

# ODE and State Space

## Control Theory, Lecture 1

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# ORDINARY DIFFERENTIAL EQUATIONS, 1ST ORDER

Let us remember the normal form of first-order *ordinary differential equations (ODEs)*:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \quad (1)$$

where  $\mathbf{x} = \mathbf{x}(t)$  is the solution of the equation and  $t$  is a free variable (usually - time).

## Definition

We can call this equation (same as any other ODEs) a *dynamical system*, and  $\mathbf{x}$  is called the *state* of the dynamical system.

## Example

$$\dot{x} = -3x^3 - 7 \quad (2)$$

*State* of a dynamical system is a minimal set of variables that describe the system, in the sense that knowing current state and all future inputs you can predict the behavior of the system.

## Example

For a spring-damper system, the state variables could be position and velocity of the mass.

## Example

For a double pendulum, the state variables could be joint angles and joint velocities.

# ODEs, N-TH ORDER

The normal form of an  $n$ -th order ordinary differential equation is:

$$y^{(n)} = f(y^{(n-1)}, y^{(n-2)}, \dots, \dot{y}, y, t) \quad (3)$$

where  $y = y(t)$  is the solution of the equation. Same as before, it is a *dynamical system*, but this time we need more variables to describe the state of this system, for example we can use the set  $\{y, \dot{y}, \dots, y^{(n-1)}\}$ .

Example (Pendulum)

$$\ddot{y} = -0.1\dot{y} - 7 \sin(y) \quad (4)$$

Example (DC motor under constant voltage)

$$\begin{cases} \dot{y}_1 = -100\dot{y}_2 - 2y_1 + 10 \\ \ddot{y}_2 = -0.1\dot{y}_2 + 100y_1 \end{cases} \quad (5)$$

# LINEAR ODE, 1ST ORDER

Linear ODEs of the first order have normal form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (6)$$

Example

$$\begin{cases} \dot{x}_1 = -20x_1 + 7x_2 \\ \dot{x}_2 = 10.5x_1 - 3x_2 \end{cases} \quad (7)$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -8 & 5 & 2 \\ 0.5 & -10 & -2 \\ 1 & -1 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (8)$$

# LINEAR DIFFERENTIAL EQUATIONS, N-TH ORDER

A single linear ODE of the n-th order are often written in the form:

$$a_n y^{(n)} + \dots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = 0 \quad (9)$$

Example

$$12 \ddot{y} - 3\ddot{y} + 5.5\dot{y} + 2y = 0 \quad (10)$$

Example

$$5\ddot{y} - 2\dot{y} + 10y = 0 \quad (11)$$

Sometimes it is convenient to write an ODE in the form with an *input*, for example:

$$a_2\ddot{y} + a_1\dot{y} + a_0y = u(t) \quad (12)$$

In this equation  $u(t)$  is a function of time. This form offers us many uses:

- We can use  $u(t)$  to model *control input*, (e.g. voltage, motor torque) that we directly control.
- We can use  $u(t)$  to model external forces acting on the system.
- We can substitute particular function instead of  $u(t)$ , e.g. sine wave or step function, to study how the system behaves with such an input.

Some examples of linear ODEs with one input:

Example

$$\begin{cases} \dot{y}_1 = -20y_1 + 7y_2 + u \\ \dot{y}_2 = 10.5y_1 - 3y_2 \end{cases} \quad (13)$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -8 & 5 & 2 \\ 0.5 & -10 & -2 \\ 1 & -1 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \quad (14)$$



General form of an  $n$ -th order linear ODE with an input can be presented as follows:

$$a_n y^{(n)} + \dots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \quad (15)$$

State-space representation of a linear system with an input is:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (16)$$

Note that in latter,  $\mathbf{u}$  can be either scalar or a vector.

# EQUATIONS WITH AN OUTPUT

Equations can also have an output. The meaning of what is an output of an equation depends on the particular use-case - it is not a mathematical issue, it is a question of interpretation. For example, an output can mean:

- What we measure (position and orientation of a quadrotor, angular velocity of motor's rotor, etc.).
- What we care about and/or what we want to control (height of a quadrotor, velocity of a car, etc.)
- etc.

We often denote output as  $y$ , and it depends on the state of the system:  $y = g(\mathbf{x})$

State-space representation of a linear system with an input and an output is:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} = \mathbf{Cx} \end{cases} \quad (17)$$

If  $\mathbf{u} \in \mathbb{R}$  and  $\mathbf{y} \in \mathbb{R}$  (i.e. if they are scalars) and you want to represent the system with an output as a single ODE, it is typical to treat the output as the ODE variable:

$$a_n y^{(n)} + \dots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \quad (18)$$

In this course we will focus entirely on linear dynamical systems, expressed as ODEs:

$$a_n y^{(n)} + \dots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \quad (19)$$

or in state-space form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \quad (20)$$

If  $\mathbf{u}$  and  $\mathbf{y}$  are scalars, the system is called *single-input single-output (SISO)*, if they are vectors - *multi-input multi-output (MIMO)*.

We can always express a SISO system in either form - ODE or state-space.

# ODE TO STATE-SPACE CONVERSION

Consider eq.  $\ddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0y = u$ .

Make a substitution:  $x_1 = y$ ,  $x_2 = \dot{y}$ ,  $x_3 = \ddot{y}$ . We get:

$$\dot{x}_1 = \dot{y} = x_2 \quad (21)$$

$$\dot{x}_2 = \ddot{y} = x_3 \quad (22)$$

$$\dot{x}_3 = u - a_2\ddot{y} - a_1\dot{y} - a_0y = u - a_2x_3 - a_1x_2 - a_0x_1 \quad (23)$$

Which can be directly put in the state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \quad (24)$$

Consider State-Space system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \\ y = \mathbf{C}\mathbf{x} \end{cases} \quad (25)$$

We want to find an equivalent representation in the ODE form:

$$y^{(n)} = d_{n-1}y^{(n-1)} + \dots + d_1\dot{y} + d_0y \quad (26)$$

Defining  $\mathbf{d}^\top = [d_0 \ d_1 \ \dots \ d_{n-1}]$  and

$\mathbf{y} = [y \ \dot{y} \ \dots \ y^{(n-1)}]^\top$ , we can re-write the ODE as:

$$y^{(n)} = \mathbf{d}^\top \mathbf{y} \quad (27)$$

Thus, if we can find  $\mathbf{d}$ , we can solve the problem.

## STATE-SPACE TO ODE CONVERSION, 2

We can differentiate  $y = \mathbf{C}\mathbf{x}$   $n$  times:

$$y = \mathbf{C}\mathbf{x} \quad (28)$$

$$\dot{y} = \mathbf{C}\dot{\mathbf{x}} = \mathbf{C}\mathbf{A}\mathbf{x} \quad (29)$$

$$\dots \quad (30)$$

$$y^{(n)} = \mathbf{C}\mathbf{x}^{(n)} = \mathbf{C}\mathbf{A}^n\mathbf{x} \quad (31)$$

This gives us relation between  $\mathbf{y}$  and  $\mathbf{x}$ :

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \dots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix} \mathbf{x} = \mathcal{O}\mathbf{x} \quad (32)$$

where matrix  $\mathcal{O}$  is called observability matrix.

## STATE-SPACE TO ODE CONVERSION, 3

As long as the observability matrix  $\mathcal{O}$  is full rank, we can express the state as:

$$\mathbf{x} = \mathcal{O}^{-1}\mathbf{y} \quad (33)$$

Then we re-write  $y^{(n)} = \mathbf{CA}^n\mathbf{x}$  as:

$$y^{(n)} = \mathbf{CA}^n\mathcal{O}^{-1}\mathbf{y} \quad (34)$$

Thus,  $\mathbf{d}^\top = \mathbf{CA}^n\mathcal{O}^{-1}$  and the ODE takes the form:

$$y^{(n)} = \mathbf{CA}^n\mathcal{O}^{-1} \begin{bmatrix} y \\ \dot{y} \\ \dots \\ y^{(n-1)} \end{bmatrix} \quad (35)$$

You can see an example in the appendix A.



- 2.14 Analysis and Design of Feedback Control Systems:
  - ▶ [State-Space Representation of LTI Systems](#)
  - ▶ [Time-Domain Solution of LTI State Equations](#)
- [Linear Physical Systems Analysis:](#)
  - ▶ [State Space Representations of Linear Physical Systems](#)  
[lpsa.swarthmore.edu/Representations/SysRepSS.html](https://lpsa.swarthmore.edu/Representations/SysRepSS.html)
  - ▶ [Transformation: Differential Equation to State Space](#)  
[lpsa.swarthmore.edu/.../DE2SS.html](https://lpsa.swarthmore.edu/.../DE2SS.html)

Lecture slides are available via Github:

[github.com/SergeiSa/Control-Theory-2024](https://github.com/SergeiSa/Control-Theory-2024)



## Appendix A

# STATE SPACE TO ODE CONVERSION, 1

(extra)

Consider a system in state-space form:

$$\left\{ \begin{array}{l} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} \right. \iff \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \\ y = \mathbf{C}\mathbf{x} \end{cases} \quad (36)$$

We want to rewrite it as a linear ODE:

$$\ddot{y} + b_2\dot{y} + b_1y = 0 \quad (37)$$

Note that initial conditions of both equation need to agree.

Since  $y = \mathbf{C}\mathbf{x}$ , its derivative is  $\dot{y} = \mathbf{C}\dot{\mathbf{x}}$ :

$$\dot{y} = \mathbf{C}\mathbf{A}\mathbf{x} \quad (38)$$

$$\dot{y} = \begin{bmatrix} (a_{11}c_1 + a_{21}c_2) & (a_{12}c_1 + a_{22}c_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (39)$$

Analogous for  $\ddot{y}$ :

$$\ddot{y} = \mathbf{C}\mathbf{A}\mathbf{A}\mathbf{x} \quad (40)$$

Combining our results we find the linear transformation between the variables  $x_1$ ,  $x_2$  and  $y$ ,  $\dot{y}$ :

$$\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ (a_{11}c_1 + a_{21}c_2) & (a_{12}c_1 + a_{22}c_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (41)$$

Resulting transformation matrix is:

$$\mathbf{T} = \begin{bmatrix} c_1 & c_2 \\ (a_{11}c_1 + a_{21}c_2) & (a_{12}c_1 + a_{22}c_2) \end{bmatrix} \quad (42)$$

$$\mathbf{x} = \mathbf{T}^{-1} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \quad (43)$$

Remember that:

$$\ddot{y} = \mathbf{CAAx} \quad (44)$$

$$\ddot{y} = \mathbf{CAAT}^{-1} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \quad (45)$$

So, we obtained  $\ddot{y}$  as a linear function of  $y, \dot{y}$ . From this it is clear how the same can be generalized to higher dimensions.

# STATE SPACE TO ODE

Check out the code implementation.

