Change of Contact

Contact-aware Control, Lecture 9

by Sergei Savin

Fall 2023

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Point contact and unilateral constraints

Let us consider unilateral point contact - a constraint $\mathbf{r}(\mathbf{q}) = 0$, where $\mathbf{r} \in \mathbb{R}^3$, with associated constraint jacobian $\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}}$ and reaction force $\mathbf{f} \in \mathbb{R}^3$.

Reaction force lies in a friction cone $\mathbf{f} \in \mathcal{C}$, with normal direction $\mathbf{n}$ and friction coefficient μ.

We refer to such contact / constraint *unilateral* because its associated reaction force can "push" but not "pull" on the support surface.

Outside of friction cone

There are a few basic scenarios of how a friction force can be broken:

1. Friction force (tangent component of the reaction force) can lie on the boundary of the friction cone leading to sliding. Length of the friction force is equal to $\mu \mathbf{n}^\top \mathbf{f}$, motion in tangent direction becomes possible.
2. Normal force can become zero, allowing the contact to be broken.

Sliding friction

In general, it is good to assume that accurate modeling of sliding regime for dry friction is hard. Possible models include Coulomb friction, "stiction" and others.

Accuracy of a static dry friction model is relatively stable for all hard surfaces. Sliding friction model on the other hand will not only radically change for any new surface, but also will be influenced by orientation of the surface and other small-scale details, hard to account for.

A rule of thumb is to avoid building precise model-based control schemes that require sliding.

Breaking contact

Breaking contact occurs when the normal reaction goes to zero and point of contact acquires acceleration projection onto the surface normal.

To describe the process of acquiring and breaking contact (locally) we can use *complementarity* constraint:

$$\begin{cases} f_n r_n = 0 \\ f_n \geq 0 \\ r_n \geq 0 \end{cases} \tag{1}$$

where $f_n = \mathbf{n}^\top \mathbf{f}$ and $r_n = \mathbf{n}^\top \mathbf{r}$. We can describe this constraint in the following way: either $f_n = 0$ and then r_n can assume any positive number, meaning the contact can be broken; or $r_n = 0$, contact is maintained and therefore normal reaction force is positive.

Complementarity - frictionless case

Consider the case when full reaction force at an i-th contact point is equal to normal reaction force: $\mathbf{f}_i = \mathbf{f}_{n,i} = f_{n,i}\mathbf{n}_i$. Then generalized constraint ϕ is constructed by concatenation:

$$\phi = \begin{bmatrix} \mathbf{n}_1^\top \left(\mathbf{r}_1(\mathbf{q}) - \mathbf{r}_1^0\right) \\ \dots \\ \mathbf{n}_m^\top \left(\mathbf{r}_m(\mathbf{q}) - \mathbf{r}_m^0\right) \end{bmatrix} = 0 \tag{2}$$

In this case complementarity constraint becomes:

$$\begin{cases} \phi^\top \lambda = 0 \\ \phi \geq 0 \\ \lambda \geq 0 \end{cases} \tag{3}$$

As we can see, this formulation includes all constraints (all contact points) at the same time, and can be directly used in conjunction with manipulator equations.

Acquiring contact - impact

Acquiring contact can mean fast (mathematically - instantaneous) change of momentum (generalized momentum) for the mechanical system. This is called *collision*.

Simple models of collision include *inelastic collision* (some of the energy is lost during the collision) and *elastic collision* (energy is conserved).

For a 1-dimensional point-mass the inelastic collision is described with the use of *coefficient of restitution* ξ:

$$v^+ = -\xi v^- \tag{4}$$

where v^- and v^+ are velocity of the point-mass before and after the collision.

Impact and walking, 1

During walking, contact has to be acquired (when the foot lands on the ground). Assuming point foot with coordinates $\mathbf{r}_f = \mathbf{r}_f(\mathbf{q})$, stepping on a foothold $\mathbf{r}_0$ and normal to the supporting surface $\mathbf{n}$, we can observe that:

- Before the step (before impact) $\mathbf{d} = \mathbf{r}_f(\mathbf{q}) - \mathbf{r}_f^0 \neq 0$ the velocity of the foot is non-zero: $\dot{\mathbf{r}}_f \neq 0$.
- After the step (after impact) $\mathbf{r}_f(\mathbf{q}) = \mathbf{r}_f^0$ the velocity of the foot is zero: $\dot{\mathbf{r}}_f = 0$.

If on the point of impact $\dot{\mathbf{r}}_f \neq 0$, an inelastic collision will take place, with "instantaneous" change of velocity.

Impact and walking, 2

Assume that impact took place at the moment of time t_1. We will describe velocity right before the impact as $\dot{\mathbf{r}}_f^-$ and right after the impact as $\dot{\mathbf{r}}_f^+$:

$$\dot{\mathbf{r}}_f^- = \lim_{\Delta t \to (-0)} \mathbf{J}_f \dot{\mathbf{q}}(t + \Delta t) \tag{5}$$

$$\dot{\mathbf{r}}_f^+ = \lim_{\Delta t \to (+0)} \mathbf{J}_f \dot{\mathbf{q}}(t + \Delta t) \tag{6}$$

where $\dot{\mathbf{r}}_f = \frac{\partial \mathbf{r}_f}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J}_f \dot{\mathbf{q}}$. As we can see, instantaneous change from $\dot{\mathbf{r}}_f^-$ to $\dot{\mathbf{r}}_f^+$ implies instantaneous change in generalized velocity:

$$\dot{\mathbf{q}}^- = \lim_{\Delta t \to (-0)} \dot{\mathbf{q}}(t + \Delta t) \tag{7}$$

$$\dot{\mathbf{q}}^+ = \lim_{\Delta t \to (+0)} \dot{\mathbf{q}}(t + \Delta t) \tag{8}$$

Impact and walking, 3

If $\dot{\mathbf{r}}_f^+ = 0$ and $\dot{\mathbf{r}}_f^- \neq 0$ it implies instantaneous change from $\dot{\mathbf{q}}^-$ to $\dot{\mathbf{q}}^+$. This can be done with an impact model:

$$\dot{\mathbf{q}}^+ = \mathbf{h}(\dot{\mathbf{q}}^-) \tag{9}$$

Alternatively, if $\dot{\mathbf{r}}_f^- = 0$, there is no need for an instantaneous change in generalized velocity. This can be achieved with, e.g. s-like curve (n-th degree polynomial):

$$\mathbf{r}_f(\mathbf{q}(t)) = \sum_{p=0}^{n} a_p t^p \tag{10}$$

$$\sum_{p=0}^{n} (a_p t_1^p) = \mathbf{r}_0 \tag{11}$$

$$\sum_{p=1}^{n} \left(p a_p t_1^{p-1} \right) = 0 \tag{12}$$

Changing contact on schedule, 1

Assume that during the time interval $t_0 < t < t_1$ contact is described by constraint $\phi(\mathbf{q}) = 0$, while during the time interval $t_1 < t < t_2$ contact is described by constraint $\psi(\mathbf{q}) = 0$. Dynamics then is described by separate sets of equations during different time intervals:

$$\begin{bmatrix} \mathbf{H} & -\mathbf{J}_1^\top \\ \mathbf{J}_1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}_1\dot{\mathbf{q}} \end{bmatrix} \quad \text{if} \quad t_0 < t < t_1 \tag{13}$$

$$\begin{bmatrix} \mathbf{H} & -\mathbf{J}_2^\top \\ \mathbf{J}_2 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}_2\dot{\mathbf{q}} \end{bmatrix} \quad \text{if} \quad t_1 < t < t_2 \tag{14}$$

where $\mathbf{J}_1 = \partial\phi/\partial\mathbf{q}$ and $\mathbf{J}_2 = \partial\psi/\partial\mathbf{q}$.

Changing contact on schedule, 2

There are a number of observations to be made here:

- ϕ and ψ may have different dimensions (and hence $\mathbf{J}_1$ and $\mathbf{J}_2$, as well as λ_1 and λ_2).
- This model does not describe the moment of change of contact. It can be described by a separate equation, e.g. by a collision equation.
- Walking implies change of contact - periodic loss of contact (foot lifted off the ground) and acquisition of contact (foot lands on the ground).

Read more

- Posa, M., Cantu, C. and Tedrake, R., 2014. A direct method for trajectory optimization of rigid bodies through contact. The International Journal of Robotics Research, 33(1), pp.69-81. dspace.mit.edu/bitstream/handle/1721.1/90907/Tedrake.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Contact-Aware-Control-Fall-2023