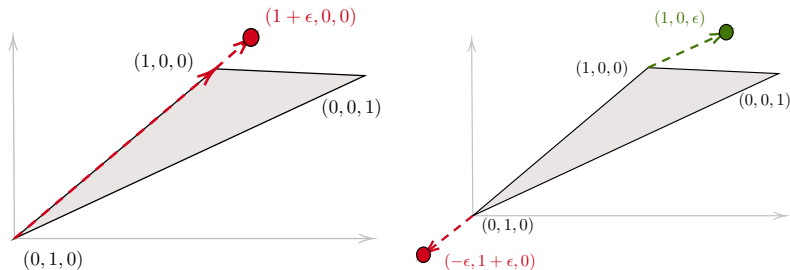
$$\mathbf{p} = \gamma_1 \mathbf{z} + \gamma_2 \mathbf{x}_2 = \gamma_1(\beta_1 \mathbf{x}_1 + \beta_3 \mathbf{x}_3) + \gamma_2 \mathbf{x}_2 \quad (8)$$

We can define $\alpha_1 = \gamma_1 \beta_1$, $\alpha_2 = \gamma_2$ and $\alpha_3 = \gamma_1 \beta_3$. Since $\gamma_i \geq 0$ and $\beta_i \geq 0$, we conclude that $\alpha_i \geq 0$.

We can show that $e = \alpha_1 + \alpha_2 + \alpha_3 = 1$:

$$e = \gamma_1(\beta_1 + \beta_3) + \gamma_2 = \gamma_1 + \gamma_2 = 1 \quad (9)$$

APPENDIX A - CONVEX HULL, 3



Previously we illustrated sufficiency of the formula's constraints. Now let us illustrate their necessity.

Dropping requirement $\alpha_i \leq 1$, and/or $\alpha_i \geq 0$ and/or $\sum_{i=1}^3 \alpha_i = 1$, leads to inclusion of points out the convex hull, as illustrated on the figures.