LMI, Control design

Computational Intelligence, Lecture 8

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Spring 2023

Content

- LMI
- Control design
- Schur complement
- Appendix A

Linear matrix inequalities (LMI)

A linear matrix inequality (LMI) is a semidefinite constraint placed on a matrix:

$$\mathbf{S} \succ 0 \tag{1}$$

We assume (and this is true!) that there exist *solvers* that can solve problems with such constraints.

Example

Given $\mathbf{A}$, find such $\mathbf{S} \succ 0$ that $\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} \prec 0$.

Notice that the last example is continious-time Lyapunov eq. for LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, and if such $\mathbf{S}$ exists the system is stable.

Consider a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, control $\mathbf{u} = \mathbf{K}\mathbf{x}$ and a Lyapunov function $V = \mathbf{x}^\top \mathbf{S}\mathbf{x}$, $\mathbf{S} \succ 0$.

Closed-form of the system is $\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}$, and full derivative of the Lyapunov function:

$$\dot{V} = \mathbf{x}^\top (\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x} < 0 \tag{2}$$

This can be re-written as an LMI:

$$(\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S} + \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K}) \prec 0 \tag{3}$$

This is *not linear* in decision variables ($\mathbf{S}$ and $\mathbf{K}$), and can't be solved directly using popular solvers.

Introducing new variable $\mathbf{P} = \mathbf{S}^{-1}$ and multiplying (3) by $\mathbf{P}$ on both sides (we can do it, as both $\mathbf{P}$ and $\mathbf{S}$ are full rank, and thus it is a congruence transformation which preserves definiteness, see appendix) we get:

$$\mathbf{P}(\mathbf{A} + \mathbf{B}\mathbf{K})^\top + (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{P} \prec 0 \tag{4}$$

Now we introduce one more variable $\mathbf{L} = \mathbf{K}\mathbf{P}$ and get an LMI constraint:

$$\mathbf{P}\mathbf{A}^\top + \mathbf{A}\mathbf{P} + \mathbf{L}^\top\mathbf{B}^\top + \mathbf{B}\mathbf{L} \prec 0 \tag{5}$$

Solving (5) gives us $\mathbf{P}$ and $\mathbf{L}$, from which we can compute $\mathbf{K} = \mathbf{L}\mathbf{P}^{-1}$ and $\mathbf{S} = \mathbf{P}^{-1}$, solving the original problem.

A discrete dynamical system $\mathbf{x}_{i+1} = \mathbf{f}(\mathbf{x}_i)$ is stable if there exists a Lyapunov function $V(\mathbf{x}_i)$, such that:

- $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) < 0$;
- $V(\mathbf{x}_i) > 0$ for all $\mathbf{x}_i \neq 0$.

Consider a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$, control $\mathbf{u}_i = \mathbf{K}\mathbf{x}_i$ and a Lyapunov function $V(\mathbf{x}_i) = \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i$, $\mathbf{S} \succ 0$.

Closed-form of the system is $\mathbf{x}_{i+1} = (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}_i$, and discrete dynamics of the Lyapunov function is:

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) < 0 \tag{6}$$

$$\mathbf{x}_{i+1}^\top \mathbf{S}\mathbf{x}_{i+1} - \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i < 0 \tag{7}$$

$$\mathbf{x}_i^\top (\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}_i - \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i < 0 \tag{8}$$

$$\mathbf{x}_i^\top ((\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K}) - \mathbf{S})\mathbf{x}_i < 0 \tag{9}$$

The last equation is equivalent to the following semidefinite inequality:

$$(\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K}) - \mathbf{S} \prec 0 \tag{10}$$

Semidefinite inequality $(\mathbf{A}+\mathbf{BK})^\top \mathbf{S}(\mathbf{A}+\mathbf{BK}) - \mathbf{S} \prec 0$ can be re-written:

$$-\mathbf{S} - (\mathbf{A}+\mathbf{BK})^\top(-\mathbf{S})(\mathbf{A}+\mathbf{BK}) \prec 0 \tag{11}$$

Define $\mathbf{P}^{-1} = \mathbf{S}$. Note that $\mathbf{P} \succ 0$. We can multiply (11) by $\mathbf{P}$ on both sides (congruence transformation preserves definiteness, see Appendix):

$$-\mathbf{P} - \mathbf{P}(\mathbf{A}+\mathbf{BK})^\top(-\mathbf{P}^{-1})(\mathbf{A}+\mathbf{BK})\mathbf{P} \prec 0 \tag{12}$$

$$-\mathbf{P} - (\mathbf{AP}+\mathbf{BKP})^\top(-\mathbf{P}^{-1})(\mathbf{AP}+\mathbf{BKP}) \prec 0 \tag{13}$$

We define $\mathbf{L} = \mathbf{KP}$:

$$-\mathbf{P} - (\mathbf{AP}+\mathbf{BL})^\top(-\mathbf{P}^{-1})(\mathbf{AP}+\mathbf{BL}) \prec 0 \tag{14}$$

Theorem (Schur complement)

Given matrix $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{C} \end{bmatrix}$, *where* $\mathbf{C} \prec 0$, *the following statements are equivalent:*

1. $\mathbf{A} - \mathbf{B}^\top \mathbf{C}^{-1} \mathbf{B} \prec 0$
2. $\mathbf{M} \prec 0$

Semidefinite inequality
$-\mathbf{P} - (\mathbf{AP} + \mathbf{BL})^\top(-\mathbf{P}^{-1})(\mathbf{AP} + \mathbf{BL}) \prec 0$ can transformed with Schur inequality:

$$\begin{bmatrix} -\mathbf{P} & \mathbf{AP} + \mathbf{BL} \\ (\mathbf{AP} + \mathbf{BL})^\top & -\mathbf{P} \end{bmatrix} \prec 0 \tag{15}$$

This is a linear matrix inequality.

Consider matrices $\mathbf{P} \succ 0$, and $\mathbf{V} \in \mathbb{R}^{n,n}$ is full rank. We can prove that:

$$\mathbf{P} \succ 0 \implies \mathbf{V}^\top \mathbf{P} \mathbf{V} \succ 0 \tag{16}$$

Proof: $\mathbf{x}^\top \mathbf{V}^\top \mathbf{P} \mathbf{V} \mathbf{x} = \mathbf{z}^\top \mathbf{P} \mathbf{z}$, where $\mathbf{z} = \mathbf{V}\mathbf{x}$. Since $\mathbf{P} \succ 0$, $\mathbf{z}^\top \mathbf{P} \mathbf{z} \geq 0$, hence $\mathbf{x}^\top \mathbf{V}^\top \mathbf{P} \mathbf{V} \mathbf{x} \geq 0$.

Definition

Congruence transformation preserves semi-definiteness: $\det(\mathbf{V}) \neq 0, \ \mathbf{P} \succ 0 \implies \mathbf{V}^\top \mathbf{P} \mathbf{V} \succ 0$

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2023