

# Stability

## Control Theory, Lecture 2

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- Critical point (node)
- Stability
- Asymptotic stability
- Stability vs Asymptotic stability
- LTI and autonomous LTI
- Stability of autonomous LTI
- Read more

# CRITICAL POINT (NODE)

Consider the following ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \quad (1)$$

Let  $\mathbf{x}_0$  be such a state that:

$$\mathbf{f}(\mathbf{x}_0, t) = 0 \quad (2)$$

Then such state  $\mathbf{x}_0$  is called a *node* or a *critical point*.

Node  $\mathbf{x}_0$  is called *stable* iff for any constant  $\delta$  there exists constant  $\varepsilon$  such that:

$$\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta \longrightarrow \|\mathbf{x}(t) - \mathbf{x}_0\| < \varepsilon \quad (3)$$

Think of it as "for any initial point that lies at most  $\delta$  away from  $\mathbf{x}_0$ , the rest of the trajectory  $\mathbf{x}(t)$  will be at most  $\varepsilon$  away from  $\mathbf{x}_0$ ".

Equivalently we can say "the solutions starting from  $\delta$ -sized ball do not diverge".

Node  $\mathbf{x}_0$  is called *asymptotically stable* iff for any constant  $\delta$  it is true that:

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \longrightarrow \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}_0 \quad (4)$$

Think of it as "for any initial point that lies at most  $\delta$  away from  $\mathbf{x}_0$ , the trajectory  $\mathbf{x}(t)$  will asymptotically approach the point  $\mathbf{x}_0$ ".

Equivalently we can say "the solutions starting from  $\delta$ -sized ball converge to the node".

# STABILITY VS ASYMPTOTIC STABILITY

## Example

Consider dynamical system  $\dot{x} = 0$ , and solution  $x = 7$ . This solution is stable, but not asymptotically stable (a solution corresponding to  $x(0) = 7 + \delta$  does not diverge, but does not converge to  $x = 7$  either).

## Example

Consider dynamical system  $\dot{x} = -x$ , and solution  $x = 0$ . This solution is stable and asymptotically stable (all solutions converge to  $x = 0$ ).

## Example

Consider dynamical system  $\dot{x} = x$ , and solution  $x = 0$ . This solution is unstable (all other solutions diverge from  $x = 0$ ).

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (5)$$

This is called a *linear time-invariant system (LTI)*, indicating that  $\mathbf{A}$  and  $\mathbf{B}$  are constant.

Removing the input we find an even simpler equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (6)$$

This LTI is an *autonomous system*, since its evolution depends only on the state of the system.

# STABILITY OF AUTONOMOUS LTI

## Real eigenvalues

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{x} \quad (7)$$

where  $\mathbf{D} = \text{diag}(d_1, \dots, d_n)$  is a diagonal matrix. This is the same as a system of independent equations:

$$\begin{cases} \dot{x}_1 = d_1 x_1 \\ \dots \\ \dot{x}_n = d_n x_n \end{cases} \quad (8)$$

Each of these equations has an exact solution  $x_i = C_i e^{d_i t}$ . It diverges from 0 if  $d_i > 0$ , it does not diverge if  $d_i \leq 0$  and it converges to 0 if  $d_i < 0$ .



# STABILITY OF AUTONOMOUS LTI

## Real eigenvalues

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (9)$$

where  $\mathbf{A}$  can be decomposed via eigen-decomposition as  $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ , where  $\mathbf{D}$  is a diagonal matrix.

$$\dot{\mathbf{x}} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x} \quad (10)$$

Multiplying it by  $\mathbf{V}^{-1}$  we get:  $\mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x}$ .  
Defining  $\mathbf{z} = \mathbf{V}^{-1}\mathbf{x}$  we transform the equation:  $\dot{\mathbf{z}} = \mathbf{D}\mathbf{z}$ .

Since elements of  $\mathbf{D}$  are real, we can clearly see, that iff they are *all negative* will the system be asymptotically stable. If they are non-positive, the system is stable. And those elements are eigenvalues of  $\mathbf{A}$ .

Eigenvalues of upper triangular matrices

Eigenvalues of upper triangular matrices are the diagonal elements of these matrices.

Examples of upper triangular matrices are:

$$\begin{bmatrix} 1 & 5 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \quad \begin{bmatrix} -2 & 0 & 8 \\ 0 & -2 & 8 \\ 0 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} \quad (11)$$

# UPPER TRIANGULAR MATRICES

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{M}\mathbf{x} \quad (12)$$

where  $\mathbf{M}$  is an upper triangular matrices with negative eigenvalues  $m_{1,1}, \dots, m_{n,n}$ .

The last equation is  $\dot{x}_n = m_{n,n}x_n$ , and since  $m_{n,n} < 0$  we can observe that  $\lim_{t \rightarrow \infty} x_n(t) = 0$ .

The equation #  $n-1$  is  $\dot{x}_{n-1} = m_{n-1,n-1}x_{n-1} + m_{n-1,n}x_n$ , and since  $m_{n-1,n-1} < 0$  and  $\lim_{t \rightarrow \infty} x_n(t) = 0$  we can observe that  $\lim_{t \rightarrow \infty} x_{n-1}(t) = 0$ .

This can be repeated for all equations, proving asymptotic stability for the system.

# STABILITY OF AUTONOMOUS LTI

## Complex eigenvalues, 2-dimensional case (1)

Let us consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (13)$$

The eigenvalues of the system are  $\alpha \pm i\beta$ . We denote  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}$ .

We start by claiming that the system will be stable iff the  $\dot{\mathbf{x}}^\top \mathbf{x} < 0$ . Indeed, vector  $\dot{\mathbf{x}}$  can always be decomposed into two components,  $\dot{\mathbf{x}}_\parallel$  parallel to  $\mathbf{x}$ , and  $\dot{\mathbf{x}}_\perp$  perpendicular to  $\mathbf{x}$ . By definition  $\dot{\mathbf{x}}_\perp^\top \mathbf{x} = 0$ , and is responsible for the change in orientation of  $\mathbf{x}$ . The value of  $\dot{\mathbf{x}}_\parallel$  is responsible for the change in the length of  $\mathbf{x}$ ; the length would shrink iff  $\dot{\mathbf{x}}_\parallel$  is of opposite direction to  $\mathbf{x}$ , giving negative value of the dot product  $\dot{\mathbf{x}}^\top \mathbf{x}$ .

# STABILITY OF AUTONOMOUS LTI

## Complex eigenvalues, 2-dimensional case (2)

Let us compute  $\dot{\mathbf{x}}^\top \mathbf{x}$ :

$$\dot{\mathbf{x}}^\top \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (14)$$

$$\dot{\mathbf{x}}^\top \mathbf{x} = \alpha(x_1^2 + x_2^2) \quad (15)$$

From this it is clear that the product  $\dot{\mathbf{x}}^\top \mathbf{x} < 0$  is negative iff  $\alpha < 0$ .

### Definition

As long as the *real parts of the eigenvalues* of the system are *strictly negative*, the system is *asymptotically stable*. If the real parts of the eigenvalues of the system are zero, the system is *marginally stable*.

# STABILITY OF AUTONOMOUS LTI

## Complex eigenvalues, 2-dimensional case (3)

Vector field of  $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  is shown below:



# STABILITY OF AUTONOMOUS LTI

## General case (1)

Given  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , where  $\mathbf{A}$  can be decomposed via eigen-decomposition as  $\mathbf{A} = \mathbf{U}\mathbf{C}\mathbf{U}^{-1}$ , where  $\mathbf{C}$  is a complex-valued diagonal matrix and  $\mathbf{U}$  is a complex-valued invertible matrix.

We multiply both sides by  $\mathbf{U}^{-1}$ , then define  $\mathbf{z} = \mathbf{U}^{-1}\mathbf{x}$  to arrive at:

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} \quad (16)$$

which falls into a set of independent equations, with complex coefficients  $c_j$ :

$$\dot{z}_j = c_j z_j \quad (17)$$

# STABILITY OF AUTONOMOUS LTI

## General case (2)

Expanding  $c_j = \alpha + i\beta$ , and  $z_j = u + iv$  (we dismiss subscripts for clarity), we find that  $\dot{z}_j = c_j z_j$  can be expanded as:

$$\dot{u} + i\dot{v} = \dot{z}_j = c_j z_j = (\alpha + i\beta)(u + iv) \quad (18)$$

$$\dot{u} + i\dot{v} = \alpha u + i\beta u + i\alpha v - \beta v \quad (19)$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad (20)$$

As we can see,  $\dot{z}_j = c_j z_j$  is asymptotically stable iff  $\text{Re}(c_j) < 0$ , and marginally stable if  $\alpha = \text{Re}(c_j) = 0$ . Same is true for  $\dot{\mathbf{z}} = \mathbf{C}\mathbf{z}$  and hence, for  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , as  $\mathbf{U}$  is invertible.



# STABILITY OF AUTONOMOUS LTI

## Condition

Consider an autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (21)$$

### Definition

Eq. (21) is stable iff real parts of eigenvalues of  $\mathbf{A}$  are non-positive.

### Definition

Eq. (21) is asymptotically stable iff real parts of eigenvalues of  $\mathbf{A}$  are negative.

# STABILITY OF AUTONOMOUS LTI

## Illustration

Here is an illustration of *phase portraits* of two-dimensional LTIs with different types of stability:

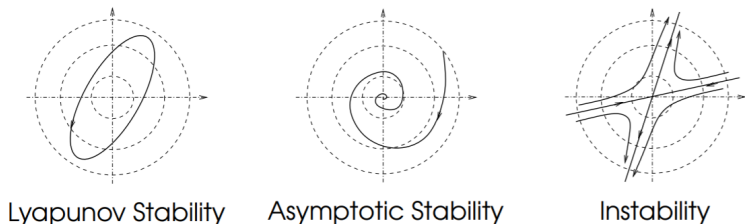


Figure 1: phase portraits for different types of stability

Credit: [staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf](http://staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf)



- Control Systems Design, by Julio H. Braslavsky  
[staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf](http://staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf)
- Stability and Eigenvalues, Steve Brunton  
[youtu.be/h7nJ6ZL4Lf0](https://youtu.be/h7nJ6ZL4Lf0)
- MAE509 (LMIs in Control): Lecture 4, part A - Stability and Eigenvalues [youtu.be/8zYOJbpiT38](https://youtu.be/8zYOJbpiT38)

Lecture slides are available via Github:

[github.com/SergeiSa/Control-Theory-2024](https://github.com/SergeiSa/Control-Theory-2024)

