

# Semidefinite Programming

## Computational Intelligence, Lecture 7

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- Schur compliment
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- Volume of an ellipsoid
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# SEMIDEFINITE PROGRAMMING (SDP)

## General form

General form of a semidefinite program is:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{c}^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \mathbf{G} + \sum \mathbf{F}_i x_i \preceq 0, \\ \mathbf{A}\mathbf{x} = \mathbf{b}. \end{cases} \end{aligned} \tag{1}$$

where  $\mathbf{F}_i \succeq 0$  and  $\mathbf{G} \succeq 0$  (meaning they are positive semidefinite).

Constraint  $\mathbf{G} + \sum \mathbf{F}_i x_i \preceq 0$  is called *linear matrix inequality* or *LMI*.

# SEMIDEFINITE PROGRAMMING (SDP)

## Multiple LMI

SDP can have several LMIs. Assume you have:

$$\begin{cases} \mathbf{G} + \sum \mathbf{F}_i x_i \preceq 0 \\ \mathbf{D} + \sum \mathbf{H}_i x_i \preceq 0 \end{cases} \quad (2)$$

This is equivalent to:

$$\begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} + \sum \begin{bmatrix} \mathbf{F}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_i \end{bmatrix} x_i \preceq 0 \quad (3)$$

# SEMIDEFINITE PROGRAMMING (SDP)

## SDP decision variable

Sometimes it is easier to directly think of semidefinite matrices as of decision variables. This leads to programs with such formulation:

$$\begin{array}{ll} \underset{\mathbf{X}}{\text{minimize}} & \text{tr}(\mathbf{E}\mathbf{X}), \\ \text{subject to} & \begin{cases} \text{tr}(\mathbf{A}_i\mathbf{X}) = \mathbf{b}_i, \\ \mathbf{C}\mathbf{X} \preceq \mathbf{D}. \end{cases} \end{array} \quad (4)$$

where cost and constraints should adhere to SDP limitations.

Schur compliment. Given  $\mathbf{M}$

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{C} \end{bmatrix} \quad (5)$$

with full-rank  $\mathbf{A}$ , we can make the following statements:

- $\mathbf{M} \succ 0$  iff  $\mathbf{A} \succ 0$  and  $\mathbf{C} - \mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B} \succ 0$
- $\mathbf{A} \succ 0 \implies \mathbf{M} \succeq 0$  iff  $\mathbf{C} - \mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B} \succeq 0$

If  $\mathbf{C}$  is full-rank, we can make the following statements:

- $\mathbf{M} \succ 0$  iff  $\mathbf{C} \succ 0$  and  $\mathbf{A} - \mathbf{B} \mathbf{C}^{-1} \mathbf{B}^\top \succ 0$
- $\mathbf{C} \succ 0 \implies \mathbf{M} \succeq 0$  iff  $\mathbf{A} - \mathbf{B} \mathbf{C}^{-1} \mathbf{B}^\top \succeq 0$

Let us prove that SOCP is a sub-set of SDP. SOC constraint is:

$$\|\mathbf{Ax} + \mathbf{b}\| \leq \mathbf{c}^\top \mathbf{x} + d \quad (6)$$

where  $\mathbf{c}^\top \mathbf{x} + d \geq 0$ , and we can rewrite the SOC as:

$(\mathbf{Ax} + \mathbf{b})^\top (\mathbf{Ax} + \mathbf{b}) = (\mathbf{c}^\top \mathbf{x} + d)^2$ , and assuming  $\mathbf{c}^\top \mathbf{x} + d > 0$  we can write it as:

$$\frac{(\mathbf{Ax} + \mathbf{b})^\top (\mathbf{Ax} + \mathbf{b})}{\mathbf{c}^\top \mathbf{x} + d} \leq \mathbf{c}^\top \mathbf{x} + d \quad (7)$$

which is equivalent to:

$$-\frac{(\mathbf{Ax} + \mathbf{b})^\top (\mathbf{Ax} + \mathbf{b})}{-(\mathbf{c}^\top \mathbf{x} + d)} \leq \mathbf{c}^\top \mathbf{x} + d \quad (8)$$

Note that  $-\frac{(\mathbf{Ax}+\mathbf{b})^\top(\mathbf{Ax}+\mathbf{b})}{-(\mathbf{c}^\top\mathbf{x}+d)} \leq \mathbf{c}^\top\mathbf{x} + d$  is equivalent to:

$$\frac{(\mathbf{Ax} + \mathbf{b})^\top (\mathbf{Ax} + \mathbf{b})}{-(\mathbf{c}^\top \mathbf{x} + d)} + (\mathbf{c}^\top \mathbf{x} + d) \geq 0 \quad (9)$$

Using Schur we can re-write it as:

$$\begin{bmatrix} (\mathbf{c}^\top \mathbf{x} + d) & (\mathbf{Ax} + \mathbf{b}) \\ (\mathbf{Ax} + \mathbf{b})^\top & (\mathbf{c}^\top \mathbf{x} + d) \end{bmatrix} \succeq 0 \quad (10)$$

which is an LMI constraint.



Consider the following constraint, where  $\mathbf{X} \succeq 0$ :

$$\|\mathbf{X}\mathbf{v} + \mathbf{b}\| \leq \mathbf{c}^\top \mathbf{x} + d \quad (11)$$

Can we re-write it as an LMI? Using the same process as before we get:

$$\begin{bmatrix} (\mathbf{c}^\top \mathbf{x} + d) & (\mathbf{X}\mathbf{v} + \mathbf{b}) \\ (\mathbf{X}\mathbf{v} + \mathbf{b})^\top & (\mathbf{c}^\top \mathbf{x} + d) \end{bmatrix} \succeq 0 \quad (12)$$

So, (11) is an admissible constraint in an SDP.

Consider the problem: minimize the largest eigenvalue of  $A$ .  
The solution is:

$$\begin{aligned} & \underset{\mathbf{A}, t}{\text{minimize}} && t, \\ & \text{subject to} && \mathbf{A} \preceq t\mathbf{I} \end{aligned} \tag{13}$$

Proof. If  $\lambda$  is an eigenvalue of  $\mathbf{A}$ , then  $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ , hence  $(\mathbf{A} - t\mathbf{I})\mathbf{v} = (\lambda - t)\mathbf{v}$ , meaning  $\lambda - t$  is eigenvalue of  $(\mathbf{A} - t\mathbf{I})$ . Thus, if  $(\mathbf{A} - t\mathbf{I})$  is negative semi-definite, then  $\lambda - t \leq 0$  and  $\lambda \leq t$ . □

# CONTINUOUS LYAPUNOV EQ. AS SDP/LMI (1)

In control theory, Lyapunov equation is a condition of whether or not a continuous LTI system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  is stable:

$$\begin{cases} \mathbf{A}^\top \mathbf{P} + \mathbf{P}\mathbf{A} \preceq -\mathbf{Q} \\ \mathbf{P} \succeq 0 \end{cases} \quad (14)$$

where  $\mathbf{Q} \succeq 0$  is a constant and decision variable is  $\mathbf{P}$ . This can be represented as an SDP:

$$\begin{aligned} & \underset{\mathbf{P}}{\text{minimize}} && 0, \\ & \text{subject to} && \begin{cases} \mathbf{P} \succeq 0, \\ \mathbf{A}^\top \mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} \preceq 0. \end{cases} \end{aligned} \quad (15)$$

## CONTINUOUS LYAPUNOV EQ. AS SDP/LMI (2)

```
0 n = 7; A = randn(n, n) - 3*rand*eye(n);  
  Q = eye(n);  
2  
  cvx_begin sdp  
4      variable P(n, n) symmetric  
      minimize 0  
6      subject to  
          P >= 0;  
          A'*P + P*A + Q <= 0;  
8  cvx_end  
10  
  if strcmp(cvx_status, 'Solved')  
12      [eig(A), eig(A*P + P*A' + Q), eig(P)]  
  else  
14      eig(A)  
  end
```

In control theory, Discrete Lyapunov equation is a condition of whether or not a discrete LTI system  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$  is stabilizable:

$$\begin{cases} \mathbf{A}^\top \mathbf{P} \mathbf{A} - \mathbf{P} + \mathbf{Q} \preceq 0 \\ \mathbf{P} \succeq 0 \end{cases} \quad (16)$$

where  $\mathbf{Q} \succeq 0$  is a constant and decision variable is  $\mathbf{P}$ . This can be represented as an SDP:

$$\begin{aligned} & \underset{\mathbf{P}}{\text{minimize}} && 0, \\ & \text{subject to} && \begin{cases} \mathbf{P} \succeq 0, \\ \mathbf{A}^\top \mathbf{P} \mathbf{A} - \mathbf{P} + \mathbf{Q} \preceq 0. \end{cases} \end{aligned} \quad (17)$$

## DISCRETE LYAPUNOV EQ. AS SDP/LMI (2)

```
0 n = 7; A = 0.35*randn(n, n);  
  Q = eye(n);  
2  
  cvx_begin sdp  
4      variable P(n, n) symmetric  
      minimize 0  
6      subject to  
          P >= 0;  
          A'*P*A - P + Q <= 0;  
8      cvx_end  
10  
  if strcmp(cvx_status, 'Solved')  
12      [abs(eig(A)), eig(A'*P*A - P), eig(P)]  
  else  
14      abs(eig(A))  
  end
```

# HOW TO DESCRIBE AN ELLIPSOID

## Unit sphere transformation

Let us first remember how we describe a unit sphere:

$$\mathcal{S} = \{\mathbf{x} : \|\mathbf{x}\| \leq 1\} \quad (18)$$

An ellipsoid can be seen as a linear transformation of a unit sphere:

$$\mathcal{E} = \{\mathbf{Ax} + \mathbf{b} : \|\mathbf{x}\| \leq 1\} \quad (19)$$

# HOW TO DESCRIBE AN ELLIPSOID

## A dual description

Let us introduce a change of variables  $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{b}$ . Assuming  $\mathbf{A}$  is invertible, we get:

$$\mathbf{x} = \mathbf{A}^{-1}(\mathbf{z} - \mathbf{b}) \quad (20)$$

So, we can describe the exact same ellipsoid using an alternative formula:

$$\mathcal{E} = \{\mathbf{z} : \|\mathbf{B}\mathbf{z} + \mathbf{c}\| \leq 1\} \quad (21)$$

where  $\mathbf{B} = \mathbf{A}^{-1}$  and  $\mathbf{c} = -\mathbf{A}^{-1}\mathbf{b}$ .



# VOLUME OF AN ELLIPSOID (1)

For an ellipsoid of the form

$$\mathcal{E} = \{\mathbf{Ax} + \mathbf{b} : \|\mathbf{x}\| \leq 1\} \quad (22)$$

the "bigger" the  $\mathbf{A}$ , the bigger the ellipsoid. This concept can be made concrete by talking about the determinant of  $\mathbf{A}$ .

Thus, maximizing the volume of this ellipsoid is the same as maximizing  $\det(\mathbf{A})$ . Or, it is the same as *minimizing* the  $\det(\mathbf{A}^{-1})$ , since  $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$ .

Finally, note that  $\log \det(\mathbf{A})$  is a concave function and  $\log \det(\mathbf{A}^{-1})$  is a convex function.

For an ellipsoid of the form

$$\mathcal{E} = \{\mathbf{z} : \|\mathbf{B}\mathbf{z} + \mathbf{c}\| \leq 1\} \quad (23)$$

the "bigger" the  $\mathbf{B}$ , the *smaller* the ellipsoid. We can make it obvious by thinking that increasing  $\mathbf{B}$  leaves less room for valid  $\mathbf{z}$ , and it is the volume of valid  $\mathbf{z}$  that makes the volume of the ellipsoid in this case.

This concept can be made concrete by talking about the determinant of  $\mathbf{B}$ . Thus, maximizing the volume of this ellipsoid is the same as *minimizing*  $\det(\mathbf{B})$ . Or, it is the same as *maximizing* the  $\det(\mathbf{B}^{-1})$ .

Consider the problem: given V-polytope, defined by its vertices  $\mathbf{v}_i$ , find minimum-volume ellipsoid  $\mathcal{E}$  containing the polytope.

We will start with defining the ellipsoid as

$\mathcal{E} = \{\mathbf{z} : \|\mathbf{B}\mathbf{z} + \mathbf{c}\| \leq 1\}$ . The ellipsoid is smaller when  $\|\mathbf{B}\|$  is bigger, and thus we can write the minimization as minimizing  $\det(\mathbf{B}^{-1})$ .

$$\begin{aligned} & \underset{\mathbf{B}, \mathbf{c}}{\text{minimize}} && \log(\det(\mathbf{B}^{-1})), \\ & \text{subject to} && \begin{cases} \mathbf{B} \succeq 0, \\ \|\mathbf{B}\mathbf{v}_i + \mathbf{c}\| \leq 1. \end{cases} \end{aligned} \tag{24}$$

The solution gives us Löwner-John ellipsoid.

# MAX VOLUME INSCRIBED ELLIPSOID (1)

Consider the problem: given H-polytope, defined by its half-spaces  $\mathbf{a}_i^\top \mathbf{x} \leq b_i$ , find maximum-volume ellipsoid  $\mathcal{E}$  contained in the polytope. We will start with defining the ellipsoid as  $\mathcal{E} = \{\mathbf{C}\mathbf{x} + \mathbf{d} : \|\mathbf{x}\| \leq 1\}$ . The ellipsoid is larger when  $\|\mathbf{C}\|$  is bigger, and thus we can write the minimization as minimizing  $\det(\mathbf{C}^{-1})$ .

Let us write down the constraint requiring that  $\mathcal{E}$  lies in the polytope. We know that  $\mathbf{a}_i^\top (\mathbf{C}\mathbf{x} + \mathbf{d}) \leq b_i$  holds for all  $\|\mathbf{x}\| \leq 1$ . The worst-case scenario is when  $\mathbf{x}$  aligned with  $\mathbf{a}_i^\top \mathbf{C}$  and has length 1:

$$\mathbf{x} = \frac{\mathbf{a}_i^\top \mathbf{C}}{\|\mathbf{a}_i^\top \mathbf{C}\|} \quad (25)$$

Thus the constraint becomes

$$\|\mathbf{a}_i^\top \mathbf{C}\| + \mathbf{a}_i^\top \mathbf{d} \leq b_i \quad (26)$$

# MAX VOLUME INSCRIBED ELLIPSOID (2)

Here is the resulting problem:

$$\begin{array}{ll} \underset{\mathbf{C}, \mathbf{d}}{\text{minimize}} & \log(\det(\mathbf{C}^{-1})), \\ \text{subject to} & \begin{cases} \mathbf{C} \succeq 0, \\ \|\mathbf{a}_i^\top \mathbf{C}\| + \mathbf{a}_i^\top \mathbf{d} \leq b_i. \end{cases} \end{array} \quad (27)$$

The solution gives us inscribed (inner) Löwner-John ellipsoid.

Implement both examples from page 2 of the LMI CVX documents.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2023](https://github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2023)

