Lyapunov Theory, Lyapunov equations

Control Theory, Lecture 12

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Content

Asymptotic stability criteria

Autonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is assymptotically stable, if there exists a scalar function $V = V(\mathbf{x}) > 0$, whose time derivative is negative $\dot{V}(\mathbf{x}) < 0$, except $V(\mathbf{0}) = 0$, $\dot{V}(\mathbf{0}) = 0$.

Marginal stability criteria

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is stable in the sense of Lyapunov, $\exists V(\mathbf{x}) > 0$, $\dot{V}(\mathbf{x}) \leq 0$.

Definition

Function $V(\mathbf{x}) > 0$ in this case is called *Lyapunov function*.

This is not the only type of stability as you remember, you are invited to study criteria for other stability types on your own.

Lyapunov method: Example 1

Take dynamical system $\dot{x} = -x$.

We propose a *Lyapunov function candidate* $V(x) = x^2 \geq 0$. Let's find its derivative:

$$\dot{V}(x) = \frac{\partial V}{\partial x}(-x) = 2x(-x) = -x^2 \leq 0 \tag{1}$$

This satisfies the Lyapunov criteria, so the system is stable. It is in fact asymptotically stable, because $\dot{V}(x) \neq 0$ if $x \neq 0$.

Consider pendulum $\ddot{q} = f(q, \dot{q}) = -\dot{q} - \sin(q)$.

We propose a *Lyapunov function candidate* $V(q, \dot{q}) = E(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + 1 - \cos(q) \geq 0$, where $E(q, \dot{q})$ is total energy of the system. Let's find its derivative:

$$\dot{V}(q, \dot{q}) = \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}} f(q, \dot{q}) = \dot{q}\sin(q) + \dot{q}(-\dot{q} - \sin(q)) = -\dot{q}^2 \leq 0 \tag{2}$$

This satisfies the Lyapunov criteria, so the system is stable. It is not proven to be asymptotically stable, because $\dot{V}(q, \dot{q}) = 0$ for any q, as long as $\dot{q} = 0$.

LaSalle's invariance principle

Autonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is asymptotically stable, if there exists a scalar function $V = V(\mathbf{x}) > 0$, whose time derivative is negative $\dot{V}(\mathbf{x}) \leq 0$, except $V(\mathbf{0}) = 0$, where the set $\{\mathbf{x} : \ \dot{V}(\mathbf{x}) = 0\}$ does not contain non-trivial trajectories.

A trivial trajectory is $\mathbf{x}(t) = 0$. Unlike Lyapunov condition, LaSalle's principle allows us to prove asymptotic stability even for systems with $\dot{V}(\mathbf{x}) = 0$.

Local version of LaSalle's invariance principle has the following form:

Local LaSalle's invariance principle

Autonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is asymptotically stable in the neighborhood $\mathcal{D}$ of the origin, if there exists a scalar function $V = V(\mathbf{x}) > 0$, whose time derivative is negative $\dot{V}(\mathbf{x}) \leq 0$, except $V(\mathbf{0}) = 0$, where the set $\mathcal{M} = \{\mathbf{x} : \ \dot{V}(\mathbf{x}) = 0\} \cap \mathcal{D}$ does not contain non-trivial trajectories.

In our previous example $\dot{V}(q, \dot{q}) = 0$ for any q, as long as $\dot{q} = 0$. But the set $\{(q, \dot{q}) : \ \dot{q} = 0\}$ contains no trajectories of the system $\ddot{q} = -\dot{q} - \sin(q)$ other than $q(t) = 0$ in the region $-\frac{\pi}{2} < q < \frac{\pi}{2}$. So, LaSalle principle proves local asymptotic stability.

Consider oscillator $\ddot{q} = f(q, \dot{q}) = -\dot{q}$.

We propose a *Lyapunov function candidate* $V(q, \dot{q}) = T(q, \dot{q}) = \frac{1}{2}\dot{q}^2 \geq 0$, where $T(q, \dot{q})$ is kinetic energy of the system. Let's find its derivative:

$$\dot{V}(q, \dot{q}) = \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}} f(q, \dot{q}) = \dot{q}(-\dot{q}) = -\dot{q}^2 \leq 0 \tag{3}$$

This satisfies the Lyapunov criteria, so the system is stable. Note that $\dot{V}(q, \dot{q}) = 0$ for any q as long as $\dot{q} = 0$. But the set $\{(q, \dot{q}) : \ \dot{q} = 0\}$ contains infinitely many trajectories of the system $\ddot{q} = -\dot{q}$ other than $q(t) = 0$, for example $q(t) = 1$ or $q(t) = -2$. So, LaSalle principle does not prove asymptotic stability in this case.

As you saw, Lyapunov method allows you to deal with nonlinear systems, as well as linear ones. But for linear ones there are additional properties we can use.

Observation 1

For a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ we can always pick Lyapunov function candidate in the form $V = \mathbf{x}^\top \mathbf{S}\mathbf{x} > 0$, where $\mathbf{S}$ is a positive definite matrix.

Next slides will shows where this leads us.

Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and $V = \mathbf{x}^\top \mathbf{S}\mathbf{x} \geq 0$, let's find its derivative:

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}\dot{\mathbf{x}} \tag{4}$$

$$\dot{V}(\mathbf{x}) = (\mathbf{A}\mathbf{x})^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}\mathbf{A}\mathbf{x} = \mathbf{x}^\top (\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A})\mathbf{x} \tag{5}$$

Notice that $\dot{V}(x)$ should be negative for all $\mathbf{x}$ for the system to be stable, meaning that $\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A}$ should be negative definite. A more strict form of this requirement is *Lyapunov equation*:

$$\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} = -\mathbf{Q} \tag{6}$$

where $\mathbf{Q}$ is a positive-definite matrix.

Asymptotic stability criteria, discrete case

Given $\mathbf{x}_{i+1} = \mathbf{f}(\mathbf{x}_i)$, if $V(\mathbf{x}_i) > 0$, and $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) < 0$, the system is stable.

Same as before, for linear systems we will be choosing *positive-definite quadratic forms* as Lyapunov function candidates.

Consider dynamics $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$ and $V = \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i \geq 0$, let's find $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i)$:

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) = (\mathbf{A}\mathbf{x}_i)^\top \mathbf{S}\mathbf{A}\mathbf{x}_i - \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i \tag{7}$$

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) = \mathbf{x}_i^\top (\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S})\mathbf{x}_i \tag{8}$$

Notice that $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i)$ should be negative for all $\mathbf{x}_i$ for the system to be stable, meaning that $\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S}$ should be negative definite, giving us *Discrete Lyapunov equation*:

$$\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S} = -\mathbf{Q} \tag{9}$$

where $\mathbf{Q}$ is a positive-definite matrix.

In practice, you can easily use Lyapunov equations for stability verification. Python and MATLAB have built-in functionality to solve it:

- scipy: `linalg.solve_continuous_lyapunov(A, Q)`
- MATLAB: `lyap(A,Q)`

- 3.9 Liapunov's direct method
- Universita degli studi di Padova Dipartimento di Ingegneria dell'Informazione, Nicoletta Bof, Ruggero Carli, Luca Schenato, Technical Report, Lyapunov Theory for Discrete Time Systems

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2024