Forward Dynamics, Projectors

Contact-aware Control, Lecture 3

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Manipulator DAE

The differential-algebraic manipulator equations:

$$\begin{bmatrix} \mathbf{H} & -\mathbf{J}^\top \\ \mathbf{J} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}\dot{\mathbf{q}} \end{bmatrix} \tag{1}$$

have explicit solution if the Schur compliment $\mathbf{J}^\top \mathbf{H}^{-1} \mathbf{J}$ is full rank:

$$\begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{H}^{-1} - \mathbf{H}^{-1}\mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1} & \mathbf{H}^{-1}\mathbf{J}^\top \mathbf{H}_J \\ -\mathbf{H}_J \mathbf{J} \mathbf{H}^{-1} & \mathbf{H}_J \end{bmatrix} \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ -\dot{\mathbf{J}}\dot{\mathbf{q}} \end{bmatrix} \tag{2}$$

where $\mathbf{H}_J = (\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)^{-1}$.

The solution can be written out block-wise:

Analytic solution:

$$\ddot{\mathbf{q}} = (\mathbf{I} - \mathbf{H}^{-1}\mathbf{J}^{\top}\mathbf{H}_J\mathbf{J})\mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) - \mathbf{H}^{-1}\mathbf{J}^{\top}\mathbf{H}_J\dot{\mathbf{J}}\dot{\mathbf{q}} \quad (3)$$

$$\lambda = -\mathbf{H}_J\mathbf{J}\mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) - \mathbf{H}_J\dot{\mathbf{J}}\dot{\mathbf{q}} \quad (4)$$

Substituting expression for λ into dynamical equation $\mathbf{H}\ddot{\mathbf{q}} - \mathbf{J}^\top \lambda = -\mathbf{h}$, where $\mathbf{h} = \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} - \tau$ we get:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1}(-\mathbf{h}) + \mathbf{J}^\top \mathbf{H}_J \dot{\mathbf{J}}\dot{\mathbf{q}} = (-\mathbf{h}) \tag{5}$$

$$\mathbf{H}\ddot{\mathbf{q}} + (\mathbf{I} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1})\mathbf{h} + \mathbf{J}^\top \mathbf{H}_J \dot{\mathbf{J}}\dot{\mathbf{q}} = 0 \tag{6}$$

Let us consider the following equation:

$$(\mathbf{I} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1})(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = 0 \tag{7}$$

$$\mathbf{H}\ddot{\mathbf{q}} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J}\ddot{\mathbf{q}} + (\mathbf{I} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1})\mathbf{h} = 0 \tag{8}$$

Since we know $\mathbf{J}\ddot{\mathbf{q}} = -\dot{\mathbf{J}}\dot{\mathbf{q}}$:

$$\mathbf{H}\ddot{\mathbf{q}} + (\mathbf{I} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1})\mathbf{h} + \mathbf{J}^\top \mathbf{H}_J \dot{\mathbf{J}}\dot{\mathbf{q}} = 0 \tag{9}$$

Thus, the dynamics can be re-written as:

$$\mathbf{P}_S(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = 0 \tag{10}$$

where $\mathbf{P}_S = \mathbf{I} - \mathbf{J}^\top \mathbf{H}_J \mathbf{J} \mathbf{H}^{-1} = \mathbf{I} - \mathbf{J}^\top (\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)^{-1} \mathbf{J}\mathbf{H}^{-1}$ is what we can call a Schur projector.

The projector $\mathbf{P}_S$ does not change the number of equations, but it projects the dynamic onto the constraint manifold; it allows us to discard the reaction forces.

Let us remember that a solution to any non-homogeneous (affine) linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in \mathbb{R}^n$ is a sum of a *particular solution* and a *null-space solution*:

$$\mathbf{x} = \mathbf{A}^{+}\mathbf{b} + (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{x}_0 \tag{11}$$

where $\mathbf{x}_0 \in \mathbb{R}^n$ is an arbitrary number.

- $\mathbf{A}^{+}\mathbf{b}$ is a particular solution. It is a smallest-norm solution to the original equation. There is one and only one particular solution, it lies in the row space of $\mathbf{A}$.
- $(\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{x}_0$ is a null space solution. There exist a k-dimensional space of null space solutions, where k is the dimension of the null space of $\mathbf{A}$. If $\mathbf{A} \in \mathbb{R}^{n,n}$, its rank is $n - k$. The matrix $(\mathbf{I} - \mathbf{A}^{+}\mathbf{A})$ is a null space projector.

$$\ddot{\mathbf{q}} = (\mathbf{I} - \mathbf{H}^{-1}\mathbf{J}^\top(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)^{-1}\mathbf{J})\mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) - \\ - \mathbf{H}^{-1}\mathbf{J}^\top(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)^{-1}\dot{\mathbf{J}}\dot{\mathbf{q}}$$

Looking at the gen. acceleration solution, we recognize:

- Quantity $\mathbf{a} = \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g})$ is the solution (only one exists) to the unconstrained manipulator dynamics $\mathbf{H}\mathbf{a} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau$.
- Quantity $\mathbf{J}^\# = \mathbf{H}^{-1}\mathbf{J}^\top(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)^{-1}$ is a weighted pseudo-inverse of $\mathbf{J}$ (prove it for extra points).
- Quantity $\mathbf{J}^\#\dot{\mathbf{J}}\dot{\mathbf{q}}$ is related to particular solution of the constraint equation $\mathbf{J}\ddot{\mathbf{q}} = -\dot{\mathbf{J}}\dot{\mathbf{q}}$.

$$\ddot{\mathbf{q}} = (\mathbf{I} - \mathbf{J}^\#\mathbf{J})\mathbf{a} - \mathbf{J}^\#\dot{\mathbf{J}}\dot{\mathbf{q}} \tag{12}$$

Consider constrained dynamics:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h} = \mathbf{J}^\top \lambda \tag{13}$$

where $\mathbf{h} = \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} - \tau$. We can represent constraint Jacobian $\mathbf{J}^\top$ as its QR decomposition: $\mathbf{J}^\top = \mathbf{Q}\begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}$, where $\mathbf{Q}^\top\mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$ and $\mathbf{R}$ is convertible.

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h} = \mathbf{Q}\begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}\lambda \tag{14}$$

Let us multiply the equation by $\mathbf{Q}^\top$:

$$\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \lambda \tag{15}$$

Introducing switching variables (to divide upper and lower part of the equations) $\mathbf{S}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$ and $\mathbf{S}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix}$ and multiplying equations by one and the other we get two systems:

$$\begin{cases} \mathbf{S}_1\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = \mathbf{R}\lambda \\ \mathbf{S}_2\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = 0 \end{cases} \tag{16}$$

The main advantage we achieved is that now we can calculate both $\ddot{\mathbf{q}}$ and λ

QR projector

Considering equation $\mathbf{S}_2\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = 0$ we can re-write it as:

$$\mathbf{P}_{QR}(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{h}) = 0 \tag{17}$$

where $\mathbf{P}_{QR} = \mathbf{S}_2\mathbf{Q}^\top$ is a QR projector. Note a similarity of the way $\mathbf{P}_{QR}$ and $\mathbf{P}_S$ act on the mechanical equations.

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Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Contact-Aware-Control-Fall-2023