Geometry of Second-Order Cones

Contact-aware Control, Lecture 7

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- 2-norm
- Cone
- Second-order cone (linear, affine)
- The role of the free constant
- Plotting level sets

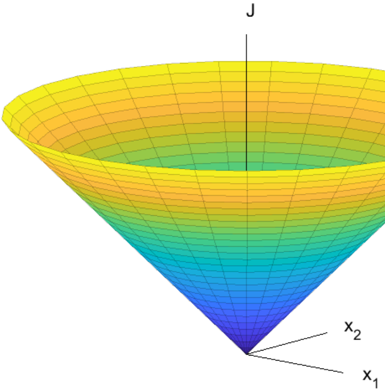
Let us consider a 2-norm as a function $f(\mathbf{x}): \mathbb{R}^n \to \mathbb{R}$:

$$f(\mathbf{x}) = ||\mathbf{x}||_2 \tag{1}$$

$$f(\mathbf{x}) = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{2}$$

We can describe 2-norm as a surface in the $\mathcal{S} \subset \mathbb{R}^{n+1}$ space:

$$\mathcal{S} = \{(J, \mathbf{x}) : \; J = ||\mathbf{x}||_2\} \tag{3}$$



Cone

The shape of the surface $\mathcal{S} = \{(J, \mathbf{x}) : \; J = ||\mathbf{x}||_2\}$ is a *cone*. We observe the following properties of a cone:

- There is a single tip point τ and a normal direction.
- Slicing cone with planes orthogonal to the normal direction, we produce ellipsoids (we can call it tangent sets).
- For any point p on the cone, the half-line from the tip point τ through p lies on the cone.

The tip point for $\mathcal{S}$ is $(0, \mathbf{0})$. Normal direction is $(n, \mathbf{0})$. Tangent sets are circles $||\mathbf{x}||_2 = h$, parameterized by h.
For any point $(J(\mathbf{p}), \mathbf{p})$ we can write half-line as
$\mathcal{L} = \{(J(\gamma \mathbf{p}), \gamma \mathbf{p}) : \; \gamma > 0\}$.

$$J(\gamma \mathbf{p}) = ||\gamma \mathbf{p}|| = \gamma ||\mathbf{p}|| = \gamma J(\mathbf{p}) \tag{4}$$

A second-order cone constraint has the following form:

$$||\mathbf{A}\mathbf{x} + \mathbf{b}|| \leq \mathbf{c}^\top \mathbf{x} + d \tag{5}$$

where $\mathbf{A} \in \mathbb{R}^{n,n}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $d \in \mathbb{R}$.

This constraint describes interior of a cone. We can prove that surface of this set is is a cone described by the following equality:

$$||\mathbf{A}\mathbf{x} + \mathbf{b}|| = \mathbf{c}^\top \mathbf{x} + d \tag{6}$$

Let us consider the following simplified case:

$$||\mathbf{A}\mathbf{x}|| = \mathbf{c}^\top \mathbf{x} \tag{7}$$

The tip of this cone is the point $\mathbf{x} = 0$, and the normal direction is $\mathbf{c}$.

Let us consider a point $\mathbf{p}$ that lies on the surface (7): $||\mathbf{A}\mathbf{p}|| - \mathbf{c}^\top \mathbf{p} = 0$. Clearly, a point $\gamma\mathbf{p}$ for $\gamma \geq 0$ lies on the same surface:

$$\begin{aligned} e = ||\mathbf{A}(\gamma\mathbf{p})|| - \mathbf{c}^\top(\gamma\mathbf{p}) = \gamma||\mathbf{A}\mathbf{p}|| - \gamma\mathbf{c}^\top\mathbf{p} = \\ = \gamma(||\mathbf{A}\mathbf{p}|| - \mathbf{c}^\top\mathbf{p}) = 0 \end{aligned}$$

Therefore, the half-line going from tip $\mathbf{x} = 0$ through $\mathbf{p}$ lies on the cone.

We can consider plane $\mathcal{P}$ perpendicular to $\mathbf{c}$, distance $h/||\mathbf{c}||$ away from the tip. All points on the plane are described as follows:

$$\mathbf{x} = \frac{h}{||\mathbf{c}||^2}\mathbf{c} + \mathbf{T}\mathbf{z}, \quad \forall \mathbf{z} \tag{8}$$

where $\mathbf{T} = \mathrm{null}(\mathbf{c}^\top)$, so $\mathbf{c}^\top \mathbf{T} = 0$. Then conic surface becomes:

$$\left|\left|\mathbf{A}\frac{h}{||\mathbf{c}||^2}\mathbf{c} + \mathbf{A}\mathbf{T}\mathbf{z}\right|\right| = \mathbf{c}^\top \left(\frac{h}{||\mathbf{c}||^2}\mathbf{c} + \mathbf{T}\mathbf{z}\right) \tag{9}$$

$$||\mathbf{b}_0 + \mathbf{A}\mathbf{T}\mathbf{z}|| = h \tag{10}$$

where $\mathbf{b}_0 = \frac{h}{||\mathbf{c}||^2}\mathbf{A}\mathbf{c}$. This is an ellipsoid. □

For a full second-order cone (SOC):

$$||\mathbf{A}\mathbf{x} + \mathbf{b}|| = \mathbf{c}^\top \mathbf{x} + d \tag{11}$$

we can find a tip point; it corresponds to both right-hand side and left-hand side becoming zero:

$$\begin{cases} \mathbf{A}\mathbf{x} + \mathbf{b} = 0 \\ \mathbf{c}^\top \mathbf{x} + d = 0 \end{cases} \tag{12}$$

Given full rank matrix $\mathbf{A}$, the solution is $\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$. The system would hold if:

$$-\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b} + d = 0 \tag{13}$$

Consider a point $\mathbf{p}_2$ on the surface $||\mathbf{A}\mathbf{p}_2 + \mathbf{b}|| = \mathbf{c}^\top \mathbf{p}_2 + d$. Let us consider an interval from tip $\mathbf{p}_1 = -\mathbf{A}^{-1}\mathbf{b}$ to $\mathbf{p}_2$:

$$\mathbf{x} = \gamma \mathbf{p}_1 + (1-\gamma)\mathbf{p}_2, \quad \gamma \geq 0 \tag{14}$$

We can check if the points on the interval lie on the surface:

$$e = ||\mathbf{A}\gamma\mathbf{p}_1 + \mathbf{b} + \mathbf{A}(1-\gamma)\mathbf{p}_2|| - (\gamma\mathbf{c}^\top\mathbf{p}_1 + d + (1-\gamma)\mathbf{c}^\top\mathbf{p}_2)$$

$$\begin{gathered} e = ||\gamma\mathbf{A}\mathbf{p}_1 + \gamma\mathbf{b} + (1-\gamma)\mathbf{b} + \mathbf{A}(1-\gamma)\mathbf{p}_2|| - \\ -(\gamma\mathbf{c}^\top\mathbf{p}_1 + \gamma d + (1-\gamma)d + (1-\gamma)\mathbf{c}^\top\mathbf{p}_2) \end{gathered}$$

We know that $\mathbf{A}\gamma\mathbf{p}_1 + \mathbf{b} = 0$ and $\mathbf{c}^\top\mathbf{p}_1 + d = 0$:

$$e = ||(1-\gamma)\mathbf{b} + \mathbf{A}(1-\gamma)\mathbf{p}_2|| - (1-\gamma)d - (1-\gamma)\mathbf{c}^\top\mathbf{p}_2 \tag{15}$$

$$e = ||\mathbf{b} + \mathbf{A}\mathbf{p}_2|| - (d + \mathbf{c}^\top\mathbf{p}_2) = 0 \tag{16}$$

Thus, the interval (and therefore, the half-line on which it lies) lies on the surface.

Let us consider plane $\mathcal{P}$ perpendicular to $\mathbf{c}$, distance $h/||\mathbf{c}||$ away from the tip. All points on the plane are described as follows:

$$\mathbf{x} = \frac{h}{||\mathbf{c}||^2}\mathbf{c} + \mathbf{T}\mathbf{z}, \quad \forall \mathbf{z} \tag{17}$$

We can substitute it into the second-order cone:

$$||\mathbf{A}(\frac{h}{||\mathbf{c}||^2}\mathbf{c} + \mathbf{T}\mathbf{z}) + \mathbf{b}|| = \mathbf{c}^\top(\frac{h}{||\mathbf{c}||^2}\mathbf{c} + \mathbf{T}\mathbf{z}) + d \tag{18}$$

$$||\mathbf{b}_0 + \mathbf{A}\mathbf{T}\mathbf{z}|| = h + d \tag{19}$$

where $\mathbf{b}_0 = \frac{h}{||\mathbf{c}||^2}\mathbf{A}\mathbf{c} + \mathbf{b}$. This is an ellipsoid. $\square$

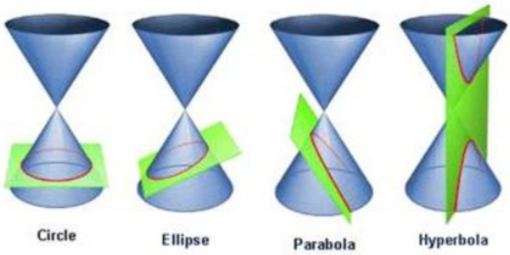
Free constant d plays a specific role in the SOC. As we said before, there is a condition $d = \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b} + d$. We can explain it by looking at COS as intersection of two surfaces:

$$J = ||\mathbf{A}\mathbf{x} + \mathbf{b}|| \tag{20}$$

$$P = \mathbf{c}^\top \mathbf{x} + d \tag{21}$$

First is a cone and second is a plane. Their intersection is called a *conic section*.

Typical conic sections are shown below:



As we can see, they represent ellipsoid and parabola. In order for them to represent a cone, the plane S needs to pass through the tip of the cone J. This can be achieved with the appropriate choice of constant d, which shifts S up or down.

To plot a cone it is convenient to first use change of coordinates $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$, meaning $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b})$, giving us SOC:

$$||\mathbf{y}|| = \mathbf{c}^\top \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b}) + d \tag{22}$$

Note that $d - \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b} = 0$ for a cone with a tip; so SOC becomes:

$$||\mathbf{y}|| = \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{y} \tag{23}$$

To plot level sets of this cone we choose height of the level set h and pick point $\mathbf{y}_h = h\frac{\mathbf{A}^{-T}\mathbf{c}}{\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{A}^{-T}\mathbf{c}}$; we note that $\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{y}_h = h$. Then we consider points on the plane $\mathcal{P}$ orthogonal to $\mathbf{c}^\top \mathbf{A}^{-1}$ and passing through $\mathbf{y}_h$:

$$\mathcal{P} = \mathbf{y}_h + \mathbf{T}\mathbf{z} : \ \forall \mathbf{z} \tag{24}$$

where $\mathbf{T} = \text{null}(\mathbf{c}^\top \mathbf{A}^{-1})$, so $\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{T} = 0$.

Since SOC becomes:

$$||\mathbf{y}_h + \mathbf{T}\mathbf{z}|| = h \tag{25}$$

Since $\mathbf{y}_h$ and $\mathbf{T}\mathbf{z}$ are orthogonal, it is equivalent to:

$$||\mathbf{T}\mathbf{z}|| = g \tag{26}$$

where $g = \sqrt{h^2 - \mathbf{y}_h^\top \mathbf{y}_h}$. In the 3D case, this is a circle with radius g. We can find N consecutive evenly spaced points of this circle, resulting in the next sequence of $\mathbf{y}_l$:

$$\mathbf{y}_l = \mathbf{y}_h + \mathbf{T}\begin{bmatrix} g\cos(\varphi) \\ -g\sin(\varphi) \end{bmatrix}, \quad \varphi = 0,\ \frac{2\pi}{N},\ 2\frac{2\pi}{N},\ ...,\ 2\pi \tag{27}$$

$$\mathbf{x}_l = \mathbf{A}^{-1}(\mathbf{y}_l - \mathbf{b}) \tag{28}$$

The center of the ellipsoid representing this level set lies at the point $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{y}_h - \mathbf{b})$.

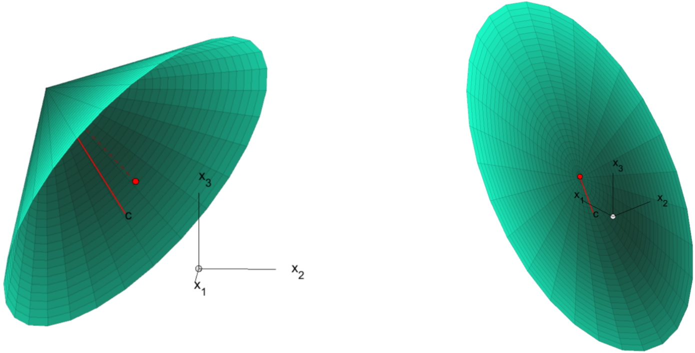


Figure 1: Cone. Dashed line - centers of level-sets.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Contact-Aware-Control-Fall-2023