

# Shortest Path Planning

## Computational Intelligence, Lecture 12

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# SHORTEST PATH ON A GRAPH

If we want to plan a path on a 2D map, we can represent obstacle-free space regions as a nodes, and possible transitions between the obstacle-free space regions as graph edges.



**Figure 1:** Path planning as graph search; Credit:  
<https://demonstrations.wolfram.com/ProbabilisticRoadmapMethod/>

# SHORTEST PATH ON A GRAPH

Consider a directed graph (each edge has a direction assigned to it):

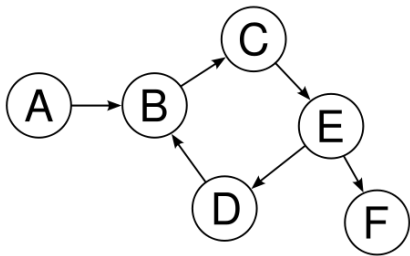


Figure 2: Directed graph; Credit:  
<https://github.com/HQarroum/directed-graph>

How can we find a shortest path from a start node to a finish node on it?

# SHORTEST PATH (1)

We assign index variable  $x_i$  to  $i$ -th edge; each index variable is positive  $x_i \geq 0$ .

If  $x_i = 1$  the edge is part of the path. We assume that otherwise  $x_i = 0$  (which will be enforced by the other constraints).

Adding a cost  $d_i$  associated with each edge (e.g. Euclidean distance) we get a linear cost  $l(\mathbf{x})$ :

$$l(\mathbf{x}) = \mathbf{x}^\top \mathbf{d} \tag{1}$$

## SHORTEST PATH (2)

Since each edge connects one node (e.g. node  $u$ ) to another (e.g. node  $v$ ), we can label all index variables  $x$  with superscripts, denoting nodes that they connect -  $x^{u,v}$ .

Our goal will be to count how many path segments enter and leave each node. For any normal node the number will be equal:

$$-\sum_{\forall i} x^{i,v} + \sum_{\forall j} x^{v,j} = 0 \quad (2)$$

We know that for the starting node  $s$ , there will only be one path segment leaving it:

$$-\sum_{\forall i} x^{i,s} + \sum_{\forall j} x^{s,j} = 1 \quad (3)$$

For the finishing node  $f$  we have only one path segment entering it:

$$-\sum_{\forall i} x^{i,f} + \sum_{\forall j} x^{f,j} = -1 \quad (4)$$

Together the problem becomes:

$$\begin{aligned}
 & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{d}, \\
 & \text{subject to} && \begin{cases} -\sum_{\forall i} x^{i,v} + \sum_{\forall j} x^{v,j} = 0, & \forall v \\ -\sum_{\forall i} x^{i,s} + \sum_{\forall j} x^{s,j} = 1, \\ -\sum_{\forall i} x^{i,f} + \sum_{\forall j} x^{f,j} = -1, \\ \mathbf{x} \geq 0. \end{cases}
 \end{aligned} \tag{5}$$

And with that, the problem can be solved as an LP.



# SPP CODE (1)

```
0 n = 5; V = randn(n, 2);  
% Connectivity:  
2 C = [1, 2; %edge 1  
      1, 3; %edge 2  
      2, 3; %edge 3  
      2, 4; %edge 4  
      3, 5; %edge 5  
      4, 5]; %edge 6  
8 nc = size(C, 1);  
d = zeros(nc, 1); %cost - distance  
10 for i = 1:nc  
    d(i) = norm(V(C(i, 2), :) - V(C(i, 1), :));  
12 end
```

## SPP CODE (2)

```
0 cvx_begin
  variable x(nc, 1)
2 minimize( dot(d, x) )
  subject to
4 x >= zeros(nc, 1);
  x(1) + x(2) == 1;%v 1
6 -x(5) - x(6) == -1;%v 5

8 -x(1) + x(3) + x(4) == 0;%v 2
  -x(2) - x(3) + x(5) == 0;%v 3
10 -x(4) + x(6) == 0;%v 4
  cvx_end
```

Another popular shortest path planning method for graphs is *A star* ( $A^*$ ). Unlike the previous method it does not involve optimization, but it requires a *heuristic*.

To study  $A^*$  we once more consider a graph whose edges have cost associated with them.

Let  $p$  be a node of the graph that the program found a path to. Point  $p$  has a predecessor point  $a(p)$  - the last node in the path towards  $p$ . Since each predecessor knows its predecessor, it means we can recursively reconstruct the path from the point  $p$  to the start.

Finding a path from the start to the point  $p$  we construct a sequence of edges that we need to travel through -  $e_1, e_2, \dots, e_n$ . Each of these edges has a cost associated with them -  $c_1, c_2, \dots, c_n$ . So, the cost of reaching a node  $p$  is  $g(p) = \sum_{i=1}^n c_i$ .

If we have a heuristic  $h(p)$  that (while more or less accurate) always *underestimates* the cost to reaching goal from the node  $p$ , we can use  $A^*$  to choose the next node in the path. We choose the node that minimizes the following cost function:

$$p_{next} = \underset{p}{\operatorname{argmin}}(g(p) + h(p)) \quad (6)$$

In practice, when we can compute  $g(p)$  much simpler. Given a new node  $p_{next}$  and its predecessor  $p_a$ , and the cost associated with the edge connecting them  $c_a$ , we can assign the value of  $g(p_{next})$  as:

$$g(p_{next}) := g(p_a) + c_a \quad (7)$$

Heuristic might be difficult to formulate in general, but as long as each node has coordinates on a plane associated with it, Euclidean distance provides a suitable heuristics.

# A STAR SEARCH - IMPLEMENTATION

A grid can easily be seen as a graph, where adjacency implies connection.

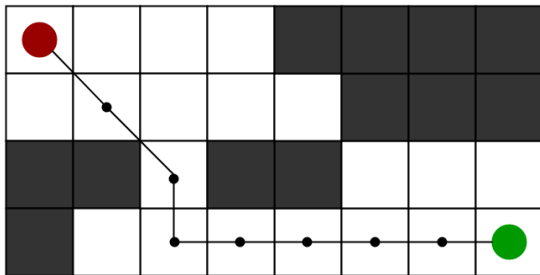


Figure 3: Example of a grid with obstacles. Credit: [geeksforgeeks.org](https://www.geeksforgeeks.org)

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2023](https://github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2023)

