

# Mechanical Equations with Constraints

## Contact-aware Control, Lecture 2

by Sergei Savin

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- Lagrange equations
- Generalized forces
- Lagrange equations with constraints
- Constraints
- Manipulator equations
- Constraints differentiation
- Solution to DAE

Lagrange equations have form:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \boldsymbol{\tau} \quad (1)$$

where  $T$  is kinetic energy,  $\mathbf{q}$  is a vector of generalized coordinates and  $\boldsymbol{\tau}$  are generalized torques.

Note that kinetic energy can be described as  $T = \frac{1}{2} \dot{\mathbf{q}}^\top \mathbf{H} \dot{\mathbf{q}}$ , where  $\mathbf{H}$  is generalized inertia matrix. Matrix  $\mathbf{H}$  is symmetric, positive-definite and full rank.

Generalized forces  $\tau$  can be generated by Cartesian forces or Cartesian torques. We can describe relations between Cartesian force  $\mathbf{f}$  and associated generalized force:

$$\tau_i = \left( \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}} \right)^\top \mathbf{f}_i \quad (2)$$

where  $\mathbf{r}_i = \mathbf{r}_i(\mathbf{q})$  is the vector describing position of the point of application of the force  $\mathbf{f}_i$ , as a function of generalized coordinates  $\mathbf{q}$ .

If we define jacobian  $\mathbf{J}_i = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}}$  we can re-write the relation above as:

$$\tau_i = (\mathbf{J}_i^r)^\top \mathbf{f}_i \quad (3)$$

We can describe relations between Cartesian torque  $\mathbf{m}$  and associated generalized force:

$$\tau_i = \left( \frac{\partial \omega_i}{\partial \dot{\mathbf{q}}} \right)^\top \mathbf{m}_i \quad (4)$$

where  $\omega_i = \omega_i(\mathbf{q}, \dot{\mathbf{q}})$  is the angular velocity of the body to which the Cartesian torque  $\mathbf{m}$  is applied.

If we define jacobian  $\mathbf{J}_i^\omega = \frac{\partial \omega_i}{\partial \dot{\mathbf{q}}}$  we can re-write the relation above as:

$$\tau_i = (\mathbf{J}_i^\omega)^\top \mathbf{m}_i \quad (5)$$

We can define a *wrench*  $\mathbf{w}$ :

$$\mathbf{w} = \begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix} \quad (6)$$

We can describe relations between a wrench  $\mathbf{w}$  and associated generalized force:

$$\tau_i = \begin{bmatrix} \mathbf{J}_i^r \\ \mathbf{J}_i^\omega \end{bmatrix}^\top \mathbf{w}_i = \mathbf{J}_i^\top \mathbf{w}_i \quad (7)$$

Note that the total generalized force can be computed as:

$$\tau = \sum \mathbf{J}_i^\top \mathbf{w}_i \quad (8)$$

Lagrange equations with constraints have form:

$$\begin{cases} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \boldsymbol{\tau} + \left( \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^{\top} \boldsymbol{\lambda} \\ \mathbf{r}(\mathbf{q}) = 0 \end{cases} \quad (9)$$

where  $\mathbf{r}(\mathbf{q}) = 0$  are constraints and  $\boldsymbol{\lambda}$  are reaction forces.

We can think of  $\boldsymbol{\lambda}$  as concatenation of all reaction forces associated with constraints.

Let us consider a planar three link mechanism, whose end-effector is described as:

$$\begin{cases} x_e = l_1 \cos q_1 + l_2 \cos q_2 + l_3 \cos q_3 \\ y_e = l_1 \sin q_1 + l_2 \sin q_2 + l_3 \sin q_3 \end{cases} \quad (10)$$

Then, if the end-effector is attached to the ground at a point  $x_e^* = 1, y_e^* = 0$ , the constraints look like the following:

$$\begin{cases} l_1 \cos q_1 + l_2 \cos q_2 + l_3 \cos q_3 - 1 = 0 \\ l_1 \sin q_1 + l_2 \sin q_2 + l_3 \sin q_3 = 0 \end{cases} \quad (11)$$



In general, we distinguish between constraints *expression* and constraints *value*. For example, a point K on the end-effector is described by radius-vector  $\mathbf{r}_K = \mathbf{r}_K(\mathbf{q})$ . If we affix K at a particular value  $\mathbf{r}_K^*$ :

- $\mathbf{r}_K(\mathbf{q})$  is the constraint expression;
- $\mathbf{r}_K^*$  is the constraint value.

The constraint will take a form  $\mathbf{r}_K(\mathbf{q}) - \mathbf{r}_K^* = 0$ . Note that constraint value does not influence the constraint jacobians; therefore, as long as we only need constraint jacobians, we do not need to know constraint values.

Manipulator equations have form:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \boldsymbol{\tau} \quad (12)$$

where  $\mathbf{H} = \mathbf{H}(\mathbf{q})$  is generalized inertia matrix,  $\mathbf{C}\dot{\mathbf{q}}$  is generalized inertial forces and  $\mathbf{g} = \mathbf{g}(\mathbf{q})$  are generalized gravitational forces.

Manipulator equations have form:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau + \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^\top \lambda \\ \mathbf{r}(\mathbf{q}) = 0 \end{cases} \quad (13)$$

Defining constraint jacobian  $\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}}$  we can re-write the equations:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau + \mathbf{J}^\top \lambda \\ \mathbf{r}(\mathbf{q}) = 0 \end{cases} \quad (14)$$

Differentiating constraint  $\mathbf{r}(\mathbf{q})$  we get:

$$\frac{\partial \mathbf{r}}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial t} = 0 \quad (15)$$

This can be written as:

$$\mathbf{J}\dot{\mathbf{q}} = 0 \quad (16)$$

Differentiating this once more we get:

$$\mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} = 0 \quad (17)$$

We can replace constraints with their second derivatives:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau + \mathbf{J}^\top \lambda \\ \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} = 0 \end{cases} \quad (18)$$

This is a DAE in variables  $\mathbf{q}$  and  $\lambda$ . We can re-write it as a vector-matrix form:

$$\begin{bmatrix} \mathbf{H} & -\mathbf{J}^\top \\ \mathbf{J} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}\dot{\mathbf{q}} \end{bmatrix} \quad (19)$$

$$\begin{bmatrix} \mathbf{H} & -\mathbf{J}^\top \\ \mathbf{J} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}\dot{\mathbf{q}} \end{bmatrix} \quad (20)$$

The matrix-vector equation above can be solved given the following condition: the Shur complement  $(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)$  needs to be full-rank.

Given that  $\mathbf{H}$  is positive-definite,  $\mathbf{H}^{-1}$  is also positive-definite. Therefore  $\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top$  is symmetric and positive-semidefinite.

For  $\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top$  to be positive-definite (full rank), jacobian  $\mathbf{J}$  has to be full row-rank.

The row rank of the constraint jacobian  $\mathbf{J}$  depends on linear independence of constraints. Constraints are not linearly independent, we call the system *overconstrained* or *overdetermined*.

If constraint jacobian  $\mathbf{J}$  has linearly dependent rows, we can define a new jacobian  $\mathbf{J}_o = \text{row}(\mathbf{J})$  as an orthonormal basis in the row space of the original one, giving us relation  $\mathbf{J} = \mathbf{T}\mathbf{J}_o$ . We can then define  $\gamma = \mathbf{T}^\top \lambda$  and re-write the dynamics as:

$$\begin{bmatrix} \mathbf{H} & -\mathbf{J}_o^\top \\ \mathbf{J}_o & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \gamma \end{bmatrix} = \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}_o\dot{\mathbf{q}} \end{bmatrix} \quad (21)$$

This will not let us recover  $\lambda$ , but we can find  $\ddot{\mathbf{q}}$ .

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:

[github.com/SergeiSa/Contact-Aware-Control-Fall-2023](https://github.com/SergeiSa/Contact-Aware-Control-Fall-2023)

