

# Minimax

## Computational Intelligence, Lecture 13

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## ■ Homework

# MINIMAX PROBLEMS

## Example

Consider the following problem:

### Example

Find smallest  $x \in \mathbb{R}$ , such that  $x + y \geq 1$ , where  $|y| \leq 2$ .

In that example we need to find optimal value of  $x$  subject to a constraint where another unknown variable is present; the constraint has to be satisfied for the *worst-case scenario*, in this case it is  $y = -2$ . Solution is  $x = 3$

This is closely related to *minimax optimization*

# MINIMAX: LINEAR CONSTRAINT

## Part 1

Consider the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x}||, \\ \text{subject to} \quad & \mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{1}$$

It is clear that worst-case scenario corresponds to the largest value of  $\mathbf{d}^\top \mathbf{y}$ , meaning that  $\mathbf{y}$  should align with  $\mathbf{d}$  and have its maximum possible length  $p$ . From that we conclude that  $\mathbf{y} = p \frac{\mathbf{d}}{||\mathbf{d}||}$ .

# MINIMAX: LINEAR CONSTRAINT

## Part 2

Therefor  $\mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \leq h$  becomes:

$$\mathbf{c}^\top \mathbf{x} + p \frac{\mathbf{d}^\top \mathbf{d}}{\|\mathbf{d}\|} \leq h \quad (2)$$

$$\mathbf{c}^\top \mathbf{x} + p \|\mathbf{d}\| \leq h \quad (3)$$

Thus our problem becomes:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x}\|, \\ \text{subject to} \quad & \mathbf{c}^\top \mathbf{x} \leq h - p \|\mathbf{d}\| \end{aligned} \quad (4)$$

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 1

## Part 1

Consider the following problem, where  $\mathbf{x}^*$  is the desired value of  $\mathbf{x}$ :

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} \quad & \mathbf{y}^\top \mathbf{D}\mathbf{x} \leq h, \\ & \|\mathbf{y}\| \leq p \end{aligned} \tag{5}$$

This time worst-case scenario corresponds to  $\mathbf{y}$  aligned with  $\mathbf{D}\mathbf{x}$  and having its maximum possible length  $p$ . From that we conclude that  $\mathbf{y} = p \frac{\mathbf{D}\mathbf{x}}{\|\mathbf{D}\mathbf{x}\|}$ . Let us substitute it to  $\mathbf{y}^\top \mathbf{D}\mathbf{x}$ :

$$p \left( \frac{\mathbf{D}\mathbf{x}}{\|\mathbf{D}\mathbf{x}\|} \right)^\top \mathbf{D}\mathbf{x} = p \frac{\mathbf{x}^\top \mathbf{D}^\top \mathbf{D}\mathbf{x}}{\|\mathbf{D}\mathbf{x}\|} = p \frac{\|\mathbf{D}\mathbf{x}\|^2}{\|\mathbf{D}\mathbf{x}\|} = p \|\mathbf{D}\mathbf{x}\| \tag{6}$$

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 1

## Part 2

Thus our problem becomes:

$$\begin{array}{ll} \min_{\mathbf{x}} & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} & ||\mathbf{D}\mathbf{x}|| \leq \frac{h}{p} \end{array} \quad (7)$$

which is an SOCP.

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 2

## Part 1

A more general case of the previous problem is:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h, \\ & \|\mathbf{y}\| \leq p \end{aligned} \tag{8}$$

We can rewrite  $(\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h$  as:

$$\mathbf{y}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{9}$$

With that we see that the worse case scenario is  $\mathbf{y}$  is aligned with  $\mathbf{D}(\mathbf{x} - \mathbf{b})$  and has length  $p$ :

$$\mathbf{y} = p \frac{\mathbf{D}(\mathbf{x} - \mathbf{b})}{\|\mathbf{D}(\mathbf{x} - \mathbf{b})\|} \tag{10}$$



# MINIMAX: QUADRATIC CONSTRAINT, TYPE 2

## Part 2

Then  $\mathbf{y}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h$  becomes:

$$p \frac{(\mathbf{x} - \mathbf{b})^\top \mathbf{D}^\top \mathbf{D}(\mathbf{x} - \mathbf{b})}{\|\mathbf{D}(\mathbf{x} - \mathbf{b})\|} - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \quad (11)$$

which is the same as:

$$p \|\mathbf{D}(\mathbf{x} - \mathbf{b})\| - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \quad (12)$$

$$\|\mathbf{D}(\mathbf{x} - \mathbf{b})\| \leq \frac{1}{p} \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \frac{h}{p} \quad (13)$$

which is an SOCP constraint.

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 2

## Part 2

And thus we get:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} \quad & \|\mathbf{D}(\mathbf{x} - \mathbf{b})\| \leq \frac{1}{p} \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \frac{h}{p} \end{aligned} \tag{14}$$

which is SOCP.

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 3

## Part 1

A more general case of the previous problem is:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h, \\ & ||\mathbf{H}\mathbf{y} + \mathbf{f}|| \leq p \end{aligned} \tag{15}$$

where  $\mathbf{H}$  is has an inverse. We start by making substitution:

$$\mathbf{v} = \mathbf{H}\mathbf{y} + \mathbf{f} \tag{16}$$

meaning  $\mathbf{y} = \mathbf{H}^{-1}(\mathbf{v} - \mathbf{f})$ :

$$(\mathbf{H}^{-1}(\mathbf{v} - \mathbf{f}) - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{17}$$

$$\mathbf{v}^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) - (\mathbf{H}^{-1}\mathbf{f} + \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{18}$$

$$\mathbf{v}^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) - (\mathbf{H}\mathbf{a} + \mathbf{f})^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{19}$$

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 3

## Part 2

We can introduce notation:

$$\mathbf{M} = \mathbf{H}^{-\top} \mathbf{D} \quad (20)$$

$$\mathbf{g} = \mathbf{H}\mathbf{a} + \mathbf{f} \quad (21)$$

With that we can re-write our constraint:

$$\mathbf{v}^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) - \mathbf{g}^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h \quad (22)$$

$$(\mathbf{v} - \mathbf{g})^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h \quad (23)$$

And now we formulated type 3 problem as type 2:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{v}} \quad & \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} \quad & (\mathbf{v} - \mathbf{g})^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h, \\ & \|\mathbf{v}\| \leq p \end{aligned} \quad (24)$$

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 4

Try solving this problem on your own:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \mathbf{s}^\top \mathbf{y} + \mathbf{q}^\top \mathbf{x} \leq h, \\ & ||\mathbf{H}\mathbf{y} + \mathbf{f}|| \leq p \end{aligned} \tag{25}$$

# CONTROL WITH PARAMETER UNCERTAINTY

## Part 1

Consider the system:

$$\dot{\mathbf{x}} = \mathbf{A}_p \mathbf{x} + \mathbf{B}_p \mathbf{u} \quad (26)$$

where  $\mathbf{A}_p$  and  $\mathbf{B}_p$  are linear functions of parameters  $\mathbf{p}$ . We want to stabilize the origin.

Assume we use control law:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^* \quad (27)$$

With that we get:

$$\dot{\mathbf{x}} = (\mathbf{A}_p - \mathbf{B}_p \mathbf{K}) \mathbf{x} + \mathbf{B}_p \mathbf{u}^* \quad (28)$$

# CONTROL WITH PARAMETER UNCERTAINTY

## Part 2

Let us write Lyapunov function for the system:

$$V = \mathbf{x}^\top \mathbf{S} \mathbf{x} \quad (29)$$

$$\dot{V} = \dot{\mathbf{x}}^\top \mathbf{S} \mathbf{x} + \mathbf{x}^\top \mathbf{S} \dot{\mathbf{x}} = \quad (30)$$

$$= \mathbf{x}^\top \mathbf{S} (\mathbf{A}_p - \mathbf{B}_p \mathbf{K}) \mathbf{x} + \mathbf{x}^\top (\mathbf{A}_p - \mathbf{B}_p \mathbf{K})^\top \mathbf{S} \mathbf{x} + \quad (31)$$

$$+ \mathbf{x}^\top \mathbf{S} \mathbf{B}_p \mathbf{u}^* + \mathbf{u}^{*\top} \mathbf{B}_p^\top \mathbf{S} \mathbf{x} \quad (32)$$

Let us define:

$$a = 2\mathbf{x}^\top \mathbf{S} (\mathbf{A}_p - \mathbf{B}_p \mathbf{K}) \mathbf{x} \quad (33)$$

$$\mathbf{b} = 2\mathbf{x}^\top \mathbf{S} \mathbf{B}_p \quad (34)$$

With that we can find Jacobians:

$$\mathbf{a}_x = \frac{\partial a}{\partial \mathbf{p}} \quad \mathbf{B}_x = \frac{\partial \mathbf{b}}{\partial \mathbf{p}} \quad (35)$$

# CONTROL WITH PARAMETER UNCERTAINTY

## Part 3

Thus we get minimax constraint on the Lyapunov function

$$\dot{V} = \mathbf{a}_x^\top \mathbf{p}_t + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{p}_t \quad (36)$$

where  $\mathbf{p}_t$  are true values of parameters  $\mathbf{p}$ . Assuming:

$$\mathbf{p}_t = \mathbf{p} + \mathbf{p}_0 \quad (37)$$

we get:

$$\dot{V} = \mathbf{a}_x^\top (\mathbf{p} + \mathbf{p}_0) + \mathbf{u}^{*\top} \mathbf{B}_x (\mathbf{p} + \mathbf{p}_0) \quad (38)$$

which is a minimax constraint. Let's solve it for the case  $\|\mathbf{p}\| \leq 1$ .



# CONTROL WITH PARAMETER UNCERTAINTY

## Part 4

Taking derivative of  $\dot{V}$  with respect to  $\mathbf{p}$  we get

$$\frac{\partial \dot{V}}{\partial \mathbf{p}} = \mathbf{a}_x^\top + \mathbf{u}^{*\top} \mathbf{B}_x \quad (39)$$

this is the direction where the function grow the most. But we know its length is 1, so we conclude that:

$$\mathbf{p} = \frac{\mathbf{a}_x^\top + \mathbf{u}^{*\top} \mathbf{B}_x}{\|\mathbf{a}_x^\top + \mathbf{u}^{*\top} \mathbf{B}_x\|} \quad (40)$$

So:

$$\dot{V} = \|\mathbf{a}_x^\top + \mathbf{u}^{*\top} \mathbf{B}_x\| + (\mathbf{a}_x^\top + \mathbf{u}^{*\top} \mathbf{B}_x) \mathbf{p}_0 \quad (41)$$

# ELLIPTICAL PARAMETER UNCERTAINTY

## Part 1

Let's do the same, but for the case when  $\|\mathbf{G}\mathbf{p}\| \leq 1$ :

$$\begin{cases} \dot{V} = \mathbf{a}_x^\top (\mathbf{p} + \mathbf{p}_0) + \mathbf{u}^{*\top} \mathbf{B}_x (\mathbf{p} + \mathbf{p}_0) \leq 0 \\ \|\mathbf{G}\mathbf{p}\| \leq 1 \end{cases} \quad (42)$$

First step is to introduce new variable:

$$\rho = \mathbf{G}\mathbf{p} \quad (43)$$

from which it follows that  $\mathbf{p} = \mathbf{G}^{-1}\rho$  ( $\mathbf{G}$  should be invertible for the parameters to be bounded). Hence we get:

$$\begin{cases} \dot{V} = \mathbf{a}_x^\top (\mathbf{G}^{-1}\rho + \mathbf{p}_0) + \mathbf{u}^{*\top} \mathbf{B}_x (\mathbf{G}^{-1}\rho + \mathbf{p}_0) \leq 0 \\ \|\rho\| \leq 1 \end{cases} \quad (44)$$

We can find gradient:

$$\frac{\partial \dot{V}}{\partial \mathbf{p}} = \mathbf{a}_x^\top \mathbf{G}^{-1} + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{G}^{-1} \quad (45)$$

We know that length of  $\rho$  is bounded, so:

$$\rho = \frac{\mathbf{a}_x^\top \mathbf{G}^{-1} + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{G}^{-1}}{\|\mathbf{a}_x^\top \mathbf{G}^{-1} + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{G}^{-1}\|} \quad (46)$$

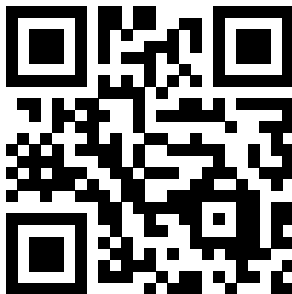
And thus we get SOCP constraint:

$$\dot{V} = \|\mathbf{a}_x^\top \mathbf{G}^{-1} + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{G}^{-1}\| + \mathbf{a}_x^\top \mathbf{p}_0 + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{p}_0 \leq 0 \quad (47)$$

Lecture slides are available via Moodle.

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[github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2021](https://github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2021)



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