

# Orthogonal Observer

## Contact-aware Control, Lecture 6

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- recap
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- Subspace representation
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- Separation Principle

LTI system with explicit constraints (EC-LTI) can be presented in the following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\lambda \\ \mathbf{G}\dot{\mathbf{x}} = 0 \end{cases} \quad (1)$$

It is equivalent to LTI system with implicit constraints (IC-LTI):

$$\dot{\mathbf{x}} = \mathbf{A}_c\mathbf{x} + \mathbf{B}_c\mathbf{u} \quad (2)$$

where  $\mathbf{A}_c = (\mathbf{I} - \mathbf{F}(\mathbf{G}\mathbf{F})^+\mathbf{G})\mathbf{A}$  and  $\mathbf{B}_c = (\mathbf{I} - \mathbf{F}(\mathbf{G}\mathbf{F})^+\mathbf{G})\mathbf{B}$ .

We define  $\mathbf{N} = \text{null}(\mathbf{G})$  and  $\mathbf{R} = \text{col}(\mathbf{G}^\top)$ , giving us  $\mathbf{x} = \mathbf{N}\mathbf{z} + \mathbf{R}\zeta$  and  $\dot{\mathbf{x}} = \mathbf{N}\dot{\mathbf{z}}$ . Multiplying the IC-LTI by  $\mathbf{N}^\top$  we get:

$$\dot{\mathbf{z}} = \mathbf{N}^\top \mathbf{A}_c (\mathbf{N}\mathbf{z} + \mathbf{R}\zeta) + \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \quad (3)$$

Orthogonal LQR is solved just as the regular one, but with different quadruple  $(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R})$ :

$$\mathbf{K}_z = \text{lqr}((\mathbf{N}^\top \mathbf{A}_c \mathbf{N}), (\mathbf{N}^\top \mathbf{B}_c), \mathbf{Q}_N, \mathbf{R}) \quad (4)$$

which minimizes the cost function:

$$J = \int \left( \mathbf{z}^\top \mathbf{Q}_N \mathbf{z} + \mathbf{u}^\top \mathbf{R} \mathbf{u} \right) dt \quad (5)$$

We can define IC-LTI with observation output  $\mathbf{y}$ :

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_c \mathbf{x} + \mathbf{B}_c \mathbf{u} \\ \mathbf{y} = \mathbf{C} \mathbf{x} \end{cases} \quad (6)$$

It can be observable or not; however, we can take advantage of the constraints to lower the number of states we need to observe.

As before, we multiply the dynamics by  $\mathbf{N}^\top$  and replace  $\mathbf{x}$  by pair  $(\mathbf{z}, \zeta)$ :

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{N}^\top \mathbf{A}_c \mathbf{N} \mathbf{z} + \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \zeta + \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \\ \mathbf{y} = \mathbf{C} \mathbf{N} \mathbf{z} + \mathbf{C} \mathbf{R} \zeta \end{cases} \quad (7)$$

We can re-write the dynamics in terms of the pair  $(\mathbf{z}, \zeta)$ :

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} \mathbf{N}^\top \mathbf{A}_c \mathbf{N} & \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \zeta \end{bmatrix} + \begin{bmatrix} \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \\ \mathbf{0} \end{bmatrix} \\ \mathbf{y} = \mathbf{C} \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \zeta \end{bmatrix} \end{cases} \quad (8)$$

We will call it *subspace representation*.

Using this representation we can propose the following state observer:

$$\begin{bmatrix} \dot{\hat{\mathbf{z}}} \\ \dot{\hat{\zeta}} \end{bmatrix} = \begin{bmatrix} \mathbf{N}^\top \mathbf{A}_c \mathbf{N} & \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\zeta} \end{bmatrix} + \begin{bmatrix} \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \\ \mathbf{0} \end{bmatrix} + \mathbf{L} \left( \mathbf{y} - \mathbf{C} [\mathbf{N} \quad \mathbf{R}] \begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\zeta} \end{bmatrix} \right)$$

where  $\hat{\mathbf{z}}$  and  $\hat{\zeta}$  are estimates of  $\mathbf{z}$  and  $\zeta$ .

## Theorem

*Orthogonal Observer and IC-LTI dynamics, with control law  $\mathbf{u} = -\mathbf{K}_z \hat{\mathbf{z}} - \mathbf{K}_\zeta \hat{\zeta}$  form a stable system as the following condition: are satisfied:*

$$\begin{bmatrix} \mathbf{N}^\top \\ \mathbf{R}^\top \end{bmatrix} \left( \begin{bmatrix} \mathbf{A}_c^\top \mathbf{N} \\ \mathbf{0} \end{bmatrix} - \mathbf{C}^\top \mathbf{L}^\top \right) \in \mathbb{H} \quad (9)$$

Here we assume that  $\mathbf{K}_z$  and  $\mathbf{K}_\zeta$  are chosen such that:

$$\mathbf{N}^\top \mathbf{A}_c \mathbf{N} - \mathbf{N}^\top \mathbf{B}_c \mathbf{K}_z \in \mathbb{H} \quad (10)$$

$$\mathbf{K}_\zeta = (\mathbf{N}^\top \mathbf{B}_c)^+ \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \quad (11)$$



# SEPARATION PRINCIPLE, 1

Defining observation error  $\mathbf{e} = \begin{bmatrix} \mathbf{z} - \hat{\mathbf{z}} \\ \zeta - \hat{\zeta} \end{bmatrix}$  and subtracting observer from system dynamics we get observer error dynamics:

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} \mathbf{N}^\top \mathbf{A}_c \mathbf{N} & \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \zeta \end{bmatrix} + \begin{bmatrix} \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \\ \mathbf{0} \end{bmatrix} \\ \mathbf{y} = \mathbf{C} \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \zeta \end{bmatrix} \end{cases}$$

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\mathbf{z}}} \\ \dot{\hat{\zeta}} \end{bmatrix} &= \begin{bmatrix} \mathbf{N}^\top \mathbf{A}_c \mathbf{N} & \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\zeta} \end{bmatrix} + \begin{bmatrix} \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \\ \mathbf{0} \end{bmatrix} + \mathbf{L} \left( \mathbf{y} - \mathbf{C} \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\zeta} \end{bmatrix} \right) \\ \dot{\mathbf{e}} &= \left( \begin{bmatrix} \mathbf{N}^\top \mathbf{A}_c \mathbf{N} & \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \mathbf{L} \mathbf{C} \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix} \right) \mathbf{e} \quad (12) \end{aligned}$$

Substituting control law, we find state dynamics as:

$$\begin{aligned}\dot{\mathbf{z}} = & (\mathbf{N}^\top \mathbf{A}_c \mathbf{N} - \mathbf{N}^\top \mathbf{B}_c \mathbf{K}_z) \mathbf{z} + \\ & + \mathbf{N}^\top \mathbf{B}_c \mathbf{K} \mathbf{e} + (\mathbf{N}^\top \mathbf{A}_c \mathbf{R} - \mathbf{N}^\top \mathbf{B}_c \mathbf{K}_\zeta) \zeta.\end{aligned}$$

where  $\mathbf{K} = [\mathbf{K}_z \quad \mathbf{K}_\zeta]$ .

With that we can write the combined state and observer dynamics:

$$\begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} (\mathbf{N}^\top \mathbf{A}_c \mathbf{N} - \mathbf{N}^\top \mathbf{B}_c \mathbf{K}_z) & \mathbf{N}^\top \mathbf{B}_c \mathbf{K} \\ \mathbf{0} & (\bar{\mathbf{N}}^\top \mathbf{A}_c - \mathbf{L} \mathbf{C}) \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{e} \end{bmatrix} + \text{const} \quad (13)$$

where  $\bar{\mathbf{N}} = [\mathbf{N} \quad \mathbf{0}_{n \times n}]$ , and  $\mathbf{E} = [\mathbf{N} \quad \mathbf{R}]$ . Since the state matrix here is upper triangular, we only need the diagonal blocks  $(\mathbf{N}^\top \mathbf{A}_c \mathbf{N} - \mathbf{N}^\top \mathbf{B}_c \mathbf{K}_z)$  and  $(\bar{\mathbf{N}}^\top \mathbf{A}_c - \mathbf{L} \mathbf{C}) \mathbf{E}$  to be stable for the system to be stable. Transpose of the last one gives us proof of the theorem.  $\square$

Note that we derived a *separation principle* for the system with constraints.

- Savin, S., Balakhnov, O., Khusainov, R. and Klimchik, A., 2021. State observer for linear systems with explicit constraints: Orthogonal decomposition method. Sensors, 21(18), p.6312

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:

[github.com/SergeiSa/Contact-Aware-Control-Fall-2023](https://github.com/SergeiSa/Contact-Aware-Control-Fall-2023)

