$$\min_x \max_y \quad \|x - x^*\|$$

$$\text{subject to} \quad (y-a)^T D(x-b) + s^T y + q^T x \le h\,, \qquad (1)$$

$$\|Hy + f\| \le p \qquad (2)$$

An approach for solving this Minimax problem is to re-formulate a Minimax problem that takes the following form:

$$\min_x \max_y \quad \|x - x^*\|$$

$$\text{subject to} \quad (y-a)^T D(x-b) \le h\,, \qquad (3)$$

$$\|y\| \le p \qquad (4)$$

And re-formulate it into a *SOCP constrained problem* which can be then given to any convex Optimization solver (e.g. CVX) in the following SOCP form:

$$\min_x \quad \|x - x^*\|$$

$$\text{subject to} \quad \|D(x-b)\| \le \frac{1}{p}\, a^T D(x-b) + \frac{h}{p} \qquad (5)$$

First, we need to transform the ellipse constraint (2) into a circle constraint (4) by introducing new variable z

$$z = Hy + f$$

$$y = H^{-1}(z - f) = H^{-1}z - H^{-1}f \qquad (6)$$

$$y^T = z^T H^{-T} - f^T H^{-T} \qquad (7)$$

Re-writing (2):

$$y^T D(x-b) - a^T D(x-b) + s^T y + q^T x \le h$$

Substituting (6) & (7) in (1):

$$(z^T H^{-T} - f^T H^{-T}) D(x-b) - a^T D(x-b) + s^T H^{-1}(z-f) + q^T x \le h \qquad (8)$$

Re-arranging (8):

$$z^T H^{-T} D(x-b) - (a^T + f^T H^{-T}) D(x-b) + \left(s^T H^{-1}(z-f)\right)^T + q^T x \le h$$

$$z^T H^{-T} D(x-b) - (a^T H^T + f^T) H^{-T} D(x-b) + \left(s^T H^{-1}(z-f)\right)^T + q^T x \le h$$

$$z^T H^{-T} D(x-b) - (Ha + f)^T H^{-T} D(x-b) + (z-f)^T H^{-T} s + q^T x \le h \qquad (9)$$

Re-arranging (9)

$$z^T H^{-T}(D(x-b)+s) - (Ha+f)^T H^{-T} D(x-b) - f^T H^{-T} s + q^T x \leq h \qquad (10)$$

Thus, from (10) we can deduce that worst case scenario (in terms of z) would be if vector z is aligned with vector $H^{-T}(D(x-b)+s)$.

And since constraint (2) states that:

$$\|Hy+f\| \leq p$$

$$\|z\| \leq p$$

Therefore, we can say that z at the worst case scenario will be:

$$z = p\frac{H^{-T}(D(x-b)+s)}{\|H^{-T}(D(x-b)+s)\|} \qquad (11)$$

Substituting (11) in (10):

*Since p is scalar, $p^T = p$

$$p\frac{(D(x-b)+s)^T\ H^{-1}H^{-T}(D(x-b)+s)}{\|H^{-T}(D(x-b)+s)\|} - (Ha+f)^T H^{-T} D(x-b) - f^T H^{-T} s + q^T x \leq h$$

*Since $(D(x-b)+s)^T\ H^{-1}H^{-T}(D(x-b)+s) = \|H^{-T}(D(x-b)+s)\|^2$

Therefore,

$$p\left\|H^{-T}(D(x-b)+s)\right\| - (Ha+f)^T H^{-T} D(x-b) - f^T H^{-T} s + q^T x \leq h \qquad (12)$$

Finally, the problem is re-formulated into:

$$\min_x \quad \|x-x^*\|$$

$$\textit{subject to} \quad p\|H^{-T}(D(x-b)+s)\| \leq (Ha+f)^T H^{-T} D(x-b) + f^T H^{-T} s - q^T x$$

Which is a SOCP Problem

Abdelaziz Shaarawy

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