

Linear Programming

Computational Intelligence, Lecture 6

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- Linear Programming
- Convex piece-wise linear functions
- Chebyshev center of a polyhedron
- Linear-Fractional Programming
- Homework

A linear program (LP) is an optimization problem of the form:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{Ax} \leq \mathbf{b}, \\ \mathbf{Cx} = \mathbf{d}. \end{cases} \end{array} \quad (1)$$

It is one of the older and widely used classes of convex optimization problems.

Note that the solution of such problem will always lie on the boundary of its domain.

Inequality $\mathbf{Ax} \leq \mathbf{b}$ can be re-written as a combination of two constraints: $\mathbf{Ax} - \mathbf{b} = -\mathbf{z}$ and $\mathbf{z} \geq 0$. Thus, we can re-write the LP as:

$$\begin{aligned} & \underset{\mathbf{x}, \mathbf{z}}{\text{minimize}} && \mathbf{f}^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \mathbf{Ax} - \mathbf{z} = \mathbf{b}, \\ \mathbf{Cx} = \mathbf{d}, \\ \mathbf{z} \geq 0 \end{cases} \end{aligned} \tag{2}$$

Domain of the variable \mathbf{x} is $\mathbb{X} = \{\mathbf{x} : \mathbb{R}^n\}$ and the domain of the variable \mathbf{z} is $\mathbb{Z} = \{\mathbf{z} : \mathbf{z} \geq 0\}$.

Domain of the entire problem can be described as direct sum $\mathbb{X} \oplus \mathbb{Z}$ intersected by the hyperplane $\begin{bmatrix} \mathbf{A} & -\mathbf{I} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{d} \end{bmatrix}$.

Equality $\mathbf{C}\mathbf{x} = \mathbf{d}$ can be solved as $\mathbf{x} = \mathbf{C}^+\mathbf{d} + \mathbf{N}\mathbf{y}$. Thus, we can re-write the inequality $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ as:

$$\mathbf{A}(\mathbf{C}^+\mathbf{d} + \mathbf{N}\mathbf{y}) \leq \mathbf{b} \quad (3)$$

$$\mathbf{A}\mathbf{N}\mathbf{y} \leq \mathbf{b} - \mathbf{A}\mathbf{C}^+\mathbf{d} \quad (4)$$

$$\mathbf{A}\mathbf{N}\mathbf{y} \leq \mathbf{b}_0 \quad (5)$$

where $\mathbf{b}_0 = \mathbf{b} - \mathbf{A}\mathbf{C}^+\mathbf{d}$. Thus we get LP in the following form:

$$\begin{aligned} & \underset{\mathbf{y}}{\text{minimize}} && \mathbf{f}^\top \mathbf{N}^\top \mathbf{y}, \\ & \text{subject to} && \mathbf{A}\mathbf{N}\mathbf{y} \leq \mathbf{b}_0 \end{aligned} \quad (6)$$

Domain of this problem is a polytope $\mathbf{A}\mathbf{N}\mathbf{y} \leq \mathbf{b}_0$.

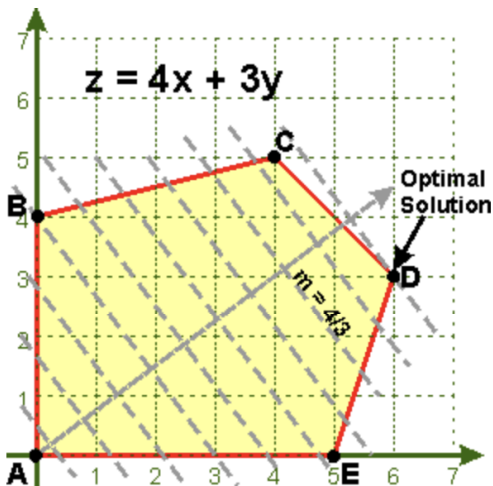


Figure 1: Geometry of an LP problem - example. [Credit](#)

LINEAR PROGRAMMING

LP with no solution - examples

Here are some examples of LP which have no solutions:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (7)$$

This one has no boundaries at all, hence no solution. Next one has boundaries, but they do not restrict motion along the descent direction for the cost function.

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \quad \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ &\text{subject to} \quad \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 1 \end{aligned} \quad (8)$$

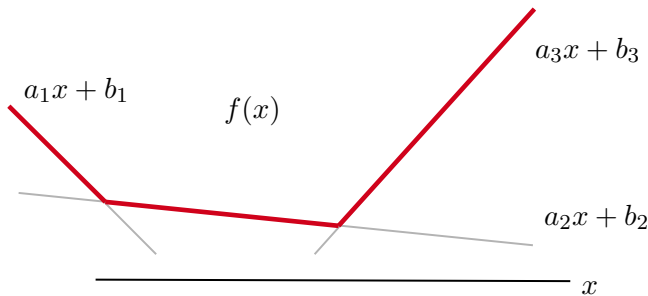
CONVEX PIECE-WISE LINEAR FUNCTIONS

Problem statement

Convex piece-wise linear functions have the form:

$$f(\mathbf{x}) = \max(\mathbf{a}_i^\top \mathbf{x} + b_i) \quad (9)$$

Figure below shows geometric interpretation of such function for a one-dimensional case.



CONVEX PIECE-WISE LINEAR FUNCTIONS

Solution as LP

We can formulate a minimization problem using convex piece-wise linear functions:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \max(\mathbf{a}_i^\top \mathbf{x} + b_i) \quad (10)$$

Which can be equivalently transformed into the following LP:

$$\begin{aligned} &\underset{\mathbf{x}, t}{\text{minimize}} \quad t \\ &\text{subject to} \quad \mathbf{a}_i^\top \mathbf{x} + b_i \leq t \end{aligned} \quad (11)$$

We can observe that optimal (minimal) t will have to lie on one of the linear functions $\mathbf{a}_i^\top \mathbf{x} + b_i$, i.e. on the original piece-wise linear function $f(\mathbf{x})$. And optimal value on t corresponds to the smallest value of the original function $f(\mathbf{x})$.

SUM OF PIECE-WISE LINEAR FUNCTIONS

Solution as LP

Sum of convex piece-wise linear functions have the form:

$$f(\mathbf{x}) + g(\mathbf{x}) = \max(\mathbf{a}_i^\top \mathbf{x} + b_i) + \max(\mathbf{c}_i^\top \mathbf{x} + d_i) \quad (12)$$

Their representation as LP is:

$$\begin{array}{ll} \underset{\mathbf{x}, t_1, t_2}{\text{minimize}} & t_1 + t_2 \\ \text{subject to} & \begin{cases} \mathbf{a}_i^\top \mathbf{x} + b_i \leq t_1 \\ \mathbf{c}_i^\top \mathbf{x} + d_i \leq t_2 \end{cases} \end{array} \quad (13)$$

CONVEX PIECE-WISE LINEAR FUNCTIONS

Code

```
0 func = @(t) t^2;
  derivative_func = @(t) 2*t;
2
  approx_points = [-1, -0.3, 0, 0.3, 1];
4 n = length(approx_points);
  a = zeros(n, 1);
6 b = zeros(n, 1);

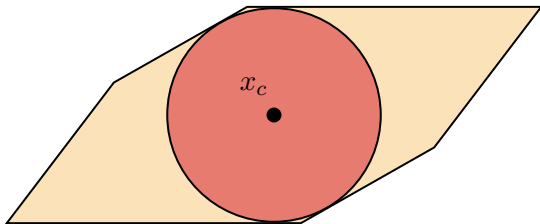
8 for i = 1:n
    t = approx_points(i);
10    a(i) = derivative_func(t);
    b(i) = func(t) - a(i)*t ;
12 end

14 f = [1; 0];
  lin_A = [-ones(n, 1), a];
16 lin_b = -b;
  x = linprog(f, lin_A, lin_b, [], []);
```

Chebyshev center of a polyhedron

Problem statement

Chebyshev center of a polyhedron is the center of the largest ball inscribed in a polyhedron:



Equation describing this ball can be written as:

$$\mathcal{B} = \{\mathbf{x}_c + \mathbf{u} : \|\mathbf{u}\|_2 \leq r\} \quad (14)$$

where r is the radius of the ball and \mathbf{x}_c is its center.

Before we move towards solving the problem, let us consider the following maximization:

$$\sup\{\mathbf{a}^\top \mathbf{u} : \|\mathbf{u}\|_2 \leq r\} \quad (15)$$

We can re-write the expression:

$$\sup\{\mathbf{a}^\top \mathbf{u} : \|\mathbf{u}\|_2 \leq r\} = \sup\{\|\mathbf{a}\| \cdot \|\mathbf{u}\| \cos(\varphi) : \|\mathbf{u}\|_2 \leq r\} \quad (16)$$

where φ is the angle between \mathbf{a} and \mathbf{u} . Since \mathbf{a} is constant, $\max(\|\mathbf{u}\|) = r$, and $\max(\cos(\varphi)) = 1$, we get:

$$\sup\{\mathbf{a}^\top \mathbf{u} : \|\mathbf{u}\|_2 \leq r\} = \|\mathbf{a}\|r \quad (17)$$

For the ball \mathcal{B} to be inscribed in a polygon $\mathcal{P} = \{\mathbf{x} : \mathbf{Ax} \leq \mathbf{b}\}$, the following should hold:

$$\sup\{\mathbf{a}_i^\top (\mathbf{x}_c + \mathbf{u}) : \|\mathbf{u}\|_2 \leq r\} \leq b_i \quad (18)$$

Note that the largest value of $\mathbf{a}_i^\top \mathbf{u}$ under condition $\|\mathbf{u}\|_2 \leq r$ is $r\|\mathbf{a}_i\|$: it can indeed achieve this value if \mathbf{a}_i and \mathbf{u} are co-directional, but a larger one is not possible. Therefore:

$$\sup\{\mathbf{a}_i^\top (\mathbf{x}_c + \mathbf{u}) : \|\mathbf{u}\|_2 \leq r\} = \mathbf{a}_i^\top \mathbf{x}_c + r\|\mathbf{a}_i\| \leq b_i \quad (19)$$

Finally, we can write down the solution of the problem as a linear optimization:

$$\begin{array}{ll} \underset{r, \mathbf{x}_c}{\text{maximize}} & r \\ \text{subject to} & \mathbf{a}_i^\top \mathbf{x}_c + r \|\mathbf{a}_i\| \leq b_i \end{array} \quad (20)$$

Chebyshev Center of a Polyhedron

Code

Below we can see MATLAB code for solving the problem:

```
0 V = randn(10, 2);
2 k = convhull(V);
  P = V(k, :);
4
  [domain_A, domain_b] = vert2con(P);
6 norm_A = vecnorm(domain_A');

8 f = [-1; 0; 0];
  A = [reshape(norm_A, [], 1), domain_A];
10 b = domain_b;

12 x = linprog(f, A, b, [], []);

14 center = [x(2), x(3)];
   r = x(1);
```


The following is the Linear-Fractional Programming problem:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{maximize}} & \frac{\mathbf{c}^\top \mathbf{x} + d}{\mathbf{e}^\top \mathbf{x} + f} \\ \text{subject to} & \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathbf{A}_e\mathbf{x} = \mathbf{b}_e \end{cases} \end{array} \quad (21)$$

This doesn't look like an LP, but let us see if we can try to bring the problem to the LP form.

LINEAR-FRACTIONAL PROGRAMMING

The following is the Linear-Fractional Programming problem in LP form:

$$\begin{array}{ll} \underset{\mathbf{y}, z}{\text{maximize}} & \mathbf{c}^\top \mathbf{y} + zd \\ \text{subject to} & \begin{cases} \mathbf{A}\mathbf{y} \leq z\mathbf{b} \\ \mathbf{A}_e\mathbf{y} = z\mathbf{b}_e \\ \mathbf{e}^\top \mathbf{y} + zf = 1 \\ z \geq 0 \end{cases} \end{array} \quad (22)$$

Here the variables \mathbf{y} and z are related to \mathbf{x} as follows.

$$\mathbf{y} = \frac{\mathbf{x}}{\mathbf{e}^\top \mathbf{x} + f} \quad (23)$$

$$z = \frac{1}{\mathbf{e}^\top \mathbf{x} + f} \quad (24)$$

We assumed that the domain of the previous problem is limited to $\mathbf{e}^\top \mathbf{x} + f \geq 0$. With that we have:

$$\mathbf{c}^\top \mathbf{y} + zd = \mathbf{c}^\top \frac{\mathbf{x}}{\mathbf{e}^\top \mathbf{x} + f} + \frac{1}{\mathbf{e}^\top \mathbf{x} + f} d = \frac{\mathbf{c}^\top \mathbf{x} + d}{\mathbf{e}^\top \mathbf{x} + f} \quad (25)$$

$$\mathbf{A}\mathbf{y} \leq z\mathbf{b} \implies \mathbf{A} \frac{\mathbf{x}}{\mathbf{e}^\top \mathbf{x} + f} \leq \frac{1}{\mathbf{e}^\top \mathbf{x} + f} \mathbf{b} \implies \mathbf{A}\mathbf{x} \leq \mathbf{b} \quad (26)$$

Implement linear approximation of a convex function and solve it as LP

Linear Programming and simplex algorithm, fmin.

Lecture slides are available via Github, links are on Moodle:

github.com/SergeiSa/Computational-Intelligence-2024

