

Constrained LQR

Contact-aware Control, Lecture 13

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Consider equations in the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \quad (1)$$

where \mathbf{A} is the state matrix, \mathbf{B} is the control matrix and \mathbf{c} is the affine term of the affine dynamics model.

For systems with constraints the same linearization takes form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{S}\lambda + \mathbf{c} \\ \mathbf{G}\dot{\mathbf{x}} = 0 \end{cases} \quad (2)$$

where \mathbf{S} is linearized constraint Jacobian and $\mathbf{G} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \dot{\mathbf{F}} & \mathbf{F} \end{bmatrix}$.

Implicit (minimal) representation of a constrained system

Part 1

We can observe that constraint $\mathbf{G}\dot{\mathbf{x}} = 0$ implies that all feasible state velocities $\dot{\mathbf{x}}$ lie in the null space of \mathbf{G} . This means that we can introduce a new lower dimensional variable \mathbf{z} to describe $\dot{\mathbf{x}}$ (assuming initial value of \mathbf{x} lies in the column space of \mathbf{N}):

$$\mathbf{N}\dot{\mathbf{z}} = \dot{\mathbf{x}} \tag{3}$$

where $\mathbf{N} = \text{null}(\mathbf{G})$ - orthonormal basis in the null space of \mathbf{G} .

Implicit (minimal) representation of a constrained system

Part 2

Let us re-express dynamics (2) in terms of \mathbf{z} by multiplying it by \mathbf{N}^\top on the left:

$$\mathbf{N}^\top \dot{\mathbf{x}} = \mathbf{N}^\top \mathbf{A} \mathbf{x} + \mathbf{N}^\top \mathbf{B} \mathbf{u} + \mathbf{N}^\top \mathbf{S} \lambda + \mathbf{N}^\top \mathbf{c} \quad (4)$$

We can prove that $\mathbf{N}^\top \mathbf{S} = 0$ for all mechanical systems (for example, by observing that mechanical constraints do not do work) or check that our particular \mathbf{S} lies in the row space of our \mathbf{G} .

Noting that $\dot{\mathbf{z}} = \mathbf{N}^\top \dot{\mathbf{x}}$ and $\mathbf{x} = \mathbf{N} \mathbf{z}$ we get:

$$\dot{\mathbf{z}} = \mathbf{N}^\top \mathbf{A} \mathbf{N} \mathbf{z} + \mathbf{N}^\top \mathbf{B} \mathbf{u} + \mathbf{N}^\top \mathbf{c} \quad (5)$$

Defining $\mathbf{A}_N = \mathbf{N}^\top \mathbf{A} \mathbf{N}$, $\mathbf{B}_N = \mathbf{N}^\top \mathbf{B}$ and $\mathbf{c}_N = \mathbf{N}^\top \mathbf{c}$ we get:

$$\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{B}_N \mathbf{u} + \mathbf{c}_N \quad (6)$$

Implicit (minimal) representation of a constrained system

Part 3

Since we achieved that our constrained dynamics is written in the standard LTI form:

$$\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{B}_N \mathbf{u} + \mathbf{c}_N, \quad (7)$$

we can use standard LTI control methods on it, for example finding optimal feedback gains via pole placement or LQR:

$$\mathbf{K}_N = \text{lqr}(\mathbf{A}_N, \mathbf{B}_N, \mathbf{Q}, \mathbf{R}) \quad (8)$$

where \mathbf{Q} and \mathbf{R} are matrices defining cost function for the LQR problem.

For any LTI system, including the LTI form of a constrained system we saw previously, inverse dynamics can be solved precisely by a pseudo-inverse, as long as there exist a solution. The following condition verifies it:

$$(\mathbf{I} - \mathbf{B}\mathbf{B}^+)(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c}) = 0, \quad (9)$$

The condition checks if vector $(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c})$ lies in the column space of \mathbf{B} . If it holds, precise solution to inverse kinematics can be found as:

$$\mathbf{u}_{ID} = \mathbf{B}^+(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c}). \quad (10)$$

Inverse dynamics

Manipulator equations

For a constrained mechanical system we can solve inverse dynamics without the need for linearization. Consider the following dynamics:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{F}^\top \lambda \quad (11)$$

We can represent constraint Jacobian \mathbf{F}^\top as its QR decomposition: $\mathbf{F}^\top = \mathbf{Q} \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}$, where $\mathbf{Q}^\top \mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$ and \mathbf{R} is convertible.

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{Q} \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \lambda \quad (12)$$

Inverse dynamics

Manipulator equations, part 2

Let us multiply the equation by \mathbf{Q}^\top :

$$\mathbf{Q}^\top (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{Q}^\top \mathbf{T}\mathbf{u} + \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \lambda \quad (13)$$

Introducing switching variables (to divide upper and lower part of the equations) $\mathbf{S}_1 = [\mathbf{I} \ \mathbf{0}]$ and $\mathbf{S}_2 = [\mathbf{0} \ \mathbf{I}]$ and multiplying equations by one and the other we get two systems:

$$\begin{cases} \mathbf{S}_1 \mathbf{Q}^\top (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{S}_1 \mathbf{Q}^\top \mathbf{T}\mathbf{u} + \mathbf{R}\lambda \\ \mathbf{S}_2 \mathbf{Q}^\top (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{S}_2 \mathbf{Q}^\top \mathbf{T}\mathbf{u} \end{cases} \quad (14)$$

The main advantage we achieved is that now we can calculate both \mathbf{u} and λ

Inverse dynamics

Manipulator equations, part 3

Resulting expression for \mathbf{u} is:

$$\mathbf{u} = (\mathbf{S}_2 \mathbf{Q}^\top \mathbf{T})^+ \mathbf{S}_2 \mathbf{Q}^\top (\mathbf{H} \ddot{\mathbf{q}} + \mathbf{C} \dot{\mathbf{q}} + \mathbf{g}) \quad (15)$$

Expression for λ is:

$$\lambda = \mathbf{R}^{-1} \mathbf{S}_1 \mathbf{Q}^\top (\mathbf{H} \ddot{\mathbf{q}} + \mathbf{C} \dot{\mathbf{q}} + \mathbf{g} - \mathbf{T} \mathbf{u}) \quad (16)$$

We can notice a pseudo-inverse, implying that the no-residual solution does not have to exist.

Inverse dynamics

Quadratic program

We can easily write inverse dynamics as a QP:

$$\begin{array}{ll} \underset{\mathbf{u}, \lambda}{\text{minimize}} & ||\mathbf{u}||, \\ \text{subject to} & \begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{F}^\top \lambda \\ \mathbf{F}\ddot{\mathbf{q}} + \dot{\mathbf{F}}\dot{\mathbf{q}} = 0 \end{cases} \end{array} \quad (17)$$

If there are some constraints or limits on the control input (torque limits, for instance) or the reaction forces are restricted (by friction cones, for instance), those can be directly added.

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Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Contact-Aware-Control-Slides-Fall-2020

Check Moodle for additional links, videos, textbook suggestions.