

# Barrier functions, Duality, Sensitivity

## Computational Intelligence, Lecture 13

by Sergei Savin

Spring 2022

- Barrier functions
- Analytic center of linear inequalities
- Lagrange dual function
- Duality gap, strong and weak duality
- Sensitivity
- Homework

# LINEAR INEQUALITIES

Consider linear inequality constraints:

$$\mathbf{Ax} \leq \mathbf{b} \quad (1)$$

Remember that we can rewrite it as:

$$\mathbf{a}_i^\top \mathbf{x} \leq b_i \quad (2)$$

$$\mathbf{a}_i^\top \mathbf{x} - b_i \leq 0 \quad (3)$$

Instead of *hard constraints* in (3) we can turn these into a cost function component:

$$J = - \sum_{i=1}^n \log(b_i - \mathbf{a}_i^\top \mathbf{x}) \quad (4)$$

Which is called a *barrier function*.

Let us consider barrier functions  $J = - \sum_{i=1}^n \log(b_i - \mathbf{a}_i^\top \mathbf{x})$ :

- It removes the constraint, but modifies the cost.
- When  $b_i - \mathbf{a}_i^\top \mathbf{x}$  is a very small positive number,  $\log(b_i - \mathbf{a}_i^\top \mathbf{x})$  is a very big negative number, hence the minus sign in front.
- Barrier function does not behave well outside of the domain, when  $b_i - \mathbf{a}_i^\top \mathbf{x} < 0$ .

Hence the following QP:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b}, \\ \mathbf{C}(\mathbf{x}) = \mathbf{d}. \end{cases} \end{aligned} \tag{5}$$

...can be approximated as:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x} - \sum_{i=1}^n \log(b_i - \mathbf{a}_i^\top \mathbf{x}), \\ & \text{subject to} && \mathbf{C}(\mathbf{x}) = \mathbf{d} \end{aligned} \tag{6}$$

We can define *analytic center of linear inequalities* as a minimum of the function  $J = - \sum_{i=1}^n \log(b_i - \mathbf{a}_i^\top \mathbf{x})$ . And that can be solved as a convex optimization:

$$\mathbf{x}_a = \underset{\mathbf{x}}{\operatorname{argmin}} \quad - \sum_{i=1}^n \log(b_i - \mathbf{a}_i^\top \mathbf{x})$$

At the analytic center of linear inequalities the shape of contour lines can be analysed as a local quadratic approximation of the function  $J$ :

$$\mathcal{C} = \{\mathbf{x} : (\mathbf{x} - \mathbf{x}_a)^\top \frac{\partial^2 J}{\partial \mathbf{x}^2} (\mathbf{x} - \mathbf{x}_a) = \epsilon\} \quad (7)$$

where  $\epsilon$  is a small number.

# ILLUSTRATION OF A BARRIER FUNCTIONS



Figure 1: Barrier functions

Pink is the domain. The ellipsoids represent the shape of the hessian  $\frac{\partial^2 J}{\partial \mathbf{x}^2}$  at different points on the domain. Green dot is  $\mathbf{x}_a$ .

Consider an optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & f_0(\mathbf{x}), \\ \text{subject to} & \begin{cases} f_i(\mathbf{x}) \leq 0, \\ h_j(\mathbf{x}) = 0. \end{cases} \end{array} \quad (8)$$

It's *Lagrangian* is given as:

$$L(\mathbf{x}, \lambda_i, \nu_j) = f_0(\mathbf{x}) + \sum_i \lambda_i f_i(\mathbf{x}) + \sum_j \nu_j h_j(\mathbf{x}) \quad (9)$$

where  $\lambda_i$  and  $\nu_j$  are Lagrange multipliers; they are sometimes called *dual variables*.



# LAGRANGE DUAL FUNCTION

Given *Lagrangian*  $L(\mathbf{x}, \lambda_i, \nu_j) = f_0(\mathbf{x}) + \sum_i \lambda_i f_i(\mathbf{x}) + \sum_j \nu_j h_j(\mathbf{x})$ ,  
the associated *Lagrange dual function* is given as:

$$g(\lambda_i, \nu_j) = \inf_{\mathbf{x}} L(\mathbf{x}, \lambda_i, \nu_j). \quad (10)$$

Lagrange dual function is always concave. If  $p^*$  is the optimal value of the cost function of the original problem, then  $g(\lambda_i, \nu_j)$  gives as a *lower bound* on its possible values. In fact, substituting any  $\nu_j$  and  $\lambda_i > 0$  gives us a valid lower bound on the cost. Maximum of  $g(\lambda_i, \nu_j)$  over the domain given by  $\lambda_i > 0$  provides us optimal (largest) lower bound of the problem, denoted as  $g^*$ .

# DUALITY GAP, STRONG AND WEAK DUALITY

If  $p^*$  is the optimal value of the cost function of the original problem and  $g^*$  is the optimal lower bound of the problem, then  $p^* - g^*$  is called optimal *duality gap*.

If optimal duality gap is zero, the problem is said to have *strong duality*. If optimal duality gap greater than zero, the problem is said to have *weak duality*.

# LAGRANGE DUAL FUNCTION FOR A QP, 1

Consider the following QP:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{H} \mathbf{x}, \\ & \text{subject to} && \mathbf{A} \mathbf{x} \leq \mathbf{b}. \end{aligned} \tag{11}$$

Its Lagrangian is:

$$L(\mathbf{x}, \lambda) = \mathbf{x}^\top \mathbf{H} \mathbf{x} + \lambda^\top (\mathbf{A} \mathbf{x} - \mathbf{b}) \tag{12}$$

In order to minimize the Lagrangian with respect to  $\mathbf{x}$  we find the gradient and set it to zero:

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = 2\mathbf{x}^\top \mathbf{H} + \lambda^\top \mathbf{A} = 0 \tag{13}$$

With that we can compute  $\mathbf{x}$  as a function of  $\lambda$ :

$$\mathbf{x} = -0.5\mathbf{H}^{-1}\mathbf{A}^\top \lambda \tag{14}$$

## LAGRANGE DUAL FUNCTION FOR A QP, 2

Knowing that  $\mathbf{x} = -0.5\mathbf{H}^{-1}\mathbf{A}^\top\lambda$  we can compute  $g(\lambda)$  by substituting the  $\mathbf{x}$  we found into the Lagrangian:

$$g(\lambda) = \frac{1}{4}\lambda^\top \mathbf{A}\mathbf{H}^{-1}\mathbf{H}\mathbf{H}^{-1}\mathbf{A}^\top\lambda - \frac{1}{2}\lambda^\top \mathbf{A}\mathbf{H}^{-1}\mathbf{A}^\top\lambda - \lambda^\top \mathbf{b} \quad (15)$$

$$g(\lambda) = -\frac{1}{4}\lambda^\top \mathbf{A}\mathbf{H}^{-1}\mathbf{A}^\top\lambda - \lambda^\top \mathbf{b} \quad (16)$$

In order to find the optimal lower bound we solve the following problem:

$$\begin{aligned} & \underset{\lambda}{\text{maximize}} && -\frac{1}{4}\lambda^\top \mathbf{A}\mathbf{H}^{-1}\mathbf{A}^\top\lambda - \lambda^\top \mathbf{b}, \\ & \text{subject to} && \lambda \geq 0. \end{aligned} \quad (17)$$

Note that optimal values of  $\lambda$  determine local sensitivity of the system with respect to small perturbations of constraints.

## EXAMPLE, SENSITIVITY

Consider minimizing  $(\mathbf{x} - \mathbf{c})^\top (\mathbf{x} - \mathbf{c})$  when the domain is the second quadrant:  $x_1 \geq 0$  and  $x_2 \leq 0$ . Find sensitivity of the problem as a function of  $\mathbf{c}$ .

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & (\mathbf{x} - \mathbf{c})^\top (\mathbf{x} - \mathbf{c}), \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq 0. \end{array} \quad (18)$$

where  $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ .

The dual Lagrange function is:

$$g(\lambda) = -\frac{1}{4}\lambda^\top \mathbf{A}\mathbf{A}^\top \lambda + \lambda^\top \mathbf{A}\mathbf{c} \quad (19)$$

# EXAMPLE, ILLUSTRATION OF THE SENSITIVITY



Figure 2: Sensitivity

Turquoise on the left is the domain. The arrows on the right show the values of  $\lambda$ .

# HOMEWORK

Visualize contours of a quadratic program of your choice.  
Compute its optimal lower bound and duality gap.

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022](https://github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022)



Check Moodle for additional links, videos, textbook suggestions.