

Minimax

Computational Intelligence, Lecture 10

by Sergei Savin

Spring 2022

- MiniMax problems
- MiniMax: Linear constraint
- MiniMax: Quadratic constraint
- Homework

MINIMAX PROBLEMS

Example

Consider the following problem:

Example

Find smallest $x \in \mathbb{R}$, such that $x + y \geq 1$, where $|y| \leq 2$.

In that example we need to find optimal value of x subject to a constraint where another unknown variable is present; the solution we find has to satisfy the constraint for any allowed value of y . The solution here is $x = 3$

This is closely related to *minimax optimization*

MINIMAX: LINEAR CONSTRAINT

Part 1

Consider the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \max_{\mathbf{y}} \quad ||\mathbf{x}||, \\ \text{subject to} \quad & \mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{1}$$

It is clear that worst-case scenario corresponds to the largest value of $\mathbf{d}^\top \mathbf{y}$, meaning that \mathbf{y} should align with \mathbf{d} and have its maximum possible length p . From that we conclude that $\mathbf{y} = p \frac{\mathbf{d}}{||\mathbf{d}||}$.

MINIMAX: LINEAR CONSTRAINT

Part 2

Therefor $\mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \leq h$ becomes:

$$\mathbf{c}^\top \mathbf{x} + p \frac{\mathbf{d}^\top \mathbf{d}}{\|\mathbf{d}\|} \leq h \quad (2)$$

$$\mathbf{c}^\top \mathbf{x} + p \|\mathbf{d}\| \leq h \quad (3)$$

Thus our problem becomes:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x}\|, \\ \text{subject to} \quad & \mathbf{c}^\top \mathbf{x} \leq h - p \|\mathbf{d}\| \end{aligned} \quad (4)$$

MINIMAX: QUADRATIC CONSTRAINT, TYPE 1

Part 1

Consider the following problem, where \mathbf{x}^* is the desired value of \mathbf{x} :

$$\begin{aligned} \min_{\mathbf{x}} \quad & \min_{\mathbf{y}} \quad ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & \mathbf{y}^\top \mathbf{D}\mathbf{x} \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{5}$$

This time worst-case scenario corresponds to \mathbf{y} aligned with $\mathbf{D}\mathbf{x}$ and having its maximum possible length p . From that we conclude that $\mathbf{y} = p \frac{\mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||}$. Let us substitute it to $\mathbf{y}^\top \mathbf{D}\mathbf{x}$:

$$p \left(\frac{\mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||} \right)^\top \mathbf{D}\mathbf{x} = p \frac{\mathbf{x}^\top \mathbf{D}^\top \mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||} = p \frac{||\mathbf{D}\mathbf{x}||^2}{||\mathbf{D}\mathbf{x}||} = p ||\mathbf{D}\mathbf{x}|| \tag{6}$$

MINIMAX: QUADRATIC CONSTRAINT, TYPE 1

Part 2

Thus our problem becomes:

$$\begin{array}{ll} \min_{\mathbf{x}} & \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} & \|\mathbf{D}\mathbf{x}\| \leq \frac{h}{p} \end{array} \quad (7)$$

which is an SOCP.

MINIMAX: QUADRATIC CONSTRAINT, TYPE 2

Part 1

A more general case of the previous problem is:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \max_{\mathbf{y}} \quad ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{8}$$

We can rewrite $(\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h$ as:

$$\mathbf{y}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{9}$$

With that we see that the worse case scenario is \mathbf{y} is aligned with $\mathbf{D}(\mathbf{x} - \mathbf{b})$ and has length p :

$$\mathbf{y} = p \frac{\mathbf{D}(\mathbf{x} - \mathbf{b})}{||\mathbf{D}(\mathbf{x} - \mathbf{b})||} \tag{10}$$

MINIMAX: QUADRATIC CONSTRAINT, TYPE 2

Part 2

Then $\mathbf{y}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h$ becomes:

$$p \frac{(\mathbf{x} - \mathbf{b})^\top \mathbf{D}^\top \mathbf{D}(\mathbf{x} - \mathbf{b})}{\|\mathbf{D}(\mathbf{x} - \mathbf{b})\|} - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \quad (11)$$

which is the same as:

$$p \|\mathbf{D}(\mathbf{x} - \mathbf{b})\| - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \quad (12)$$

$$\|\mathbf{D}(\mathbf{x} - \mathbf{b})\| \leq \frac{1}{p} \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \frac{h}{p} \quad (13)$$

which is an SOCP constraint.

MINIMAX: QUADRATIC CONSTRAINT, TYPE 2

Part 2

And thus we get:

$$\begin{array}{ll} \min_{\mathbf{x}} & \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} & \|\mathbf{D}(\mathbf{x} - \mathbf{b})\| \leq \frac{1}{p} \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \frac{h}{p} \end{array} \quad (14)$$

which is SOCP.

MINIMAX: QUADRATIC CONSTRAINT, TYPE 3

Part 1

A more general case of the previous problem is:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \max_{\mathbf{y}} \quad ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h, \\ & ||\mathbf{H}\mathbf{y} + \mathbf{f}|| \leq p \end{aligned} \tag{15}$$

where \mathbf{H} is has an inverse. We start by making substitution:

$$\mathbf{v} = \mathbf{H}\mathbf{y} + \mathbf{f} \tag{16}$$

meaning $\mathbf{y} = \mathbf{H}^{-1}(\mathbf{v} - \mathbf{f})$:

$$(\mathbf{H}^{-1}(\mathbf{v} - \mathbf{f}) - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{17}$$

$$\mathbf{v}^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) - (\mathbf{H}^{-1}\mathbf{f} + \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{18}$$

$$\mathbf{v}^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) - (\mathbf{H}\mathbf{a} + \mathbf{f})^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{19}$$

MINIMAX: QUADRATIC CONSTRAINT, TYPE 3

Part 2

We can introduce notation:

$$\mathbf{M} = \mathbf{H}^{-\top} \mathbf{D} \quad (20)$$

$$\mathbf{g} = \mathbf{H}\mathbf{a} + \mathbf{f} \quad (21)$$

With that we can re-write our constraint:

$$\mathbf{v}^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) - \mathbf{g}^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h \quad (22)$$

$$(\mathbf{v} - \mathbf{g})^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h \quad (23)$$

And now we formulated type 3 problem as type 2:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \max_{\mathbf{v}} \quad \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} \quad & (\mathbf{v} - \mathbf{g})^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h, \\ & \|\mathbf{v}\| \leq p \end{aligned} \quad (24)$$

MINIMAX: QUADRATIC CONSTRAINT, TYPE 4

Try solving this problem on your own:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \mathbf{s}^\top \mathbf{y} + \mathbf{q}^\top \mathbf{x} \leq h, \\ & ||\mathbf{H}\mathbf{y} + \mathbf{f}|| \leq p \end{aligned} \tag{25}$$

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022



Check Moodle for additional links, videos, textbook suggestions.

MAX OVER NORM OF A SUM OF VECTORS

Consider the problem $\max_{\mathbf{x}}(\|\mathbf{a} + \mathbf{x}\|)$ and $\|\mathbf{x}\| \leq p$. Let us open the norm:

$$\|\mathbf{a} + \mathbf{x}\| = \sqrt{(\mathbf{a} + \mathbf{x})^\top (\mathbf{a} + \mathbf{x})} = \mathbf{a}^\top \mathbf{a} + 2\mathbf{a}^\top \mathbf{x} + \mathbf{x}^\top \mathbf{x} \quad (26)$$

If \mathbf{a} is a constant, then the expression $\mathbf{a}^\top \mathbf{a} + 2\mathbf{a}^\top \mathbf{x} + \mathbf{x}^\top \mathbf{x}$ attains a maximum when $\mathbf{x}^\top \mathbf{x} = p^2$ and $\mathbf{a}^\top \mathbf{x} = \|\mathbf{a}\|p$. This implies that:

$$\mathbf{x} = p \frac{\mathbf{a}}{\|\mathbf{a}\|} \quad (27)$$

And thus:

$$\max_{\mathbf{x}}(\|\mathbf{a} + \mathbf{x}\|) = \|\mathbf{a} + p \frac{\mathbf{a}}{\|\mathbf{a}\|}\| = \|\mathbf{a}\| \left(1 + \frac{p}{\|\mathbf{a}\|}\right) = \|\mathbf{a}\| + p$$

MINIMAX: CONIC CONSTRAINT

Part 1

Consider the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \min_{\mathbf{y}} \quad \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} \quad & \|\mathbf{Ax} + \mathbf{b} + \mathbf{y}\| \leq \mathbf{c}^\top \mathbf{x} + d, \\ & \|\mathbf{y}\| \leq p \end{aligned} \tag{28}$$

As we discussed in the last slide, expression $\|\mathbf{Ax} + \mathbf{b} + \mathbf{y}\|$ is biggest when $\mathbf{y} = p \frac{\mathbf{Ax} + \mathbf{b}}{\|\mathbf{Ax} + \mathbf{b}\|}$, and the conic constraint becomes:

$$\|\mathbf{Ax} + \mathbf{b} + p \frac{\mathbf{Ax} + \mathbf{b}}{\|\mathbf{Ax} + \mathbf{b}\|}\| \leq \mathbf{c}^\top \mathbf{x} + d \tag{29}$$

$$\|\mathbf{Ax} + \mathbf{b}\| \left(1 + \frac{p}{\|\mathbf{Ax} + \mathbf{b}\|} \right) \leq \mathbf{c}^\top \mathbf{x} + d \tag{30}$$

MINIMAX: CONIC CONSTRAINT

Part 2

Continue the derivation:

$$\|\mathbf{Ax} + \mathbf{b}\| \left(1 + \frac{p}{\|\mathbf{Ax} + \mathbf{b}\|} \right) \leq \mathbf{c}^\top \mathbf{x} + d \quad (31)$$

$$\|\mathbf{Ax} + \mathbf{b}\| + p \leq \mathbf{c}^\top \mathbf{x} + d \quad (32)$$

Finally the problem becomes:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x} - \mathbf{x}^*\|, \\ \text{subject to} \quad & \|\mathbf{Ax} + \mathbf{b}\| \leq \mathbf{c}^\top \mathbf{x} + d - p \end{aligned} \quad (33)$$