Quadratically constrained quadratic programming, Second-order cone programming

Computational Intelligence, Lecture 6

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Remember the general form of a quadratic program:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{1}$$

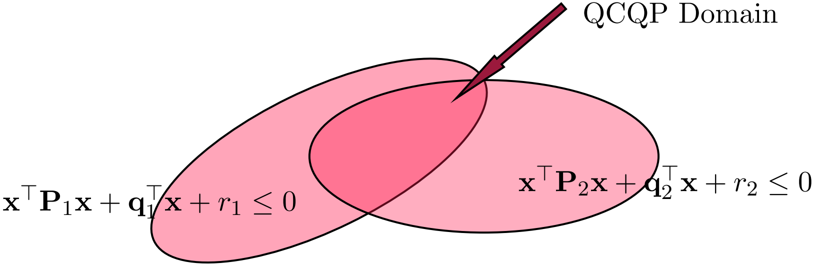
where $\mathbf{H}$ is positive-definite and $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ describe a convex region.

General form of a quadratically constrained quadratic program (QCQP) is given below:

$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^\top \mathbf{x}, \\
\text{subject to} \quad & \begin{cases} \mathbf{x}^\top \mathbf{P}_i \mathbf{x} + \mathbf{q}_i^\top \mathbf{x} + r_i \leq 0, \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases}
\end{aligned} \tag{2}
$$

where $\mathbf{P}_i$ are positive-definite.

Domain of a QCQP without equality constraints and with no degenerate inequality constraints is an intersection of ellipses:



Set $\mathbf{P}_i = \mathbf{0}$ and you get a QP.

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \begin{bmatrix} \mathbf{q}_1^\top \\ ... \\ \mathbf{q}_n^\top \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} -r_1 \\ ... \\ -r_n \end{bmatrix} \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{3}$$

Set $\mathbf{P}_0 = \mathbf{0}$ and you get an LP.

Turning ellipsoid to the canonical form (1)

Can we re-write the expression $\mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r \leq 0$ as a canonical form ellipsoid:

$$\frac{z_1^2}{m_1^2} + \frac{z_2^2}{m_2^2} + ... + \frac{z_n^2}{m_n^2} \leq 1 \tag{4}$$

We start by proposing a substitution $\mathbf{x}_0 = -\frac{1}{2}\mathbf{P}^{-1}\mathbf{q}$ and $-d = r - \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0$. We can prove that:
$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P} (\mathbf{x} - \mathbf{x}_0) - d^2 = \mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P} (\mathbf{x} - \mathbf{x}_0) - d^2 = \tag{5}$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} - 2\mathbf{x}_0^\top \mathbf{P} \mathbf{x} + \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 - d^2 = \tag{6}$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} + 2\left(\frac{1}{2}\mathbf{P}^{-1}\mathbf{q}\right)^\top \mathbf{P} \mathbf{x} + \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 + r - \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 = \tag{7}$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r. \quad \square \tag{8}$$

Turning ellipsoid to the canonical form (1)

Thus our original expression became:

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P} (\mathbf{x} - \mathbf{x}_0) - d^2 \leq 0 \tag{9}$$

We define $\mathbf{A} = \sqrt{\mathbf{P}}$ and give its SVD $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$. Defining $\mathbf{z} = \mathbf{V}^\top(\mathbf{x} - \mathbf{x}_0)$ we get:

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{A}^\top \mathbf{A} (\mathbf{x} - \mathbf{x}_0) - d^2 \leq 0 \tag{10}$$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{V}\Sigma\mathbf{U}^\top \mathbf{U}\Sigma\mathbf{V}^\top (\mathbf{x} - \mathbf{x}_0) - d^2 \leq 0 \tag{11}$$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{V}\Sigma^2\mathbf{V}^\top (\mathbf{x} - \mathbf{x}_0) - d^2 \leq 0 \tag{12}$$

$$\mathbf{z}^\top \Sigma^2 \mathbf{z} - d^2 \leq 0 \tag{13}$$

$$\sum z_i^2 \sigma_i^2 \leq d^2 \tag{14}$$

Defining $1/m_i^2 = \sigma_i^2/d^2$ we get:

$$\frac{z_1^2}{m_1^2} + \frac{z_2^2}{m_2^2} + ... + \frac{z_n^2}{m_n^2} \leq 1 \tag{15}$$

The general form of a Second-order cone program (SOCP) is:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{f}^\top \mathbf{x}, \qquad \text{subject to} \quad \begin{cases} ||\mathbf{A}_i \mathbf{x} + \mathbf{b}_i||_2 \leq \mathbf{c}_i^\top \mathbf{x} + d_i, \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \tag{16}$$

LP, QP and QCQP are subsets of SOCP.

Consider the following SOC constraint:

$$||\mathbf{A}\mathbf{x} + \mathbf{b}||_2 \leq \mathbf{c}^\top \mathbf{x} + d \tag{17}$$

Let us consider a special case when $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{(n-1)\times n}$ and $\text{rank}\left(\begin{bmatrix}\mathbf{A} \\ \mathbf{c}^\top\end{bmatrix}\right) = n$. Then we can introduce the following substitution:

$$\xi = \begin{bmatrix}\mathbf{A} \\ \mathbf{c}^\top\end{bmatrix}\mathbf{x} + \begin{bmatrix}\mathbf{b} \\ d\end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix}\mathbf{E} \\ \mathbf{e}^\top\end{bmatrix} \tag{18}$$

where $\mathbf{I} \in \mathbb{R}^{n,n}$ is an identity matrix. Then constraint (17) becomes:

$$||\mathbf{E}\xi||_2 \leq \mathbf{e}^\top \xi \tag{19}$$

SOC constraints, 2

Notice that $||\mathbf{E}\xi||_2 \leq \mathbf{e}^\top \xi$ is equivalent to:

$$\sum_{i=1}^{n-1} \xi_i^2 \leq \xi_n^2 \tag{20}$$

which is a standard form of a cone. A map back from ξ to $\mathbf{x}$ is given as:

$$\mathbf{x} = \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix}^{-1} \left(\xi - \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix} \right) \tag{21}$$

We can write problem where our domain is a ball as SOCP:

$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\
\text{subject to} \quad & ||\mathbf{x}||_2 \leq d_i
\end{aligned}
\tag{22}
$$

Same for ellipsoidal constraints:

$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\
\text{subject to} \quad & ||\mathbf{A}_i \mathbf{x}||_2 \leq d_i
\end{aligned}
\tag{23}
$$

Set $\mathbf{c}_i = 0$ and recognize that $||\mathbf{A}_i\mathbf{x} + \mathbf{b}_i||_2 \leq d_i$ is the same as $(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i)^\top(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i) \leq d_i^2$ (since the first implies that d_i is non-negative).

$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top\mathbf{x}, \\
\text{subject to} \quad & \begin{cases} \mathbf{x}^\top\mathbf{A}_i^\top\mathbf{A}_i\mathbf{x} + 2\mathbf{b}_i^\top\mathbf{A}_i\mathbf{x} + \mathbf{b}_i^\top\mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases}
\end{aligned} \tag{24}
$$

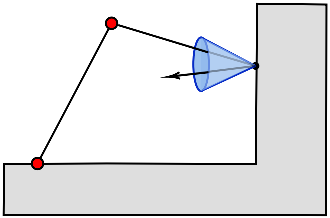
Now to make the cost quadratic:

$$\begin{aligned} \underset{\mathbf{x},t}{\text{minimize}} \quad & t, \\ \text{subject to} \quad & \begin{cases} \mathbf{x}^\top \mathbf{A}_0^\top \mathbf{A}_0 \mathbf{x} + 2\mathbf{b}_0^\top \mathbf{A}_0 \mathbf{x} + \mathbf{b}_0^\top \mathbf{b}_0 \leq t \\ \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{25}$$

Which is the same as:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{26}$$

As long as $\mathbf{A}_0 = \sqrt{\mathbf{H}}$, and $\mathbf{b}_0 = 0.5\mathbf{A}_0^{-1}\mathbf{f}$.



Let $\mathbf{f}$ be total reaction force, $\mathbf{f}_n$ be its normal component (perpendicular to the surface locally), also known as normal reaction; and let $\mathbf{f}_{fr}$ be its tangential component (a vector lying in the tangent plane to the surface, constructed at the contact point), or friction force. Let $\mathbf{e}_n$ be a unit vector, normal to the surface at the point of contact.

$$\mathbf{f} = \mathbf{f}_n + \mathbf{f}_{fr} \tag{27}$$

Defining $\mathbf{E}_t = [\mathbf{e}_{t,1},\ \mathbf{e}_{t,2}] = \text{null}(\mathbf{e}_n^\top)$ be an orthonormal basis in the tangential space to the surface, we can write:

$$\mathbf{f} = \mathbf{e}_n n + \mathbf{E}_t \mathbf{t}$$
$$\mathbf{f}_n = \mathbf{e}_n n$$
$$\mathbf{f}_{fr} = \mathbf{E}_t \mathbf{t}$$
$$\mathbf{t} = [t_1,\ t_2]$$

The friction cone conditions could be written in any of the following ways:

$$\sqrt{t_1^2 + t_2^2} < \mu n \tag{28}$$

$$||\mathbf{E}_t^\top \mathbf{f}|| \leq \mu \mathbf{e}_n^\top \mathbf{f} \tag{29}$$

where μ is a friction coefficient.

Implement a program that finds right-most point of an intersection of two ellipsoids; visualise the problem and the solution.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2023