ODEs and DAEs

Contact-aware Control, Lecture 1

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Ordinary Differential Equations

A general form of an *n-th order ordinary differential equation* (ODE) is:

$$F\left(\frac{d^n x}{dt^n}, ..., \frac{dx}{dt}, x, t\right) = 0 \tag{1}$$

Notice that $x(t)$ is a scalar variable. A normal form of an ODE is:

$$\frac{d^n x}{dt^n} = f\left(\frac{d^{n-1} x}{dt^{n-1}}, ..., \frac{dx}{dt}, x, t\right) \tag{2}$$

Ordinary Differential Equations - examples

Below we see two examples of an ODE in the general form:

$$\dot{x} + \sin(x+1) = \sin(2t) \tag{3}$$

$$\dot{x}^3 + x = 0 \tag{4}$$

...and two examples of ODEs in the normal form:

$$\dot{x} = \sin(2t) - \sin(x+1) \tag{5}$$

$$\ddot{x} = -5\dot{x} - 2x \tag{6}$$

Linear ODE

A general form of a *linear* n-th order ODE is:

$$a_n \frac{d^n x}{dt^n} + ... + a_1 \frac{dx}{dt} + a_0 x = f(t) \tag{7}$$

Note that it is trivial to transform general form linear ODE to the normal form:

$$\frac{d^n x}{dt^n} = -\frac{a_{n-1}}{a_n} \frac{d^{n-1} x}{dt^{n-1}} - ... - \frac{a_1}{a_n} \frac{dx}{dt} - \frac{a_0}{a_n} x + \frac{1}{a_n} f(t) \tag{8}$$

ODE matrix form

A general form of a first order matrix ODE is:

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t) = 0 \tag{9}$$

Normal form is:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{10}$$

If the ODE (9) is linear with respect $\dot{\mathbf{x}}$ we could transform it into the normal form:

$$\dot{\mathbf{x}} = -\mathbf{M}^{-1}\mathbf{F}(0, \mathbf{x}, t) \tag{11}$$

where $\mathbf{M} = d\mathbf{F}/d\dot{\mathbf{x}}$.

DAE matrix form

A general form of a first order matrix *differential-algebraic equations* (DAE) is:

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t) = 0 \tag{12}$$

where $\mathbf{M} = d\mathbf{F}/d\dot{\mathbf{x}}$ and it is either rectangular or $\det(\mathbf{M}) = 0$.

Let us observe how similar are DAE and general form ODEs.

DAE examples

DAE examples:

$$\begin{cases} \dot{x}_1 + \dot{x}_2 + x_1 x_2 + \sin(t) = 0 \\ x_1 - x_2 = 0 \end{cases} \tag{13}$$

$$\begin{cases} \dot{x}_1 + \dot{x}_2 - 7 = 0 \\ \dot{x}_1 + \dot{x}_2 + x_1 + \cos(x_2) = 0 \end{cases} \tag{14}$$

DAE with explicit algebraic variables

An normal form DAE with algebraic variables is:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda, t) \\ \mathbf{g}(\mathbf{x}, t) = 0 \end{cases} \tag{15}$$

In this equation, λ are algebraic variables, and $\mathbf{g}(\mathbf{x}, t) = 0$ are *constraints*.

Nothing prevents us from defining constraints as $\mathbf{g}(\dot{\mathbf{x}}, \mathbf{x}, t) = 0$, but the clarity of the definition will be lost.

DAE linear in algebraic variables, 1

A DAE linear in algebraic variables is:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{J}(\mathbf{x}, t)\lambda \\ \mathbf{g}(\mathbf{x}, t) = 0 \end{cases} \tag{16}$$

We could solve it by differentiating constraints and expressing $\dot{\mathbf{x}}$ and λ:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{J}(\mathbf{x}, t)\lambda \\ \mathbf{G}(\mathbf{x}, t)\dot{\mathbf{x}} + \mathbf{g}_0(\mathbf{x}, t) = 0 \end{cases} \tag{17}$$

where $\mathbf{G} = d\mathbf{g}/d\mathbf{x}$ and $\frac{d}{dt}\mathbf{g}(\mathbf{x}, t) - \mathbf{G}\dot{\mathbf{x}}$.

DAE linear in algebraic variables, 2

DAE can be re-written as:

$$\begin{bmatrix} \mathbf{I} & -\mathbf{J}(\mathbf{x},t) \\ \mathbf{G}(\mathbf{x},t) & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\mathbf{x},t) \\ -\mathbf{g}_0(\mathbf{x},t) \end{bmatrix} \tag{18}$$

If the matrix on the left-hand side is invertible, we can find $\dot{\mathbf{x}}$ and integrate it forward:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{I} - \mathbf{J}(\mathbf{GJ})^{-1}\mathbf{G} & \mathbf{J}(\mathbf{GJ})^{-1} \\ -(\mathbf{GJ})^{-1}\mathbf{G} & (\mathbf{GJ})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{f}(\mathbf{x},t) \\ -\mathbf{g}_0(\mathbf{x},t) \end{bmatrix} \tag{19}$$

For the lhs matrix to me invertible, the matrix $(\mathbf{GJ})$ needs to be full rank.

DAE linear in algebraic variables, 3

Invertability of the lhs matrix has implications:

- There exists one and only one $(\dot{\mathbf{x}}, \lambda)$ pair that satisfy both the differential and algebraic constraints.
- We can exclude algebraic variables λ from the problem.

DAE linear in algebraic variables, 4

We could propose a more general form of the DAE linear in derivatives and algebraic variables:

$$\begin{bmatrix} \mathbf{M}(\mathbf{x}, t) & -\mathbf{J}(\mathbf{x}, t) \\ \mathbf{G}(\mathbf{x}, t) & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_0(\mathbf{x}, t) \\ -\mathbf{g}_0(\mathbf{x}, t) \end{bmatrix} \tag{20}$$

where $\det(\mathbf{M}) \neq 0$.

We will find these types of systems in robotics. Inclusion of inertia matrix $\mathbf{M}$ does not change the process of solving the equation. It does influence the analytic form of the block matrix inverse though.

more DAE examples

DAE with algebraic variables - examples:

$$\begin{cases} \dot{x}_1 = x_1 + 5x_2 - \lambda \\ \dot{x}_2 = -2x_1 + x_2 + \lambda \\ 2x_1 - x_2 = 0 \end{cases} \tag{21}$$

$$\begin{cases} \dot{x}_1 = 10x_1 + 15x_2 - \lambda_1 + \lambda_2 \\ \dot{x}_2 = x_1 + 5x_2 + \lambda_2 \\ 3x_1 + 7x_2 = 0 \end{cases} \tag{22}$$

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Contact-Aware-Control-Fall-2023