

# Quadratic programming

## Computational Intelligence, Lecture 7

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- Quadratic programming
- Quadratically constrained quadratic programming (QCQP)
  - ▶ QCQP to LP; QCQP to LP
  - ▶ Ellipsoidal constraints  $\rightarrow$  to canonical form

# POSITIVE-DEFINITE MATRICES

Positive-definite matrices the following properties:

## Eigenvalues

All eigenvalues of a positive-definite (PD) matrix are real and positive.

## Eigenvectors

Eigenvectors of a PD matrix with form an orthogonal basis.

Proof (for the case of distinct eigenvalues): let  $\mathbf{M}$  be a PD matrix, and  $\mathbf{M}\mathbf{v} = \lambda\mathbf{v}$  and  $\mathbf{M}\mathbf{u} = \gamma\mathbf{u}$ . Then:

$$\mathbf{u}^\top \mathbf{M}\mathbf{v} = \mathbf{u}^\top (\mathbf{M}\mathbf{v}) = \lambda \mathbf{u}^\top \mathbf{v} \quad (1)$$

$$\mathbf{u}^\top \mathbf{M}\mathbf{v} = (\mathbf{u}^\top \mathbf{M})\mathbf{v} = \gamma \mathbf{u}^\top \mathbf{v} \quad (2)$$

Therefore,  $\lambda \mathbf{u}^\top \mathbf{v} = \gamma \mathbf{u}^\top \mathbf{v}$ , and since  $\lambda$  and  $\gamma$  are eigenvalues and eigenvalues are distinct, so  $\lambda \neq \gamma$ . This implies that  $\mathbf{u}^\top \mathbf{v} = 0$ .

Remember the general form of a quadratic program:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b}, \\ \mathbf{F} \mathbf{x} = \mathbf{g}. \end{cases} \end{array} \quad (3)$$

where  $\mathbf{H}$  is positive-definite and  $\mathbf{A} \mathbf{x} \leq \mathbf{b}$  describe a convex region.

# GEOMETRY OF A QP

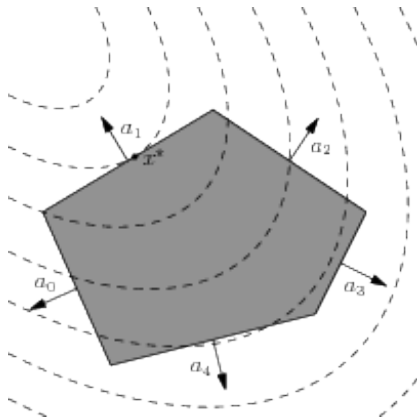


Figure 1: Geometry of a QP. [Source](#)

# COST FUNCTION OF A QP, 1

The cost function of a QP has the form  $c(\mathbf{x}) = \mathbf{x}^\top \mathbf{H}\mathbf{x} + \mathbf{f}^\top \mathbf{x}$ . Let us show that the requirement that  $\mathbf{H}$  is positive-definite does not limit the range of convex problems that can be solved as a QP.

Let  $\mathbf{M}$  be a non-symmetric matrix. Quadratic form  $q(\mathbf{x}) = \mathbf{x}^\top \mathbf{M}\mathbf{x}$  is a scalar and is equal to its transpose:

$$q(\mathbf{x}) = \mathbf{x}^\top \mathbf{M}\mathbf{x} \quad (4)$$

$$q(\mathbf{x}) = 0.5(\mathbf{x}^\top \mathbf{M}\mathbf{x} + \mathbf{x}^\top \mathbf{M}\mathbf{x}) \quad (5)$$

$$q(\mathbf{x}) = 0.5(\mathbf{x}^\top \mathbf{M}\mathbf{x} + \mathbf{x}^\top \mathbf{M}^\top \mathbf{x}) \quad (6)$$

$$q(\mathbf{x}) = 0.5\mathbf{x}^\top (\mathbf{M} + \mathbf{M}^\top) \mathbf{x} \quad (7)$$

Equivalently prove that the cost function  $c(\mathbf{x})$  is always equivalent to the cost function  $c(\mathbf{x}) = 0.5\mathbf{x}^\top (\mathbf{H} + \mathbf{H}^\top) \mathbf{x} + \mathbf{f}^\top \mathbf{x}$ . Because of that, without a loss of generality we can assume  $\mathbf{H}$  to be symmetric.

Let us prove that  $\mathbf{H}$  needs to be positive semi-definite in order for  $c(\mathbf{x})$  to be convex.

Assume that one of the eigenvalues of  $\mathbf{H}$  is negative:  $\mathbf{H}\mathbf{v} = \lambda\mathbf{v}$ , where  $\lambda < 0$  and  $\|\mathbf{v}\| = 1$ . We can find values of  $c(0)$ ,  $c(\mathbf{v})$  and  $c(2\mathbf{v})$ :

$$c(0) = 0 \tag{8}$$

$$c(\mathbf{v}) = \mathbf{v}^\top \mathbf{H} \mathbf{v} = \lambda \mathbf{v}^\top \mathbf{v} = \lambda \tag{9}$$

Note that  $c((1 - \beta)0 + \beta\mathbf{v}) = c(\beta\mathbf{v}) = \lambda\beta^2$  and  $(1 - \beta)c(0) + \beta c(\mathbf{v}) = \lambda\beta$ . Since  $0 < \beta < 1$  we conclude  $\beta^2 < \beta$ ; since  $\lambda < 0$ ,  $\lambda\beta^2 > \lambda\beta$ . So  $c((1 - \beta)0 + \beta\mathbf{v}) > (1 - \beta)c(0) + \beta c(\mathbf{v})$ . Thus, such  $c(\mathbf{x})$  is not convex. □

Let us consider a QP with degenerate matrix  $\mathbf{H} = \mathbf{R}\Sigma\mathbf{R}^\top$  (without a loss of generality  $\mathbf{H}$  can be assumed to be symmetric), where  $\mathbf{R} \in \mathbb{R}^{n \times k}$ .

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{R}\Sigma\mathbf{R}^\top \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{10}$$

Matrix  $\mathbf{R}$  is a row-space basis of  $\mathbf{H}$ , and  $\mathbf{N}$  is its orthogonal complement. Then we can decompose  $\mathbf{x}$  as  $\mathbf{x} = \mathbf{R}\zeta + \mathbf{N}\mathbf{z}$ :

$$\begin{aligned} & \underset{\zeta, \mathbf{z}}{\text{minimize}} && \zeta^\top \Sigma \zeta + \mathbf{f}^\top \mathbf{R}\zeta + \mathbf{f}^\top \mathbf{N}\mathbf{z}, \\ & \text{subject to} && \begin{cases} \mathbf{A}\mathbf{R}\zeta + \mathbf{A}\mathbf{N}\mathbf{z} \leq \mathbf{b}, \\ \mathbf{F}\mathbf{R}\zeta + \mathbf{F}\mathbf{N}\mathbf{z} = \mathbf{g}. \end{cases} \end{aligned} \tag{11}$$



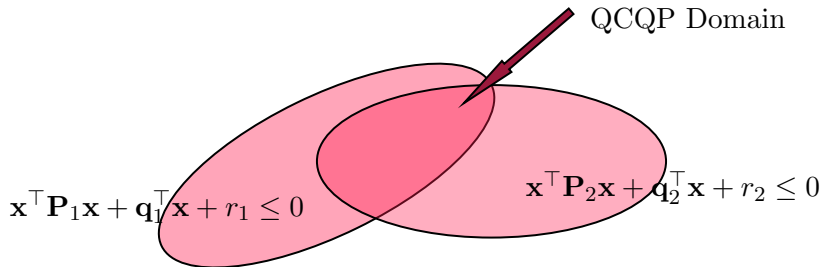
# QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING

General form of a quadratically constrained quadratic program (QCQP) is given below:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^\top \mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{x}^\top \mathbf{P}_i \mathbf{x} + \mathbf{q}_i^\top \mathbf{x} + r_i \leq 0, \\ \mathbf{F} \mathbf{x} = \mathbf{g}. \end{cases} \end{array} \quad (12)$$

where  $\mathbf{P}_i$  are positive-definite.

Domain of a QCQP without equality constraints and with no degenerate inequality constraints is an intersection of ellipses:



Set  $\mathbf{P}_i = \mathbf{0}$  and you get a QP.

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \begin{bmatrix} \mathbf{q}_1^\top \\ \dots \\ \mathbf{q}_n^\top \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} -r_1 \\ \dots \\ -r_n \end{bmatrix} \\ \mathbf{F} \mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{13}$$

Set  $\mathbf{P}_0 = \mathbf{0}$  and you get an LP.

# TURNING ELLIPSOID TO THE CANONICAL FORM (1)

Can we re-write the expression  $\mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r \leq 0$  as a canonical form ellipsoid:

$$\frac{z_1^2}{m_1^2} + \frac{z_2^2}{m_2^2} + \dots + \frac{z_n^2}{m_n^2} \leq 1 \quad (14)$$

We start by proposing a substitution  $\mathbf{x}_0 = -\frac{1}{2}\mathbf{P}^{-1}\mathbf{q}$  and  $-d = r - \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0$ . We can prove that:

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P} (\mathbf{x} - \mathbf{x}_0) - d = \mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r$$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P} (\mathbf{x} - \mathbf{x}_0) - d = \quad (15)$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} - 2\mathbf{x}_0^\top \mathbf{P} \mathbf{x} + \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 - d = \quad (16)$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} + 2 \left( \frac{1}{2} \mathbf{P}^{-1} \mathbf{q} \right)^\top \mathbf{P} \mathbf{x} + \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 + r - \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 = \quad (17)$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r. \quad (18)$$

## TURNING ELLIPSOID TO THE CANONICAL FORM (2)

Thus our original expression became:

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P}(\mathbf{x} - \mathbf{x}_0) - d \leq 0 \quad (19)$$

We define  $\mathbf{A} = \sqrt{\mathbf{P}}$  with SVD decomposition  $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$ .

Defining  $\mathbf{z} = \mathbf{V}^\top(\mathbf{x} - \mathbf{x}_0)$  we get:

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{A}^\top \mathbf{A}(\mathbf{x} - \mathbf{x}_0) - d \leq 0 \quad (20)$$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{V}\Sigma\mathbf{U}^\top \mathbf{U}\Sigma\mathbf{V}^\top (\mathbf{x} - \mathbf{x}_0) - d \leq 0 \quad (21)$$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{V}\Sigma^2\mathbf{V}^\top (\mathbf{x} - \mathbf{x}_0) - d \leq 0 \quad (22)$$

$$\mathbf{z}^\top \Sigma^2 \mathbf{z} - d \leq 0 \quad (23)$$

$$\sum z_i^2 \sigma_i^2 \leq d \quad (24)$$

Defining  $1/m_i^2 = \sigma_i^2/d$  we get:

$$\frac{z_1^2}{m_1^2} + \frac{z_2^2}{m_2^2} + \dots + \frac{z_n^2}{m_n^2} \leq 1 \quad (25)$$

Implement a program that finds right-most point of an intersection of two ellipsoids; visualise the problem and the solution.

- Symmetric Matrices and Eigendecomposition. Robert M. Freund, MIT, 2014.
- MOSEK, QP and QCQP.

Lecture slides are available via Github, links are on Moodle:

[github.com/SergeiSa/Computational-Intelligence-2024](https://github.com/SergeiSa/Computational-Intelligence-2024)

