Mixed-Integer Convex Programming

Computational Intelligence, Lecture 12

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Content

Mixed Integer Linear Programming (MILP)

A general form of a mixed-integer linear program is:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ \mathbf{C}\mathbf{x} = \mathbf{d}, \\ \mathbf{x}_1, ..., \mathbf{x}_m \in \mathbb{R}, \\ \mathbf{x}_{m+1}, ..., \mathbf{x}_n \in \mathbb{N}. \end{cases} \end{aligned} \tag{1}$$

In other words, the only difference is that some of the variables are only allowed to assume pure integer values.

Mixed Integer Convex Programming (MICP)

A general form of a mixed-integer convex program is:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & l(\mathbf{x}) \quad \text{convex cost} \\ \text{subject to} \quad & \begin{cases} g(\mathbf{x}) \leq 0, & \text{convex domain} \\ \mathbf{C}\mathbf{x} = \mathbf{d}, & \\ x_1, ..., x_m \in \mathbb{R}, & \text{continuous variables} \\ x_{m+1}, ..., x_n \in \mathbb{N}. & \text{integer variables} \end{cases} \end{aligned} \tag{2}$$

Remarks

- Mixed-integer convex programs are not convex, even if the name seems to suggest otherwise.
- The "convex program" in the phrase "mixed-integer convex program" should be understood as "if we fix values of the integer variables, or relax them into reals, the result will be a convex program".
- Mixed integer programs are usually solved using *branch and bound algorithms*; in worse case scenario, the algorithms performs exhaustive search.
- In robotics applications, integer variables in mixed integer programs are often restricted to binary values, i.e. $\mathbf{y} \in \{0, 1\}^n$.. It is still called "mixed-integer" in the literature, although phrases such as "mixed-binary" or "binary constraints" are not rare.

Branch and bound

Branch and bound is searches for the nest solution by solving a series of convex optimization problems.

1. We start by relaxing all binary variables z_1, z_2, ... to real numbers on the interval $0 \leq z_i \leq 1$. We solve the resulting convex problem and obtain node cost c.
2. We grow 2 branches by fixing one of the z_i to be $z_i = 0$ for one branch and $z_i = 1$ for another. For each feasible branch we find node cost.
3. The minimal node cost among nodes without branches is the current lower bound on the optimal cost (we cannot do better than that by growing more branches / fixing more z_i).
4. The minimal node cost among nodes with all z_i fixed is the current upper bound (the optimal solution will have to be at least as good as that).
5. The gap between lower and upper bounds becomes smaller as the algorithm progresses; it can be used to terminate the procedure.

Example: Footstep planning
Problem statement

Given N convex regions defined by linear inequalities $\{\mathbf{x}: \ \mathbf{A}_i\mathbf{x} \leq \mathbf{b}_i\}$ (which is called H-polytope representation), find a sequence of K points (footsteps) from the given starting point to the given goal point, such that all footsteps lie in one of the convex regions.

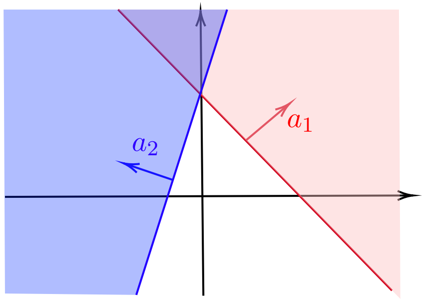
$$\begin{aligned} \underset{\mathbf{x}_1,\ ...,\ \mathbf{x}_K}{\text{minimize}} \quad & ||\mathbf{x}_1 - \mathbf{x}_{\text{start}}|| + ||\mathbf{x}_K - \mathbf{x}_{\text{goal}}||, \\ \text{subject to} \quad & \exists\{\mathbf{A}_j, \mathbf{b}_j\} \in \Omega \text{ s.t. } \mathbf{A}_j\mathbf{x}_i \leq \mathbf{b}_j \end{aligned} \tag{3}$$

where $\Omega = \{\{\mathbf{A}_1, \mathbf{b}_1\},\ \{\mathbf{A}_2, \mathbf{b}_2\},\ ...,\ \{\mathbf{A}_N, \mathbf{b}_N\}\}$. This is not a convex program.

Big-M method, 1

One of the key methods associated with the use of mixed-integer programming in robotics is the big-M method.

Assume you have two inequalities, $\mathbf{a}_1^\top \mathbf{x} \leq b_1$ and $\mathbf{a}_2^\top \mathbf{x} \leq b_2$, and you are happy if at least one of them holds. Define new binary variables $c_1, c_2 \in \{0,\ 1\}$ and find a big enough constant M, such that $\mathbf{a}_1^\top \mathbf{x} \leq b_1 + M$ and $\mathbf{a}_2^\top \mathbf{x} \leq b_2 + M$ holds for all of your domain (of the part of it you are interested in).



Big-M method, 2

Let us put the following constraint on the variables c_1, c_2:

$$c_1 + c_2 = 1 \tag{4}$$

The pair of variables (c_1, c_2) together can assume only two values - (1, 0) and (0, 1). We can modify the inequalities $\mathbf{a}_1^\top \mathbf{x} \leq b_1$ and $\mathbf{a}_2^\top \mathbf{x} \leq b_2$:

$$\begin{cases} \mathbf{a}_1^\top \mathbf{x} \leq b_1 + M \cdot c_1 \\ \mathbf{a}_2^\top \mathbf{x} \leq b_2 + M \cdot c_2 \end{cases} \tag{5}$$

If $(c_1, c_2) = (1, 0)$, then modified inequalities become:

$$\begin{cases} \mathbf{a}_1^\top \mathbf{x} \leq b_1 + M & \text{- true for all } \mathbf{x} \\ \mathbf{a}_2^\top \mathbf{x} \leq b_2 & \text{- original constraint} \end{cases} \tag{6}$$

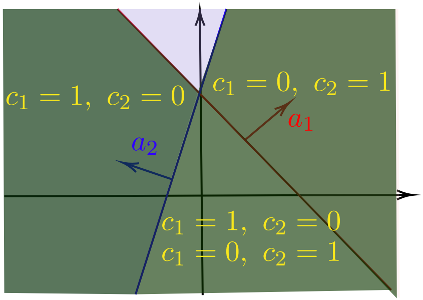
If $(c_1, c_2) = (0, 1)$, then modified inequalities become:

$$\begin{cases} \mathbf{a}_1^\top \mathbf{x} \leq b_1 & \text{- original constraint} \\ \mathbf{a}_2^\top \mathbf{x} \leq b_2 + M & \text{- true for all } \mathbf{x} \end{cases} \tag{7}$$

Big-M method, 3

With that, we propose the following constraint:

$$\begin{cases} \mathbf{a}_1^\top \mathbf{x} \le b_1 + M \cdot c_1 \\ \mathbf{a}_2^\top \mathbf{x} \le b_2 + M \cdot c_2 \\ c_1 + c_2 = 1 \\ c_i \in \{0,\ 1\} \end{cases} \tag{8}$$



Choosing between polytopes, 1

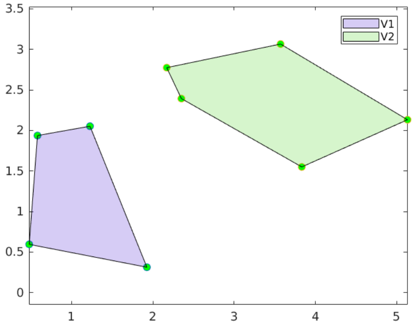
It works the same way for the case when you have three (or two or more) systems of inequalities $\mathbf{A}_1\mathbf{x} \leq \mathbf{b}_1$, $\mathbf{A}_2\mathbf{x} \leq \mathbf{b}_2$, $\mathbf{A}_3\mathbf{x} \leq \mathbf{b}_3$ and are happy if at least one holds:

$$\begin{cases} \mathbf{A}_1\mathbf{x} \leq \mathbf{b}_1 + M \cdot \mathbf{1} \cdot (1 - c_1) \\ \mathbf{A}_2\mathbf{x} \leq \mathbf{b}_2 + M \cdot \mathbf{1} \cdot (1 - c_2) \\ \mathbf{A}_3\mathbf{x} \leq \mathbf{b}_3 + M \cdot \mathbf{1} \cdot (1 - c_3) \\ c_1 + c_2 + c_3 = 1 \\ c_i \in \{0,\ 1\} \end{cases} \tag{9}$$

where $\mathbf{1}$ is a vector of all ones. Notice that constraint $c_1 + c_2 + c_3 = 1$ can be replaced with $c_1 + c_2 + c_3 >= 1$, allowing avoid relaxing more than one region.

Choosing between polytopes, 2

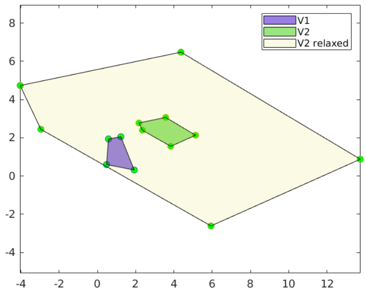
Below are two H-polytops (convex regions represented by systems of inequalities):

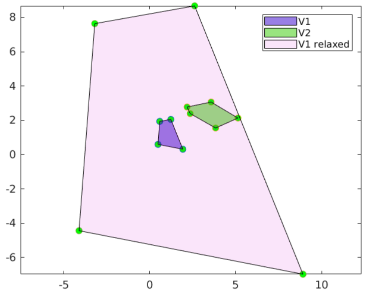


As we can see, their union represents a non-convex domain, and they have no intersection.

Choosing between polytopes, 3

Now, one of them is relaxed as described above. Notice that the intersection of the two polytopes is non-relaxed polytope.





Choosing between polytopes, 4
Multiple variables

If you have multiple variables $\mathbf{x}_1$, ..., $\mathbf{x}_K$, and each should belong to at least one of the H-polytopes $\{\mathbf{A}_i, \mathbf{b}_i\}$, this can also be represented using big-M method:

$$\begin{cases} \mathbf{A}_1\mathbf{x}_k \leq \mathbf{b}_1 + M \cdot \mathbf{1} \cdot (1 - c_{1,k}) \\ \mathbf{A}_2\mathbf{x}_k \leq \mathbf{b}_2 + M \cdot \mathbf{1} \cdot (1 - c_{2,k}) \\ \mathbf{A}_3\mathbf{x}_k \leq \mathbf{b}_3 + M \cdot \mathbf{1} \cdot (1 - c_{3,k}) \\ c_{1,k} + c_{2,k} + c_{3,k} = 1 \\ c_{i,k} \in \{0,\ 1\} \\ k = 1, ..., K \end{cases} \tag{10}$$

Notice that the only difference from the previous example is that now we have K sets of binary variables $c_{1,k}$, $c_{2,k}$ and $c_{3,k}$.

Example: Footstep planning
Formulation as MIQP

Using big-M relaxation we can now formulate the problem as follows:

$$\begin{aligned}
&\underset{\mathbf{x}_k,\ \mathbf{c}_{i,k}}{\text{minimize}} && ||\mathbf{x}_1 - \mathbf{x}_{\text{start}}|| + ||\mathbf{x}_K - \mathbf{x}_{\text{goal}}||, \\
&\text{subject to} && \begin{cases} \mathbf{A}_i \mathbf{x}_k \leq \mathbf{b}_i + M \cdot \mathbf{1} \cdot (1 - c_{i,k}),\ i = 1, ..., N \\ \sum_{i=1}^{N} c_{i,k} = 1 \\ c \in \{0,\ 1\}^{N,\ K} \\ k = 1, ..., K \end{cases}
\end{aligned} \tag{11}$$

Example: Footstep planning

Evenly spaced steps

In order to make the footsteps evenly spaced we add cost on the distance between consequent steps:

$$\underset{\mathbf{x}_k,\ \mathbf{c}_{i,k}}{\text{minimize}} \quad ||\mathbf{x}_1 - \mathbf{x}_{\text{start}}||^2 + ||\mathbf{x}_K - \mathbf{x}_{\text{goal}}||^2 + w \cdot \sum_{k=1}^{K-1} ||\mathbf{x}_{i+1} - \mathbf{x}_i||^2,$$

$$\text{subject to} \quad \begin{cases} \mathbf{A}_i \mathbf{x}_k \leq \mathbf{b}_i + M \cdot \mathbf{1} \cdot (1 - c_{i,k}),\ \ i = 1, ..., N \\ \sum_{i=1}^{N} c_{i,k} = 1 \\ c \in \{0,\ 1\}^{N,\ K} \\ k = 1, ..., K \end{cases}$$

where w is a weight, a coefficient we can tune to adjust relative importance of our primary objective (starting from the point $\mathbf{x}_{\text{start}}$ and finishing at the point $\mathbf{x}_{\text{goal}}$ and our secondary objective (making the footsteps evenly spaced).

Example: Footstep planning

Code, part 1

```
n = 2;
shift_1 = 1*rand(n, 1);
shift_2 = 1*rand(n, 1);
V1 = randn(n, 6);
V2 = randn(n, 6) + shift_1;
V3 = randn(n, 6) + shift_1 + shift_2;

[A1, b1] = vert2con(V1');
[A2, b2] = vert2con(V2');
[A3, b3] = vert2con(V3');

number_of_steps = 7;
start_point = sum(V1, 2) / size(V1, 2);
finish_point = sum(V3, 2) / size(V3, 2);
weight = 5;
bigM = 15;
```

Example: Footstep planning

Code, part 2

```
cvx_begin
    variable x(n, number_of_steps)
    binary variable c(3, number_of_steps);
    cost = 0;
    for i = 1:(number_of_steps-1)
        cost = cost + norm(x(:, i) - x(:, i+1));     end
    cost = cost + (norm(x(:, 1) - start_point) + norm(x
   (:, number_of_steps) - finish_point))*weight;
    minimize( cost )
    subject to
for i = 1:number_of_steps
  A1*x(:, i) <= b1 + (1 - c(1, i))*bigM;
  A2*x(:, i) <= b2 + (1 - c(2, i))*bigM;
  A3*x(:, i) <= b3 + (1 - c(3, i))*bigM;
  c(1, i) + c(2, i) + c(3, i) == 1;
end
cvx_end
plot(x(1, :)', x(2, :)', '^', 'MarkerEdgeColor', 'k', '
    MarkerSize', 10, 'LineWidth', 2); hold on;
```

Example: switching control
Multiple variables

Consider the case when you have a control system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u}_1 + \mathbf{B}_2\mathbf{u}_2$, but you are required to use either $\mathbf{B}_1\mathbf{u}_1$ or $\mathbf{B}_2\mathbf{u}_2$ at any time, not both.

This can be cast as a mixed-integer constraint:

$$\begin{cases} ||\mathbf{u}_1|| \leq M(1 - c_1) \\ ||\mathbf{u}_2|| \leq M(1 - c_2) \\ c_1 + c_2 = 1 \end{cases} \tag{12}$$

Homework

Implement footstep planning for a biped, making sure every pair of steps lands in the same H-polytope.

Lecture slides are available via Github, links are on Moodle:

github.com/SergeiSa/Computational-Intelligence-2025