

# Forward Dynamics, Projectors

## Contact-aware Control, Lecture 3

by Sergei Savin

Fall 2023

- Solution to Manipulator DAE
- Schur Projector
- QR Projector

The differential-algebraic manipulator equations:

$$\begin{bmatrix} \mathbf{H} & -\mathbf{J}^\top \\ \mathbf{J} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}\dot{\mathbf{q}} \end{bmatrix} \quad (1)$$

have explicit solution if the Schur complement  $\mathbf{J}^\top \mathbf{H}^{-1} \mathbf{J}$  is full rank:

$$\begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{H}^{-1} - \mathbf{H}^{-1} \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1} & \mathbf{H}^{-1} \mathbf{J}^\top \mathbf{H}_J \\ -\mathbf{H}_J \mathbf{J} \mathbf{H}^{-1} & \mathbf{H}_J \end{bmatrix} \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}\dot{\mathbf{q}} \end{bmatrix} \quad (2)$$

where  $\mathbf{H}_J = (\mathbf{J} \mathbf{H}^{-1} \mathbf{J}^\top)^{-1}$ .

The solution can be written out block-wise:

Analytic solution:

$$\ddot{\mathbf{q}} = (\mathbf{I} - \mathbf{H}^{-1}\mathbf{J}^\top \mathbf{H}_J \mathbf{J})\mathbf{H}^{-1}(\boldsymbol{\tau} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) - \mathbf{H}^{-1}\mathbf{J}^\top \mathbf{H}_J \dot{\mathbf{J}}\dot{\mathbf{q}} \quad (3)$$

$$\boldsymbol{\lambda} = -\mathbf{H}_J \mathbf{J} \mathbf{H}^{-1}(\boldsymbol{\tau} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) - \mathbf{H}_J \dot{\mathbf{J}}\dot{\mathbf{q}} \quad (4)$$

Substituting expression for  $\lambda$  into dynamical equation  $\mathbf{H}\ddot{\mathbf{q}} - \mathbf{J}^\top \lambda = -\mathbf{h}$ , where  $\mathbf{h} = \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} - \tau$  we get:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1} (-\mathbf{h}) + \mathbf{J}^\top \mathbf{H}_J \dot{\mathbf{J}} \dot{\mathbf{q}} = (-\mathbf{h}) \quad (5)$$

$$\mathbf{H}\ddot{\mathbf{q}} + (\mathbf{I} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1}) \mathbf{h} + \mathbf{J}^\top \mathbf{H}_J \dot{\mathbf{J}} \dot{\mathbf{q}} = 0 \quad (6)$$

Let us consider the following equation:

$$(\mathbf{I} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1}) (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = 0 \quad (7)$$

$$\mathbf{H}\ddot{\mathbf{q}} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \ddot{\mathbf{q}} + (\mathbf{I} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1}) \mathbf{h} = 0 \quad (8)$$

Since we know  $\mathbf{J}\ddot{\mathbf{q}} = -\dot{\mathbf{J}}\dot{\mathbf{q}}$ :

$$\mathbf{H}\ddot{\mathbf{q}} + (\mathbf{I} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1}) \mathbf{h} + \mathbf{J}^\top \mathbf{H}_J \dot{\mathbf{J}} \dot{\mathbf{q}} = 0 \quad (9)$$

Thus, the dynamics can be re-written as:

$$\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = 0 \quad (10)$$

where  $\mathbf{P}_S = \mathbf{I} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1} = \mathbf{I} - \mathbf{J}^\top (\mathbf{J} \mathbf{H}^{-1} \mathbf{J}^\top)^{-1} \mathbf{J} \mathbf{H}^{-1}$  is what we can call a Schur projector.

The projector  $\mathbf{P}_S$  does not change the number of equations, but it projects the dynamic onto the constraint manifold; it allows us to discard the reaction forces.

# LINEAR ALGEBRAIC EQUATIONS

Let us remember that a solution to any non-homogeneous (affine) linear equation  $\mathbf{Ax} = \mathbf{b}$ ,  $\mathbf{x} \in \mathbb{R}^n$  is a sum of a *particular solution* and a *null-space solution*:

$$\mathbf{x} = \mathbf{A}^+\mathbf{b} + (\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{x}_0 \quad (11)$$

where  $\mathbf{x}_0 \in \mathbb{R}^n$  is an arbitrary number.

- $\mathbf{A}^+\mathbf{b}$  is a particular solution. It is a smallest-norm solution to the original equation. There is one and only one particular solution, it lies in the row space of  $\mathbf{A}$ .
- $(\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{x}_0$  is a null space solution. There exist a  $k$ -dimensional space of null space solutions, where  $k$  is the dimension of the null space of  $\mathbf{A}$ . If  $\mathbf{A} \in \mathbb{R}^{n,n}$ , its rank is  $n - k$ . The matrix  $(\mathbf{I} - \mathbf{A}^+\mathbf{A})$  is a null space projector.

$$\ddot{\mathbf{q}} = (\mathbf{I} - \mathbf{H}^{-1}\mathbf{J}^\top(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)^{-1}\mathbf{J})\mathbf{H}^{-1}(\boldsymbol{\tau} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) - \mathbf{H}^{-1}\mathbf{J}^\top(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)^{-1}\dot{\mathbf{J}}\dot{\mathbf{q}}$$

Looking at the gen. acceleration solution, we recognize:

- Quantity  $\mathbf{a} = \mathbf{H}^{-1}(\boldsymbol{\tau} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g})$  is the solution (only one exists) to the unconstrained manipulator dynamics  $\mathbf{H}\mathbf{a} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \boldsymbol{\tau}$ .
- Quantity  $\mathbf{J}^\# = \mathbf{H}^{-1}\mathbf{J}^\top(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)^{-1}$  is a weighted pseudo-inverse of  $\mathbf{J}$  (prove it for extra points).
- Quantity  $\mathbf{J}^\#\dot{\mathbf{J}}\dot{\mathbf{q}}$  is related to particular solution of the constraint equation  $\mathbf{J}\ddot{\mathbf{q}} = -\dot{\mathbf{J}}\dot{\mathbf{q}}$ .

$$\ddot{\mathbf{q}} = (\mathbf{I} - \mathbf{J}^\#\mathbf{J})\mathbf{a} - \mathbf{J}^\#\dot{\mathbf{J}}\dot{\mathbf{q}} \tag{12}$$



Consider constrained dynamics:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h} = \mathbf{J}^\top \lambda \quad (13)$$

where  $\mathbf{h} = \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} - \boldsymbol{\tau}$ . We can represent constraint Jacobian

$\mathbf{J}^\top$  as its QR decomposition:  $\mathbf{J}^\top = \mathbf{Q} \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}$ , where

$\mathbf{Q}^\top \mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$  and  $\mathbf{R}$  is invertible.

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h} = \mathbf{Q} \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \lambda \quad (14)$$

Let us multiply the equation by  $\mathbf{Q}^\top$ :

$$\mathbf{Q}^\top (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \lambda \quad (15)$$

Introducing switching variables (to divide upper and lower part of the equations)  $\mathbf{S}_1 = [\mathbf{I} \ \mathbf{0}]$  and  $\mathbf{S}_2 = [\mathbf{0} \ \mathbf{I}]$  and multiplying equations by one and the other we get two systems:

$$\begin{cases} \mathbf{S}_1 \mathbf{Q}^\top (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = \mathbf{R}\lambda \\ \mathbf{S}_2 \mathbf{Q}^\top (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = 0 \end{cases} \quad (16)$$

The main advantage we achieved is that now we can calculate both  $\ddot{\mathbf{q}}$  and  $\lambda$

Considering equation  $\mathbf{S}_2 \mathbf{Q}^\top (\mathbf{H} \ddot{\mathbf{q}} + \mathbf{h}) = 0$  we can re-write it as:

$$\mathbf{P}_{QR}(\mathbf{H} \ddot{\mathbf{q}} + \mathbf{h}) = 0 \quad (17)$$

where  $\mathbf{P}_{QR} = \mathbf{S}_2 \mathbf{Q}^\top$  is a QR projector. Note a similarity of the way  $\mathbf{P}_{QR}$  and  $\mathbf{P}_S$  act on the mechanical equations.

- Righetti, L., Buchli, J., Mistry, M. and Schaal, S., 2011, May. Inverse dynamics control of floating-base robots with external constraints: A unified view. In 2011 IEEE international conference on robotics and automation (pp. 1085-1090). IEEE. - [Inverse Dynamics Control of Floating-Base Robots with External Constraints: a Unified View.](#)
- Mistry, M., Buchli, J. and Schaal, S., 2010, May. Inverse dynamics control of floating base systems using orthogonal decomposition. In 2010 IEEE international conference on robotics and automation (pp. 3406-3412). IEEE. - [cite-seerx.ist.psu.edu/viewdoc/download?doi=10.1.1.212.3601.](#)

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:

[github.com/SergeiSa/Contact-Aware-Control-Fall-2023](https://github.com/SergeiSa/Contact-Aware-Control-Fall-2023)

