

Geometry of Second-Order Cones

Contact-aware Control, Lecture 7

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- 2-norm
- Cone
- Second-order cone (linear, affine)
- The role of the free constant
- Plotting level sets

Let us consider a 2-norm as a function $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$:

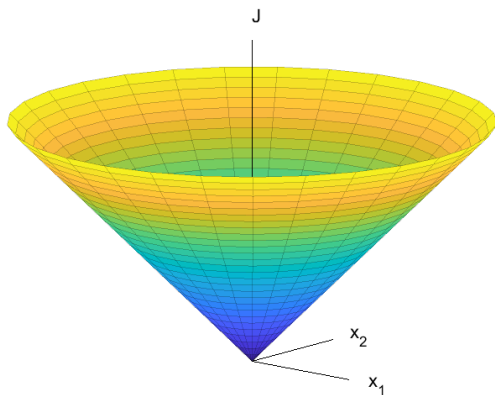
$$f(\mathbf{x}) = \|\mathbf{x}\|_2 \tag{1}$$

$$f(\mathbf{x}) = \sqrt{\sum_{i=1}^n x_i^2} \tag{2}$$

2-NORM, 2

We can describe 2-norm as a surface in the $\mathcal{S} \subset \mathbb{R}^{n+1}$ space:

$$\mathcal{S} = \{(J, \mathbf{x}) : J = \|\mathbf{x}\|_2\} \quad (3)$$



The shape of the surface $\mathcal{S} = \{(J, \mathbf{x}) : J = \|\mathbf{x}\|_2\}$ is a *cone*. We observe the following properties of a cone:

- There is a single tip point τ and a normal direction.
- Slicing cone with planes orthogonal to the normal direction, we produce ellipsoids (we can call it tangent sets).
- For any point p on the cone, the half-line from the tip point τ through p lies on the cone.

The tip point for \mathcal{S} is $(0, \mathbf{0})$. Normal direction is $(n, \mathbf{0})$. Tangent sets are circles $\|\mathbf{x}\|_2 = h$, parameterized by h .

For any point $(J(\mathbf{p}), \mathbf{p})$ we can write half-line as $\mathcal{L} = \{(J(\gamma\mathbf{p}), \gamma\mathbf{p}) : \gamma > 0\}$.

$$J(\gamma\mathbf{p}) = \|\gamma\mathbf{p}\| = \gamma\|\mathbf{p}\| = \gamma J(\mathbf{p}) \quad (4)$$

A second-order cone constraint has the following form:

$$||\mathbf{Ax} + \mathbf{b}|| \leq \mathbf{c}^\top \mathbf{x} + d \quad (5)$$

where $\mathbf{A} \in \mathbb{R}^{n,n}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $d \in \mathbb{R}$.

This constraint describes interior of a cone. We can prove that surface of this set is a cone described by the following equality:

$$||\mathbf{Ax} + \mathbf{b}|| = \mathbf{c}^\top \mathbf{x} + d \quad (6)$$

Let us consider the following simplified case:

$$||\mathbf{Ax}|| = \mathbf{c}^\top \mathbf{x} \quad (7)$$

The tip of this cone is the point $\mathbf{x} = 0$, and the normal direction is \mathbf{c} .

Let us consider a point \mathbf{p} that lies on the surface (7):

$||\mathbf{Ap}|| - \mathbf{c}^\top \mathbf{p} = 0$. Clearly, a point $\gamma\mathbf{p}$ for $\gamma \geq 0$ lies on the same surface:

$$\begin{aligned} e &= ||\mathbf{A}(\gamma\mathbf{p})|| - \mathbf{c}^\top (\gamma\mathbf{p}) = \gamma||\mathbf{Ap}|| - \gamma\mathbf{c}^\top \mathbf{p} = \\ &= \gamma(||\mathbf{Ap}|| - \mathbf{c}^\top \mathbf{p}) = 0 \end{aligned}$$

Therefore, the half-line going from tip $\mathbf{x} = 0$ through \mathbf{p} lies on the cone.

We can consider plane \mathcal{P} perpendicular to \mathbf{c} , distance $h/||\mathbf{c}||$ away from the tip. All points on the plane are described as follows:

$$\mathbf{x} = \frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{Tz}, \quad \forall \mathbf{z} \quad (8)$$

where $\mathbf{T} = \text{null}(\mathbf{c}^\top)$, so $\mathbf{c}^\top \mathbf{T} = 0$. Then conic surface becomes:

$$\left\| \mathbf{A} \frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{ATz} \right\| = \mathbf{c}^\top \left(\frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{Tz} \right) \quad (9)$$

$$||\mathbf{b}_0 + \mathbf{ATz}|| = h \quad (10)$$

where $\mathbf{b}_0 = \frac{h}{||\mathbf{c}||^2} \mathbf{Ac}$. This is an ellipsoid. \square

For a full second-order cone (SOC):

$$\|\mathbf{Ax} + \mathbf{b}\| = \mathbf{c}^\top \mathbf{x} + d \quad (11)$$

we can find a tip point; it corresponds to both right-hand side and left-hand side becoming zero:

$$\begin{cases} \mathbf{Ax} + \mathbf{b} = 0 \\ \mathbf{c}^\top \mathbf{x} + d = 0 \end{cases} \quad (12)$$

Given full rank matrix \mathbf{A} , the solution is $\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$. The system would hold if:

$$-\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b} + d = 0 \quad (13)$$

Consider a point \mathbf{p}_2 on the surface $\|\mathbf{A}\mathbf{p}_2 + \mathbf{b}\| = \mathbf{c}^\top \mathbf{p}_2 + d$. Let us consider an interval from tip $\mathbf{p}_1 = -\mathbf{A}^{-1}\mathbf{b}$ to \mathbf{p}_2 :

$$\mathbf{x} = \gamma \mathbf{p}_1 + (1 - \gamma) \mathbf{p}_2, \quad \gamma \geq 0 \quad (14)$$

We can check if the points on the interval lie on the surface:

$$e = \|\mathbf{A}\gamma \mathbf{p}_1 + \mathbf{b} + \mathbf{A}(1 - \gamma) \mathbf{p}_2\| - (\gamma \mathbf{c}^\top \mathbf{p}_1 + d + (1 - \gamma) \mathbf{c}^\top \mathbf{p}_2)$$

$$e = \|\gamma \mathbf{A}\mathbf{p}_1 + \gamma \mathbf{b} + (1 - \gamma) \mathbf{b} + \mathbf{A}(1 - \gamma) \mathbf{p}_2\| - \\ - (\gamma \mathbf{c}^\top \mathbf{p}_1 + \gamma d + (1 - \gamma) d + (1 - \gamma) \mathbf{c}^\top \mathbf{p}_2)$$

We know that $\mathbf{A}\gamma \mathbf{p}_1 + \mathbf{b} = 0$ and $\mathbf{c}^\top \mathbf{p}_1 + d = 0$:

$$e = \|(1 - \gamma) \mathbf{b} + \mathbf{A}(1 - \gamma) \mathbf{p}_2\| - (1 - \gamma) d - (1 - \gamma) \mathbf{c}^\top \mathbf{p}_2 \quad (15)$$

$$e = \|\mathbf{b} + \mathbf{A}\mathbf{p}_2\| - (d + \mathbf{c}^\top \mathbf{p}_2) = 0 \quad (16)$$

Thus, the interval (and therefore, the half-line on which it lies) lies on the surface.

Let us consider plane \mathcal{P} perpendicular to \mathbf{c} , distance $h/||\mathbf{c}||$ away from the tip. All points on the plane are described as follows:

$$\mathbf{x} = \frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{Tz}, \quad \forall \mathbf{z} \quad (17)$$

We can substitute it into the second-order cone:

$$||\mathbf{A}(\frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{Tz}) + \mathbf{b}|| = \mathbf{c}^\top (\frac{h}{||\mathbf{c}||^2} \mathbf{c} + \mathbf{Tz}) + d \quad (18)$$

$$||\mathbf{b}_0 + \mathbf{ATz}|| = h + d \quad (19)$$

where $\mathbf{b}_0 = \frac{h}{||\mathbf{c}||^2} \mathbf{Ac} + \mathbf{b}$. This is an ellipsoid. \square

Free constant d plays a specific role in the SOC. As we said before, there is a condition $d = \mathbf{c}^\top \mathbf{A}^{-1} \mathbf{b} + d$. We can explain it by looking at COS as intersection of two surfaces:

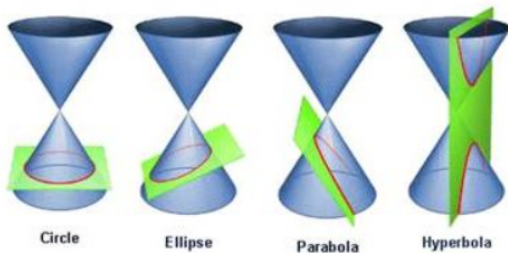
$$J = \|\mathbf{Ax} + \mathbf{b}\| \tag{20}$$

$$P = \mathbf{c}^\top \mathbf{x} + d \tag{21}$$

First is a cone and second is a plane. Their intersection is called a *conic section*.

THE ROLE OF THE FREE CONSTANT, 2

Typical conic sections are shown below:



As we can see, they represent ellipsoid and parabola. In order for them to represent a cone, the plane S needs to pass through the tip of the cone J . This can be achieved with the appropriate choice of constant d , which shifts S up or down.

PLOTTING LEVEL SETS, 1

To plot a cone it is convenient to first use change of coordinates $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$, meaning $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b})$, giving us SOC:

$$\|\mathbf{y}\| = \mathbf{c}^\top \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b}) + d \quad (22)$$

Note that $d - \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b} = 0$ for a cone with a tip; so SOC becomes:

$$\|\mathbf{y}\| = \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{y} \quad (23)$$

To plot level sets of this cone we choose height of the level set h and pick point $\mathbf{y}_h = h \frac{\mathbf{A}^{-T}\mathbf{c}}{\mathbf{c}^\top \mathbf{A}^{-1} \mathbf{A}^{-T} \mathbf{c}}$; we note that $\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{y}_h = h$. Then we consider points on the plane \mathcal{P} orthogonal to $\mathbf{c}^\top \mathbf{A}^{-1}$ and passing through \mathbf{y}_h :

$$\mathcal{P} = \mathbf{y}_h + \mathbf{T}\mathbf{z} : \forall \mathbf{z} \quad (24)$$

where $\mathbf{T} = \text{null}(\mathbf{c}^\top \mathbf{A}^{-1})$, so $\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{T} = 0$.

PLOTTING LEVEL SETS, 2

Since SOC becomes:

$$||\mathbf{y}_h + \mathbf{T}\mathbf{z}|| = h \quad (25)$$

Since \mathbf{y}_h and $\mathbf{T}\mathbf{z}$ are orthogonal, it is equivalent to:

$$||\mathbf{T}\mathbf{z}|| = g \quad (26)$$

where $g = \sqrt{h^2 - \mathbf{y}_h^\top \mathbf{y}_h}$. In the 3D case, this is a circle with radius g . We can find N consecutive evenly spaced points of this circle, resulting in the next sequence of \mathbf{y}_l :

$$\mathbf{y}_l = \mathbf{y}_h + \mathbf{T} \begin{bmatrix} g \cos(\varphi) \\ -g \sin(\varphi) \end{bmatrix}, \quad \varphi = 0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, \dots, 2\pi \quad (27)$$

$$\mathbf{x}_l = \mathbf{A}^{-1}(\mathbf{y}_l - \mathbf{b}) \quad (28)$$

The center of the ellipsoid representing this level set lies at the point $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{y}_h - \mathbf{b})$.

PLOTTING LEVEL SETS, 3

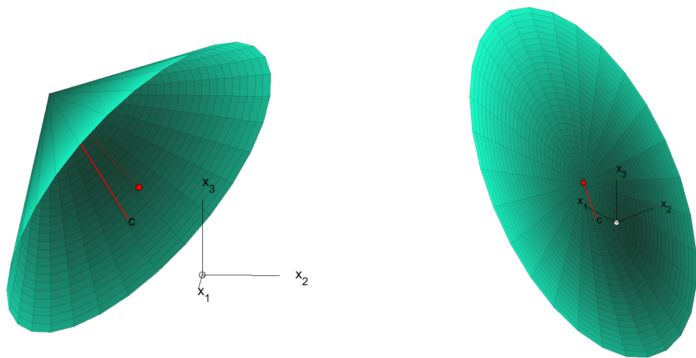


Figure 1: Cone. Dashed line - centers of level-sets.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:

github.com/SergeiSa/Contact-Aware-Control-Fall-2023

