

Quadratic programming

Computational Intelligence, Lecture 7

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- Quadratic programming
- Quadratically constrained quadratic programming (QCQP)
 - ▶ QCQP to LP; QCQP to LP
 - ▶ Ellipsoidal constraints \rightarrow to canonical form

POSITIVE-DEFINITE MATRICES

Positive-definite matrices the following properties:

Eigenvalues

All eigenvalues of a positive-definite (PD) matrix are real and positive.

Eigenvectors

Eigenvectors of a PD matrix with form an orthogonal basis.

Proof (for the case of distinct eigenvalues): let \mathbf{M} be a PD matrix, and $\mathbf{M}\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{M}\mathbf{u} = \gamma\mathbf{u}$. Then:

$$\mathbf{u}^\top \mathbf{M}\mathbf{v} = \mathbf{u}^\top (\mathbf{M}\mathbf{v}) = \lambda \mathbf{u}^\top \mathbf{v} \quad (1)$$

$$\mathbf{u}^\top \mathbf{M}\mathbf{v} = (\mathbf{u}^\top \mathbf{M})\mathbf{v} = \gamma \mathbf{u}^\top \mathbf{v} \quad (2)$$

Therefore, $\lambda \mathbf{u}^\top \mathbf{v} = \gamma \mathbf{u}^\top \mathbf{v}$, and since λ and γ are eigenvalues and eigenvalues are distinct, so $\lambda \neq \gamma$. This implies that $\mathbf{u}^\top \mathbf{v} = 0$.

Remember the general form of a quadratic program:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b}, \\ \mathbf{F} \mathbf{x} = \mathbf{g}. \end{cases} \end{array} \quad (3)$$

where \mathbf{H} is positive-definite and $\mathbf{A} \mathbf{x} \leq \mathbf{b}$ describe a convex region.

GEOMETRY OF A QP

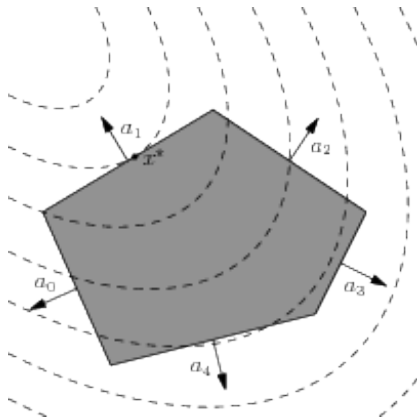


Figure 1: Geometry of a QP. [Source](#)

COST FUNCTION OF A QP, 1

The cost function of a QP has the form $c(\mathbf{x}) = \mathbf{x}^\top \mathbf{H}\mathbf{x} + \mathbf{f}^\top \mathbf{x}$. Let us show that the requirement that \mathbf{H} is positive-definite does not limit the range of convex problems that can be solved as a QP.

Let \mathbf{M} be a non-symmetric matrix. Quadratic form $q(\mathbf{x}) = \mathbf{x}^\top \mathbf{M}\mathbf{x}$ is a scalar and is equal to its transpose:

$$q(\mathbf{x}) = \mathbf{x}^\top \mathbf{M}\mathbf{x} \quad (4)$$

$$q(\mathbf{x}) = 0.5(\mathbf{x}^\top \mathbf{M}\mathbf{x} + \mathbf{x}^\top \mathbf{M}\mathbf{x}) \quad (5)$$

$$q(\mathbf{x}) = 0.5(\mathbf{x}^\top \mathbf{M}\mathbf{x} + \mathbf{x}^\top \mathbf{M}^\top \mathbf{x}) \quad (6)$$

$$q(\mathbf{x}) = 0.5\mathbf{x}^\top (\mathbf{M} + \mathbf{M}^\top) \mathbf{x} \quad (7)$$

Equivalently prove that the cost function $c(\mathbf{x})$ is always equivalent to the cost function $c(\mathbf{x}) = 0.5\mathbf{x}^\top (\mathbf{H} + \mathbf{H}^\top) \mathbf{x} + \mathbf{f}^\top \mathbf{x}$. Because of that, without a loss of generality we can assume \mathbf{H} to be symmetric.

Let us prove that \mathbf{H} needs to be positive semi-definite in order for $c(\mathbf{x})$ to be convex.

Assume that one of the eigenvalues of \mathbf{H} is negative: $\mathbf{H}\mathbf{v} = \lambda\mathbf{v}$, where $\lambda < 0$ and $\|\mathbf{v}\| = 1$. We can find values of $c(0)$, $c(\mathbf{v})$ and $c(2\mathbf{v})$:

$$c(0) = 0 \tag{8}$$

$$c(\mathbf{v}) = \mathbf{v}^\top \mathbf{H} \mathbf{v} = \lambda \mathbf{v}^\top \mathbf{v} = \lambda \tag{9}$$

Note that $c((1 - \beta)0 + \beta\mathbf{v}) = c(\beta\mathbf{v}) = \lambda\beta^2$ and $(1 - \beta)c(0) + \beta c(\mathbf{v}) = \lambda\beta$. Since $0 < \beta < 1$ we conclude $\beta^2 < \beta$; since $\lambda < 0$, $\lambda\beta^2 > \lambda\beta$. So $c((1 - \beta)0 + \beta\mathbf{v}) > (1 - \beta)c(0) + \beta c(\mathbf{v})$. Thus, such $c(\mathbf{x})$ is not convex. □

Let us consider a QP with degenerate matrix $\mathbf{H} = \mathbf{R}\Sigma\mathbf{R}^\top$ (without a loss of generality \mathbf{H} can be assumed to be symmetric), where $\mathbf{R} \in \mathbb{R}^{n \times k}$.

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{R}\Sigma\mathbf{R}^\top \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{10}$$

Matrix \mathbf{R} is a row-space basis of \mathbf{H} , and \mathbf{N} is its orthogonal complement. Then we can decompose \mathbf{x} as $\mathbf{x} = \mathbf{R}\zeta + \mathbf{N}\mathbf{z}$:

$$\begin{aligned} & \underset{\zeta, \mathbf{z}}{\text{minimize}} && \zeta^\top \Sigma \zeta + \mathbf{f}^\top \mathbf{R}\zeta + \mathbf{f}^\top \mathbf{N}\mathbf{z}, \\ & \text{subject to} && \begin{cases} \mathbf{A}\mathbf{R}\zeta + \mathbf{A}\mathbf{N}\mathbf{z} \leq \mathbf{b}, \\ \mathbf{F}\mathbf{R}\zeta + \mathbf{F}\mathbf{N}\mathbf{z} = \mathbf{g}. \end{cases} \end{aligned} \tag{11}$$

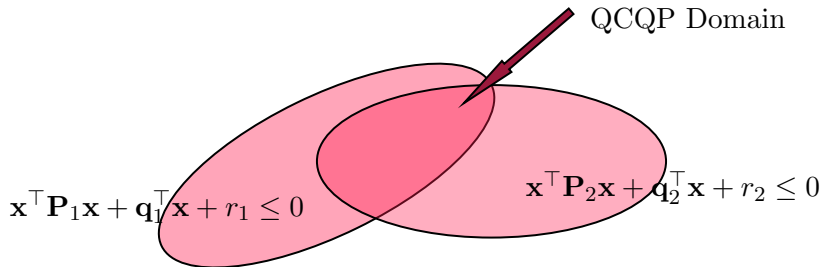
QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING

General form of a quadratically constrained quadratic program (QCQP) is given below:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^\top \mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{x}^\top \mathbf{P}_i \mathbf{x} + \mathbf{q}_i^\top \mathbf{x} + r_i \leq 0, \\ \mathbf{F} \mathbf{x} = \mathbf{g}. \end{cases} \end{array} \quad (12)$$

where \mathbf{P}_i are positive-definite.

Domain of a QCQP without equality constraints and with no degenerate inequality constraints is an intersection of ellipses:



Set $\mathbf{P}_i = \mathbf{0}$ and you get a QP.

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \begin{bmatrix} \mathbf{q}_1^\top \\ \dots \\ \mathbf{q}_n^\top \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} -r_1 \\ \dots \\ -r_n \end{bmatrix} \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{13}$$

Set $\mathbf{P}_0 = \mathbf{0}$ and you get an LP.

TURNING ELLIPSOID TO THE CANONICAL FORM (1)

Can we re-write the expression $\mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r \leq 0$ as a canonical form ellipsoid:

$$\frac{z_1^2}{m_1^2} + \frac{z_2^2}{m_2^2} + \dots + \frac{z_n^2}{m_n^2} \leq 1 \quad (14)$$

We start by proposing a substitution $\mathbf{x}_0 = -\frac{1}{2}\mathbf{P}^{-1}\mathbf{q}$ and $-d = r - \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0$. We can prove that:

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P} (\mathbf{x} - \mathbf{x}_0) - d = \mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r$$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P} (\mathbf{x} - \mathbf{x}_0) - d = \quad (15)$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} - 2\mathbf{x}_0^\top \mathbf{P} \mathbf{x} + \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 - d = \quad (16)$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} + 2 \left(\frac{1}{2} \mathbf{P}^{-1} \mathbf{q} \right)^\top \mathbf{P} \mathbf{x} + \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 + r - \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 = \quad (17)$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r. \quad (18)$$

TURNING ELLIPSOID TO THE CANONICAL FORM (2)

Thus our original expression became:

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P}(\mathbf{x} - \mathbf{x}_0) - d \leq 0 \quad (19)$$

We define $\mathbf{A} = \sqrt{\mathbf{P}}$ with SVD decomposition $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$.

Defining $\mathbf{z} = \mathbf{V}^\top(\mathbf{x} - \mathbf{x}_0)$ we get:

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{A}^\top \mathbf{A}(\mathbf{x} - \mathbf{x}_0) - d \leq 0 \quad (20)$$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{V}\Sigma\mathbf{U}^\top \mathbf{U}\Sigma\mathbf{V}^\top (\mathbf{x} - \mathbf{x}_0) - d \leq 0 \quad (21)$$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{V}\Sigma^2\mathbf{V}^\top (\mathbf{x} - \mathbf{x}_0) - d \leq 0 \quad (22)$$

$$\mathbf{z}^\top \Sigma^2 \mathbf{z} - d \leq 0 \quad (23)$$

$$\sum z_i^2 \sigma_i^2 \leq d \quad (24)$$

Defining $1/m_i^2 = \sigma_i^2/d$ we get:

$$\frac{z_1^2}{m_1^2} + \frac{z_2^2}{m_2^2} + \dots + \frac{z_n^2}{m_n^2} \leq 1 \quad (25)$$

Implement a program that finds right-most point of an intersection of two ellipsoids; visualise the problem and the solution.

- Symmetric Matrices and Eigendecomposition. Robert M. Freund, MIT, 2014.
- MOSEK, QP and QCQP.

Lecture slides are available via Github, links are on Moodle:

github.com/SergeiSa/Computational-Intelligence-2024

