Discrete Dynamics

Control Theory, Lecture 6

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Spring 2024

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The following dynamical system is called *discrete*:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \tag{1}$$

- There are no derivatives in the equation;
- It is easy to simulate.

The affine control for this system can be written as:

$$\mathbf{u}_i = -\mathbf{K}\mathbf{x}_i + \mathbf{u}_i^* \tag{2}$$

Let us consider stability of the discrete dynamical system where matrix $\mathbf{A}$ has purely real eigenvalues:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i \tag{3}$$

With eigendecomposition $\mathbf{A} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}$ (where $\mathbf{D}$ is a diagonal matrix with eigenvalues λ_j of $\mathbf{A}$ on its diagonal) and introducing notation $\mathbf{z}_i = \mathbf{V}\mathbf{x}_i$ we get:

$$\mathbf{x}_{i+1} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}\mathbf{x}_i \tag{4}$$

$$\mathbf{z}_{i+1} = \mathbf{D}\mathbf{z}_i \tag{5}$$

Meaning that the dynamics became a system of independent scalar equations $z_{j,i+1} = \lambda_j z_{j,i}$.

Considering individual equations $z_{j,i+1} = \lambda_j z_{j,i}$ we find that the absolute value of the scalars z_j will dwindle with time iff $|\lambda_j| < 1$:

$$\left| \frac{z_{j,i+1}}{z_{j,i}} \right| = |\lambda_j| \tag{6}$$

Let us consider stability of the discrete dynamical system with a 2-by-2 matrix $\mathbf{A}$:

$$\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \tag{7}$$

Let us find norms of $\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix}$ and $\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}$:

$$\left|\left| \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \right|\right|^2 = x_{1,i}^2 + x_{2,i}^2 \tag{8}$$

$$\left|\left| \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} \right|\right|^2 = (\alpha^2 + \beta^2)(x_{1,i}^2 + x_{2,i}^2) \tag{9}$$

We can find the ratio of the norms of $\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix}$ and $\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}$:

$$\left|\left|\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix}\right|\right|^2 / \left|\left|\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}\right|\right|^2 = \alpha^2 + \beta^2 \tag{10}$$

Remembering that eigenvalues of the system are $\lambda = \alpha \pm j\beta$, we can rewrite the expression above as:

$$\left|\left|\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix}\right|\right|^2 / \left|\left|\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}\right|\right|^2 = |\lambda|^2 \tag{11}$$

We can see that the norm of the variable $\mathbf{x}$ will dwindle iff $|\lambda| < 1$.

General stability criterion is given below:

Stability criterion

Discrete system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$ is stable as long as the eigenvalues of $\mathbf{A}$ are smaller than 1 by absolute value: $|\lambda_i(\mathbf{A})| \leq 1, \ \forall i$.

CT-LTI: Analytical solution

Exponential e^a is defined as a series:

$$e^a = 1 + a + \frac{1}{2}a^2 + \frac{1}{6}a^3 + ... = \sum_{n=0}^{\infty} \frac{1}{n!} a^n \tag{12}$$

Matrix exponential $e^{\mathbf{A}}$ is defined as a series:

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}\mathbf{A} + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A} + ... = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n \tag{13}$$

An ODE of the form $\dot{x} = ax$ has analytical solution $x(t) = e^{at}x(0)$.

An ODE of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ has analytical solution $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$.

Let us check that this is a solution:

$$\mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + ...\right)\mathbf{x}(0) \tag{14}$$

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} + \mathbf{A}\mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}\mathbf{A}t^2 + ...\right)\mathbf{x}(0) \tag{15}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + ...\right)\mathbf{x}(0) \tag{16}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) \tag{17}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \qquad \square \tag{18}$$

An ODE of the form $\dot{x} = ax + bu(t)$ also has analytical solution. To find it, we first find the following derivative:

$$\frac{d}{dt}\left(e^{-at}x(t)\right) = e^{-at}\dot{x}(t) - ae^{-at}x(t) \tag{19}$$

Multiplying $\dot{x} = ax + bu(t)$ by e^{-at} we see:

$$e^{-at}\dot{x} = e^{-at}ax + e^{-at}bu(t) \tag{20}$$

$$e^{-at}\dot{x} - e^{-at}ax = e^{-at}bu(t) \tag{21}$$

$$\frac{d}{dt}\left(e^{-at}x(t)\right) = e^{-at}bu(t) \tag{22}$$

$$\int_0^t \frac{d}{d\tau}\left(e^{-a\tau}x(\tau)\right)d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \tag{23}$$

Continuing the derivation:

$$\int_0^t \frac{d}{d\tau}\left(e^{-a\tau}x(\tau)\right)d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \tag{24}$$

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau \tag{25}$$

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau}bu(\tau)d\tau \tag{26}$$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \tag{27}$$

Forced State Response (LTI) (3)

State-space equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t)$ also has an analytical solution:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{28}$$

The same can be re-written as:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{29}$$

Discretization

From analytical solution to discrete dynamics

Given a solution to a state-space system we can consider how the system evolves from the point $t = 0$ to the point $t = \Delta t$, assuming that $\mathbf{u}(t) = \mathbf{u}_0 = \text{const}, \; t \in [0, \; \Delta t]$, and denoting $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(\Delta t) = \mathbf{x}_1$:

$$\mathbf{x}_1 = e^{\mathbf{A}\Delta t}\mathbf{x}_0 + e^{\mathbf{A}\Delta t} \int_0^{\Delta t} e^{-\mathbf{A}\tau} \mathbf{B}\mathbf{u}_0 d\tau \tag{30}$$

Now we denote:

$$\bar{\mathbf{A}} = e^{\mathbf{A}\Delta t}, \quad \bar{\mathbf{B}} = e^{\mathbf{A}\Delta t} \int_0^{\Delta t} e^{-\mathbf{A}\tau} \mathbf{B} d\tau \tag{31}$$

We get:

$$\mathbf{x}_1 = \bar{\mathbf{A}}\mathbf{x}_0 + \bar{\mathbf{B}}\mathbf{u}_0 \tag{32}$$

Consider linear time-invariant autonomous system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{33}$$

The time derivative $\dot{\mathbf{x}}$ can be replaces with a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + \Delta t) - \mathbf{x}(t)) \tag{34}$$

Note that we could have also used other definitions of a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + 0.5\Delta t) - \mathbf{x}(t - 0.5\Delta t)) \tag{35}$$

or

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t) - \mathbf{x}(t - \Delta t)) \tag{36}$$

Zero-order hold (1)

We can introduce notation:

$$\begin{cases} \mathbf{x}_0 = \mathbf{x}(0) \\ \mathbf{x}_1 = \mathbf{x}(\Delta t) \\ \mathbf{x}_2 = \mathbf{x}(2\Delta t) \\ \dots \\ \mathbf{x}_n = \mathbf{x}(n\Delta t) \end{cases} \tag{37}$$

We say that $\mathbf{x}_i$ is the value of $\mathbf{x}$ at the time step i. Then the finite difference can be written, for example, as follows:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) \tag{38}$$

Zero-order hold (2)

We can rewrite our original autonomous LTI as follows:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_i \tag{39}$$

Isolating $\mathbf{x}_{i+1}$ on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{A}\Delta t + \mathbf{I})\mathbf{x}_i \tag{40}$$

Or alternatively:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_{i+1} \tag{41}$$

Isolating $\mathbf{x}_{i+1}$ on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{I} - \mathbf{A}\Delta t)^{-1}\mathbf{x}_i \tag{42}$$

Defining *discrete state space matrix* $\bar{\mathbf{A}}$ and *discrete control matrix* $\bar{\mathbf{B}}$ as follows:

$$\bar{\mathbf{A}} = \mathbf{A}\Delta t + \mathbf{I} \tag{43}$$

$$\bar{\mathbf{B}} = \mathbf{B}\Delta t \tag{44}$$

We get discrete dynamics:

$$\mathbf{x}_{i+1} = \bar{\mathbf{A}}\mathbf{x}_i + \bar{\mathbf{B}}\mathbf{u}_i \tag{45}$$

This way of defining discrete dynamics is called *zero-order hold (ZOH).*

Graphically, we can understand what zero order hold is, by comparing it to the first order hold:

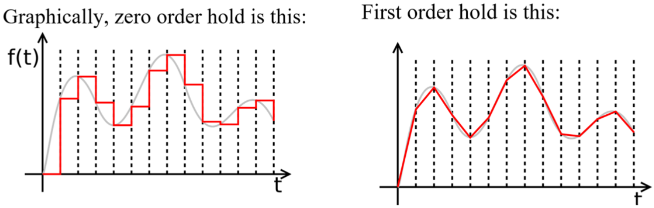

Figure 1: Different types of discretization

Read more

- Automatic Control 1 Discrete-time linear systems, Prof. Alberto Bemporad, University of Trento
- MIT 2.14, State Space Response

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2024