

# Linearization

## Control Theory, Lecture 12

by Sergei Savin

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- Taylor expansion
- Linearization
- Linearization of Manipulator equations

# TAYLOR EXPANSION AROUND NODE, 1

Consider a non-linear dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \quad (1)$$

If  $\mathbf{x}_0$  and  $\mathbf{u}_0$  represent a *node*, i.e.  $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) = 0$ ,  $\mathbf{x}_0 = \text{const}$ ,  $\mathbf{u}_0 = \text{const}$ , we can consider a Taylor expansion around that node:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0) + \text{H.O.T.} \quad (2)$$

Where  $\mathbf{x}_0$  and  $\mathbf{u}_0$  are expansion point. We define new variables  $\mathbf{e}$  and  $\mathbf{v}$  as distance from the expansion point:

$$\mathbf{e} = \mathbf{x} - \mathbf{x}_0, \quad \dot{\mathbf{e}} = \dot{\mathbf{x}}, \quad \mathbf{v} = \mathbf{u} - \mathbf{u}_0. \quad (3)$$

With that, we can re-write the Taylor expansion:

$$\dot{\mathbf{e}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \mathbf{e} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \mathbf{v} + \text{H.O.T.} \quad (4)$$

$$\dot{\mathbf{e}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \mathbf{e} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \mathbf{v} + \text{H.O.T.} \quad (5)$$

We can introduce notation:

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}, \quad \mathbf{B} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}. \quad (6)$$

If we drop the higher order terms from the Taylor expansion, we obtain *linearization* of the system dynamics:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \quad (7)$$

In this context,  $\mathbf{x}_0$  and  $\mathbf{u}_0$  is the *linearization point*.

# TAYLOR EXPANSION ALONG A TRAJECTORY

Consider a non-linear dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \quad (8)$$

and a trajectory  $\dot{\mathbf{x}}_0 = \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)$ . We can consider a Taylor expansion along this trajectory:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0) + \text{H.O.T.} \quad (9)$$

Since  $\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_0$ , we re-write:

$$\dot{\mathbf{e}} \sim \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} + \text{H.O.T.} \quad (10)$$

As before, we drop higher order terms and obtain linearization:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \quad (11)$$

If we want to maintain our original variables, we can still use Taylor expansion:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) + \mathbf{A}(\mathbf{x} - \mathbf{x}_0) + \mathbf{B}(\mathbf{u} - \mathbf{u}_0) \quad (12)$$

Denoting  $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{A}\mathbf{x}_0 - \mathbf{B}\mathbf{u}_0 = \mathbf{c}$  and dropping H.O.T. we approximate the system as affine:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \quad (13)$$

# MANIPULATOR EQUATION LINEARIZATION, 1

Consider Manipulator equation:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau \quad (14)$$

We will attempt to linearize it.

We begin by proposing the following new variables:

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} - \mathbf{q}_0 \\ \dot{\mathbf{q}} - \dot{\mathbf{q}}_0 \end{bmatrix}, \quad \mathbf{u} = \tau - \tau_0 \quad (15)$$

where  $\tau_0$  is chosen such that  $\mathbf{C}(\dot{\mathbf{q}}_0, \mathbf{q}_0)\dot{\mathbf{q}}_0 + \mathbf{g}(\mathbf{q}_0) = \tau_0$ .

Defining selector matrices  $\mathbf{S}_q = [\mathbf{I} \ 0]$  and  $\mathbf{S}_v = [0 \ \mathbf{I}]$  we can express conversion from the state to generalized position and velocity:

$$\mathbf{q}(\mathbf{x}) = \mathbf{S}_q \mathbf{x} + \mathbf{q}_0, \quad \dot{\mathbf{q}}(\mathbf{x}) = \mathbf{S}_v \mathbf{x} + \dot{\mathbf{q}}_0 \quad (16)$$

We introduce function  $\phi(\dot{\mathbf{q}}, \mathbf{q}, \tau) = \ddot{\mathbf{q}}$ , expressed as:

$$\phi(\dot{\mathbf{q}}, \mathbf{q}, \tau) = \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \quad (17)$$

Next, we write our dynamics as a first order ODE:

$$\frac{d}{dt} \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} = \begin{bmatrix} \dot{\mathbf{q}} \\ \phi(\dot{\mathbf{q}}, \mathbf{q}, \tau) \end{bmatrix} \quad (18)$$

$$\dot{\mathbf{x}} = \varphi(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \dot{\mathbf{q}}(\mathbf{x}) \\ \phi(\mathbf{x}, \mathbf{u}) \end{bmatrix} \quad (19)$$

With that, we can find matrices  $\mathbf{A}$  and  $\mathbf{B}$ .



In this case, state matrices  $\mathbf{A}$  and  $\mathbf{B}$  become:

$$\mathbf{A} = \begin{bmatrix} \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{q}} & \frac{\partial \dot{\mathbf{q}}}{\partial \dot{\mathbf{q}}} \\ \frac{\partial \phi}{\partial \mathbf{q}} & \frac{\partial \phi}{\partial \dot{\mathbf{q}}} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ \frac{\partial \phi}{\partial \mathbf{q}} & \frac{\partial \phi}{\partial \dot{\mathbf{q}}} \end{bmatrix} \quad (20)$$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial \dot{\mathbf{q}}}{\partial \tau} \\ \frac{\partial \phi}{\partial \tau} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{H}^{-1} \end{bmatrix} \quad (21)$$

Thus, our task is to find the following jacobians:  $\frac{\partial \phi}{\partial \mathbf{q}}$  and  $\frac{\partial \phi}{\partial \dot{\mathbf{q}}}$ .

Let us find  $\frac{\partial \phi}{\partial \mathbf{q}}$ :

$$\frac{\partial \phi}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}} (\mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g})) = \quad (22)$$

$$= \frac{\partial \mathbf{H}^{-1}}{\partial \mathbf{q}} (\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \mathbf{H}^{-1} \frac{\partial}{\partial \mathbf{q}} (\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \quad (23)$$

If we evaluate  $\frac{\partial \phi}{\partial \mathbf{q}}$  at the point  $\mathbf{q} = \mathbf{q}_0$ ,  $\dot{\mathbf{q}} = \dot{\mathbf{q}}_0$ ,  $\tau = \tau_0$ , we can use the fact that  $\mathbf{C}(\dot{\mathbf{q}}_0, \mathbf{q}_0)\dot{\mathbf{q}}_0 + \mathbf{g}(\mathbf{q}_0) = \tau_0$  to avoid computing derivative  $\frac{\partial \mathbf{H}^{-1}}{\partial \mathbf{q}}$ :

$$\frac{\partial \phi}{\partial \mathbf{q}} = \mathbf{H}^{-1} \left( -\frac{\partial \mathbf{C}\dot{\mathbf{q}}}{\partial \mathbf{q}} - \frac{\partial \mathbf{g}}{\partial \mathbf{q}} \right) \quad (24)$$

Let us find  $\frac{\partial \phi}{\partial \dot{\mathbf{q}}}$ :

$$\frac{\partial \phi}{\partial \dot{\mathbf{q}}} = \frac{\partial}{\partial \dot{\mathbf{q}}} (\mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g})) = \quad (25)$$

$$= -\mathbf{H}^{-1} \frac{\partial \mathbf{C}\dot{\mathbf{q}}}{\partial \dot{\mathbf{q}}} \quad (26)$$

With that, we expressed all jacobians. The rest is the same as in the general case we studied in the first slides.

Lecture slides are available via Github:

[github.com/SergeiSa/Control-Theory-2024](https://github.com/SergeiSa/Control-Theory-2024)

