Mechanical Equations with Constraints

Contact-aware Control, Lecture 2

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Lagrange equations

Lagrange equations have form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} = \tau \tag{1}$$

where T is kinetic energy, $\mathbf{q}$ is a vector of generalized coordinates and τ are generalized torques.

Note that kinetic energy can be described as $T = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{H}\dot{\mathbf{q}}$, where $\mathbf{H}$ is generalized inertia matrix. Matrix $\mathbf{H}$ is symmetric, positive-definite and full rank.

Generalized forces, 1

Generalized forces τ can be generated by Cartesian forces or Cartesian torques. We can describe relations between Cartesian force $\mathbf{f}$ and associated generalized force:

$$\tau_i = \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}}\right)^\top \mathbf{f}_i \tag{2}$$

where $\mathbf{r}_i = \mathbf{r}_i(\mathbf{q})$ is the vector describing position of the point of application of the force $\mathbf{f}_i$, as a function of generalized coordinates $\mathbf{q}$.

If we define jacobian $\mathbf{J}_i = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}}$ we can re-write the relation above as:

$$\tau_i = (\mathbf{J}_i^r)^\top \mathbf{f}_i \tag{3}$$

Generalized forces, 2

We can describe relations between Cartesian torque $\mathbf{m}$ and associated generalized force:

$$\tau_i = \left(\frac{\partial \omega_i}{\partial \dot{\mathbf{q}}}\right)^\top \mathbf{m}_i \tag{4}$$

where $\omega_i = \omega_i(\mathbf{q}, \dot{\mathbf{q}})$ is the angular velocity of the body to which the Cartesian torque $\mathbf{m}$ is applied.

If we define jacobian $\mathbf{J}_i^\omega = \frac{\partial \omega_i}{\partial \dot{\mathbf{q}}}$ we can re-write the relation above as:

$$\tau_i = (\mathbf{J}_i^\omega)^\top \mathbf{m}_i \tag{5}$$

Generalized forces, 3

We can define a *wrench* $\mathbf{w}$:

$$\mathbf{w} = \begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix} \tag{6}$$

We can describe relations between a wrench $\mathbf{w}$ and associated generalized force:

$$\tau_i = \begin{bmatrix} \mathbf{J}_i^r \\ \mathbf{J}_i^\omega \end{bmatrix}^\top \mathbf{w}_i = \mathbf{J}_i^\top \mathbf{w}_i \tag{7}$$

Note that the total generalized force can be computed as:

$$\tau = \sum \mathbf{J}_i^\top \mathbf{w}_i \tag{8}$$

Lagrange equations with constraints

Lagrange equations with constraints have form:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} = \tau + \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^{\top} \lambda \\ \mathbf{r}(\mathbf{q}) = 0 \end{cases} \tag{9}$$

where $\mathbf{r}(\mathbf{q}) = 0$ are constraints and λ are reaction forces.

We can think of λ as concatenation of all reaction forces associated with constraints.

Constraints, Example

Let us consider a planar three link mechanism, whose end-effector is described as:

$$\begin{cases} x_e = l_1 \cos q_1 + l_2 \cos q_2 + l_3 \cos q_3 \\ y_e = l_1 \sin q_1 + l_2 \sin q_2 + l_3 \sin q_3 \end{cases} \tag{10}$$

Then, if the end-effector is attached to the ground at a point $x_e^* = 1$, $y_e^* = 0$, the constraints look like the following:

$$\begin{cases} l_1 \cos q_1 + l_2 \cos q_2 + l_3 \cos q_3 - 1 = 0 \\ l_1 \sin q_1 + l_2 \sin q_2 + l_3 \sin q_3 = 0 \end{cases} \tag{11}$$

Constraints

In general, we distinguish between constraints *expression* and constraints *value*. For example, a point K on the end-effector is described by radius-vector $\mathbf{r}_K = \mathbf{r}_K(\mathbf{q})$. If we affix K at a particular value $\mathbf{r}_K^*$:

- $\mathbf{r}_K(\mathbf{q})$ is the constraint expression;
- $\mathbf{r}_K^*$ is the constraint value.

The constraint will take a form $\mathbf{r}_K(\mathbf{q}) - \mathbf{r}_K^* = 0$. Note that constraint value does not influence the constraint jacobians; therefore, as long as we only need constraint jacobians, we do not need to know constraint values.

Manipulator equations

Manipulator equations have form:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau \tag{12}$$

where $\mathbf{H} = \mathbf{H}(\mathbf{q})$ is generalized inertia matrix, $\mathbf{C}\dot{\mathbf{q}}$ is generalized inertial forces and $\mathbf{g} = \mathbf{g}(\mathbf{q})$ are generalized gravitational forces.

Manipulator equations with constraints

Manipulator equations have form:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau + \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^{\top} \lambda \\ \mathbf{r}(\mathbf{q}) = 0 \end{cases} \tag{13}$$

Defining constraint jacobian $\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}}$ we can re-write the equations:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau + \mathbf{J}^{\top} \lambda \\ \mathbf{r}(\mathbf{q}) = 0 \end{cases} \tag{14}$$

Constraint differentiation

Differentiating constraint $\mathbf{r}(\mathbf{q})$ we get:

$$\frac{\partial \mathbf{r}}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial t} = 0 \tag{15}$$

This can be written as:

$$\mathbf{J}\dot{\mathbf{q}} = 0 \tag{16}$$

Differentiating this once more we get:

$$\mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} = 0 \tag{17}$$

Second-order constraints

We can replace constraints with their second derivatives:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau + \mathbf{J}^\top \lambda \\ \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} = 0 \end{cases} \tag{18}$$

This is a DAE in variables $\mathbf{q}$ and λ. We can re-write it as a vector-matrix form:

$$\begin{bmatrix} \mathbf{H} & -\mathbf{J}^\top \\ \mathbf{J} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}\dot{\mathbf{q}} \end{bmatrix} \tag{19}$$

Solution to the DAE

$$\begin{bmatrix} \mathbf{H} & -\mathbf{J}^\top \\ \mathbf{J} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}\dot{\mathbf{q}} \end{bmatrix} \tag{20}$$

The matrix-vector equation above can be solved given the following condition: the Shur compliment ($\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top$) needs to be full-rank.

Given that $\mathbf{H}$ is positive-definite, $\mathbf{H}^{-1}$ is also positive-definite. Therefore $\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top$ is symmetric and positive-semidefinite.

For $\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top$ to be positive-definite (full rank), jacobian $\mathbf{J}$ has to be full row-rank.

Constraint jacobian rank

The row rank of the constraint jacobian $\mathbf{J}$ depends on linear independence of constraints. Constraints are not linearly independent, we call the system *overconstrained* or *overdetermined*.

If constraint jacobian $\mathbf{J}$ has linearly dependent rows, we can define a new jacobian $\mathbf{J}_o = \text{row}(\mathbf{J})$ as an orthonormal basis in the row space of the original one, giving us relation $\mathbf{J} = \mathbf{T}\mathbf{J}_o$. We can then define $\gamma = \mathbf{T}^\top \lambda$ and re-write the dynamics as:

$$\begin{bmatrix} \mathbf{H} & -\mathbf{J}_o^\top \\ \mathbf{J}_o & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \gamma \end{bmatrix} = \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}_o \dot{\mathbf{q}} \end{bmatrix} \tag{21}$$

This will not let us recover λ, but we can find $\ddot{\mathbf{q}}$.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Contact-Aware-Control-Fall-2023