Friction Cone

Contact-aware Control, Lecture 8

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Dry friction

Dry friction is a physical effect, appearing as a force preventing sliding of two rigid bodies. It is typically modeled as a constraint with a condition of changing from static contact to sliding.

There are a number of friction models, different in conditions for contact regime change. One of the popular models is friction cone - requiring friction force to stay inside a cone, normal to the surface.

Friction cone in 2D

In two dimensional case, the dry friction model with a friction cone can be described as:

$$|f_\tau| \leq \mu f_n \tag{1}$$

where μ is friction coefficient, f_τ is the magnitude of the friction force and f_n is the magnitude of the normal reaction force.

Friction in 3D, 1

Friction force $\mathbf{f}_\tau$ together with normal reaction force $\mathbf{f}_n$ together form contact reaction force $\mathbf{f}_R$:

$$\mathbf{f}_R = \mathbf{f}_n + \mathbf{f}_\tau \tag{2}$$

We can choose to represent the reaction force in a basis $\mathbf{B}$ formed by concatenating normal direction $\mathbf{n}$ and two tangent directions $\mathbf{t}_1$, $\mathbf{t}_2$.

$$\mathbf{f}_R = \mathbf{B} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} \tag{3}$$

Friction in 3D, 2

We can prove that $f_n = \mathbf{n}^\top \mathbf{f}_R$:

$$\mathbf{n}^\top \mathbf{f}_R = \mathbf{n}^\top \begin{bmatrix} \mathbf{n} & \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = f_n \tag{4}$$

We can prove that $\begin{bmatrix} f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}^\top \mathbf{f}_R$:

$$\begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}^\top \mathbf{f}_R = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}^\top \begin{bmatrix} \mathbf{n} & \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \tag{5}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \begin{bmatrix} f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} \tag{6}$$

Friction cone representations, 1

We can write friction cone constraint as follows:

$$\sqrt{f_{\tau,1}^2 + f_{\tau,2}^2} \leq \mu f_n \tag{7}$$

where μ is friction coefficient, f_τ is the magnitude of the friction force and f_n is the magnitude of the normal reaction force.

We can describe it as *element-wise description*. The simplicity of this description makes it quite attractive.

Friction cone representations, 2

It is possible to re-write the same constraint as:

$$\left|\left|\begin{bmatrix}\mathbf{t}_1 & \mathbf{t}_2\end{bmatrix}^\top \mathbf{f}_R\right|\right| \leq \mu \mathbf{n}^\top \mathbf{f}_R \tag{8}$$

We can describe it as a *vector description*. The advantage of this description is the use of a single vector variable $\mathbf{f}_R$. It takes the form of a second-order cone (SOC) constraint.

Note that $\mathbf{t}_1$ and $\mathbf{t}_2$ are usually not given, and can be chosen arbitrarily, up to rotation. We can find them as a left null space of the normal vector: $\mathbf{T} = \begin{bmatrix}\mathbf{t}_1 & \mathbf{t}_2\end{bmatrix} = \text{null}(\mathbf{n}^\top)$:

$$||\mathbf{T}^\top \mathbf{f}_R|| \leq \mu \mathbf{n}^\top \mathbf{f}_R \tag{9}$$

Friction cone representations, 3

We can do the same with projectors:

$$||\mathbf{P}_\tau \mathbf{f}_R|| \leq \mu ||\mathbf{P}_n \mathbf{f}_R|| \tag{10}$$

where:

$$\mathbf{P}_\tau = \mathbf{I} - \mathbf{n}\mathbf{n}^\top \tag{11}$$

$$\mathbf{P}_n = \mathbf{n}\mathbf{n}^\top \tag{12}$$

We can describe it as a *projector description*. Here the advantage is the fact that we do not need to compute SVD decomposition (which is needed to compute the null space). We can combine vector and projector descriptors:

$$||(\mathbf{I} - \mathbf{n}\mathbf{n}^\top)\mathbf{f}_R|| \leq \mu \mathbf{n}^\top \mathbf{f}_R \tag{13}$$

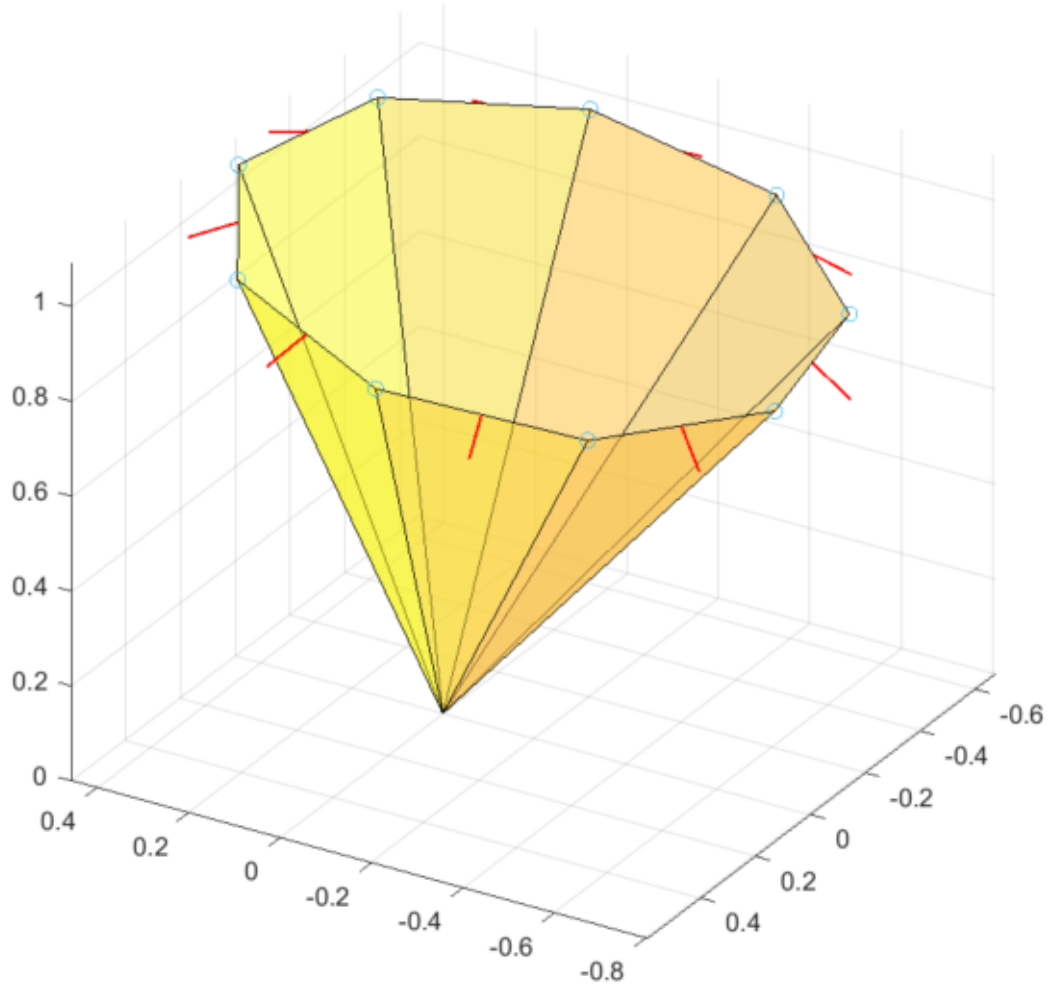
Linearization, 1

SOC constraints in convex optimization imply SOC problems, which is harder to solve than, e.g. quadratic programs. Replacing cones with linear constraints allows to turn SOCP to QP.

Geometrically, a linearized cone is a polytope. The cost of turning SOCP to QP is over- (or under-) approximation.

Linearization, 2

Here is how a linear approximation of a friction cone can look like (red are normals to half-spaces / faces):



Under- and over-approximation, 1

When approximating friction cone, we aim to retain some of the properties of the original geometry.

Under-approximation (inner approximation) implies that the approximate cone $\mathcal{C}_a$ lies inside the original cone $\mathcal{C}$. Meaning any point $\mathbf{x} \in \mathcal{C}_a$ is also in $\mathcal{C}$, but $\exists \mathbf{y} : \mathbf{y} \in \mathcal{C}, \mathbf{y} \notin \mathcal{C}_a$.

Over-approximation (outer approximation) implies that the original cone $\mathcal{C}$ lies inside the approximate cone $\mathcal{C}_a$. Meaning any point $\mathbf{x} \in \mathcal{C}$ is also in $\mathcal{C}_a$, but $\exists \mathbf{y} : \mathbf{y} \in \mathcal{C}_a, \mathbf{y} \notin \mathcal{C}$.

Usually we assume that approximation is as tight as possible, given number of faces allocated for the approximate cone $\mathcal{C}_a$.

Under- and over-approximation, 2

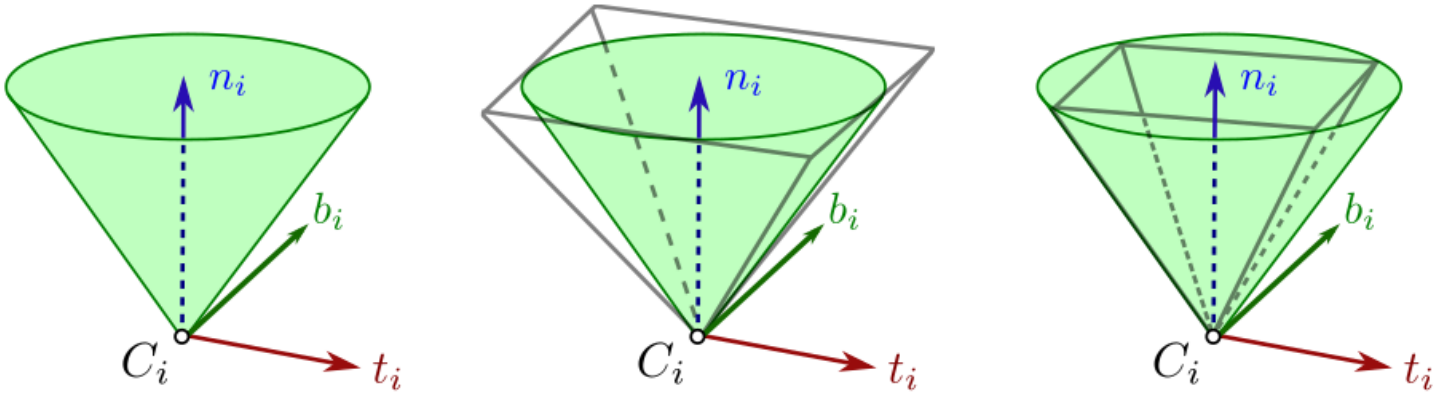


Figure 1: Original cone, inner and outer approximations. Image credit.

H-representation, 1

One of the ways to encode a linearized friction cone is H-representation - presenting the polytope as a collection of linear inequalities, which represent faces of the polytope. Let us compare cone and H-polytope as sets:

- $\mathcal{C} = \{\mathbf{x} : \ ||(\mathbf{I} - \mathbf{n}\mathbf{n}^\top)\mathbf{f}_R|| \leq \mu \mathbf{n}^\top \mathbf{f}_R\}$, - cone;
- $\mathcal{C}_a = \{\mathbf{x} : \ \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, - H-polytope.

H-representation, 2

There are many way to generate matrix $\mathbf{A}$ and vector $\mathbf{b}$ in H-representation.

Assuming the only restriction imposed on the normal reaction force is its positive value (it pushes and not pulls on the ground), all faces of the friction polytope pass through the origin. Each face lies on a boundary of halfspace represented as:

$$\mathbf{a}_i^\top \mathbf{x} \leq b_i \tag{14}$$

where $\mathbf{a}_i$ is the i-th row of the matrix $\mathbf{A}$. The Plane of which the face lies is represented by equality $\mathbf{a}_i^\top \mathbf{x} \leq b_i$. But since $\mathbf{x} = 0$ lies on each of these faces, it implies $\mathbf{b} = \mathbf{0}$.

H-representation, 3

The key requirement in generating friction polytope is doing it with less computation. One of the ways to do it is by starting with generating a number of equally spaced points $\mathbf{v}_i$ on a level set of the cone (such that $\mathbf{n}^\top \mathbf{v}_i = \mathbf{n}^\top \mathbf{v}_j \ \forall i, j$).

We can describe a plane passing through points $\mathbf{0}, \mathbf{v}_i, \mathbf{v}_{i+1}$ by its normal vector $\mathbf{m}_i$:

$$\mathbf{m}_i = \mathbf{v}_i \times \mathbf{v}_{i+1} \tag{15}$$

These normals $\mathbf{n}_i$ are equivalent to $\mathbf{a}_i$ up to a sign. To make sure the vectors are pointed "out" of the cone, we can use the following trick:

$$\mathbf{a}_i = \mathbf{m}_i \ \mathrm{sign}(\mathbf{m}_i^\top \mathbf{P}(\mathbf{v}_i + \mathbf{v}_{i+1})) \tag{16}$$

H-representation, 4

Sometimes we want to add limits of minimum and maximum normal reaction force, e.g. to avoid forces that test the limits of our friction models. This can be done by appending the following inequalities to our H-representation:

$$\mathbf{n}^\top \mathbf{x} \leq f_{\max} \tag{17}$$

$$-\mathbf{n}^\top \mathbf{x} \leq -f_{\min} \tag{18}$$

V-representation

Alternatively we could use *vertex representation* of a friction cone, that requires a friction force to lie inside a polytope generated as a convex combination of its vertices $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots$; Unlike in the previous example, these should include the origin.

We can represent such polytope as a set in the following way:

$$\mathcal{C}_a = \{\mathbf{x}:\ \mathbf{x} = \sum \alpha_i \mathbf{v}_i;\ \sum \alpha_i = 1;\ \alpha_i \geq 0\} \tag{19}$$

V-rep vs H-rep

- In H-rep solving containment problem comes to evaluating linear inequalities. In V-rep solving containment problem requires solving an optimization problem.
- V-rep naturally handles upper limit on normal reaction force; in fact, the choice of vertices "height" determines the upper limit.
- V-rep skips the step of constructing the half-space normal.

V-representation without a cap, 1

We can lift the cap on the maximum normal reaction by excluding the condition $\sum \alpha_i = 1$:

$$\mathcal{C}_a = \{\mathbf{x} : \ \mathbf{x} = \sum \alpha_i \mathbf{v}_i; \ \alpha_i \geq 0\} \tag{20}$$

Assume that $\mathbf{v}_1 = 0$ and $\mathbf{v}_i, \forall i \neq 1$ lie on a level-set, meaning $\mathbf{n}^\top \mathbf{v}_i = h, \ \forall i \neq 1$. Let us consider the subset $\mathcal{C}_d$:

$$\mathcal{C}_d = \{\mathbf{x} : \ \mathbf{x} = \sum \alpha_i \mathbf{w}_i; \ \sum \alpha_i = 1; \ \alpha_i \geq 0\} \tag{21}$$

where $\mathbf{w}_i = k\mathbf{v}_i$. Defining $\beta_i = k\alpha_i$:

$$\mathcal{C}_d = \{\mathbf{x} : \ \mathbf{x} = \sum \beta_i \mathbf{v}_i; \ \sum \beta_i = k; \ \beta_i \geq 0\} \tag{22}$$

We can see that choice of k decided the size of $\mathcal{C}_d$; without the constraint $\sum \alpha_i = 1$ (and hence $\sum \beta_i = k$) we get a set $\mathcal{C}_a$ which is a union of all sets $\mathcal{C}_d$.

Example

A simple example of linear approximation of a friction cone is:

$$\begin{cases} |f_{\tau,1}| \leq \mu f_n \\ |f_{\tau,2}| \leq \mu f_n \end{cases} \tag{23}$$

This approximation has 4 faces.

Read more

- Friction cones, notes by Stéphane Caron.
  scaron.info/robotics/friction-cones.html

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Contact-Aware-Control-Fall-2023