Linear inequalities and polytopes

Computational Intelligence, Lecture 5

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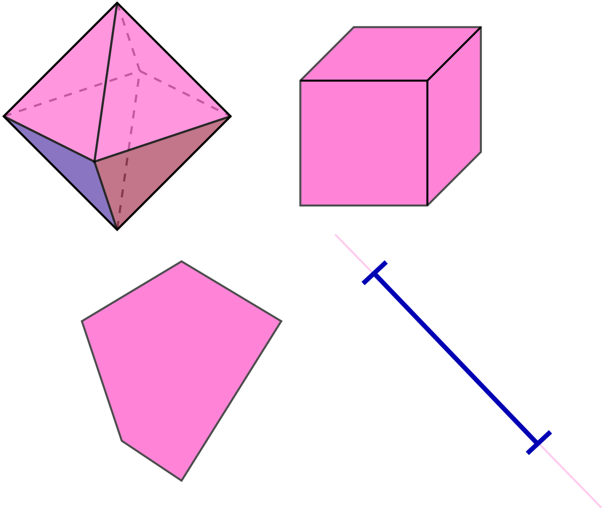
Content

- Convex polytopes
- Half-spaces
- H-representation
- V-representation
- G-representation (Zonotopes)
- Linear approximation of convex regions

Before defining what a convex polytope is, let us look at examples:

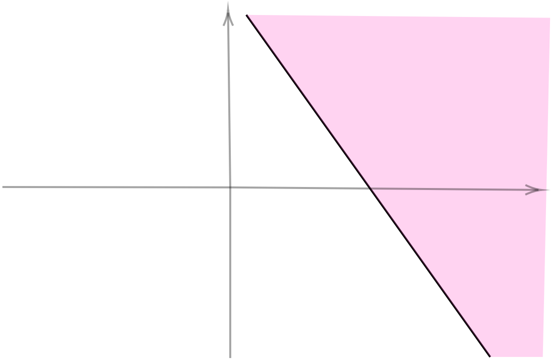


You can think of polytopes as geometric figures (or continuous sets of points) with linear edges, faces and higher-dimensional analogues.

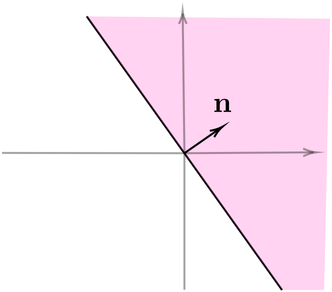
Definition

Convex polytopes are polytopes whose every two points can be connected with a line that would lie in the polytope. They can be bounded or unbounded.

We can define half-space as a set of all points $\mathbf{x}$, such that $\mathbf{a}^\top \mathbf{x} \leq b$. It has a very clear geometric interpretation. In the following image, the filled space is **not** in the half space.

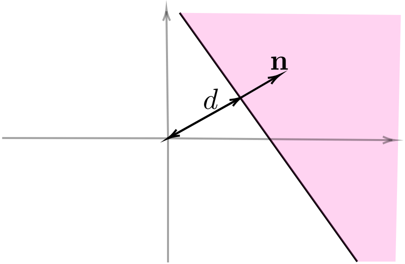


Consider half-space that passes through the origin, and defined by its normal vector $\mathbf{n}$:



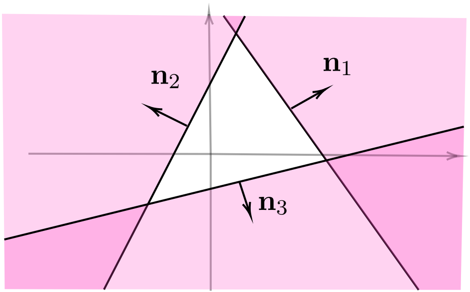
It is easy to see that this half-space can be defined as "all vectors $\mathbf{x}$, such that $\mathbf{n} \cdot \mathbf{x} \leq 0$", which is the same as using $\mathbf{n}$ instead of $\mathbf{a}$ in our original definition, setting $b = 0$.

In the general case there is some distance between the boundary of the half-space and the origin, let's say d.



Here the half space can be defined as "all vectors $\mathbf{x}$, such that $\mathbf{x}^\top \frac{\mathbf{n}}{||\mathbf{n}||} \leq d$". This is the same as making $\mathbf{a} = \mathbf{n}$ and $b = d||\mathbf{a}||$.

We can define a region of space as an *intersection* of half-spaces $\mathbf{a}_i^\top \mathbf{x} \le b_i$:



Resulting region will be easily described as $\begin{bmatrix} \mathbf{a}_1^\top \\ ... \\ \mathbf{a}_k^\top \end{bmatrix} \mathbf{x} \le \begin{bmatrix} b_1 \\ ... \\ b_k \end{bmatrix}$

The last result allows us to write any convex polytope as a matrix inequality:

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{1}$$

And conversely, any matrix inequality (1) represents either an empty set or a convex polytope.

Definition

$\mathbf{A}\mathbf{x} \leq \mathbf{b}$ is called *H-representation* (half-space representation) of a polytope.

We can use containment in an H-polytope as a part of convex optimation problem. For example, the following QP includes such constraint:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}. \end{aligned} \tag{2}$$

V-representation

Convex polytopes have alternative representations, such as *V-representation*. It amounts to representing polytope as a set of its vertices.

Example

$V = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$ is a V-representation of a square.

Example

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an H-representation of the same square.

Convex hull

Given points $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_N$ their convex hull is represented as:

$$\mathcal{P} = \left\{ \mathbf{x} = \sum_{i=1}^{N} \alpha_i \mathbf{x}_i : \ \sum_{i=1}^{N} \alpha_i = 1, \ \alpha_i \in [0 \ 1] \right\} \tag{3}$$

See Appendix for an illustration of this formula.

We can use containment in an V-polytope as a part of convex optimation problem. For example, the following QP includes such constraint:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i, \\ \sum_{i=1}^{n} \alpha_i = 1, \\ \alpha_i \geq 0. \end{cases} \end{aligned} \tag{4}$$

Notice that the constraint amounts to equating $\mathbf{x}$ to a convex combination of the vertices of the V-polytope.

H and V-representations

To transfer from H-representation to V-representation, you need to solve *vertex enumeration* problem, which is computationally expensive.

It is also possible to construct H-representation out of V-representation. Both algorithms are not convex.

A zonotope $\mathcal{Z}$ is a symmetric polytope defined by its *center* $\mathbf{c}$ and *generator* $\mathbf{G}$:

$$\mathcal{Z} = \{\mathbf{x} : \ \mathbf{x} = \mathbf{G}\beta + \mathbf{c}, \ ||\beta||_\infty \leq 1\} \tag{5}$$

The set $\{\beta : \ ||\beta||_\infty \leq 1\}$ is a hypercube and zonotope $\mathcal{Z}$ is a projection (shadow) of this hypercube onto a lower-dimensional space; the projection is defined by the matrix $\mathbf{G}$.

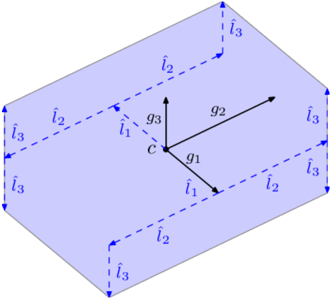
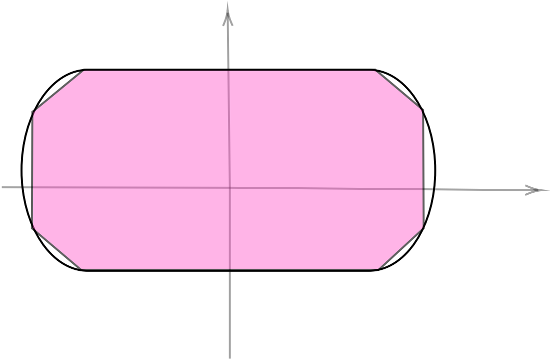


Figure 1: Zonotope (Source)

We can use containment in an G-polytope as a part of convex optimation problem. For example, the following QP includes such constraint:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{x} = \mathbf{G}\beta + \mathbf{c}, \\ -1 \geq \beta_i \geq 1. \end{cases} \end{aligned} \tag{6}$$

Some convex regions can be easily approximated using polytopes.



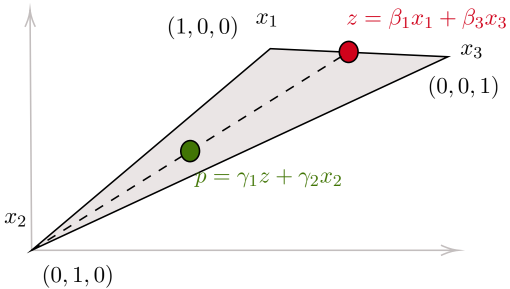
Which allows to represent constraints on $\mathbf{x}$ to belong in such a region as a matrix inequality

Write H-representation of the following polytopes:

- Equilateral triangle
- Square
- Parallelepiped
- Trapezoid

Lecture slides are available via Github, links are on Moodle:

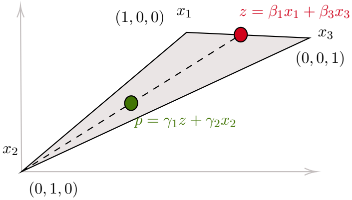
github.com/SergeiSa/Computational-Intelligence-2025



Let us illustrate the convex combination formula. Let $\mathcal{P}$ be convex hull of points $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$:

$$\mathcal{P} = \left\{ \mathbf{x} = \sum_{i=1}^{3} \alpha_i \mathbf{x}_i : \ \sum_{i=1}^{3} \alpha_i = 1, \ \alpha_i \in [0 \ 1] \right\} \tag{7}$$

Let $\mathbf{z}$ be a convex combination of $\mathbf{x}_1$ and $\mathbf{x}_3$: $\mathbf{z} = \beta_1 \mathbf{x}_1 + \beta_3 \mathbf{x}_3$. Then any $\mathbf{p} \in \mathcal{P}$ is expressed as a convex combination of $\mathbf{z}$ and $\mathbf{x}_2$: $\mathbf{p} = \gamma_1 \mathbf{z} + \gamma_2 \mathbf{x}_2$.



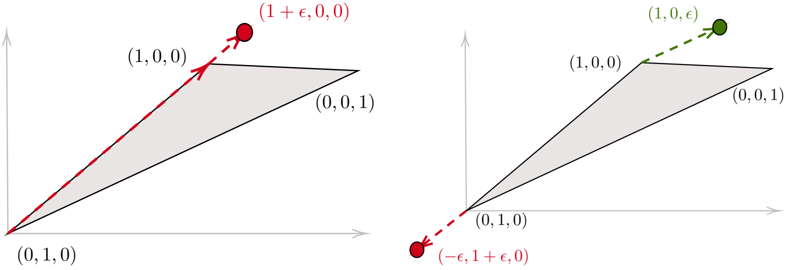
We can express $\mathbf{p}$ as:

$$\mathbf{p} = \gamma_1 \mathbf{z} + \gamma_2 \mathbf{x}_2 = \gamma_1(\beta_1 \mathbf{x}_1 + \beta_3 \mathbf{x}_3) + \gamma_2 \mathbf{x}_2 \tag{8}$$

We can define $\alpha_1 = \gamma_1 \beta_1$, $\alpha_2 = \gamma_2 \mathbf{x}_2$ and $\alpha_3 = \gamma_1 \beta_3$. Since $\gamma_i \geq 0$ and $\beta_i \geq 0$, we conclude that $\alpha_i \geq 0$.

We can show that $e = \alpha_1 + \alpha_2 + \alpha_3 = 1$:

$$e = \gamma_1(\beta_1 + \beta_3) + \gamma_2 = \gamma_1 + \gamma_2 = 1 \tag{9}$$



Previously we illustrated sufficiency of the formula's constraints. Now let us illustrate their necessity.

Dropping requirement $\alpha_i \leq 1$, and/or $\alpha_i \geq 0$ and/or $\sum_{i=1}^{3} \alpha_i = 1$, leads to inclusion of points out the convex hull, as illustrated on the figures.