Linearization, Orthogonal LQR

Contact-aware Control, Lecture 5

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LTI system with explicit constraints (EC-LTI) can be presented in the following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\lambda \\ \mathbf{G}\dot{\mathbf{x}} = 0 \end{cases} \tag{1}$$

where $\mathbf{G}\dot{\mathbf{x}} = 0$ is the constraints equation.

Substituting $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\lambda$ into the constraints equation $\mathbf{G}\dot{\mathbf{x}} = 0$ we get:

$$\mathbf{G}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\lambda) = 0 \tag{2}$$

$$\mathbf{G}\mathbf{F}\lambda = -\mathbf{G}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \tag{3}$$

$$\lambda = -(\mathbf{G}\mathbf{F})^{+}\mathbf{G}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \tag{4}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - \mathbf{F}(\mathbf{G}\mathbf{F})^{+}\mathbf{G}(\mathbf{A}\mathbf{x} + \mathbf{B} \tag{5}$$

$$\dot{\mathbf{x}} = (\mathbf{I} - \mathbf{F}(\mathbf{G}\mathbf{F})^{+}\mathbf{G})(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \tag{6}$$

Defining $\mathbf{A}_c = (\mathbf{I} - \mathbf{F}(\mathbf{G}\mathbf{F})^{+}\mathbf{G})\mathbf{A}$ and $\mathbf{B}_c = (\mathbf{I} - \mathbf{F}(\mathbf{G}\mathbf{F})^{+}\mathbf{G})\mathbf{B}$, we obtain LTI system with implicit constraints (IC-LTI):

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_c\mathbf{x} + \mathbf{B}_c\mathbf{u} \\ \mathbf{G}\dot{\mathbf{x}} = 0 \end{cases} \tag{7}$$

LTI system with implicit constraints

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_c \mathbf{x} + \mathbf{B}_c \mathbf{u} \\ \mathbf{G}\dot{\mathbf{x}} = 0 \end{cases} \tag{8}$$

Note that IC-LTI is similar in form to regular LTI $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$. This reminds us of the use of projectors in manipulator equations. Expression $(\mathbf{I} - \mathbf{F}(\mathbf{G}\mathbf{F})^{+}\mathbf{G})$ acts in a similar ways to projectors.

We can linearize dynamical system with constraints. Let us remember that linearization is done by Taylor expansion, taking into account the first term. For systems with constraints, we need to apply Taylor expansion to the expression for $\ddot{\mathbf{q}}$:

$$\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{u}) = (\mathbf{I} - \mathbf{J}^{\#}\mathbf{J})\mathbf{H}^{-1}(\mathbf{T}\mathbf{u} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) - \mathbf{J}^{\#}\dot{\mathbf{J}}\dot{\mathbf{q}} \tag{9}$$

where $\mathbf{J}^{\#} = \mathbf{H}^{-1}\mathbf{J}^{\top}(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^{\top})^{-1}$.

Taylor expansion becomes:

$$\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{u}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{q}}(\mathbf{q} - \mathbf{q}_0) + \frac{\partial \mathbf{f}}{\partial \dot{\mathbf{q}}}(\dot{\mathbf{q}} - \dot{\mathbf{q}}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0) + \text{h.o.t.}$$

Defining $\mathbf{A}_q = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}$ and $\mathbf{A}_v = \frac{\partial \mathbf{f}}{\partial \dot{\mathbf{q}}}$, $\mathbf{B} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$, and proposing state variable $\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{A}_q & \mathbf{A}_v \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \mathbf{B} \end{bmatrix} \mathbf{u} + \text{const} \tag{10}$$

Constraint equation $\mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} = 0$ becomes:

$$\begin{bmatrix} \mathbf{J} & 0 \\ \dot{\mathbf{J}} & \mathbf{J} \end{bmatrix} \dot{\mathbf{x}} = 0 \tag{11}$$

Linearized system then becomes:

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{A}_q & \mathbf{A}_v \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \mathbf{B} \end{bmatrix} \mathbf{u} + \text{const} \\ \begin{bmatrix} \mathbf{J} & 0 \\ \dot{\mathbf{J}} & \mathbf{J} \end{bmatrix} \dot{\mathbf{x}} = 0 \end{cases} \tag{12}$$

Noticed that this is an implicit constrained LTI.

We can find orthonormal basis $\mathbf{N}$ in the null space of the constraints matrix $\mathbf{G}$, and orthonormal basis $\mathbf{R}$ in the row space of the same matrix.

$$\mathbf{N} = \mathrm{null}(\mathbf{G}) \tag{13}$$

$$\mathbf{R} = \mathrm{row}(\mathbf{G}) = \mathrm{col}(\mathbf{G}^\top) \tag{14}$$

Since matrices $\mathbf{N}$ and $\mathbf{R}$ are orthogonal compliment of each other, the matrix $\begin{bmatrix}\mathbf{N} & \mathbf{R}\end{bmatrix}$ is full rank.

Orthogonal decomposition, 2

We can define new variables $\mathbf{z}$ and ζ, which have the following relation with $\mathbf{x}$:

$$\mathbf{x} = \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \zeta \end{bmatrix} \tag{15}$$

Because $\begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix}$ is full rank, there is a one-to-one correspondence between variables $\mathbf{z}$ and ζ and $\mathbf{x}$. In fact:

$$\mathbf{z} = \mathbf{N}^\top \mathbf{x} \tag{16}$$

$$\zeta = \mathbf{R}^\top \mathbf{x} \tag{17}$$

Since $\mathbf{G}\dot{\mathbf{x}} = 0$, it follows that $\dot{\mathbf{x}} \in \text{null}(\mathbf{G})$. Therefore $\mathbf{R}^\top \dot{\mathbf{x}} = \dot{\zeta} = 0$ and $\mathbf{N}^\top \dot{\mathbf{x}} = \dot{\mathbf{z}}$:

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{N} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\zeta} \end{bmatrix} = \mathbf{N}\dot{\mathbf{z}} \tag{18}$$

This implies that $\dot{\mathbf{x}}$ lies in the column space of $\mathbf{N}$.

We can multiply equation $\dot{\mathbf{x}} = \mathbf{A}_c \mathbf{x} + \mathbf{B}_c \mathbf{u}$ by $\mathbf{N}^\top$. Since $\dot{\mathbf{x}}$ lies in the column space of $\mathbf{N}$, we do not lose any component of the dynamics:

$$\mathbf{N}^\top \dot{\mathbf{x}} = \mathbf{N}^\top \mathbf{A}_c \mathbf{x} + \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \tag{19}$$

Since $\mathbf{x} = \mathbf{N}\mathbf{z} + \mathbf{R}\zeta$ and $\mathbf{N}^\top \dot{\mathbf{x}} = \dot{\mathbf{z}}$ we get:

$$\dot{\mathbf{z}} = \mathbf{N}^\top \mathbf{A}_c \mathbf{N} \mathbf{z} + \mathbf{N}^\top \mathbf{A}_c \mathbf{R} \zeta + \mathbf{N}^\top \mathbf{B}_c \mathbf{u} \tag{20}$$

We can define $\mathbf{A}_N = \mathbf{N}^\top \mathbf{A}_c \mathbf{N}$, $\mathbf{A}_R = \mathbf{N}^\top \mathbf{A}_c \mathbf{R}$ and $\mathbf{B}_N = \mathbf{N}^\top \mathbf{B}_c$:

$$\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{A}_R \zeta + \mathbf{B}_N \mathbf{u} \tag{21}$$

Feedback control

Given equation $\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{A}_R \zeta + \mathbf{B}_N \mathbf{u}$ we can find stabilizing control. Since $\mathbf{A}_R \zeta = \text{const}$, this is equivalent to stabilizing the system $\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{B}_N \mathbf{u}$.

Consider linear control law:

$$\mathbf{u} = -\mathbf{K}\mathbf{z} \tag{22}$$

Then dynamics becomes:

$$\dot{\mathbf{z}} = (\mathbf{A}_N - \mathbf{B}_N \mathbf{K})\mathbf{z} + \mathbf{A}_R \zeta \tag{23}$$

As long at the state matrix $\mathbf{A}_N - \mathbf{B}_N \mathbf{K}$ is Hurwitz, the system is stable. This can be achieved with, e.g. pole placement.

With that, we could pose LQR problem for the system. For the dynamical system:

$$\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{B}_N \mathbf{u} \tag{24}$$

find control that minimizes the next cost function:

$$J = \int \left(\mathbf{z}^\top \mathbf{Q} \mathbf{z} + \mathbf{u}^\top \mathbf{R} \mathbf{u} \right) dt \tag{25}$$

the optimal control is linear control law:

$$\mathbf{u} = -\mathbf{R}^{-1} \mathbf{B}_N^\top \mathbf{S} \mathbf{z} \tag{26}$$

where the control gain $\mathbf{K} = \mathbf{R}^{-1} \mathbf{B}_N^\top \mathbf{S}$ is obtained by solving algebraic Riccati equation:

$$\mathbf{Q} - \mathbf{S} \mathbf{B}_N \mathbf{R}^{-1} \mathbf{B}_N^\top \mathbf{S} + \mathbf{S} \mathbf{A}_N + \mathbf{A}_N^\top \mathbf{S} = 0 \tag{27}$$

We can consider LQR as a recipe of getting control gain $\mathbf{K}$:

$$\mathbf{K} = \mathrm{lqr}(\mathbf{A}_N, \mathbf{B}_N, \mathbf{Q}, \mathbf{R}) \tag{28}$$

We can do the same with pole placement:

$$\mathbf{K} = \mathrm{place}(\mathbf{A}_N, \mathbf{B}_N, \mathbf{p}) \tag{29}$$

where $\mathbf{p}$ are poles of the system.

The main outtake is that we can use standard control design methods from linear control.

Read more

- Mason, Sean, et al. "Balancing and walking using full dynamics LQR control with contact constraints." 2016 IEEE-RAS 16th International Conference on Humanoid Robots (Humanoids). IEEE, 2016.
  arxiv.org/pdf/1701.08179
- Mason, Sean, Ludovic Righetti, and Stefan Schaal. "Full dynamics LQR control of a humanoid robot: An experimental study on balancing and squatting." 2014 IEEE-RAS International Conference on Humanoid Robots. IEEE, 2014.
  ieeexplore.ieee.org/abstract/document/7041387/

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Contact-Aware-Control-Fall-2023