

# Geometry of Second-Order Cones

## Contact-aware Control, Lecture 7

by Sergei Savin

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- 2-norm
- Cone
- Second-order cone (linear, affine)
- The role of the free constant
- Plotting level sets

Let us consider a 2-norm as a function  $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ :

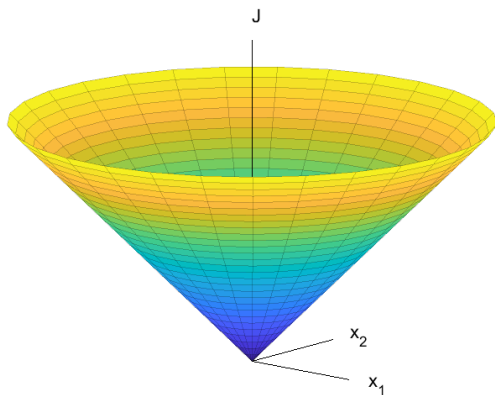
$$f(\mathbf{x}) = \|\mathbf{x}\|_2 \tag{1}$$

$$f(\mathbf{x}) = \sqrt{\sum_{i=1}^n x_i^2} \tag{2}$$

## 2-NORM, 2

We can describe 2-norm as a surface in the  $\mathcal{S} \subset \mathbb{R}^{n+1}$  space:

$$\mathcal{S} = \{(J, \mathbf{x}) : J = \|\mathbf{x}\|_2\} \quad (3)$$



The shape of the surface  $\mathcal{S} = \{(J, \mathbf{x}) : J = \|\mathbf{x}\|_2\}$  is a *cone*. We observe the following properties of a cone:

- There is a single tip point  $\tau$  and a normal direction.
- Slicing cone with planes orthogonal to the normal direction, we produce ellipsoids (we can call it tangent sets).
- For any point  $p$  on the cone, the half-line from the tip point  $\tau$  through  $p$  lies on the cone.

The tip point for  $\mathcal{S}$  is  $(0, \mathbf{0})$ . Normal direction is  $(n, \mathbf{0})$ . Tangent sets are circles  $\|\mathbf{x}\|_2 = h$ , parameterized by  $h$ .

For any point  $(J(\mathbf{p}), \mathbf{p})$  we can write half-line as  $\mathcal{L} = \{(J(\gamma\mathbf{p}), \gamma\mathbf{p}) : \gamma > 0\}$ .

$$J(\gamma\mathbf{p}) = \|\gamma\mathbf{p}\| = \gamma\|\mathbf{p}\| = \gamma J(\mathbf{p}) \quad (4)$$

A second-order cone constraint has the following form:

$$||\mathbf{Ax} + \mathbf{b}|| \leq \mathbf{c}^\top \mathbf{x} + d \quad (5)$$

where  $\mathbf{A} \in \mathbb{R}^{n,n}$ ,  $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$  and  $d \in \mathbb{R}$ .

This constraint describes interior of a cone. We can prove that surface of this set is a cone described by the following equality:

$$||\mathbf{Ax} + \mathbf{b}|| = \mathbf{c}^\top \mathbf{x} + d \quad (6)$$

Let us consider the following simplified case:

$$||\mathbf{Ax}|| = \mathbf{c}^\top \mathbf{x} \quad (7)$$

The tip of this cone is the point  $\mathbf{x} = 0$ , and the normal direction is  $\mathbf{c}$ .

Let us consider a point  $\mathbf{p}$  that lies on the surface (7):

$||\mathbf{Ap}|| - \mathbf{c}^\top \mathbf{p} = 0$ . Clearly, a point  $\gamma\mathbf{p}$  for  $\gamma \geq 0$  lies on the same surface:

$$\begin{aligned} e &= ||\mathbf{A}(\gamma\mathbf{p})|| - \mathbf{c}^\top (\gamma\mathbf{p}) = \gamma||\mathbf{Ap}|| - \gamma\mathbf{c}^\top \mathbf{p} = \\ &= \gamma(||\mathbf{Ap}|| - \mathbf{c}^\top \mathbf{p}) = 0 \end{aligned}$$

Therefore, the half-line going from tip  $\mathbf{x} = 0$  through  $\mathbf{p}$  lies on the cone.

We can consider plane  $\mathcal{P}$  perpendicular to  $\mathbf{c}$ , distance  $h/||\mathbf{c}||$  away from the tip. All points on the plane are described as follows:

$$\mathbf{x} = \frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{T}\mathbf{z}, \quad \forall \mathbf{z} \quad (8)$$

where  $\mathbf{T} = \text{null}(\mathbf{c}^\top)$ , so  $\mathbf{c}^\top \mathbf{T} = 0$ . Then conic surface becomes:

$$\left\| \mathbf{A} \frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{A}\mathbf{T}\mathbf{z} \right\| = \mathbf{c}^\top \left( \frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{T}\mathbf{z} \right) \quad (9)$$

$$||\mathbf{b}_0 + \mathbf{A}\mathbf{T}\mathbf{z}|| = h \quad (10)$$

where  $\mathbf{b}_0 = \frac{h}{||\mathbf{c}||^2} \mathbf{A}\mathbf{c}$ . This is an ellipsoid.  $\square$



For a full second-order cone (SOC):

$$\|\mathbf{Ax} + \mathbf{b}\| = \mathbf{c}^\top \mathbf{x} + d \quad (11)$$

we can find a tip point; it corresponds to both right-hand side and left-hand side becoming zero:

$$\begin{cases} \mathbf{Ax} + \mathbf{b} = 0 \\ \mathbf{c}^\top \mathbf{x} + d = 0 \end{cases} \quad (12)$$

Given full rank matrix  $\mathbf{A}$ , the solution is  $\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$ . The system would hold if:

$$-\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b} + d = 0 \quad (13)$$

Consider a point  $\mathbf{p}_2$  on the surface  $\|\mathbf{A}\mathbf{p}_2 + \mathbf{b}\| = \mathbf{c}^\top \mathbf{p}_2 + d$ . Let us consider an interval from tip  $\mathbf{p}_1 = -\mathbf{A}^{-1}\mathbf{b}$  to  $\mathbf{p}_2$ :

$$\mathbf{x} = \gamma\mathbf{p}_1 + (1 - \gamma)\mathbf{p}_2, \quad \gamma \geq 0 \quad (14)$$

We can check if the points on the interval lie on the surface:

$$e = \|\mathbf{A}\gamma\mathbf{p}_1 + \mathbf{b} + \mathbf{A}(1 - \gamma)\mathbf{p}_2\| - (\gamma\mathbf{c}^\top \mathbf{p}_1 + d + (1 - \gamma)\mathbf{c}^\top \mathbf{p}_2)$$

$$e = \|\gamma\mathbf{A}\mathbf{p}_1 + \gamma\mathbf{b} + (1 - \gamma)\mathbf{b} + \mathbf{A}(1 - \gamma)\mathbf{p}_2\| - \\ - (\gamma\mathbf{c}^\top \mathbf{p}_1 + \gamma d + (1 - \gamma)d + (1 - \gamma)\mathbf{c}^\top \mathbf{p}_2)$$

We know that  $\mathbf{A}\gamma\mathbf{p}_1 + \mathbf{b} = 0$  and  $\mathbf{c}^\top \mathbf{p}_1 + d = 0$ :

$$e = \|(1 - \gamma)\mathbf{b} + \mathbf{A}(1 - \gamma)\mathbf{p}_2\| - (1 - \gamma)d - (1 - \gamma)\mathbf{c}^\top \mathbf{p}_2 \quad (15)$$

$$e = \|\mathbf{b} + \mathbf{A}\mathbf{p}_2\| - (d + \mathbf{c}^\top \mathbf{p}_2) = 0 \quad (16)$$

Thus, the interval (and therefore, the half-line on which it lies) lies on the surface.

Let us consider plane  $\mathcal{P}$  perpendicular to  $\mathbf{c}$ , distance  $h/||\mathbf{c}||$  away from the tip. All points on the plane are described as follows:

$$\mathbf{x} = \frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{Tz}, \quad \forall \mathbf{z} \quad (17)$$

We can substitute it into the second-order cone:

$$||\mathbf{A}(\frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{Tz}) + \mathbf{b}|| = \mathbf{c}^\top (\frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{Tz}) + d \quad (18)$$

$$||\mathbf{b}_0 + \mathbf{ATz}|| = h + d \quad (19)$$

where  $\mathbf{b}_0 = \frac{h}{||\mathbf{c}||^2} \mathbf{Ac} + \mathbf{b}$ . This is an ellipsoid.  $\square$

Free constant  $d$  plays a specific role in the SOC. As we said before, there is a condition  $d = \mathbf{c}^\top \mathbf{A}^{-1} \mathbf{b} + d$ . We can explain it by looking at COS as intersection of two surfaces:

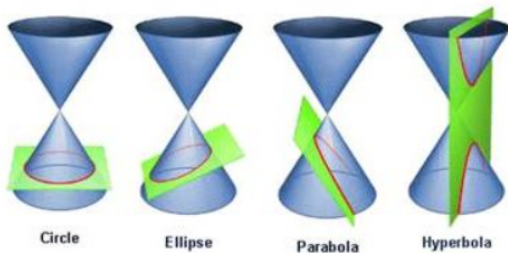
$$J = \|\mathbf{Ax} + \mathbf{b}\| \tag{20}$$

$$P = \mathbf{c}^\top \mathbf{x} + d \tag{21}$$

First is a cone and second is a plane. Their intersection is called a *conic section*.

# THE ROLE OF THE FREE CONSTANT, 2

Typical conic sections are shown below:



As we can see, they represent ellipsoid and parabola. In order for them to represent a cone, the plane  $S$  needs to pass through the tip of the cone  $J$ . This can be achieved with the appropriate choice of constant  $d$ , which shifts  $S$  up or down.

# PLOTTING LEVEL SETS, 1

To plot a cone it is convenient to first use change of coordinates  $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$ , meaning  $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b})$ , giving us SOC:

$$\|\mathbf{y}\| = \mathbf{c}^\top \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b}) + d \quad (22)$$

Note that  $d - \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b} = 0$  for a cone with a tip; so SOC becomes:

$$\|\mathbf{y}\| = \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{y} \quad (23)$$

To plot level sets of this cone we choose height of the level set  $h$  and pick point  $\mathbf{y}_h = h \frac{\mathbf{A}^{-T}\mathbf{c}}{\mathbf{c}^\top \mathbf{A}^{-1} \mathbf{A}^{-T} \mathbf{c}}$ ; we note that  $\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{y}_h = h$ . Then we consider points on the plane  $\mathcal{P}$  orthogonal to  $\mathbf{c}^\top \mathbf{A}^{-1}$  and passing through  $\mathbf{y}_h$ :

$$\mathcal{P} = \mathbf{y}_h + \mathbf{T}\mathbf{z} : \forall \mathbf{z} \quad (24)$$

where  $\mathbf{T} = \text{null}(\mathbf{c}^\top \mathbf{A}^{-1})$ , so  $\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{T} = 0$ .

## PLOTTING LEVEL SETS, 2

Since SOC becomes:

$$||\mathbf{y}_h + \mathbf{T}\mathbf{z}|| = h \quad (25)$$

Since  $\mathbf{y}_h$  and  $\mathbf{T}\mathbf{z}$  are orthogonal, it is equivalent to:

$$||\mathbf{T}\mathbf{z}|| = g \quad (26)$$

where  $g = \sqrt{h^2 - \mathbf{y}_h^\top \mathbf{y}_h}$ . In the 3D case, this is a circle with radius  $g$ . We can find  $N$  consecutive evenly spaced points of this circle, resulting in the next sequence of  $\mathbf{y}_l$ :

$$\mathbf{y}_l = \mathbf{y}_h + \mathbf{T} \begin{bmatrix} g \cos(\varphi) \\ -g \sin(\varphi) \end{bmatrix}, \quad \varphi = 0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, \dots, 2\pi \quad (27)$$

$$\mathbf{x}_l = \mathbf{A}^{-1}(\mathbf{y}_l - \mathbf{b}) \quad (28)$$

The center of the ellipsoid representing this level set lies at the point  $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{y}_h - \mathbf{b})$ .

# PLOTTING LEVEL SETS, 3

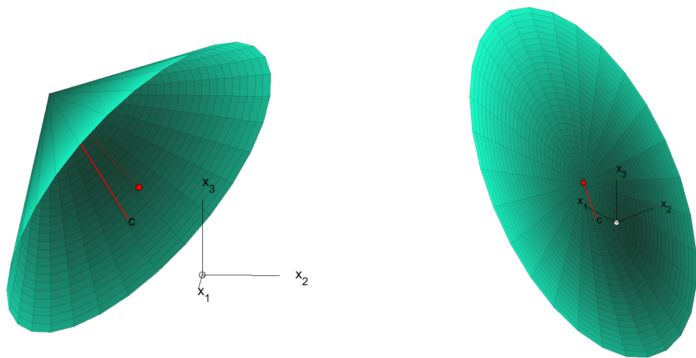


Figure 1: Cone. Dashed line - centers of level-sets.



Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:

[github.com/SergeiSa/Contact-Aware-Control-Fall-2023](https://github.com/SergeiSa/Contact-Aware-Control-Fall-2023)

