Orthogonal Observer

Contact-aware Control, Lecture 6

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LTI system with explicit constraints (EC-LTI) can be presented in the following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\lambda \\ \mathbf{G}\dot{\mathbf{x}} = 0 \end{cases} \tag{1}$$

It is equivalent to LTI system with implicit constraints (IC-LTI):

$$\dot{\mathbf{x}} = \mathbf{A}_c\mathbf{x} + \mathbf{B}_c\mathbf{u} \tag{2}$$

where $\mathbf{A}_c = (\mathbf{I} - \mathbf{F}(\mathbf{G}\mathbf{F})^{+}\mathbf{G})\mathbf{A}$ and $\mathbf{B}_c = (\mathbf{I} - \mathbf{F}(\mathbf{G}\mathbf{F})^{+}\mathbf{G})\mathbf{B}$.

We define $\mathbf{N} = \text{null}(\mathbf{G})$ and $\mathbf{R} = \text{col}(\mathbf{G}^\top)$, giving us $\mathbf{x} = \mathbf{N}\mathbf{z} + \mathbf{R}\zeta$ and $\dot{\mathbf{x}} = \mathbf{N}\dot{\mathbf{z}}$. Multiplying the IC-LTI by $\mathbf{N}^\top$ we get:

$$\dot{\mathbf{z}} = \mathbf{N}^\top \mathbf{A}_c (\mathbf{N}\mathbf{z} + \mathbf{R}\zeta) + \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \tag{3}$$

Orthogonal LQR is solved just as the regular one, but with different quadruple $(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R})$:

$$\mathbf{K}_z = \text{lqr}((\mathbf{N}^\top \mathbf{A}_c \mathbf{N}),\ (\mathbf{N}^\top \mathbf{B}_c),\ \mathbf{Q}_N,\ \mathbf{R}) \tag{4}$$

which minimizes the cost function:

$$J = \int \left(\mathbf{z}^\top \mathbf{Q}_N \mathbf{z} + \mathbf{u}^\top \mathbf{R} \mathbf{u} \right) dt \tag{5}$$

We can define IC-LTI with observation output $\mathbf{y}$:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_c \mathbf{x} + \mathbf{B}_c \mathbf{u} \\ \mathbf{y} = \mathbf{C} \mathbf{x} \end{cases} \tag{6}$$

It can be observable or not; however, we can take advantage of the constraints to lower the number of states we need to observe.

As before, we multiply the dynamics by $\mathbf{N}^\top$ and replace $\mathbf{x}$ by pair $(\mathbf{z}, \zeta)$:

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{N}^\top \mathbf{A}_c \mathbf{N} \mathbf{z} + \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \zeta + \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \\ \mathbf{y} = \mathbf{C} \mathbf{N} \mathbf{z} + \mathbf{C} \mathbf{R} \zeta \end{cases} \tag{7}$$

We can re-write the dynamics in terms of the pair $(\mathbf{z}, \zeta)$:

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} \mathbf{N}^\top \mathbf{A}_c \mathbf{N} & \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \zeta \end{bmatrix} + \begin{bmatrix} \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \\ \mathbf{0} \end{bmatrix} \\ \mathbf{y} = \mathbf{C} \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \zeta \end{bmatrix} \end{cases} \tag{8}$$

We will call it *subspace representation.*

Using this representation we can propose the following state observer:

$$\begin{bmatrix} \dot{\hat{\mathbf{z}}} \\ \dot{\hat{\zeta}} \end{bmatrix} = \begin{bmatrix} \mathbf{N}^\top \mathbf{A}_c \mathbf{N} & \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\zeta} \end{bmatrix} + \begin{bmatrix} \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \\ \mathbf{0} \end{bmatrix} + \mathbf{L} \left(\mathbf{y} - \mathbf{C} \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\zeta} \end{bmatrix} \right)$$

where $\hat{\mathbf{z}}$ and $\hat{\zeta}$ are estimates of $\mathbf{z}$ and ζ.

Theorem

Orthogonal Observer and IC-LTI dynamics, with control law $\mathbf{u} = -\mathbf{K}_z \hat{\mathbf{z}} - \mathbf{K}_\zeta \hat{\zeta}$ *form a stable system as the following condition: are satisfied:*

$$\begin{bmatrix} \mathbf{N}^\top \\ \mathbf{R}^\top \end{bmatrix} \left(\begin{bmatrix} \mathbf{A}_c^\top \mathbf{N} \\ \mathbf{0} \end{bmatrix} - \mathbf{C}^\top \mathbf{L}^\top \right) \in \mathbb{H} \tag{9}$$

Here we assume that $\mathbf{K}_z$ and $\mathbf{K}_\zeta$ are chosen such that:

$$\mathbf{N}^\top \mathbf{A}_c \mathbf{N} - \mathbf{N}^\top \mathbf{B}_c \mathbf{K}_z \in \mathbb{H} \tag{10}$$

$$\mathbf{K}_\zeta = (\mathbf{N}^\top \mathbf{B}_c)^+ \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \tag{11}$$

Defining observation error $\mathbf{e} = \begin{bmatrix} \mathbf{z} - \hat{\mathbf{z}} \\ \zeta - \hat{\zeta} \end{bmatrix}$ and subtracting observer from system dynamics we get observer error dynamics:

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} \mathbf{N}^\top \mathbf{A}_c \mathbf{N} & \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \zeta \end{bmatrix} + \begin{bmatrix} \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \\ \mathbf{0} \end{bmatrix} \\ \mathbf{y} = \mathbf{C} \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \zeta \end{bmatrix} \end{cases}$$

$$\begin{bmatrix} \dot{\hat{\mathbf{z}}} \\ \dot{\hat{\zeta}} \end{bmatrix} = \begin{bmatrix} \mathbf{N}^\top \mathbf{A}_c \mathbf{N} & \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\zeta} \end{bmatrix} + \begin{bmatrix} \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \\ \mathbf{0} \end{bmatrix} + \mathbf{L} \left(\mathbf{y} - \mathbf{C} \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\zeta} \end{bmatrix} \right)$$

$$\dot{\mathbf{e}} = \left(\begin{bmatrix} \mathbf{N}^\top \mathbf{A}_c \mathbf{N} & \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \mathbf{L} \mathbf{C} \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix} \right) \mathbf{e} \tag{12}$$

Substituting control law, we find state dynamics as:

$$\dot{\mathbf{z}} = (\mathbf{N}^\top \mathbf{A}_c \mathbf{N} - \mathbf{N}^\top \mathbf{B}_c \mathbf{K}_z)\mathbf{z} + \\ + \mathbf{N}^\top \mathbf{B}_c \mathbf{K} \mathbf{e} + (\mathbf{N}^\top \mathbf{A}_c \mathbf{R} - \mathbf{N}^\top \mathbf{B}_c \mathbf{K}_\zeta)\zeta.$$

where $\mathbf{K} = \begin{bmatrix} \mathbf{K}_z & \mathbf{K}_\zeta \end{bmatrix}$.

With that we can write the combined state and observer dynamics:

$$\begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} (\mathbf{N}^\top \mathbf{A}_c \mathbf{N} - \mathbf{N}^\top \mathbf{B}_c \mathbf{K}_z) & \mathbf{N}^\top \mathbf{B}_c \mathbf{K} \\ \mathbf{0} & (\bar{\mathbf{N}}^\top \mathbf{A}_c - \mathbf{L}\mathbf{C})\mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{e} \end{bmatrix} + \text{const} \tag{13}$$

where $\bar{\mathbf{N}} = \begin{bmatrix} \mathbf{N} & \mathbf{0}_{n \times n} \end{bmatrix}$, and $\mathbf{E} = \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix}$. Since the state matrix here is upper triangular, we only need the diagonal blocks $(\mathbf{N}^\top \mathbf{A}_c \mathbf{N} - \mathbf{N}^\top \mathbf{B}_c \mathbf{K}_z)$ and $(\bar{\mathbf{N}}^\top \mathbf{A}_c - \mathbf{L}\mathbf{C})\mathbf{E}$ to be stable for the system to be stable. Transpose of the last one gives us proof of the theorem. $\square$

Note that we derived a *separation principle* for the system with constraints.

- Savin, S., Balakhnov, O., Khusainov, R. and Klimchik, A., 2021. State observer for linear systems with explicit constraints: Orthogonal decomposition method. Sensors, 21(18), p.6312

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Contact-Aware-Control-Fall-2023