LMI: Control design and robustness

Control Theory, Lecture ??

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A linear matrix inequality (LMI) is a semidefinite constraint placed on a matrix:

$$\mathbf{S} \succ 0 \tag{1}$$

We assume (and this is true!) that there exist *solvers* that can solve problems with such constraints.

Example

Given $\mathbf{A}$, find such $\mathbf{S} \succ 0$ that $\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} \prec 0$.

Notice that the last example is continious-time Lyapunov eq. for LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, and if such $\mathbf{S}$ exists the system is stable.

Consider a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, control $\mathbf{u} = \mathbf{K}\mathbf{x}$ and a Lyapunov function $V = \mathbf{x}^\top \mathbf{S}\mathbf{x}$, $\mathbf{S} \succ 0$.

Closed-form of the system is $\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}$, and full derivative of the Lyapunov function:

$$\dot{V} = \mathbf{x}^\top(\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x} \leq 0 \tag{2}$$

This can be re-written as an LMI:

$$(\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S} + \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K}) \prec 0 \tag{3}$$

This is *not linear* in decision variables ($\mathbf{S}$ and $\mathbf{K}$), and can't be solved directly using popular solvers.

Introducing new variable $\mathbf{P} = \mathbf{S}^{-1}$ and multiplying (3) by $\mathbf{P}$ on both sides (we can do it, as both $\mathbf{P}$ and $\mathbf{S}$ are full rank, and thus it is a congruence transformation which preserves definiteness, see appendix) we get:

$$\mathbf{P}(\mathbf{A} + \mathbf{B}\mathbf{K})^\top + (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{P} \prec 0 \tag{4}$$

Now we introduce one more variable $\mathbf{L} = \mathbf{K}\mathbf{P}$ and get an LMI constraint:

$$\mathbf{P}\mathbf{A}^\top + \mathbf{A}\mathbf{P} + \mathbf{L}^\top\mathbf{B}^\top + \mathbf{B}\mathbf{L} \prec 0 \tag{5}$$

Solving (5) gives us $\mathbf{P}$ and $\mathbf{L}$, from which we can compute $\mathbf{K} = \mathbf{L}\mathbf{P}^{-1}$ and $\mathbf{S} = \mathbf{P}^{-1}$, solving the original problem.

Consider a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, but when you don't know $\mathbf{A}$ exactly. In other words, you don't know the model exactly. This is not to say that we know nothing about the model, but there is an uncertainty in our knowledge.

A good way to model is luck of model knowledge, this *uncertainty*, is this:

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{F}\boldsymbol{\Delta}\mathbf{E})\mathbf{x} \tag{6}$$

where $\mathbf{F}$ and $\mathbf{E}$ are arbitrary matrices, and Δ is a *norm-bounded* matrix: $||\Delta|| \leq 1$.

We can think of it this way: $\mathbf{A} + \mathbf{F}\boldsymbol{\Delta}\mathbf{E}$ is the true but unknown model, and the range of all possible models we can expect is bounded by the possible values of Δ.

Lets write the Lyapunov equation for the system (6):

$$\dot{V} = \mathbf{x}^\top(\mathbf{A} + \mathbf{F}\Delta\mathbf{E})^\top\mathbf{S}\mathbf{x} + \mathbf{x}^\top\mathbf{S}(\mathbf{A} + \mathbf{F}\Delta\mathbf{E})\mathbf{x} \leq 0 \tag{7}$$

Let us introduce a new variable $\mathbf{w} = \Delta\mathbf{E}\mathbf{x}$:

$$\dot{V} = \mathbf{x}^\top(\mathbf{A}^\top\mathbf{S} + \mathbf{S}\mathbf{A})\mathbf{x} + \mathbf{w}^\top\mathbf{F}^\top\mathbf{S}\mathbf{x} + \mathbf{x}^\top\mathbf{S}\mathbf{F}\mathbf{w} \leq 0 \tag{8}$$

Let us consider $\mathbf{w}^\top\mathbf{w}$:

$$\mathbf{w}^\top\mathbf{w} = \mathbf{x}^\top\mathbf{E}^\top\Delta\Delta\mathbf{E}\mathbf{x} \leq \mathbf{x}^\top\mathbf{E}^\top\mathbf{E}\mathbf{x} \tag{9}$$

which is true because $||\Delta|| \leq 1$. In fact, the only property of the norm that we need here is that the delta inequality (9) holds.

With $\mathbf{w}^\top \mathbf{w} \leq \mathbf{x}^\top \mathbf{E}^\top \mathbf{E}\mathbf{x}$ we can write:

$$\mathbf{x}^\top \mathbf{E}^\top \mathbf{E}\mathbf{x} - \mathbf{w}^\top \mathbf{w} \geq 0 \tag{10}$$

Which is the same as:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}^\top \begin{bmatrix} \mathbf{E}^\top \mathbf{E} & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \geq 0 \tag{11}$$

The same way we can rewrite (8):

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}^\top \begin{bmatrix} \mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} & \mathbf{S}\mathbf{F} \\ \mathbf{F}^\top \mathbf{S} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \leq 0 \tag{12}$$

which only need to hold while (11) holds.

There is a way to enforce constraint $\mathbf{z}^\top \mathbf{M} \mathbf{z} \leq 0$ for such $\mathbf{z}$ that $\mathbf{z}^\top \mathbf{N} \mathbf{z} \geq 0$. This is called *s-procedure*.

Theorem

If $\gamma > 0$ *and* $\mathbf{M} + \gamma \mathbf{N} \prec 0$ *then* $\mathbf{z}^\top \mathbf{N} \mathbf{z} \geq 0 \implies \mathbf{z}^\top \mathbf{M} \mathbf{z} \leq 0$

Using s-procedure we enforce (12) when (11) holds:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}^\top \begin{bmatrix} \mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} + \gamma \mathbf{E}^\top \mathbf{E} & \mathbf{S}\mathbf{F} \\ \mathbf{F}^\top \mathbf{S} & -\gamma \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \leq 0 \qquad (13)$$

In LMI form this is:

$$\begin{bmatrix} \mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} + \gamma \mathbf{E}^\top \mathbf{E} & \mathbf{S}\mathbf{F} \\ \mathbf{F}^\top \mathbf{S} & -\gamma \mathbf{I} \end{bmatrix} \prec 0 \qquad (14)$$

This is a condition that the system is stable for all values of Δ. The decision variables are $\mathbf{S}$ and γ.

Let us consider the following system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{15}$$

where $\mathbf{A} = \sum_{i=1}^{n} \alpha_i \mathbf{A}_i$, $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$ with known $\mathbf{A}_i$ but unknown coefficients α_i. Is it stable for all possible values of α_i? Note that we can't use eigenvalue analysis in this case.

Geometrically, this means $\mathbf{A}$ is in a polytope with vertices $\mathbf{A}_i$.

Quadratic stability, 2

Theorem (Quadratic stability)

$\mathbf{A}_i^\top \mathbf{S} + \mathbf{S}\mathbf{A}_i \leq 0$ *implies* $\dot{\mathbf{x}} = \sum_{i=1}^{n} \alpha_i \mathbf{A}_i \mathbf{x}$ *is stable, where* $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$

Proof: $\dot{V} = \left(\sum_{i=1}^{n} \alpha_i \mathbf{A}_i\right)^\top \mathbf{S} + \mathbf{S}\left(\sum_{i=1}^{n} \alpha_i \mathbf{A}_i\right) \leq 0$ can be re-written as: $\dot{V} = \sum_{i=1}^{n} \left(\alpha_i (\mathbf{A}_i^\top \mathbf{S} + \mathbf{S}\mathbf{A}_i)\right)$ and since $\mathbf{A}_i^\top \mathbf{S} + \mathbf{S}\mathbf{A}_i \leq 0$ and $\alpha_i \geq 0$, then $\dot{V} \leq 0$. $\square$

Let us consider the following system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x} \tag{16}$$

where $\mathbf{A} = \sum_{i=1}^{n} \alpha_i \mathbf{A}_i$, $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$ with known $\mathbf{A}_i$ but unknown coefficients α_i. How to design control law $\mathbf{u} = \mathbf{K}\mathbf{x}$ making the system stable for all possible values of α_i?

The closed-loop form of the system is:

$$\dot{\mathbf{x}} = (\sum_{i=1}^{n} \alpha_i \mathbf{A}_i + \mathbf{B}\mathbf{K})\mathbf{x} \tag{17}$$

Let us write Lyapunov eq. for the system:

$$\left(\sum_{i=1}^{n} \alpha_i (\mathbf{A}_i + \mathbf{B}\mathbf{K})\right)^\top \mathbf{S} + \mathbf{S}\left(\sum_{i=1}^{n} \alpha_i (\mathbf{A}_i + \mathbf{B}\mathbf{K})\right) \prec 0 \tag{18}$$

We can re-write it as:

$$\sum_{i=1}^{n} \alpha_i \left((\mathbf{A}_i + \mathbf{B}\mathbf{K})^\top \mathbf{S} + \mathbf{S}(\mathbf{A}_i + \mathbf{B}\mathbf{K})\right) \prec 0 \tag{19}$$

Hence if $(\mathbf{A}_i + \mathbf{B}\mathbf{K})^\top \mathbf{S} + \mathbf{S}(\mathbf{A}_i + \mathbf{B}\mathbf{K}) \prec 0$, the original system is stable.

From $(\mathbf{A}_i + \mathbf{BK})^\top \mathbf{S} + \mathbf{S}(\mathbf{A}_i + \mathbf{BK}) \prec 0$, we can go on to do control design. Introducing $\mathbf{P} = \mathbf{S}^{-1}$, we use congruence transformation multiplying by $\mathbf{P}$ on both sides:

$$\mathbf{P}(\mathbf{A}_i + \mathbf{BK})^\top + (\mathbf{A}_i + \mathbf{BK})\mathbf{P} \prec 0 \tag{20}$$

Introducing new variable $\mathbf{L} = \mathbf{KP}$ we get a problem linear in decision variables:

$$\mathbf{PA}_i^\top + \mathbf{A}_i\mathbf{P} + \mathbf{L}^\top\mathbf{B}^\top + \mathbf{BL} \prec 0 \tag{21}$$

where the decision variables are $\mathbf{P}$ and $\mathbf{L}$. The control gain matrix is found as $\mathbf{K} = \mathbf{LP}^{-1}$.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2023

Consider matrices $\mathbf{P} \succ 0$, and $\mathbf{V} \in \mathbb{R}^{n,n}$ is full rank. We can prove that:

$$\mathbf{P} \succ 0 \implies \mathbf{V}^\top \mathbf{P} \mathbf{V} \succ 0 \tag{22}$$

Proof: $\mathbf{x}^\top \mathbf{V}^\top \mathbf{P} \mathbf{V} \mathbf{x} = \mathbf{z}^\top \mathbf{P} \mathbf{z}$, where $\mathbf{z} = \mathbf{V}\mathbf{x}$. Since $\mathbf{P} \succ 0$, $\mathbf{z}^\top \mathbf{P} \mathbf{z} \geq 0$, hence $\mathbf{x}^\top \mathbf{V}^\top \mathbf{P} \mathbf{V} \mathbf{x} \geq 0$.

Definition

Congruence transformation preserves semi-definiteness: $\det(\mathbf{V}) \neq 0, \ \mathbf{P} \succ 0 \implies \mathbf{V}^\top \mathbf{P} \mathbf{V} \succ 0$