Linearization

Control Theory, Lecture 11

by Sergei Savin

Spring 2023

Content

- Taylor expansion
- Linearization
- Linearization of Manipulator equations

Consider a non-linear dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{1}$$

If $\mathbf{x}_0$ and $\mathbf{u}_0$ represent a *node*, i.e. $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) = 0$, $\mathbf{x}_0 = \text{const}$, $\mathbf{u}_0 = \text{const}$, we can consider a Taylor expansion around that node:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0) + \text{H.O.T.} \tag{2}$$

Where $\mathbf{x}_0$ and $\mathbf{u}_0$ are expansion point. We define new variables $\mathbf{e}$ and $\mathbf{v}$ as distance from the expansion point:

$$\mathbf{e} = \mathbf{x} - \mathbf{x}_0, \quad \dot{\mathbf{e}} = \dot{\mathbf{x}}, \tag{3}$$

$$\mathbf{v} = \mathbf{u} - \mathbf{u}_0. \tag{4}$$

$$\tag{5}$$

With that we can re-write the Taylor expansion:

$$\dot{\mathbf{e}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \mathbf{e} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \mathbf{v} + \text{H.O.T.} \tag{6}$$

We can introduce notation:

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}, \quad \mathbf{B} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}. \tag{7}$$

If we drop higher order terms from the Taylor expansion, we obtain *linearization* of the system dynamics:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \tag{8}$$

In this context, $\mathbf{x}_0$ and $\mathbf{u}_0$ is the *linearization point.*

Taylor expansion along a trajectory

Consider a non-linear dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{9}$$

and a trajectory $\dot{\mathbf{x}}_0 = \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)$. We can consider a Taylor expansion along this trajectory:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0) + \text{H.O.T.} \tag{10}$$

Since $\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_0$, we re-write:

$$\dot{\mathbf{e}} \sim \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} + \text{H.O.T.} \tag{11}$$

As before, we drop higher order terms and obtain linearization:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \tag{12}$$

If we want to maintain our original variables, we can still use Taylor expansion:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) + \mathbf{A}(\mathbf{x} - \mathbf{x}_0) + \mathbf{B}(\mathbf{u} - \mathbf{u}_0) \tag{13}$$

Denoting $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{A}\mathbf{x}_0 - \mathbf{B}\mathbf{u}_0 = \mathbf{c}$ and dropping H.O.T. we approximate the system as affine:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \tag{14}$$

Consider Manipulator equation:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau \tag{15}$$

We will attempt to linearize it.

We begin by proposing the following new variables:

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} - \mathbf{q}_0 \\ \dot{\mathbf{q}} - \dot{\mathbf{q}}_0 \end{bmatrix}, \quad \mathbf{u} = \tau - \tau_0 \tag{16}$$

$$\mathbf{q} = \mathbf{S}_q \mathbf{x}, \quad \dot{\mathbf{q}} = \mathbf{S}_v \mathbf{x} \tag{17}$$

where τ_0 is chosen such that $\mathbf{C}(\dot{\mathbf{q}}_0, \mathbf{q}_0)\dot{\mathbf{q}}_0 + \mathbf{g}(\mathbf{q}_0) = \tau_0$, and $\mathbf{S}_q$ and $\mathbf{S}_v$ are choice matrices.

Next, we introduce function $\phi(\dot{\mathbf{q}}, \mathbf{q}, \tau) = \ddot{\mathbf{q}}$, expressed as:

$$\phi(\dot{\mathbf{q}}, \mathbf{q}, \tau) = \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \tag{18}$$

Next, we write our dynamics as a first order ODE:

$$\frac{d}{dt}\left(\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}\right) = \begin{bmatrix} \dot{\mathbf{q}} \\ \phi(\dot{\mathbf{q}}, \mathbf{q}, \tau) \end{bmatrix} \tag{19}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{S}_v \mathbf{x} \\ \phi(\mathbf{x}, \tau) \end{bmatrix} \tag{20}$$

With that, we can find matrices **A** and **B**.

In this case, state matrices **A** and **B** become:

$$\mathbf{A} = \begin{bmatrix} \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{q}} & \frac{\partial \dot{\mathbf{q}}}{\partial \dot{\mathbf{q}}} \\ \frac{\partial \phi}{\partial \mathbf{q}} & \frac{\partial \phi}{\partial \dot{\mathbf{q}}} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ \frac{\partial \phi}{\partial \mathbf{q}} & \frac{\partial \phi}{\partial \dot{\mathbf{q}}} \end{bmatrix} \tag{21}$$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial \dot{\mathbf{q}}}{\partial \tau} \\ \frac{\partial \phi}{\partial \tau} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{H}^{-1} \end{bmatrix} \tag{22}$$

Thus. our task is to find the following jacobians: $\frac{\partial \phi}{\partial \mathbf{q}}$ and $\frac{\partial \phi}{\partial \dot{\mathbf{q}}}$.

Let us find $\frac{\partial \phi}{\partial \mathbf{q}}$:

$$\frac{\partial \phi}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}} \left(\mathbf{H}^{-1} (\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \right) = \tag{23}$$

$$= \frac{\partial \mathbf{H}^{-1}}{\partial \mathbf{q}} (\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \mathbf{H}^{-1} \frac{\partial}{\partial \mathbf{q}} (\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \tag{24}$$

If we evaluate $\frac{\partial \phi}{\partial \mathbf{q}}$ at the point $\mathbf{q} = \mathbf{q}_0$, $\dot{\mathbf{q}} = \dot{\mathbf{q}}_0$, $\tau = \tau_0$, we can use the fact that $\mathbf{C}(\dot{\mathbf{q}}_0, \mathbf{q}_0)\dot{\mathbf{q}}_0 + \mathbf{g}(\mathbf{q}_0) = \tau_0$ to avoid computing derivative $\frac{\partial \mathbf{H}^{-1}}{\partial \mathbf{q}}$:

$$\frac{\partial \phi}{\partial \mathbf{q}} = \mathbf{H}^{-1} \left(\tau - \frac{\partial \mathbf{C}\dot{\mathbf{q}}}{\partial \mathbf{q}} - \frac{\partial \mathbf{g}}{\partial \mathbf{q}} \right) \tag{25}$$

Let us find $\frac{\partial \phi}{\partial \dot{\mathbf{q}}}$:

$$\frac{\partial \phi}{\partial \dot{\mathbf{q}}} = \frac{\partial}{\partial \dot{\mathbf{q}}} \left(\mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g})\right) = \tag{26}$$

$$= -\mathbf{H}^{-1} \frac{\partial \mathbf{C}\dot{\mathbf{q}}}{\partial \dot{\mathbf{q}}} \tag{27}$$

With that, we expressed all jacobians. The rest is the same as in the general case we studied in the first slides.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2023