Frequency response, Bode

Control Theory, Lecture 5

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Frequency response

Frequency response is a steady-state output of the system, given sinusoidal input.

Consider a system $Y(s) = G(s)U(s)$. Sinusoidal input $u(t) = \sin(\omega t)$ in time domain translates to $U(s) = \frac{\omega}{\omega^2+s^2}$ in Laplace domain. So, given a sinusoidal input, the system becomes:

$$Y(s) = G(s)\frac{\omega}{\omega^2 + s^2} \tag{1}$$

Fraction expansion

If a transfer function $G(s)$ is a rational fraction, it can be represented as:

$$G(s) = \frac{n(s)}{(s+p_1)(s+p_2)\ ...\ (s+p_n)} \tag{2}$$

where p_i are the roots on the denominator - called *poles* of the transfer function.

In many cases (for example when p_i are real and non-repeating), the fraction can be expanded:

$$G(s) = \frac{n(s)}{(s+p_1)(s+p_2)\ ...\ (s+p_n)} = \frac{r_1}{s+p_1} + \frac{r_2}{s+p_2} + ... + \frac{r_n}{s+p_n}$$

We can expand the function $Y(s) = G(s)\frac{\omega}{\omega^2+s^2}$ in a similar way:

$$Y(s) = \frac{r_1}{s+p_1} + \frac{r_2}{s+p_2} + ... + \frac{r_n}{s+p_n} + \frac{\alpha}{s+j\omega} + \frac{\beta}{s-j\omega}$$

Laplace function of the form $\frac{r_i}{s+p_i}$ corresponds to the following time function:

$$y(t) = r_i e^{-p_i t} \tag{3}$$

So, for a stable transfer function as time goes to infinity, $r_i e^{-p_i t}$ goes to zero. The only components of the function $Y(s)$ that do not disappear are the last two: $\frac{\alpha}{s+j\omega} + \frac{\beta}{s-j\omega}$.

Fraction expansion

One can show that constants in the expansion $\frac{\alpha}{s+j\omega} + \frac{\beta}{s-j\omega}$ can be found in the form:

$$\alpha = -G(j\omega)g \tag{4}$$

$$\beta = G(-j\omega)g \tag{5}$$

In fact, the analysis of the frequency response will involve analyzing the transfer function $G(j\omega)$.

Laplace and Fourier transforms

- *Fourier series* can be seen as representing a periodic function as a sum of harmonics (sines and cosines). These sines and cosines can be thought of as forming a basis in a linear space. The coefficients of the series can be thought of as a discrete spectrum of the function.
- *Fourier transform* gives a continuous spectrum of the function. The "basis" is still made of harmonic functions.
- *Laplace transform* also gives a continuous spectrum of the function, but in a different basis: the basis is given by complex exponentials. I like to think of this basis as solutions of second order ODEs.

Let's compare. Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi j t \omega} dt, \quad \omega \in \mathbb{R} \tag{6}$$

Laplace transform:

$$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt, \quad s \in \mathbb{C} \tag{7}$$

We can see that Fourier looks like Laplace with purely imaginary number in the exponent.

Laplace and steady state solution

From analysing solutions of linear ODEs we know that, given harmonic input (sine, cosine, their combination) "after the transient process is over, the solution approaches a harmonic with the same frequency", but possibly different amplitude and phase.

Intuitively we can think of the imaginary part of s as having to do with this frequency response.

The kernel function of the Laplace transform is e^{-st} with $s = \sigma + j\omega$ being a complex variable. If $\sigma = 0$, the kernel becomes $e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$. You can see the similarity with Fourier transform kernel.

The first key idea of a Bode plot is substitution of purely complex variable $j\omega$ in place of Laplace variable s, which can have non-zero real part.

Given a transfer function $W(s)$, $s = \sigma + j\omega$ we can analyse its behaviour when $\sigma = 0$. We can plot its amplitude $a(\omega) = |W(j\omega)|$ and its phase $\varphi(\omega) = \text{atan2}(\text{im}(W(j\omega)),\ \text{real}(W(j\omega)))$.

Bode plot is actually two plots, 1) $20 \cdot \log(a(\omega))$ and 2) $\frac{180}{\pi}\varphi(\omega)$. The 20 and log has to do with the vertical axis being in decibels.

Consider $W(s) = \frac{1}{1+s}$. Then $W(j\omega) = \frac{1}{1+j\omega}$. We can transform it as:

$$W(j\omega) = \frac{1 - j\omega}{(1 + j\omega)(1 - j\omega)} = \frac{1 - j\omega}{1 + \omega^2} \tag{8}$$

Thus we have $\mathrm{real}(W(j\omega)) = \frac{1}{1+\omega^2}$ and $\mathrm{im}(W(j\omega)) = -\frac{\omega}{1+\omega^2}$.

Bode plot is then given as:

$$a(\omega) = \sqrt{\frac{1 + \omega^2}{(1 + \omega^2)^2}} = \frac{1}{\sqrt{(1 + \omega^2)}} \tag{9}$$

$$\varphi(\omega) = \mathrm{atan2}\left(-\frac{\omega}{1 + \omega^2},\ \frac{1}{1 + \omega^2}\right) \tag{10}$$

Before we discuss the use of Bode plot, let us remember that closed-loop transfer function has form (when simple feedback is used):

$$W(s) = \frac{G(s)}{1 + G(s)} \tag{11}$$

Substituting $s \longrightarrow j\omega$ we get:

$$W(\omega) = \frac{G(j\omega)}{1 + G(j\omega)} \tag{12}$$

From this we can see that $W(\omega)$ becomes ill-defined if $G(j\omega) = -1$. Meaning, we want to avoid two things happening simultaneously: the amplitude of $G(j\omega)$ being equal to 1, and its phase (argument) being equal to 180° (remember, phase of 0° is pure positive real number, phase of 90° is pure positive imaginary number, 180° is pure negative real number, etc.).

Let's check an illustration:

Check the colab notebook based on the example above for an illustration of how the Bode plot can be made by hand or via scipy signal library.

Read more

- Control System Lectures - Bode Plots, Introduction
- Oxford University Press. s-Domain analysis: poles, zeros, and Bode plots

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2024