

# Minimax

## Computational Intelligence, Lecture 10

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Spring 2022

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# MINIMAX PROBLEMS

## Example

Consider the following problem:

### Example

Find smallest  $x \in \mathbb{R}$ , such that  $x + y \geq 1$ , where  $|y| \leq 2$ .

In that example we need to find optimal value of  $x$  subject to a constraint where another unknown variable is present; the solution we find has to satisfy the constraint for any allowed value of  $y$ . The solution here is  $x = 3$

This is closely related to *minimax optimization*

# MINIMAX: LINEAR CONSTRAINT

## Part 1

Consider the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \max_{\mathbf{y}} \quad ||\mathbf{x}||, \\ \text{subject to} \quad & \mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{1}$$

It is clear that worst-case scenario corresponds to the largest value of  $\mathbf{d}^\top \mathbf{y}$ , meaning that  $\mathbf{y}$  should align with  $\mathbf{d}$  and have its maximum possible length  $p$ . From that we conclude that  $\mathbf{y} = p \frac{\mathbf{d}}{||\mathbf{d}||}$ .

# MINIMAX: LINEAR CONSTRAINT

## Part 2

Therefor  $\mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \leq h$  becomes:

$$\mathbf{c}^\top \mathbf{x} + p \frac{\mathbf{d}^\top \mathbf{d}}{\|\mathbf{d}\|} \leq h \quad (2)$$

$$\mathbf{c}^\top \mathbf{x} + p \|\mathbf{d}\| \leq h \quad (3)$$

Thus our problem becomes:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x}\|, \\ \text{subject to} \quad & \mathbf{c}^\top \mathbf{x} \leq h - p \|\mathbf{d}\| \end{aligned} \quad (4)$$

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 1

## Part 1

Consider the following problem, where  $\mathbf{x}^*$  is the desired value of  $\mathbf{x}$ :

$$\begin{aligned} \min_{\mathbf{x}} \quad & \min_{\mathbf{y}} \quad ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & \mathbf{y}^\top \mathbf{D}\mathbf{x} \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{5}$$

This time worst-case scenario corresponds to  $\mathbf{y}$  aligned with  $\mathbf{D}\mathbf{x}$  and having its maximum possible length  $p$ . From that we conclude that  $\mathbf{y} = p \frac{\mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||}$ . Let us substitute it to  $\mathbf{y}^\top \mathbf{D}\mathbf{x}$ :

$$p \left( \frac{\mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||} \right)^\top \mathbf{D}\mathbf{x} = p \frac{\mathbf{x}^\top \mathbf{D}^\top \mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||} = p \frac{||\mathbf{D}\mathbf{x}||^2}{||\mathbf{D}\mathbf{x}||} = p ||\mathbf{D}\mathbf{x}|| \tag{6}$$

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 1

## Part 2

Thus our problem becomes:

$$\begin{array}{ll} \min_{\mathbf{x}} & \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} & \|\mathbf{D}\mathbf{x}\| \leq \frac{h}{p} \end{array} \quad (7)$$

which is an SOCP.

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 2

## Part 1

A more general case of the previous problem is:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \max_{\mathbf{y}} \quad ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{8}$$

We can rewrite  $(\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h$  as:

$$\mathbf{y}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{9}$$

With that we see that the worse case scenario is  $\mathbf{y}$  is aligned with  $\mathbf{D}(\mathbf{x} - \mathbf{b})$  and has length  $p$ :

$$\mathbf{y} = p \frac{\mathbf{D}(\mathbf{x} - \mathbf{b})}{||\mathbf{D}(\mathbf{x} - \mathbf{b})||} \tag{10}$$



# MINIMAX: QUADRATIC CONSTRAINT, TYPE 2

## Part 2

Then  $\mathbf{y}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h$  becomes:

$$p \frac{(\mathbf{x} - \mathbf{b})^\top \mathbf{D}^\top \mathbf{D}(\mathbf{x} - \mathbf{b})}{\|\mathbf{D}(\mathbf{x} - \mathbf{b})\|} - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \quad (11)$$

which is the same as:

$$p \|\mathbf{D}(\mathbf{x} - \mathbf{b})\| - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \quad (12)$$

$$\|\mathbf{D}(\mathbf{x} - \mathbf{b})\| \leq \frac{1}{p} \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \frac{h}{p} \quad (13)$$

which is an SOCP constraint.

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 2

## Part 2

And thus we get:

$$\begin{array}{ll} \min_{\mathbf{x}} & \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} & \|\mathbf{D}(\mathbf{x} - \mathbf{b})\| \leq \frac{1}{p} \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \frac{h}{p} \end{array} \quad (14)$$

which is SOCP.

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 3

## Part 1

A more general case of the previous problem is:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \max_{\mathbf{y}} \quad ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h, \\ & ||\mathbf{H}\mathbf{y} + \mathbf{f}|| \leq p \end{aligned} \tag{15}$$

where  $\mathbf{H}$  is has an inverse. We start by making substitution:

$$\mathbf{v} = \mathbf{H}\mathbf{y} + \mathbf{f} \tag{16}$$

meaning  $\mathbf{y} = \mathbf{H}^{-1}(\mathbf{v} - \mathbf{f})$ :

$$(\mathbf{H}^{-1}(\mathbf{v} - \mathbf{f}) - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{17}$$

$$\mathbf{v}^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) - (\mathbf{H}^{-1}\mathbf{f} + \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{18}$$

$$\mathbf{v}^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) - (\mathbf{H}\mathbf{a} + \mathbf{f})^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{19}$$

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 3

## Part 2

We can introduce notation:

$$\mathbf{M} = \mathbf{H}^{-\top} \mathbf{D} \quad (20)$$

$$\mathbf{g} = \mathbf{H}\mathbf{a} + \mathbf{f} \quad (21)$$

With that we can re-write our constraint:

$$\mathbf{v}^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) - \mathbf{g}^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h \quad (22)$$

$$(\mathbf{v} - \mathbf{g})^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h \quad (23)$$

And now we formulated type 3 problem as type 2:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \max_{\mathbf{v}} \quad \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} \quad & (\mathbf{v} - \mathbf{g})^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h, \\ & \|\mathbf{v}\| \leq p \end{aligned} \quad (24)$$

# MINIMAX: QUADRATIC CONSTRAINT, TYPE 4

Try solving this problem on your own:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \mathbf{s}^\top \mathbf{y} + \mathbf{q}^\top \mathbf{x} \leq h, \\ & ||\mathbf{H}\mathbf{y} + \mathbf{f}|| \leq p \end{aligned} \tag{25}$$

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022](https://github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022)



Check Moodle for additional links, videos, textbook suggestions.

# MAX OVER NORM OF A SUM OF VECTORS

Consider the problem  $\max_{\mathbf{x}}(\|\mathbf{a} + \mathbf{x}\|)$  and  $\|\mathbf{x}\| \leq p$ . Let us open the norm:

$$\|\mathbf{a} + \mathbf{x}\| = \sqrt{(\mathbf{a} + \mathbf{x})^\top (\mathbf{a} + \mathbf{x})} = \mathbf{a}^\top \mathbf{a} + 2\mathbf{a}^\top \mathbf{x} + \mathbf{x}^\top \mathbf{x} \quad (26)$$

If  $\mathbf{a}$  is a constant, then the expression  $\mathbf{a}^\top \mathbf{a} + 2\mathbf{a}^\top \mathbf{x} + \mathbf{x}^\top \mathbf{x}$  attains a maximum when  $\mathbf{x}^\top \mathbf{x} = p^2$  and  $\mathbf{a}^\top \mathbf{x} = \|\mathbf{a}\|p$ . This implies that:

$$\mathbf{x} = p \frac{\mathbf{a}}{\|\mathbf{a}\|} \quad (27)$$

And thus:

$$\max_{\mathbf{x}}(\|\mathbf{a} + \mathbf{x}\|) = \|\mathbf{a} + p \frac{\mathbf{a}}{\|\mathbf{a}\|}\| = \|\mathbf{a}\| \left(1 + \frac{p}{\|\mathbf{a}\|}\right) = \|\mathbf{a}\| + p$$

# MINIMAX: CONIC CONSTRAINT

## Part 1

Consider the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} \quad & \|\mathbf{Ax} + \mathbf{b} + \mathbf{y}\| \leq \mathbf{c}^\top \mathbf{x} + d, \\ & \|\mathbf{y}\| \leq p \end{aligned} \tag{28}$$

As we discussed in the last slide, expression  $\|\mathbf{Ax} + \mathbf{b} + \mathbf{y}\|$  is biggest when  $\mathbf{y} = p \frac{\mathbf{Ax} + \mathbf{b}}{\|\mathbf{Ax} + \mathbf{b}\|}$ , and the conic constraint becomes:

$$\|\mathbf{Ax} + \mathbf{b} + p \frac{\mathbf{Ax} + \mathbf{b}}{\|\mathbf{Ax} + \mathbf{b}\|}\| \leq \mathbf{c}^\top \mathbf{x} + d \tag{29}$$

$$\|\mathbf{Ax} + \mathbf{b}\| \left(1 + \frac{p}{\|\mathbf{Ax} + \mathbf{b}\|}\right) \leq \mathbf{c}^\top \mathbf{x} + d \tag{30}$$



# MINIMAX: CONIC CONSTRAINT

## Part 2

Continue the derivation:

$$\|\mathbf{Ax} + \mathbf{b}\| \left( 1 + \frac{p}{\|\mathbf{Ax} + \mathbf{b}\|} \right) \leq \mathbf{c}^\top \mathbf{x} + d \quad (31)$$

$$\|\mathbf{Ax} + \mathbf{b}\| + p \leq \mathbf{c}^\top \mathbf{x} + d \quad (32)$$

Finally the problem becomes:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} \quad & \|\mathbf{Ax} + \mathbf{b}\| \leq \mathbf{c}^\top \mathbf{x} + d - p \end{aligned} \quad (33)$$