

# Linear inequalities and H-polytopes

## Computational Intelligence, Lecture 4

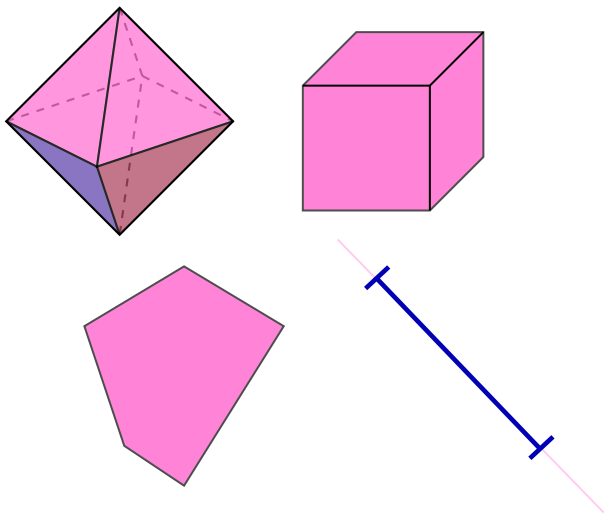
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Spring 2023

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# CONVEX POLYTOPES

Before defining what a convex polytope is, let us look at examples:



You can think of polytopes as geometric figures (or continuous sets of points) with linear edges, faces and higher-dimensional analogues.

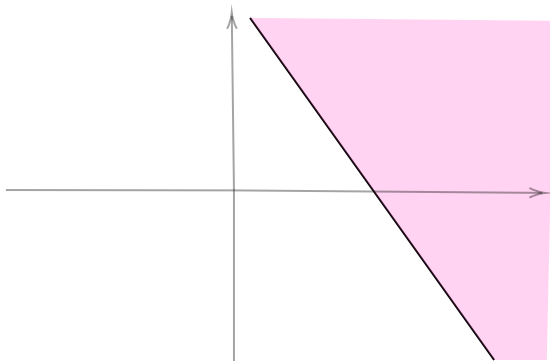
## Definition

Convex polytopes are polytopes whose every two points can be connected with a line that would lie in the polytope. They can be bounded or unbounded.

# HALF-SPACES

## Definition

We can define half-space as a set of all points  $\mathbf{x}$ , such that  $\mathbf{a}^\top \mathbf{x} \leq b$ . It has a very clear geometric interpretation. In the following image, the filled space is **not** in the half space.



# HALF-SPACES

## Construction. Simple case

Consider half-space that passes through the origin, and defined by its normal vector  $\mathbf{n}$ :



It is easy to see that this half-space can be defined as "all vectors  $\mathbf{x}$ , such that  $\mathbf{n} \cdot \mathbf{x} \leq 0$ ", which is the same as using  $\mathbf{n}$  instead of  $\mathbf{a}$  in our original definition, setting  $b = 0$ .

# HALF-SPACES

## Construction. General case

In the general case there is some distance between the boundary of the half-space and the origin, let's say  $d$ .



Here the half space can be defined as "all vectors  $\mathbf{x}$ , such that  $\mathbf{x}^\top \frac{\mathbf{n}}{\|\mathbf{n}\|} \leq d$ ". This is the same as making  $\mathbf{a} = \mathbf{n}$  and  $b = d\|\mathbf{a}\|$ .

# HALF-SPACES

## Combination

We can define a region of space as an *intersection* of half-spaces  
 $\mathbf{a}_i^\top \mathbf{x} \leq b_i$ :



Resulting region will be easily described as 
$$\begin{bmatrix} \mathbf{a}_1^\top \\ \dots \\ \mathbf{a}_k^\top \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} b_1 \\ \dots \\ b_k \end{bmatrix}$$



The last result allows us to write any convex polytope as a matrix inequality:

$$\mathbf{Ax} \leq \mathbf{b} \tag{1}$$

And conversely, any matrix inequality (1) represents either an empty set or a convex polytope.

### Definition

$\mathbf{Ax} \leq \mathbf{b}$  is called *H-representation* (half-space representation) of a polytope.

Convex polytopes have alternative representations, such as *V-representation*. In amounts to representing polytope as a set of its vertices.

## Example

$V = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$  is a V-representation of a square.

## Example

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  is an H-representation of the same square.

To transfer from H-representation to V-representation, you need to solve *vertex enumeration* problem, which is computationally expensive.

It is also possible to construct H-representation out of V-representation. Both algorithms are not convex.

# LINEAR APPROXIMATION OF CONVEX REGIONS

Some convex regions can be easily approximated using polytopes.



Which allows to represent constraints on  $\mathbf{x}$  to belong in such a region as a matrix inequality

Represent in matrix inequality form the following figures:

- Equilateral triangle
- A square
- Parallelepiped
- Trapezoid

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2023](https://github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2023)

