

# Mixed Integer Convex Programming

## Computational Intelligence, Lecture 12

by Sergei Savin

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- Mixed Integer Linear Programming (MILP)
- Mixed Integer Quadratic Programming (MIQP)
- Example: Footstep planning
- Big-M method relaxation
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# MIXED INTEGER LINEAR PROGRAMMING (MILP)

A general form of a mixed-integer linear program is:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{Ax} \leq \mathbf{b}, \\ \mathbf{Cx} = \mathbf{d}, \\ \mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}, \\ \mathbf{x}_{m+1}, \dots, \mathbf{x}_n \in \mathbb{N}. \end{cases} \end{array} \quad (1)$$

In other words, the only difference is that some of the variables are only allowed to assume pure integer values.

# MIXED INTEGER CONVEX PROGRAMMING (MICP)

A general form of a mixed-integer convex program is:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & l(\mathbf{x}) \quad \text{convex cost} \\ \text{subject to} & \left\{ \begin{array}{ll} g(\mathbf{x}) \leq 0, & \text{convex domain} \\ \mathbf{C}\mathbf{x} = \mathbf{d}, & \\ x_1, \dots, x_m \in \mathbb{R}, & \text{continuous variables} \\ x_{m+1}, \dots, x_n \in \mathbb{N}. & \text{integer variables} \end{array} \right. \end{array} \quad (2)$$

- Mixed-integer convex programs are not convex, even if the name seems to suggest otherwise.
- The "convex program" in the phrase "mixed-integer convex program" should be understood as "if we fix values of the integer variables, or relax them into reals, the result will be a convex program".
- Mixed integer programs are usually solved using *branch and bound algorithms*; in worse case scenario, the algorithms performs exhaustive search.
- In robotics applications, integer variables in mixed integer programs are often restricted to binary values, i.e.  $\mathbf{y} \in \{0, 1\}^n$ . It is still called "mixed-integer" in the literature, although phrases such as "mixed-binary" or "binary constraints" are not rare.

# EXAMPLE: FOOTSTEP PLANNING

## Problem statement

Given  $N$  convex regions defined by linear inequalities  $\{\mathbf{x} : \mathbf{A}_i \mathbf{x} \leq \mathbf{b}_i\}$  (which is called H-polytope representation), find a sequence of  $K$  points (footsteps) from the given starting point to the given goal point, such that all footsteps lie in one of the convex regions.

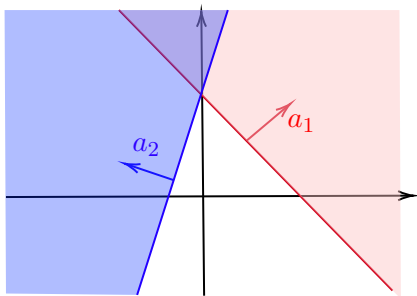
$$\begin{aligned} & \underset{\mathbf{x}_1, \dots, \mathbf{x}_K}{\text{minimize}} && \|\mathbf{x}_1 - \mathbf{x}_{\text{start}}\| + \|\mathbf{x}_K - \mathbf{x}_{\text{goal}}\|, \\ & \text{subject to} && \exists \{\mathbf{A}_j, \mathbf{b}_j\} \in \Omega \text{ s.t. } \mathbf{A}_j \mathbf{x}_i \leq \mathbf{b}_j \end{aligned} \tag{3}$$

where  $\Omega = \{\{\mathbf{A}_1, \mathbf{b}_1\}, \{\mathbf{A}_2, \mathbf{b}_2\}, \dots, \{\mathbf{A}_N, \mathbf{b}_N\}\}$ . This is not a convex program.

# BIG-M METHOD, 1

One of the key methods associated with the use of mixed-integer programming in robotics is the big-M method.

Assume you have two inequalities,  $\mathbf{a}_1^\top \mathbf{x} \leq b_1$  and  $\mathbf{a}_2^\top \mathbf{x} \leq b_2$ , and you are happy if at least one of them holds. Define a new binary variables  $c_1, c_2 \in \{0, 1\}$  and find a big enough constant  $M$ , such that  $\mathbf{a}_1^\top \mathbf{x} \leq b_1 + M$  and  $\mathbf{a}_2^\top \mathbf{x} \leq b_2 + M$  holds for all of your domain (of the part of it you are interested in).



Let us put the following constraint on the variables  $c_1, c_2$ :

$$c_1 + c_2 = 1 \quad (4)$$

The pair of variables  $(c_1, c_2)$  together can assume only two values -  $(1, 0)$  and  $(0, 1)$ . We can modify the inequalities  $\mathbf{a}_1^\top \mathbf{x} \leq b_1$  and  $\mathbf{a}_2^\top \mathbf{x} \leq b_2$ :

$$\begin{cases} \mathbf{a}_1^\top \mathbf{x} \leq b_1 + M \cdot c_1 \\ \mathbf{a}_2^\top \mathbf{x} \leq b_2 + M \cdot c_2 \end{cases} \quad (5)$$

If  $(c_1, c_2) = (1, 0)$ , then modified inequalities become:

$$\begin{cases} \mathbf{a}_1^\top \mathbf{x} \leq b_1 + M & \text{- true for all } \mathbf{x} \\ \mathbf{a}_2^\top \mathbf{x} \leq b_2 & \text{- original constraint} \end{cases} \quad (6)$$

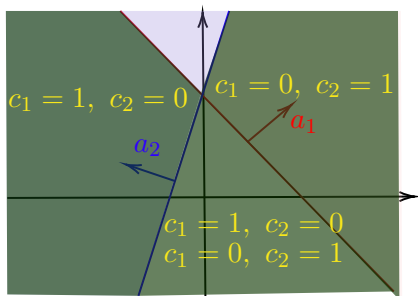
If  $(c_1, c_2) = (0, 1)$ , then modified inequalities become:

$$\begin{cases} \mathbf{a}_1^\top \mathbf{x} \leq b_1 & \text{- original constraint} \\ \mathbf{a}_2^\top \mathbf{x} \leq b_2 + M & \text{- true for all } \mathbf{x} \end{cases} \quad (7)$$



With that, we propose the following constraint:

$$\begin{cases} \mathbf{a}_1^\top \mathbf{x} \leq b_1 + M \cdot c_1 \\ \mathbf{a}_2^\top \mathbf{x} \leq b_2 + M \cdot c_2 \\ c_1 + c_2 = 1 \\ c_i \in \{0, 1\} \end{cases} \quad (8)$$



# CHOOSING BETWEEN POLYTOPES, 1

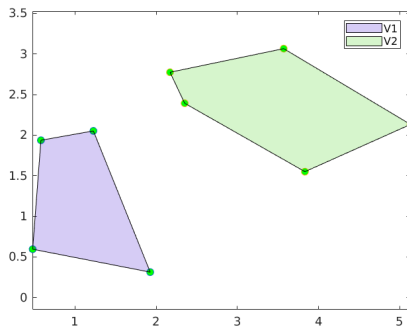
It works the same way for the case when you have three (or two or more) systems of inequalities  $\mathbf{A}_1\mathbf{x} \leq \mathbf{b}_1$ ,  $\mathbf{A}_2\mathbf{x} \leq \mathbf{b}_2$ ,  $\mathbf{A}_3\mathbf{x} \leq \mathbf{b}_3$  and are happy if at least one holds:

$$\begin{cases} \mathbf{A}_1\mathbf{x} \leq \mathbf{b}_1 + M \cdot \mathbf{1} \cdot (1 - c_1) \\ \mathbf{A}_2\mathbf{x} \leq \mathbf{b}_2 + M \cdot \mathbf{1} \cdot (1 - c_2) \\ \mathbf{A}_3\mathbf{x} \leq \mathbf{b}_3 + M \cdot \mathbf{1} \cdot (1 - c_3) \\ c_1 + c_2 + c_3 = 1 \\ c_i \in \{0, 1\} \end{cases} \quad (9)$$

where  $\mathbf{1}$  is a vector of all ones. Notice that constraint  $c_1 + c_2 + c_3 = 1$  can be replaced with  $c_1 + c_2 + c_3 \geq 1$ , allowing avoid relaxing more than one region.

## CHOOSING BETWEEN POLYTOPES, 2

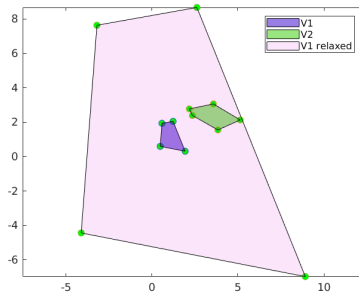
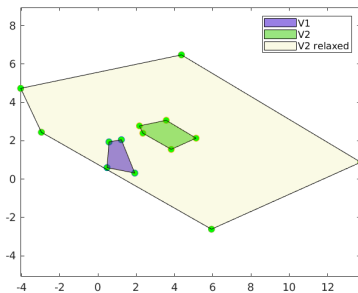
Below are two H-polytopes (convex regions represented by systems of inequalities):



As we can see, their union represents a non-convex domain, and they have no intersection.

# CHOOSING BETWEEN POLYTOPES, 3

Now, one of them is relaxed as described above. Notice that the intersection of the two polytopes is non-relaxed polytope.



# CHOOSING BETWEEN POLYTOPES, 4

## Multiple variables

If you have multiple variables  $\mathbf{x}_1, \dots, \mathbf{x}_K$ , and each should belong to at least one of the H-polytopes  $\{\mathbf{A}_i, \mathbf{b}_i\}$ , this can also be represented using big-M method:

$$\left\{ \begin{array}{l} \mathbf{A}_1 \mathbf{x}_k \leq \mathbf{b}_1 + M \cdot \mathbf{1} \cdot (1 - c_{1,k}) \\ \mathbf{A}_2 \mathbf{x}_k \leq \mathbf{b}_2 + M \cdot \mathbf{1} \cdot (1 - c_{2,k}) \\ \mathbf{A}_3 \mathbf{x}_k \leq \mathbf{b}_3 + M \cdot \mathbf{1} \cdot (1 - c_{3,k}) \\ c_{1,k} + c_{2,k} + c_{3,k} = 1 \\ c_{i,k} \in \{0, 1\} \\ k = 1, \dots, K \end{array} \right. \quad (10)$$

Notice that the only difference from the previous example is that now we have  $K$  sets of binary variables  $c_{1,k}$ ,  $c_{2,k}$  and  $c_{3,k}$ .

# EXAMPLE: FOOTSTEP PLANNING

## Formulation as MIQP

Using big-M relaxation we can now formulate the problem as follows:

$$\begin{aligned} & \underset{\mathbf{x}_k, \mathbf{c}_{i,k}}{\text{minimize}} && ||\mathbf{x}_1 - \mathbf{x}_{\text{start}}|| + ||\mathbf{x}_K - \mathbf{x}_{\text{goal}}||, \\ & \text{subject to} && \begin{cases} \mathbf{A}_i \mathbf{x}_k \leq \mathbf{b}_i + M \cdot \mathbf{1} \cdot (1 - c_{i,k}), \quad i = 1, \dots, N \\ \sum_{i=1}^N c_{i,k} = 1 \\ c \in \{0, 1\}^{N, K} \\ k = 1, \dots, K \end{cases} \end{aligned} \tag{11}$$

# EXAMPLE: FOOTSTEP PLANNING

## Evenly spaced steps

In order to make the footsteps evenly spaced we add cost on the distance between consequent steps:

$$\begin{aligned} & \underset{\mathbf{x}_k, \mathbf{c}_{i,k}}{\text{minimize}} && \|\mathbf{x}_1 - \mathbf{x}_{\text{start}}\|^2 + \|\mathbf{x}_K - \mathbf{x}_{\text{goal}}\|^2 + w \cdot \sum_{k=1}^{K-1} \|\mathbf{x}_{i+1} - \mathbf{x}_i\|^2, \\ & \text{subject to} && \begin{cases} \mathbf{A}_i \mathbf{x}_k \leq \mathbf{b}_i + M \cdot \mathbf{1} \cdot (1 - c_{i,k}), \quad i = 1, \dots, N \\ \sum_{i=1}^N c_{i,k} = 1 \\ c \in \{0, 1\}^{N, K} \\ k = 1, \dots, K \end{cases} \end{aligned}$$

where  $w$  is a weight, a coefficient we can tune to adjust relative importance of our primary objective (starting from the point  $\mathbf{x}_{\text{start}}$  and finishing at the point  $\mathbf{x}_{\text{goal}}$  and our secondary objective (making the footsteps evenly spaced).

# EXAMPLE: FOOTSTEP PLANNING

## Code, part 1

```
0 n = 2;
  shift_1 = 1*rand(n, 1);
2 shift_2 = 1*rand(n, 1);
  V1 = randn(n, 6);
4 V2 = randn(n, 6) + shift_1;
  V3 = randn(n, 6) + shift_1 + shift_2;
6
  [A1, b1] = vert2con(V1');
8 [A2, b2] = vert2con(V2');
  [A3, b3] = vert2con(V3');
10
  number_of_steps = 7;
12 start_point = sum(V1, 2) / size(V1, 2);
  finish_point = sum(V3, 2) / size(V3, 2);
14 weight = 5;
  bigM = 15;
```



# EXAMPLE: FOOTSTEP PLANNING

## Code, part 2

```
0 cvx_begin
    variable x(n, number_of_steps)
    binary variable c(3, number_of_steps);
    cost = 0;
    for i = 1:(number_of_steps-1)
        cost = cost + norm(x(:, i) - x(:, i+1));    end
    cost = cost + (norm(x(:, 1) - start_point) + norm(x
    (:, number_of_steps) - finish_point))*weight;
    minimize( cost )
    subject to
    for i = 1:number_of_steps
    10 A1*x(:, i) <= b1 + (1 - c(1, i))*bigM;
    A2*x(:, i) <= b2 + (1 - c(2, i))*bigM;
    12 A3*x(:, i) <= b3 + (1 - c(3, i))*bigM;
    c(1, i) + c(2, i) + c(3, i) == 1;
    14 end
    cvx_end
    16 plot(x(1, :)', x(2, :)', '^', 'MarkerEdgeColor', 'k', '
        MarkerSize', 10, 'LineWidth', 2); hold on;
```

# EXAMPLE: SWITCHING CONTROL

## Multiple variables

Consider the case when you have a control system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u}_1 + \mathbf{B}_2\mathbf{u}_2$ , but you are required to use either  $\mathbf{B}_1\mathbf{u}_1$  or  $\mathbf{B}_2\mathbf{u}_2$  at any time, not both.

This can be cast as a mixed-integer constraint:

$$\begin{cases} \|\mathbf{u}_1\| \leq M(1 - c_1) \\ \|\mathbf{u}_2\| \leq M(1 - c_2) \\ c_1 + c_2 = 1 \end{cases} \quad (12)$$

Implement footstep planning for a biped, making sure every pair of steps lands in the same H-polytope.

Lecture slides are available via Github, links are on Moodle:

[github.com/SergeiSa/Computational-Intelligence-2024](https://github.com/SergeiSa/Computational-Intelligence-2024)

