Constraints

Contact-aware Control, Lecture 2

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Dynamical systems

Dynamical system is this course means an ODE. A general form of an ODE is:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

Any ODE non-degenerate can be represented in this form.

Example 1

For example, consider pendulum dynamics equations:

$$\ddot{\phi} = -lsin(\phi) \tag{2}$$

Introducing a change of coordinates $\mathbf{x} = [\phi \ \dot{\phi}]$ we get:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -lsin(x_1) \end{bmatrix} \tag{3}$$

A *constraint* is an equality that must hold for a given dynamical system, as its state evolves in time.

Consider a general-form dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

A general-form algebraic constraint for this system can be written as:

$$\mathbf{g}(\mathbf{x}) = 0 \tag{4}$$

A differential constraint would have form $\mathbf{g}(\mathbf{x},\ \dot{\mathbf{x}}) = 0$. Both definitions are quite obvious.

Remember the example from the previous lecture:

$$\begin{cases} m_1\ddot{x}_1 = k_1 x_1 + k_2(x_2 - x_1 - l_{12}) + f_1(t) \\ m_2\ddot{x}_2 = -k_2(x_2 - x_1 - l_{12}) + f_2(t) \end{cases} \tag{5}$$

where $f_1(t)$ and $f_2(t)$ are external forces (we add then to make some observations later).

We can require that $x_1 - x_2 = 1$, (meaning the distance between two bodies remains to be equal to 1). The equation is then changed to:

$$\begin{cases} m_1\ddot{x}_1 = k_1 x_1 + k_2(x_2 - x_1 - l_{12}) + f_1(t) \\ m_2\ddot{x}_2 = -k_2(x_2 - x_1 - l_{12}) + f_2(t) \\ x_1 - x_2 = 1 \end{cases} \tag{6}$$

We can differentiate the constraint twice and get the following equation:

$$\begin{cases} m_1\ddot{x}_1 = k_1 x_1 + k_2(x_2 - x_1 - l_{12}) + f_1(t) \\ m_2\ddot{x}_2 = -k_2(x_2 - x_1 - l_{12}) + f_2(t) \\ \ddot{x}_1 - \ddot{x}_2 = 0 \end{cases} \tag{7}$$

which is equivalent to the original, as long as the initial condition satisfies the constraint $x_1 - x_2 = 1$.

Notice, that this equation is not guaranteed to have a solution. In fact, first two equations remained unchanged, they lack *force* that would ensure the constraint holds.

Adding a new force that acts on the first equation:

$$\begin{cases} m_1\ddot{x}_1 = k_1 x_1 + k_2(x_2 - x_1 - l_{12}) + f_1(t) + \lambda \\ m_2\ddot{x}_2 = -k_2(x_2 - x_1 - l_{12}) + f_2(t) \\ \ddot{x}_1 - \ddot{x}_2 = 0 \end{cases} \tag{8}$$

It is now *always possible* to find such λ that the constraint holds.

Let us consider the case when $k_1 = k_2 = 0$, $m_1 = m_2 = 1$. Then we have:

$$\begin{cases} \ddot{x}_1 = f_1(t) + \lambda \\ \ddot{x}_2 = f_2(t) \\ \ddot{x}_1 - \ddot{x}_2 = 0 \end{cases} \tag{9}$$

Notice that $\ddot{x}_1 = \ddot{x}_2 = f_2(t)$ and $\lambda = f_2(t) - f_1(t)$. The work produced by this force is $\int \dot{x}_1 (f_2(t) - f_1(t)) dt$, and in general does not have to be equal to 0.

Now consider that the *reaction force* is applied to both equations and in this way:

$$\begin{cases} \ddot{x}_1 = f_1(t) - \lambda \\ \ddot{x}_2 = f_2(t) + \lambda \\ \ddot{x}_1 - \ddot{x}_2 = 0 \end{cases} \tag{10}$$

Then the reaction force is $\lambda = (f_2(t) - f_1(t))/2$ (Prove it!). The work produced by this force is $\int -\dot{x}_1 \frac{f_2(t)-f_1(t)}{2} + \dot{x}_2 \frac{f_2(t)-f_1(t)}{2} dt$, which is equal to zero as long as the constraint holds, since the constraint implies that $\dot{x}_1 - \dot{x}_2 = 0$.

Thus, there are some constraints that produce no mechanical work.

Homework

(I) Prove that in (10) the value of the reaction force is $\lambda = (f_2(t) - f_1(t))/2$.

(II) For the system:

$$\begin{cases} \ddot{x}_1 = f_1(t) \\ \ddot{x}_2 = f_2(t) \\ \ddot{x}_3 = 1 \end{cases} \tag{11}$$

add reaction force λ that enforces constraint $\ddot{x}_1 + \ddot{x}_2 - \ddot{x}_3 = 0$, which would produce no work.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Contact-Aware-Control-Slides-Fall-2020

Check Moodle for additional links, videos, textbook suggestions.