Inverse Dynamics

Contact-aware Control, Lecture 4

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Let us consider a second-order dynamical system $\ddot{\mathbf{q}} = \mathcal{G}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{u})$ with state variables $(\mathbf{q}, \dot{\mathbf{q}})$ (position, velocity), highest order derivative $\ddot{\mathbf{q}}$ and control input $\mathbf{u}$.

Forward Dynamics

Forward Dynamics for a system $\mathcal{G}$ is a problem of finding the highest order derivative $\ddot{\mathbf{q}}$ given state $(\mathbf{q}, \dot{\mathbf{q}})$ and control input $\mathbf{u}$.

Inverse Dynamics

Inverse Dynamics for a system $\mathcal{G}$ is a problem of finding control input $\mathbf{u}$ given state $(\mathbf{q}, \dot{\mathbf{q}})$ and the desired value of the highest order derivative $\ddot{\mathbf{q}}$.

Let $\mathbf{q}^* = \mathbf{q}^*(t)$ be a trajectory for a system $\mathcal{G}$ - meaning that we can substitute it to the equations describing $\mathcal{G}$ and find control law $\mathbf{u}^* = \mathbf{u}^*(t)$ that would turn them into equalities.

Such trajectory $\mathbf{q}^* = \mathbf{q}^*(t)$ can be called a *nominal trajectory* and such control law $\mathbf{u}^* = \mathbf{u}^*(t)$ can be called a *nominal control law*.

A given point of time $t = t_1$ has a corresponding point on the trajectory $(\mathbf{q}_1^*, \dot{\mathbf{q}}_1^*)$, where $\mathbf{q}_1^* = \mathbf{q}^*(t_1)$, and $\dot{\mathbf{q}}_1^* = \dot{\mathbf{q}}^*(t_1)$. It also has a corresponding acceleration $\ddot{\mathbf{q}}_1^* = \ddot{\mathbf{q}}^*(t_1)$. In this case, the Inverse Dynamics problem is equivalent to finding the current value of the nominal control law $\mathbf{u}_1^* = \mathbf{u}^*(t_1)$.

Everything we said so far about dynamical systems is true both for systems with and without explicit constraints.

This is so because, as we saw previously it is possible to transform DAE into ODEs, excluding algebraic variables.

In the last lecture we saw that the dynamics of a constrained system can be written as:

$$\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = 0 \tag{1}$$

where $\mathbf{P}_S = \mathbf{I} - \mathbf{J}^\top(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)^{-1}\mathbf{J}\mathbf{H}^{-1}$.

let us assume that $\mathbf{h} = \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} - \mathbf{T}\mathbf{u}$, where $\mathbf{u}$ are is a vector of motor torques, $\mathbf{T}$ is the control map and $\mathbf{T}\mathbf{u}$ are generalized forces generated by motor torques:

$$\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{P}_S\mathbf{T}\mathbf{u} \tag{2}$$

Given dynamics in the form $\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{P}_S\mathbf{T}\mathbf{u}$, where $\ddot{\mathbf{q}}^*$ is the desired acceleration, we can find a least-square solution it for $\mathbf{u}$ using pseudo-inverse:

$$\mathbf{u} = (\mathbf{P}_S\mathbf{T})^+\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) \tag{3}$$

Inverse dynamics with Schur projector, 3

Substituting expression for $\mathbf{u}$ into the original dynamics, we get: Given dynamics in the form $\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{P}_S\mathbf{T}\mathbf{u}$, where $\ddot{\mathbf{q}}^*$ is the desired acceleration, we can find a least-square solution it for $\mathbf{u}$ using pseudo-inverse:

$$\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{P}_S\mathbf{T}(\mathbf{P}_S\mathbf{T})^+\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) \tag{4}$$

$$(\mathbf{I} - \mathbf{P}_S\mathbf{T}(\mathbf{P}_S\mathbf{T})^+)\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = 0 \tag{5}$$

As we can see, the equality will hold iff $\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g})$ lies in the column space of the matrix $(\mathbf{P}_S\mathbf{T})$.

If the equality above does not hold, we call such acceleration $\ddot{\mathbf{q}}^*$ not achievable.

Given two different projectors $\mathbf{P}_1$ and $\mathbf{P}_2$ describing the same constrained dynamics:

$$\mathbf{P}_1(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{P}_1\mathbf{T}\mathbf{u} \tag{6}$$

$$\mathbf{P}_2(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{P}_2\mathbf{T}\mathbf{u} \tag{7}$$

the following holds:

$$(\mathbf{P}_1\mathbf{T})^+\mathbf{P}_1(\mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = (\mathbf{P}_2\mathbf{T})^+\mathbf{P}_2(\mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) \tag{8}$$

Proof in Inverse Dynamics Control of Floating-Base Robots with External Constraints: a Unified View.

Consider a weighted pseudoinverse problem:

$$\text{minimize} \quad ||\mathbf{A}\mathbf{x} - \mathbf{b}||_{\mathbf{W}} \tag{9}$$

where $||\mathbf{x}||_{\mathbf{W}} = \sqrt{\mathbf{x}^\top \mathbf{W} \mathbf{x}}$ and $\mathbf{W} > 0$. We can re-write the problem as:

$$\text{minimize} \quad (\mathbf{A}\mathbf{x} - \mathbf{b})^\top \mathbf{W}^{\frac{1}{2}} \mathbf{W}^{\frac{1}{2}} (\mathbf{A}\mathbf{x} - \mathbf{b}) \tag{10}$$

But this is the same as solving least-squares problem for equality $\mathbf{W}^{\frac{1}{2}} \mathbf{A}\mathbf{x} = \mathbf{W}^{\frac{1}{2}} \mathbf{b}$, which is does via Moore-Penrose pseudoinverse:

$$\mathbf{x} = (\mathbf{W}^{\frac{1}{2}} \mathbf{A})^{+} \mathbf{W}^{\frac{1}{2}} \mathbf{b} \tag{11}$$

Consider a weighted pseudoinverse problem:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{W} \mathbf{x}, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned} \tag{12}$$

We can use Lagrange multipliers to rewrite the problem as minimization of the function $L(\mathbf{x}, \lambda) = \mathbf{x}^\top \mathbf{W} \mathbf{x} + \lambda^\top (\mathbf{A}\mathbf{x} - \mathbf{b})$; optimality conditions imply that $\frac{\partial L}{\partial \mathbf{x}} = 0$ and $\frac{\partial L}{\partial \lambda} = \mathbf{A}\mathbf{x} - \mathbf{b} = 0$, so:

$$2\mathbf{x}^\top \mathbf{W} + \lambda^\top \mathbf{A} = 0 \tag{13}$$

This implies $\mathbf{x} = \frac{1}{2}\mathbf{W}^{-1}\mathbf{A}^\top \lambda$, and since $\mathbf{A}\mathbf{x} - \mathbf{b} = 0$, we get:

$$\frac{1}{2}\mathbf{A}\mathbf{W}^{-1}\mathbf{A}^\top \lambda = \mathbf{b} \tag{14}$$

$$\lambda = 2(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}^\top)^+ \mathbf{b} \tag{15}$$

$$\mathbf{x} = \mathbf{W}^{-1}\mathbf{A}^\top (\mathbf{A}\mathbf{W}^{-1}\mathbf{A}^\top)^+ \mathbf{b} \tag{16}$$

Weighted pseudoinverse, 3

We can use the two formulas for weighted pseudo-inverse to add weights to any inverse dynamics formulation presented earlier.

For example, consider a problem:

$$\text{minimize} \quad ||\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) - \mathbf{P}_S\mathbf{T}\mathbf{u}||_{\mathbf{W}} \tag{17}$$

As we showed earlier, it is solved as:

$$\mathbf{u} = (\mathbf{W}^{\frac{1}{2}}\mathbf{P}_S\mathbf{T})^{+}\mathbf{W}^{\frac{1}{2}}\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) \tag{18}$$

Inverse dynamics with QP

Consider dynamics written in the following form:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{J}^\top \lambda \\ \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} = 0 \end{cases} \tag{19}$$

We can solve it directly, as a quadratic program:

$$\begin{aligned} \underset{\mathbf{u},\lambda}{\text{minimize}} \quad & \mathbf{u}^\top \mathbf{u}, \\ \text{subject to} \quad & \mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{J}^\top \lambda \end{aligned} \tag{20}$$

Read more

- Righetti, L., Buchli, J., Mistry, M. and Schaal, S., 2011, May. Inverse dynamics control of floating-base robots with external constraints: A unified view. In 2011 IEEE international conference on robotics and automation (pp. 1085-1090). IEEE. - Inverse Dynamics Control of Floating-Base Robots with External Constraints: a Unified View.
- Mistry, M., Buchli, J. and Schaal, S., 2010, May. Inverse dynamics control of floating base systems using orthogonal decomposition. In 2010 IEEE international conference on robotics and automation (pp. 3406-3412). IEEE. - citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.212.3601.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Contact-Aware-Control-Fall-2023