

Quadratically constrained quadratic  
programming,  
Second-order cone programming  
Computational Intelligence, Lecture 6

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- Quadratic programming: recap
- Quadratically constrained quadratic programming (QCQP)
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  - ▶ Ellipsoidal constraints  $\rightarrow$  to canonical form
- Second-order cone programming
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  - ▶ Second-order cone constraints  $\rightarrow$  to canonical form

# QUADRATIC PROGRAMMING

## General form

Remember the general form of a quadratic program:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b}, \\ \mathbf{F} \mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{1}$$

where  $\mathbf{H}$  is positive-definite and  $\mathbf{A} \mathbf{x} \leq \mathbf{b}$  describe a convex region.

# QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING

## General form

General form of a quadratically constrained quadratic program (QCQP) is given below:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^\top \mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{x}^\top \mathbf{P}_i \mathbf{x} + \mathbf{q}_i^\top \mathbf{x} + r_i \leq 0, \\ \mathbf{F} \mathbf{x} = \mathbf{g}. \end{cases} \end{array} \quad (2)$$

where  $\mathbf{P}_i$  are positive-definite.

# QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING

## Domain

Domain of a QCQP without equality constraints and with no degenerate inequality constraints is an intersection of ellipses:



Set  $\mathbf{P}_i = \mathbf{0}$  and you get a QP.

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \begin{bmatrix} \mathbf{q}_1^\top \\ \dots \\ \mathbf{q}_n^\top \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} -r_1 \\ \dots \\ -r_n \end{bmatrix} \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \quad (3)$$

Set  $\mathbf{P}_0 = \mathbf{0}$  and you get an LP.

# TURNING ELLIPSOID TO THE CANONICAL FORM (1)

Can we re-write the expression  $\mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r \leq 0$  as a canonical form ellipsoid:

$$\frac{z_1^2}{m_1^2} + \frac{z_2^2}{m_2^2} + \dots + \frac{z_n^2}{m_n^2} \leq 1 \quad (4)$$

We start by proposing a substitution  $\mathbf{x}_0 = -\frac{1}{2}\mathbf{P}^{-1}\mathbf{q}$  and  $-d = r - \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0$ . We can prove that:

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P} (\mathbf{x} - \mathbf{x}_0) - d^2 = \mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r$$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P} (\mathbf{x} - \mathbf{x}_0) - d^2 = \quad (5)$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} - 2\mathbf{x}_0^\top \mathbf{P} \mathbf{x} + \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 - d^2 = \quad (6)$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} + 2 \left( \frac{1}{2} \mathbf{P}^{-1} \mathbf{q} \right)^\top \mathbf{P} \mathbf{x} + \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 + r - \mathbf{x}_0^\top \mathbf{P} \mathbf{x}_0 = \quad (7)$$

$$= \mathbf{x}^\top \mathbf{P} \mathbf{x} + \mathbf{q}^\top \mathbf{x} + r. \quad \square \quad (8)$$

# TURNING ELLIPSOID TO THE CANONICAL FORM (1)

Thus our original expression became:

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{P}(\mathbf{x} - \mathbf{x}_0) - d^2 \leq 0 \quad (9)$$

We define  $\mathbf{A} = \sqrt{\mathbf{P}}$  and give its SVD  $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$ . Defining  $\mathbf{z} = \mathbf{V}^\top(\mathbf{x} - \mathbf{x}_0)$  we get:

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{A}^\top \mathbf{A}(\mathbf{x} - \mathbf{x}_0) - d^2 \leq 0 \quad (10)$$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{V}\Sigma\mathbf{U}^\top \mathbf{U}\Sigma\mathbf{V}^\top(\mathbf{x} - \mathbf{x}_0) - d^2 \leq 0 \quad (11)$$

$$(\mathbf{x} - \mathbf{x}_0)^\top \mathbf{V}\Sigma^2\mathbf{V}^\top(\mathbf{x} - \mathbf{x}_0) - d^2 \leq 0 \quad (12)$$

$$\mathbf{z}^\top \Sigma^2 \mathbf{z} - d^2 \leq 0 \quad (13)$$

$$\sum z_i^2 \sigma_i^2 \leq d^2 \quad (14)$$

Defining  $1/m_i^2 = \sigma_i^2/d^2$  we get:

$$\frac{z_1^2}{m_1^2} + \frac{z_2^2}{m_2^2} + \dots + \frac{z_n^2}{m_n^2} \leq 1 \quad (15)$$



# SECOND-ORDER CONE PROGRAMMING (SOCP)

## General form

The general form of a Second-order cone program (SOCP) is:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{f}^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \|\mathbf{A}_i \mathbf{x} + \mathbf{b}_i\|_2 \leq \mathbf{c}_i^\top \mathbf{x} + d_i, \\ \mathbf{F} \mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{16}$$

LP, QP and QCQP are subsets of SOCP.

Consider the following SOC constraint:

$$\|\mathbf{Ax} + \mathbf{b}\|_2 \leq \mathbf{c}^\top \mathbf{x} + d \quad (17)$$

Let us consider a special case when  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{A} \in \mathbb{R}^{(n-1) \times n}$  and  $\text{rank} \left( \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix} \right) = n$ . Then we can introduce the following substitution:

$$\xi = \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} \mathbf{E} \\ \mathbf{e}^\top \end{bmatrix} \quad (18)$$

where  $\mathbf{I} \in \mathbb{R}^{n,n}$  is an identity matrix. Then constraint (17) becomes:

$$\|\mathbf{E}\xi\|_2 \leq \mathbf{e}^\top \xi \quad (19)$$

Notice that  $\|\mathbf{E}\xi\|_2 \leq \mathbf{e}^\top \xi$  is equivalent to:

$$\sum_{i=1}^{n-1} \xi_i^2 \leq \xi_n^2 \quad (20)$$

which is a standard form of a cone. A map back from  $\xi$  to  $\mathbf{x}$  is given as:

$$\mathbf{x} = \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix}^{-1} \left( \xi - \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix} \right) \quad (21)$$

# SECOND-ORDER CONE PROGRAMMING

## Special cases

We can write problem where our domain is a ball as SOCP:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \|\mathbf{x}\|_2 \leq d_i\end{array}\tag{22}$$

Same for ellipsoidal constraints:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \|\mathbf{A}_i \mathbf{x}\|_2 \leq d_i\end{array}\tag{23}$$

Set  $\mathbf{c}_i = 0$  and recognize that  $\|\mathbf{A}_i \mathbf{x} + \mathbf{b}_i\|_2 \leq d_i$  is the same as  $(\mathbf{A}_i \mathbf{x} + \mathbf{b}_i)^\top (\mathbf{A}_i \mathbf{x} + \mathbf{b}_i) \leq d_i^2$  (since the first implies that  $d_i$  is non-negative).

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{f}^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \quad (24)$$

Now to make the cost quadratic:

$$\begin{aligned} & \underset{\mathbf{x}, t}{\text{minimize}} && t, \\ & \text{subject to} && \begin{cases} \mathbf{x}^\top \mathbf{A}_0^\top \mathbf{A}_0 \mathbf{x} + 2\mathbf{b}_0^\top \mathbf{A}_0 \mathbf{x} + \mathbf{b}_0^\top \mathbf{b}_0 \leq t \\ \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \quad (25)$$

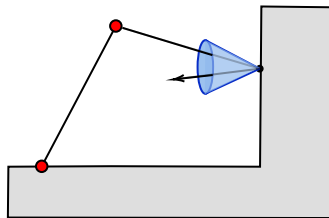
Which is the same as:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{H}\mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \quad (26)$$

As long as  $\mathbf{A}_0 = \sqrt{\mathbf{H}}$ , and  $\mathbf{b}_0 = 0.5\mathbf{A}_0^{-1}\mathbf{f}$ .

# FRICTION CONE

## Normal reaction force and friction



Let  $\mathbf{f}$  be total reaction force,  $\mathbf{f}_n$  be its normal component (perpendicular to the surface locally), also known as normal reaction; and let  $\mathbf{f}_{fr}$  be its tangential component (a vector lying in the tangent plane to the surface, constructed at the contact point), or friction force. Let  $\mathbf{e}_n$  be a unit vector, normal to the surface at the point of contact.

$$\mathbf{f} = \mathbf{f}_n + \mathbf{f}_{fr} \quad (27)$$

# SECOND-ORDER CONE PROGRAMMING

## Friction cone

Defining  $\mathbf{E}_t = [\mathbf{e}_{t,1}, \mathbf{e}_{t,2}] = \text{null}(\mathbf{e}_n^\top)$  be an orthonormal basis in the tangential space to the surface, we can write:

$$\mathbf{f} = \mathbf{e}_n n + \mathbf{E}_t \mathbf{t}$$

$$\mathbf{f}_n = \mathbf{e}_n n$$

$$\mathbf{f}_{fr} = \mathbf{E}_t \mathbf{t}$$

$$\mathbf{t} = [t_1, t_2]$$

The friction cone conditions could be written in any of the following ways:

$$\sqrt{t_1^2 + t_2^2} < \mu n \quad (28)$$

$$\|\mathbf{E}_t^\top \mathbf{f}\| \leq \mu \mathbf{e}_n^\top \mathbf{f} \quad (29)$$

where  $\mu$  is a friction coefficient.



Implement a program that finds right-most point of an intersection of two ellipsoids; visualise the problem and the solution.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2023](https://github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2023)

