ODE and State Space

Control Theory, Lecture 1

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Let us remember the normal form of first-order *ordinary differential equations (ODEs):*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where $\mathbf{x} = \mathbf{x}(t)$ is the solution of the equation and t is a free variable (usually - time).

Definition

We can call this equation (same as any other ODEs) a *dynamical system*, and $\mathbf{x}$ is called the *state* of the dynamical system.

Example

$$\dot{x} = -3x^3 - 7 \tag{2}$$

State of a dynamical system is a minimal set of variables that describe the system, in the sense that knowing current state and all future inputs you can predict the behavior of the system.

Example

For a spring-damper system, the state variables could be position and velocity of the mass.

Example

For a double pendulum, the state variables could be joint angles and joint velocities.

The normal form of an *n-th order* ordinary differential equation is:

$$y^{(n)} = f(y^{(n-1)}, y^{(n-2)}, \ldots, \dot{y}, y, t) \tag{3}$$

where $y = y(t)$ is the solution of the equation. Same as before, it is a *dynamical system*, but this time we need more variables to describe the state of this system, for example we can use the set $\{y,\ \dot{y}, \ldots, y^{(n-1)}\}$.

Example (Pendulum)

$$\ddot{y} = -0.1\dot{y} - 7\sin(y) \tag{4}$$

Example (DC motor under constant voltage)

$$\begin{cases} \dot{y}_1 = -100\dot{y}_2 - 2y_1 + 10 \\ \ddot{y}_2 = -0.1\dot{y}_2 + 100y_1 \end{cases} \tag{5}$$

Linear ODEs of the first order have normal form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$

Example

$$\begin{cases} \dot{x}_1 = -20x_1 + 7x_2 \\ \dot{x}_2 = 10.5x_1 - 3x_2 \end{cases} \tag{7}$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -8 & 5 & 2 \\ 0.5 & -10 & -2 \\ 1 & -1 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{8}$$

A single linear ODE of the n-th order are often written in the form:

$$a_n y^{(n)} + ... + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = 0 \tag{9}$$

Example

$$12\dddot{y} - 3\ddot{y} + 5.5\dot{y} + 2y = 0 \tag{10}$$

Example

$$5\ddot{y} - 2\dot{y} + 10y = 0 \tag{11}$$

Sometimes it is convenient to write an ODE in the form with an *input*, for example:

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = u(t) \tag{12}$$

In this equation $u(t)$ is a function of time. This form offers us many uses:

- We can use $u(t)$ to model *control input*, (e.g. voltage, motor torque) that we directly control.
- We can use $u(t)$ to model external forces acting on the system.
- We can substitute particular function instead of $u(t)$, e.g. sine wave or step function, to study how the system behaves with such an input.

ODEs with an input, 1

Some examples of linear ODEs with one input:

Example

$$\begin{cases} \dot{y}_1 = -20y_1 + 7y_2 + u \\ \dot{y}_2 = 10.5y_1 - 3y_2 \end{cases} \tag{13}$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -8 & 5 & 2 \\ 0.5 & -10 & -2 \\ 1 & -1 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \tag{14}$$

Equations with an input

General form of an n-th order linear ODE with an input can be presented as follows:

$$a_n y^{(n)} + ... + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \tag{15}$$

State-space representation of a linear system with an input is:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{16}$$

Note that in latter, **u** can be either scalar or a vector.

Equations can also have an output. The meaning of what is an output of an equation depends on the particular use-case - it is not a mathematical issue, it is a question of interpretation. For example, an output can mean:

- What we measure (position and orientation of a quadrotor, angular velocity of motor's rotor, etc.).
- What we care about and/or what we want to control (height of a quadrotor, velocity of a car, etc.)
- etc.

We often denote output as y, and it depends on the state of the system: $y = g(\mathbf{x})$

State-space representation of a linear system with an input and an output is:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{17}$$

If $\mathbf{u} \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}$ (i.e. if they are scalars) and you want to represent the system with an output as a single ODE, it is typical to treat the output as the ODE variable:

$$a_n y^{(n)} + ... + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \tag{18}$$

In this course we will focus entirely on linear dynamical systems, expressed as ODEs:

$$a_n y^{(n)} + ... + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \tag{19}$$

or in state-space form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{20}$$

If $\mathbf{u}$ and $\mathbf{y}$ are scalars, the system is called *single-input single-output (SISO)*, if they are vectors - *multi-input multi-output (MIMO)*.

We can always express a SISO system in either form - ODE or state-space.

Consider eq. $\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = u$.

Make a substitution: $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$. We get:

$$\dot{x}_1 = \dot{y} = x_2 \tag{21}$$

$$\dot{x}_2 = \ddot{y} = x_3 \tag{22}$$

$$\dot{x}_3 = u - a_2\ddot{y} - a_1\dot{y} - a_0 y = u - a_2 x_3 - a_1 x_2 - a_0 x_1 \tag{23}$$

Which can be directly put in the state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \tag{24}$$

Consider State-Space system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \\ y = \mathbf{C}\mathbf{x} \end{cases} \tag{25}$$

We want to find an equivalent representation in the ODE form:

$$y^{(n)} = d_{n-1} y^{(n-1)} + ... + d_1 \dot{y} + d_0 y \tag{26}$$

Defining $\mathbf{d}^\top = \begin{bmatrix} d_0 & d_1 & ... & d_{n-1} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y & \dot{y} & ... & y^{(n-1)} \end{bmatrix}^\top$, we can re-write the ODE as:

$$y^{(n)} = \mathbf{d}^\top \mathbf{y} \tag{27}$$

Thus, if we can find $\mathbf{d}$, we can solve the problem.

State-Space to ODE conversion, 2

We can differentiate $y = \mathbf{C}\mathbf{x}$ n times:

$$y = \mathbf{C}\mathbf{x} \tag{28}$$

$$\dot{y} = \mathbf{C}\dot{\mathbf{x}} = \mathbf{C}\mathbf{A}\mathbf{x} \tag{29}$$

$$... \tag{30}$$

$$y^{(n)} = \mathbf{C}\mathbf{x}^{(n)} = \mathbf{C}\mathbf{A}^n\mathbf{x} \tag{31}$$

This gives us relation between $\mathbf{y}$ and $\mathbf{x}$:

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ ... \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix} \mathbf{x} = \mathcal{O}\mathbf{x} \tag{32}$$

where matrix $\mathcal{O}$ is called observability matrix.

State-Space to ODE conversion, 3

As long as the observability matrix $\mathcal{O}$ is full rank, we can express the state as:

$$\mathbf{x} = \mathcal{O}^{-1}\mathbf{y} \tag{33}$$

Then we re-write $y^{(n)} = \mathbf{C}\mathbf{A}^n\mathbf{x}$ as:

$$y^{(n)} = \mathbf{C}\mathbf{A}^n\mathcal{O}^{-1}\mathbf{y} \tag{34}$$

Thus, $\mathbf{d}^\top = \mathbf{C}\mathbf{A}^n\mathcal{O}^{-1}$ and the ODE takes the form:

$$y^{(n)} = \mathbf{C}\mathbf{A}^n\mathcal{O}^{-1} \begin{bmatrix} y \\ \dot{y} \\ \ldots \\ y^{(n-1)} \end{bmatrix} \tag{35}$$

You can see an example in the appendix A.

Read more

- 2.14 Analysis and Design of Feedback Control Systems:
  - State-Space Representation of LTI Systems
  - Time-Domain Solution of LTI State Equations
- Linear Physical Systems Analysis:
  - State Space Representations of Linear Physical Systems
    lpsa.swarthmore.edu/Representations/SysRepSS.html
  - Transformation: Differential Equation to State Space
    lpsa.swarthmore.edu/.../DE2SS.html

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2023

Appendix A

State Space to ODE conversion, 1
(extra)

Consider a system in state-space form:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases} \iff \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \\ y = \mathbf{C}\mathbf{x} \end{cases} \tag{36}$$

We want to rewrite it as a linear ODE:

$$\ddot{y} + b_2\dot{y} + b_1 y = 0 \tag{37}$$

Note that initial conditions of both equation need to agree.

Since $y = \mathbf{C}\mathbf{x}$, its derivative is $\dot{y} = \mathbf{C}\dot{\mathbf{x}}$:

$$\dot{y} = \mathbf{C}\mathbf{A}\mathbf{x} \tag{38}$$

$$\dot{y} = \begin{bmatrix}(a_{11}c_1 + a_{21}c_2) & (a_{12}c_1 + a_{22}c_2)\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{39}$$

Analogous for $\ddot{y}$:

$$\ddot{y} = \mathbf{C}\mathbf{A}\mathbf{A}\mathbf{x} \tag{40}$$

State Space to ODE

Combining our results we find the linear transformation between the variables x_1, x_2 and y, $\dot{y}$:

$$\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ (a_{11}c_1 + a_{21}c_2) & (a_{12}c_1 + a_{22}c_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{41}$$

Resulting transformation matrix is:

$$\mathbf{T} = \begin{bmatrix} c_1 & c_2 \\ (a_{11}c_1 + a_{21}c_2) & (a_{12}c_1 + a_{22}c_2) \end{bmatrix} \tag{42}$$

$$\mathbf{x} = \mathbf{T}^{-1} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \tag{43}$$

Remember that:

$$\ddot{y} = \mathbf{CAAx} \tag{44}$$

$$\ddot{y} = \mathbf{CAAT}^{-1} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \tag{45}$$

So, we obtained $\ddot{y}$ as a linear function of y, $\dot{y}$. From this it is clear how the same can be generalized to higher dimensions.

Check out the code implementation.