# Linear Programming Computational Intelligence, Lecture 8

by Sergei Savin

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### LINEAR PROGRAMMING General form

A linear program (LP) is an optimization problem of the form:

minimize 
$$\mathbf{f}^{\top}\mathbf{x}$$
,  
subject to 
$$\begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ \mathbf{C}\mathbf{x} = \mathbf{d}. \end{cases}$$
 (1)

It is one of the older and widely used classes of convex optimization problems.

Note that the solution of such problem will always lie on the boundary of its domain.

### LINEAR PROGRAMMING

#### LP with no solution - examples

Here are some examples of LP which have no solutions:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

This one is has no boundaries at all, hence no solution. Next one has boundaries, but they do not restrict motion along the descent direction for the cost function.

minimize 
$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
,  
subject to  $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le 1$  (3)

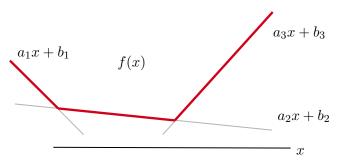
#### CONVEX PIECE-WISE LINEAR FUNCTIONS

#### Problem statement

Convex piece-wise linear functions have the form:

$$f(\mathbf{x}) = \max(\mathbf{a}_i^{\mathsf{T}} \mathbf{x} + b_i) \tag{4}$$

Figure below shows geometric interpretation of such function for a one-dimensional case.



# CONVEX PIECE-WISE LINEAR FUNCTIONS Solution as LP

We can formulate a minimization problem using convex piece-wise linear functions:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \max(\mathbf{a}_i^{\top} \mathbf{x} + b_i) \tag{5}$$

Which can be equivalently transformed into the following LP:

minimize 
$$t$$
subject to  $\mathbf{a}_i^{\top} \mathbf{x} + b_i \leq t$  (6)

We can observe that optimal (minimal) t will have to lie on one of the linear functions  $\mathbf{a}_i^{\mathsf{T}}\mathbf{x} + b_i$ , i.e. on the original piece-wise linear function  $f(\mathbf{x})$ . And optimal value on t corresponds to the smallest value of the original function  $f(\mathbf{x})$ .

## SUM OF PIECE-WISE LINEAR FUNCTIONS Solution as LP

Sum of convex piece-wise linear functions have the form:

$$f(\mathbf{x}) + g(\mathbf{x}) = \max(\mathbf{a}_i^{\mathsf{T}} \mathbf{x} + b_i) + \max(\mathbf{c}_i^{\mathsf{T}} \mathbf{x} + d_i)$$
 (7)

Their representation as LP is:

minimize 
$$t_1 + t_2$$
  
subject to 
$$\begin{cases} \mathbf{a}_i^\top \mathbf{x} + b_i \le t_1 \\ \mathbf{c}_i^\top \mathbf{x} + d_i \le t_2 \end{cases}$$
(8)

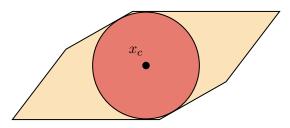
## CONVEX PIECE-WISE LINEAR FUNCTIONS Code

```
o func = @(t) t^2:
  derivative\_func = @(t) 2*t;
  approx_points = [-1, -0.3, 0, 0.3, 1];
4|n = length (approx_points);
  a = zeros(n, 1);
6 \mid b = zeros(n, 1);
s \mid for i = 1:n
      t = approx_points(i);
      a(i) = derivative\_func(t);
       b(i) = func(t) - a(i)*t ;
12 end
|14| f = [1; 0];
  \lim_{A} A = [-\operatorname{ones}(n, 1), a];
16 | \lim_{b \to b} = -b;
  x = linprog(f, lin_A, lin_b, [], []);
```

#### CHEBYSHEV CENTER OF A POLYHEDRON

#### Problem statement

Chebyshev center of a polyhedron is the center of the largest ball inscribed in a polyhedron:



Equation describing this ball can be written as:

$$\mathcal{B} = \{ \mathbf{x}_c + \mathbf{u} : ||\mathbf{u}||_2 \le r \}$$
 (9)

where r is the radius of the ball and  $\mathbf{x}_c$  is its center.

### CHEBYSHEV CENTER OF A POLYHEDRON

#### Max over the dot product

Before we move towards solving the problem, let us consider the following maximization:

$$\sup\{\mathbf{a}^{\top}\mathbf{u}: ||\mathbf{u}||_2 \le r\} \tag{10}$$

We can re-write the expression:

$$\sup\{\mathbf{a}^{\top}\mathbf{u}: ||\mathbf{u}||_{2} \le r\} = \sup\{||\mathbf{a}|| \cdot ||\mathbf{u}||\cos(\varphi): ||\mathbf{u}||_{2} \le r\}$$
(11)

where  $\varphi$  is the angle between **a** and **u**. Since **a** is constant,  $\max(||\mathbf{u}||) = r$ , and  $\max(\cos(\varphi)) = 1$ , we get:

$$\sup\{\mathbf{a}^{\mathsf{T}}\mathbf{u}: ||\mathbf{u}||_2 \le r\} = ||\mathbf{a}||r \tag{12}$$

## CHEBYSHEV CENTER OF A POLYHEDRON Solution as LP, part one

For the ball  $\mathcal{B}$  to be inscribed in a polygon  $\mathcal{P} = \{\mathbf{x}: \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ , the following should hold:

$$\sup\{\mathbf{a}_i^{\top}(\mathbf{x}_c + \mathbf{u}) : ||\mathbf{u}||_2 \le r\} \le b_i$$
 (13)

Note that the largest value of  $\mathbf{a}_i^{\top}\mathbf{u}$  under condition  $||\mathbf{u}||_2 \leq r$  is  $r||\mathbf{a}_i||$ : it can indeed achieve this value if  $\mathbf{a}_i$  and  $\mathbf{u}$  are co-directional, but a larger one is not possible. Therefore:

$$\sup\{\mathbf{a}_i^{\top}(\mathbf{x}_c + \mathbf{u}): ||\mathbf{u}||_2 \le r\} = \mathbf{a}_i^{\top}\mathbf{x}_c + r||\mathbf{a}_i|| \le b_i$$
 (14)

### CHEBYSHEV CENTER OF A POLYHEDRON Solution as LP, part two

Finally, we can write down the solution of the problem as a linear optimization:

maximize 
$$r$$
 $r, \mathbf{x}_c$ 
subject to  $\mathbf{a}_i^{\top} \mathbf{x}_c + r||\mathbf{a}_i|| \leq b_i$  (15)

# CHEBYSHEV CENTER OF A POLYHEDRON Code

Below we can see MATLAB code for solving the problem:

```
V = randn(10, 2);
_{2}|_{k} = convhull(V);
 |P = V(k, :);
  [domain_A, domain_b] = vert2con(P);
6 norm_A = vecnorm (domain_A');
| \mathbf{s} | \mathbf{f} = [-1; 0; 0];
 A = [reshape(norm_A, [], 1), domain_A];
b = domain_b;
|x| = \lim_{h \to 0} (f, A, b, [], []);
14 center = [x(2), x(3)];
  r = x(1);
```

### LINEAR-FRACTIONAL PROGRAMMING

#### Formulation

The following is the Linear-Fractional Programming problem:

maximize 
$$\frac{\mathbf{c}^{\top}\mathbf{x} + d}{\mathbf{e}^{\top}\mathbf{x} + f}$$
subject to 
$$\begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathbf{A}_{e}\mathbf{x} = \mathbf{b}_{e} \end{cases}$$
 (16)

This doesn't look like an LP, but let us see if we can try to bring this problem into this form.

# LINEAR-FRACTIONAL PROGRAMMING LP form

The following is the Linear-Fractional Programming problem in LP form:

maximize 
$$\mathbf{c}^{\top}\mathbf{y} + zd$$
subject to
$$\begin{cases}
\mathbf{A}\mathbf{y} \leq z\mathbf{b} \\
\mathbf{A}_{e}\mathbf{y} = z\mathbf{b}_{e} \\
\mathbf{e}^{\top}\mathbf{y} + zf = 1 \\
z \geq 0
\end{cases}$$
(17)

Here the variables  $\mathbf{y}$  and z are related to  $\mathbf{x}$  as follows.

$$\mathbf{y} = \frac{\mathbf{x}}{\mathbf{e}^{\mathsf{T}}\mathbf{x} + f} \tag{18}$$

$$\mathbf{z} = \frac{1}{\mathbf{e}^{\top} \mathbf{x} + f} \tag{19}$$

### LINEAR-FRACTIONAL PROGRAMMING details

We assumed that the domain of the previous problem is limited to  $\mathbf{e}^{\top}\mathbf{x} + f \geq 0$ . With that we have:

$$\mathbf{c}^{\top}\mathbf{y} + zd = \mathbf{c}^{\top}\frac{\mathbf{x}}{\mathbf{e}^{\top}\mathbf{x} + f} + \frac{1}{\mathbf{e}^{\top}\mathbf{x} + f}d = \frac{\mathbf{c}^{\top}\mathbf{x} + d}{\mathbf{e}^{\top}\mathbf{x} + f}$$
(20)

$$\mathbf{A}\mathbf{y} \le z\mathbf{b} \implies \mathbf{A}\frac{\mathbf{x}}{\mathbf{e}^{\top}\mathbf{x} + f} \le \frac{1}{\mathbf{e}^{\top}\mathbf{x} + f}\mathbf{b} \implies \mathbf{A}\mathbf{x} \le \mathbf{b}$$
 (21)

### Homework

Implement linear approximation of a convex function and solve it as LP

Lecture slides are available via Moodle.

 $You\ can\ help\ improve\ these\ slides\ at:$  github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022



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