

# Frequency response, Bode

## Control Theory, Lecture 5

by Sergei Savin

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## Frequency response

Frequency response is a steady-state output of the system, given sinusoidal input.

Consider a system  $Y(s) = G(s)U(s)$ . Sinusoidal input  $u(t) = \sin(\omega t)$  in time domain translates to  $U(s) = \frac{\omega}{\omega^2 + s^2}$  in Laplace domain. So, given a sinusoidal input, the system becomes:

$$Y(s) = G(s) \frac{\omega}{\omega^2 + s^2} \quad (1)$$

# FRACTION EXPANSION

If a transfer function  $G(s)$  is a rational fraction, it can be represented as:

$$G(s) = \frac{n(s)}{(s + p_1)(s + p_2) \dots (s + p_n)} \quad (2)$$

where  $p_i$  are the roots on the denominator - called *poles* of the transfer function.

In many cases (for example when  $p_i$  are real and non-repeating), the fraction can be expanded:

$$G(s) = \frac{n(s)}{(s + p_1)(s + p_2) \dots (s + p_n)} = \frac{r_1}{s + p_1} + \frac{r_2}{s + p_2} + \dots + \frac{r_n}{s + p_n}$$

We can expand the function  $Y(s) = G(s) \frac{\omega}{\omega^2 + s^2}$  in a similar way:

$$Y(s) = \frac{r_1}{s + p_1} + \frac{r_2}{s + p_2} + \dots + \frac{r_n}{s + p_n} + \frac{\alpha}{s + j\omega} + \frac{\beta}{s - j\omega}$$

Laplace function of the form  $\frac{r_i}{s + p_i}$  corresponds to the following time function:

$$y(t) = r_i e^{-p_i t} \quad (3)$$

So, for a stable transfer function as time goes to infinity,  $r_i e^{-p_i t}$  goes to zero. The only components of the function  $Y(s)$  that do not disappear are the last two:  $\frac{\alpha}{s + j\omega} + \frac{\beta}{s - j\omega}$ .

One can show that constants in the expansion  $\frac{\alpha}{s+j\omega} + \frac{\beta}{s-j\omega}$  can be found in the form:

$$\alpha = -G(j\omega)g \quad (4)$$

$$\beta = G(-j\omega)g \quad (5)$$

In fact, the analysis of the frequency response will involve analyzing the transfer function  $G(j\omega)$ .

# LAPLACE AND FOURIER TRANSFORMS

- *Fourier series* can be seen as representing a periodic function as a sum of harmonics (sines and cosines). These sines and cosines can be thought of as forming a basis in a linear space. The coefficients of the series can be thought of as a discrete spectrum of the function.
- *Fourier transform* gives a continuous spectrum of the function. The "basis" is still made of harmonic functions.
- *Laplace transform* also gives a continuous spectrum of the function, but in a different basis: the basis is given by complex exponentials. I like to think of this basis as solutions of second order ODEs.

# LAPLACE AND FOURIER TRANSFORMS

Let's compare. Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-2\pi j t \omega} dt, \quad \omega \in \mathbb{R} \quad (6)$$

Laplace transform:

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt, \quad s \in \mathbb{C} \quad (7)$$

We can see that Fourier looks like Laplace with purely imaginary number in the exponent.



# LAPLACE AND STEADY STATE SOLUTION

From analysing solutions of linear ODEs we know that, given harmonic input (sine, cosine, their combination) "after the transient process is over, the solution approaches a harmonic with the same frequency", but possibly different amplitude and phase.

Intuitively we can think of the imaginary part of  $s$  as having to do with this frequency response.

The kernel function of the Laplace transform is  $e^{-st}$  with  $s = \sigma + j\omega$  being a complex variable. If  $\sigma = 0$ , the kernel becomes  $e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$ . You can see the similarity with Fourier transform kernel.

# BODE PLOT

The first key idea of a Bode plot is substitution of purely complex variable  $j\omega$  in place of Laplace variable  $s$ , which can have non-zero real part.

Given a transfer function  $W(s)$ ,  $s = \sigma + j\omega$  we can analyse its behaviour when  $\sigma = 0$ . We can plot its amplitude

$a(\omega) = |W(j\omega)|$  and its phase

$\varphi(\omega) = \text{atan2}(\text{im}(W(j\omega)), \text{real}(W(j\omega)))$ .

Bode plot is actually two plots, 1)  $20 \cdot \log(a(\omega))$  and 2)  $\frac{180}{\pi} \varphi(\omega)$ . The 20 and log has to do with the vertical axis being in decibels.

## BODE PLOT - EXAMPLE

Consider  $W(s) = \frac{1}{1+s}$ . Then  $W(j\omega) = \frac{1}{1+j\omega}$ . We can transform it as:

$$W(j\omega) = \frac{1 - j\omega}{(1 + j\omega)(1 - j\omega)} = \frac{1 - j\omega}{1 + \omega^2} \quad (8)$$

Thus we have  $\text{real}(W(j\omega)) = \frac{1}{1+\omega^2}$  and  $\text{im}(W(j\omega)) = -\frac{\omega}{1+\omega^2}$ .

Bode plot is then given as:

$$a(\omega) = \sqrt{\frac{1 + \omega^2}{(1 + \omega^2)^2}} = \frac{1}{\sqrt{(1 + \omega^2)}} \quad (9)$$

$$\varphi(\omega) = \text{atan2} \left( -\frac{\omega}{1 + \omega^2}, \frac{1}{1 + \omega^2} \right) \quad (10)$$

## BODE PLOT - STABILITY MARGINS

Before we discuss the use of Bode plot, let us remember that closed-loop transfer function has form (when simple feedback is used):

$$W(s) = \frac{G(s)}{1 + G(s)} \quad (11)$$

Substituting  $s \rightarrow j\omega$  we get:

$$W(\omega) = \frac{G(j\omega)}{1 + G(j\omega)} \quad (12)$$

From this we can see that  $W(\omega)$  becomes ill-defined if  $G(j\omega) = -1$ . Meaning, we want to avoid two things happening simultaneously: the amplitude of  $G(j\omega)$  being equal to 1, and its phase (argument) being equal to  $180^\circ$  (remember, phase of  $0^\circ$  is pure positive real number, phase of  $90^\circ$  is pure positive imaginary number,  $180^\circ$  is pure negative real number, etc.).

Let's check an illustration:



Check the colab notebook based on the example above for an illustration of how the Bode plot can be made by hand or via scipy signal library.



- Control System Lectures - Bode Plots, Introduction
- Oxford University Press. s-Domain analysis: poles, zeros, and Bode plots

Lecture slides are available via Github:

[github.com/SergeiSa/Control-Theory-2024](https://github.com/SergeiSa/Control-Theory-2024)

