Linear Programming

Computational Intelligence, Lecture 8

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A linear program (LP) is an optimization problem of the form:

$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\
\text{subject to} \quad & \begin{cases} \mathbf{A}\mathbf{x} \le \mathbf{b}, \\ \mathbf{C}\mathbf{x} = \mathbf{d}. \end{cases}
\end{aligned} \tag{1}
$$

It is one of the older and widely used classes of convex optimization problems.

Note that the solution of such problem will always lie on the boundary of its domain.

Here are some examples of LP which have no solutions:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

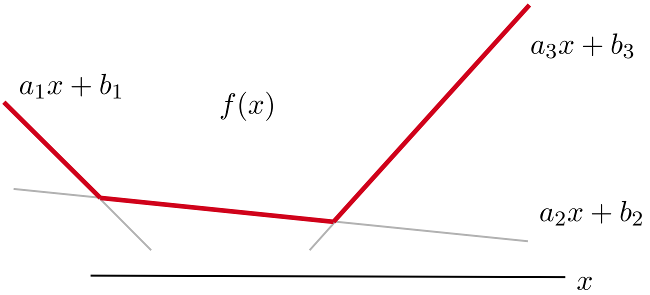
This one is has no boundaries at all, hence no solution. Next one has boundaries, but they do not restrict motion along the descent direction for the cost function.

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \quad \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ &\text{subject to} \quad \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 1 \end{aligned} \tag{3}$$

Convex piece-wise linear functions have the form:

$$f(\mathbf{x}) = \max(\mathbf{a}_i^\top \mathbf{x} + b_i) \tag{4}$$

Figure below shows geometric interpretation of such function for a one-dimensional case.



We can formulate a minimization problem using convex piece-wise linear functions:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \max(\mathbf{a}_i^\top \mathbf{x} + b_i) \tag{5}$$

Which can be equivalently transformed into the following LP:

$$\begin{aligned} \underset{\mathbf{x},t}{\text{minimize}} \quad & t \\ \text{subject to} \quad & \mathbf{a}_i^\top \mathbf{x} + b_i \leq t \end{aligned} \tag{6}$$

We can observe that optimal (minimal) t will have to lie on one of the linear functions $\mathbf{a}_i^\top \mathbf{x} + b_i$, i.e. on the original piece-wise linear function $f(\mathbf{x})$. And optimal value on t corresponds to the smallest value of the original function $f(\mathbf{x})$.

Sum of convex piece-wise linear functions have the form:

$$f(\mathbf{x}) + g(\mathbf{x}) = \max(\mathbf{a}_i^\top \mathbf{x} + b_i) + \max(\mathbf{c}_i^\top \mathbf{x} + d_i) \tag{7}$$

Their representation as LP is:

$$\begin{aligned} \underset{\mathbf{x}, t_1, t_2}{\text{minimize}} \quad & t_1 + t_2 \\ \text{subject to} \quad & \begin{cases} \mathbf{a}_i^\top \mathbf{x} + b_i \leq t_1 \\ \mathbf{c}_i^\top \mathbf{x} + d_i \leq t_2 \end{cases} \end{aligned} \tag{8}$$

Code

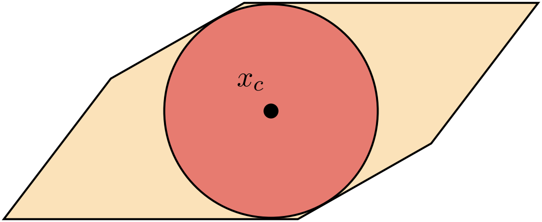
```
func = @(t) t^2;
derivative_func = @(t) 2*t;

approx_points = [-1, -0.3, 0, 0.3, 1];
n = length(approx_points);
a = zeros(n, 1);
b = zeros(n, 1);

for i = 1:n
    t = approx_points(i);
    a(i) = derivative_func(t);
    b(i) = func(t) - a(i)*t ;
end

f = [1; 0];
lin_A = [-ones(n, 1), a];
lin_b = -b;
x = linprog(f, lin_A,lin_b, [], []);
```

Chebyshev center of a polyhedron is the center of the largest ball inscribed in a polyhedron:



Equation describing this ball can be written as:

$$\mathcal{B} = \{\mathbf{x}_c + \mathbf{u} : \ ||\mathbf{u}||_2 \leq r\} \tag{9}$$

where r is the radius of the ball and $\mathbf{x}_c$ is its center.

Before we move towards solving the problem, let us consider the following maximization:

$$\sup\{\mathbf{a}^\top \mathbf{u} : \ ||\mathbf{u}||_2 \leq r\} \tag{10}$$

We can re-write the expression:

$$\sup\{\mathbf{a}^\top \mathbf{u} : \ ||\mathbf{u}||_2 \leq r\} = \sup\{||\mathbf{a}|| \cdot ||\mathbf{u}|| \cos(\varphi) : \ ||\mathbf{u}||_2 \leq r\} \tag{11}$$

where φ is the angle between $\mathbf{a}$ and $\mathbf{u}$. Since $\mathbf{a}$ is constant, $\max(||\mathbf{u}||) = r$, and $\max(\cos(\varphi)) = 1$, we get:

$$\sup\{\mathbf{a}^\top \mathbf{u} : \ ||\mathbf{u}||_2 \leq r\} = ||\mathbf{a}|| r \tag{12}$$

For the ball $\mathcal{B}$ to be inscribed in a polygon $\mathcal{P} = \{\mathbf{x} : \ \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, the following should hold:

$$\sup\{\mathbf{a}_i^\top(\mathbf{x}_c + \mathbf{u}) : \ ||\mathbf{u}||_2 \leq r\} \leq b_i \tag{13}$$

Note that the largest value of $\mathbf{a}_i^\top \mathbf{u}$ under condition $||\mathbf{u}||_2 \leq r$ is $r||\mathbf{a}_i||$: it can indeed achieve this value if $\mathbf{a}_i$ and $\mathbf{u}$ are co-directional, but a larger one is not possible. Therefore:

$$\sup\{\mathbf{a}_i^\top(\mathbf{x}_c + \mathbf{u}) : \ ||\mathbf{u}||_2 \leq r\} = \mathbf{a}_i^\top \mathbf{x}_c + r||\mathbf{a}_i|| \leq b_i \tag{14}$$

Finally, we can write down the solution of the problem as a linear optimization:

$$\begin{aligned} \underset{r,\ \mathbf{x}_c}{\text{maximize}} \quad & r \\ \text{subject to} \quad & \mathbf{a}_i^\top \mathbf{x}_c + r||\mathbf{a}_i|| \leq b_i \end{aligned} \tag{15}$$

Below we can see MATLAB code for solving the problem:

```
V = randn(10, 2);
k = convhull(V);
P = V(k, :);

[domain_A, domain_b] = vert2con(P);
norm_A = vecnorm(domain_A');

f = [-1; 0; 0];
A = [reshape(norm_A, [], 1), domain_A];
b = domain_b;

x = linprog(f, A,b, [], []);

center = [x(2),x(3)];
r = x(1);
```

The following is the Linear-Fractional Programming problem:

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \frac{\mathbf{c}^\top \mathbf{x} + d}{\mathbf{e}^\top \mathbf{x} + f} \\ \text{subject to} \quad & \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathbf{A}_e \mathbf{x} = \mathbf{b}_e \end{cases} \end{aligned} \tag{16}$$

This doesn't look like an LP, but let us see if we can try to bring this problem into this form.

The following is the Linear-Fractional Programming problem in LP form:

$$\begin{aligned} \underset{\mathbf{y},\ z}{\text{maximize}} \quad & \mathbf{c}^\top \mathbf{y} + zd \\ \text{subject to} \quad & \begin{cases} \mathbf{A}\mathbf{y} \le z\mathbf{b} \\ \mathbf{A}_e\mathbf{y} = z\mathbf{b}_e \\ \mathbf{e}^\top \mathbf{y} + zf = 1 \\ z \ge 0 \end{cases} \end{aligned} \tag{17}$$

Here the variables $\mathbf{y}$ and z are related to $\mathbf{x}$ as follows.

$$\mathbf{y} = \frac{\mathbf{x}}{\mathbf{e}^\top \mathbf{x} + f} \tag{18}$$

$$z = \frac{1}{\mathbf{e}^\top \mathbf{x} + f} \tag{19}$$

We assumed that the domain of the previous problem is limited to $\mathbf{e}^\top \mathbf{x} + f \geq 0$. With that we have:

$$\mathbf{c}^\top \mathbf{y} + zd = \mathbf{c}^\top \frac{\mathbf{x}}{\mathbf{e}^\top \mathbf{x} + f} + \frac{1}{\mathbf{e}^\top \mathbf{x} + f} d = \frac{\mathbf{c}^\top \mathbf{x} + d}{\mathbf{e}^\top \mathbf{x} + f} \tag{20}$$

$$\mathbf{A}\mathbf{y} \leq z\mathbf{b} \implies \mathbf{A}\frac{\mathbf{x}}{\mathbf{e}^\top \mathbf{x} + f} \leq \frac{1}{\mathbf{e}^\top \mathbf{x} + f}\mathbf{b} \implies \mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{21}$$

Implement linear approximation of a convex function and solve it as LP

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2023

Check Moodle for additional links, videos, textbook suggestions.