Linear Programming

Computational Intelligence, Lecture 6

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Linear Programming

A linear program (LP) is an optimization problem of the form:

$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\
\text{subject to} \quad & \begin{cases} \mathbf{A}\mathbf{x} \le \mathbf{b}, \\ \mathbf{C}\mathbf{x} = \mathbf{d}. \end{cases}
\end{aligned} \tag{1}
$$

It is one of the older and widely used classes of convex optimization problems.

Note that the solution of such problem will always lie on the boundary of its domain.

Inequality $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ can be re-written as a combination of two constraints: $\mathbf{A}\mathbf{x} - \mathbf{b} = -\mathbf{z}$ and $\mathbf{z} \geq 0$. Thus, we can re-write the LP as:

$$\begin{aligned} \underset{\mathbf{x},\mathbf{z}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{A}\mathbf{x} - \mathbf{z} = \mathbf{b}, \\ \mathbf{C}\mathbf{x} = \mathbf{d}, \\ \mathbf{z} \geq 0 \end{cases} \end{aligned} \tag{2}$$

Domain of the variable $\mathbf{x}$ is $\mathbb{X} = \{\mathbf{x} : \mathbb{R}^n\}$ and the domain of the variable $\mathbf{z}$ is $\mathbb{Z} = \{\mathbf{z} : \mathbf{z} \geq 0\}$.

Domain of the entire proble can be described as direct sum $\mathbb{X} \oplus \mathbb{Z}$ intersected by the hyperplane $\begin{bmatrix} \mathbf{A} & -\mathbf{I} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{d} \end{bmatrix}$.

LP - simpler formulations, 2

Equality $\mathbf{Cx} = \mathbf{d}$ can be solved as $\mathbf{x} = \mathbf{C}^{+}\mathbf{d} + \mathbf{Ny}$. Thus, we can re-write the inequality $\mathbf{Ax} \leq \mathbf{b}$ as:

$$\mathbf{A}(\mathbf{C}^{+}\mathbf{d} + \mathbf{Ny}) \leq \mathbf{b} \tag{3}$$

$$\mathbf{ANy} \leq \mathbf{b} - \mathbf{AC}^{+}\mathbf{d} \tag{4}$$

$$\mathbf{ANy} \leq \mathbf{b}_0 \tag{5}$$

where $\mathbf{b}_0 = \mathbf{b} - \mathbf{AC}^{+}\mathbf{d}$. Thus we get LP in the following form:

$$\begin{aligned} \underset{\mathbf{y}}{\text{minimize}} \quad & \mathbf{f}^{\top}\mathbf{N}^{\top}\mathbf{y}, \\ \text{subject to} \quad & \mathbf{ANy} \leq \mathbf{b}_0 \end{aligned} \tag{6}$$

Domain of this problem is a polytope $\mathbf{ANy} \leq \mathbf{b}_0$.

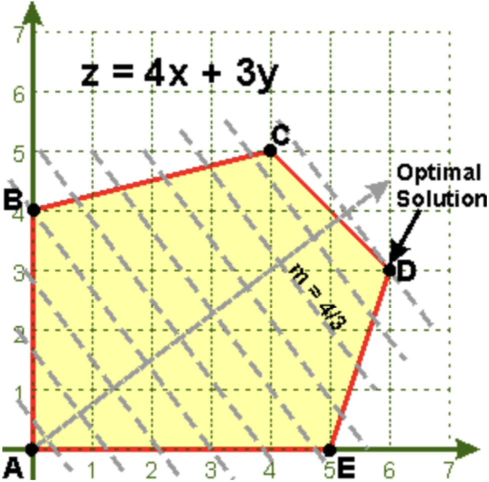


Figure 1: Geometry of an LP problem - example. Credit

Here are some examples of LP which have no solutions:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{7}$$

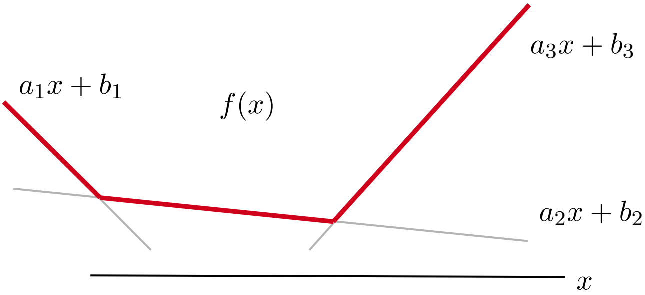
This one is has no boundaries at all, hence no solution. Next one has boundaries, but they do not restrict motion along the descent direction for the cost function.

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ \text{subject to} \quad & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 1 \end{aligned} \tag{8}$$

Convex piece-wise linear functions have the form:

$$f(\mathbf{x}) = \max(\mathbf{a}_i^\top \mathbf{x} + b_i) \tag{9}$$

Figure below shows geometric interpretation of such function for a one-dimensional case.



We can formulate a minimization problem using convex piece-wise linear functions:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \max(\mathbf{a}_i^\top \mathbf{x} + b_i) \tag{10}$$

Which can be equivalently transformed into the following LP:

$$\begin{aligned} \underset{\mathbf{x},t}{\text{minimize}} \quad & t \\ \text{subject to} \quad & \mathbf{a}_i^\top \mathbf{x} + b_i \leq t \end{aligned} \tag{11}$$

We can observe that optimal (minimal) t will have to lie on one of the linear functions $\mathbf{a}_i^\top \mathbf{x} + b_i$, i.e. on the original piece-wise linear function $f(\mathbf{x})$. And optimal value on t corresponds to the smallest value of the original function $f(\mathbf{x})$.

Sum of convex piece-wise linear functions have the form:

$$f(\mathbf{x}) + g(\mathbf{x}) = \max(\mathbf{a}_i^\top \mathbf{x} + b_i) + \max(\mathbf{c}_i^\top \mathbf{x} + d_i) \tag{12}$$

Their representation as LP is:

$$\begin{aligned} \underset{\mathbf{x}, t_1, t_2}{\text{minimize}} \quad & t_1 + t_2 \\ \text{subject to} \quad & \begin{cases} \mathbf{a}_i^\top \mathbf{x} + b_i \leq t_1 \\ \mathbf{c}_i^\top \mathbf{x} + d_i \leq t_2 \end{cases} \end{aligned} \tag{13}$$

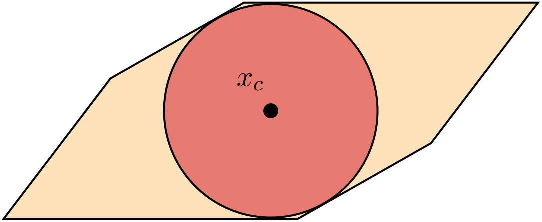
```
func = @(t) t^2;
derivative_func = @(t) 2*t;

approx_points = [-1, -0.3, 0, 0.3, 1];
n = length(approx_points);
a = zeros(n, 1);
b = zeros(n, 1);

for i = 1:n
    t = approx_points(i);
    a(i) = derivative_func(t);
    b(i) = func(t) - a(i)*t ;
end

f = [1; 0];
lin_A = [-ones(n, 1), a];
lin_b = -b;
x = linprog(f, lin_A,lin_b, [], []);
```

Chebyshev center of a polyhedron is the center of the largest ball inscribed in a polyhedron:



Equation describing this ball can be written as:

$$\mathcal{B} = \{\mathbf{x}_c + \mathbf{u} : \ ||\mathbf{u}||_2 \leq r\} \tag{14}$$

where r is the radius of the ball and $\mathbf{x}_c$ is its center.

Before we move towards solving the problem, let us consider the following maximization:

$$\sup\{\mathbf{a}^\top \mathbf{u} : \ ||\mathbf{u}||_2 \leq r\} \tag{15}$$

We can re-write the expression:

$$\sup\{\mathbf{a}^\top \mathbf{u} : \ ||\mathbf{u}||_2 \leq r\} = \sup\{||\mathbf{a}|| \cdot ||\mathbf{u}|| \cos(\varphi) : \ ||\mathbf{u}||_2 \leq r\} \tag{16}$$

where φ is the angle between $\mathbf{a}$ and $\mathbf{u}$. Since $\mathbf{a}$ is constant, $\max(||\mathbf{u}||) = r$, and $\max(\cos(\varphi)) = 1$, we get:

$$\sup\{\mathbf{a}^\top \mathbf{u} : \ ||\mathbf{u}||_2 \leq r\} = ||\mathbf{a}|| r \tag{17}$$

For the ball $\mathcal{B}$ to be inscribed in a polygon $\mathcal{P} = \{\mathbf{x} : \ \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, the following should hold:

$$\sup\{\mathbf{a}_i^\top(\mathbf{x}_c + \mathbf{u}) : \ ||\mathbf{u}||_2 \leq r\} \leq b_i \tag{18}$$

Note that the largest value of $\mathbf{a}_i^\top \mathbf{u}$ under condition $||\mathbf{u}||_2 \leq r$ is $r||\mathbf{a}_i||$: it can indeed achieve this value if $\mathbf{a}_i$ and $\mathbf{u}$ are co-directional, but a larger one is not possible. Therefore:

$$\sup\{\mathbf{a}_i^\top(\mathbf{x}_c + \mathbf{u}) : \ ||\mathbf{u}||_2 \leq r\} = \mathbf{a}_i^\top \mathbf{x}_c + r||\mathbf{a}_i|| \leq b_i \tag{19}$$

Finally, we can write down the solution of the problem as a linear optimization:

$$
\begin{aligned}
\underset{r,\ \mathbf{x}_c}{\text{maximize}} \quad & r \\
\text{subject to} \quad & \mathbf{a}_i^\top \mathbf{x}_c + r||\mathbf{a}_i|| \leq b_i
\end{aligned}
\tag{20}
$$

Below we can see MATLAB code for solving the problem:

```
V = randn(10, 2);
k = convhull(V);
P = V(k, :);

[domain_A, domain_b] = vert2con(P);
norm_A = vecnorm(domain_A');

f = [-1; 0; 0];
A = [reshape(norm_A, [], 1), domain_A];
b = domain_b;

x = linprog(f, A,b, [], []);

center = [x(2),x(3)];
r = x(1);
```

The following is the Linear-Fractional Programming problem:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{maximize}} && \frac{\mathbf{c}^\top \mathbf{x} + d}{\mathbf{e}^\top \mathbf{x} + f} \\ &\text{subject to} && \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathbf{A}_e \mathbf{x} = \mathbf{b}_e \end{cases} \end{aligned} \tag{21}$$

This doesn't look like an LP, but let us see if we can try to bring the problem to the LP form.

The following is the Linear-Fractional Programming problem in LP form:

$$\begin{aligned} & \underset{\mathbf{y},\ z}{\text{maximize}} && \mathbf{c}^\top \mathbf{y} + zd \\ & \text{subject to} && \begin{cases} \mathbf{A}\mathbf{y} \leq z\mathbf{b} \\ \mathbf{A}_e\mathbf{y} = z\mathbf{b}_e \\ \mathbf{e}^\top \mathbf{y} + zf = 1 \\ z \geq 0 \end{cases} \end{aligned} \tag{22}$$

Here the variables $\mathbf{y}$ and z are related to $\mathbf{x}$ as follows.

$$\mathbf{y} = \frac{\mathbf{x}}{\mathbf{e}^\top \mathbf{x} + f} \tag{23}$$

$$z = \frac{1}{\mathbf{e}^\top \mathbf{x} + f} \tag{24}$$

We assumed that the domain of the previous problem is limited to $\mathbf{e}^\top \mathbf{x} + f \geq 0$. With that we have:

$$\mathbf{c}^\top \mathbf{y} + zd = \mathbf{c}^\top \frac{\mathbf{x}}{\mathbf{e}^\top \mathbf{x} + f} + \frac{1}{\mathbf{e}^\top \mathbf{x} + f} d = \frac{\mathbf{c}^\top \mathbf{x} + d}{\mathbf{e}^\top \mathbf{x} + f} \tag{25}$$

$$\mathbf{A}\mathbf{y} \leq z\mathbf{b} \implies \mathbf{A}\frac{\mathbf{x}}{\mathbf{e}^\top \mathbf{x} + f} \leq \frac{1}{\mathbf{e}^\top \mathbf{x} + f}\mathbf{b} \implies \mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{26}$$

Implement linear approximation of a convex function and solve it as LP

Further reading

Linear Programming and simplex algorithm, fmin.

Lecture slides are available via Github, links are on Moodle:

github.com/SergeiSa/Computational-Intelligence-2025