

# Discrete Dynamics

## Control Theory, Lecture 6

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The following dynamical system is called *discrete*:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \quad (1)$$

Note that those:

- have no derivatives in the equation;
- are easily simulated.

The affine control for this system can be given as:

$$\mathbf{u}_i = -\mathbf{K}\mathbf{x}_i + \mathbf{u}_i^* \quad (2)$$

# STABILITY OF THE DISCRETE DYNAMICS

## Real eigenvalues

Let us consider stability of the discrete dynamical system where matrix  $\mathbf{A}$  has purely real eigenvalues:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i \quad (3)$$

With eigendecomposition  $\mathbf{A} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}$  (where  $\mathbf{D}$  is a diagonal matrix with eigenvalues  $\lambda_j$  of  $\mathbf{A}$  on its diagonal) and introducing notation  $\mathbf{z}_i = \mathbf{V}\mathbf{x}_i$  we get:

$$\mathbf{x}_{i+1} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}\mathbf{x}_i \quad (4)$$

$$\mathbf{z}_{i+1} = \mathbf{D}\mathbf{z}_i \quad (5)$$

Meaning that the dynamics became a system of independent scalar equations  $z_{j,i+1} = \lambda_j z_{j,i}$ .

# STABILITY OF THE DISCRETE DYNAMICS

## Real eigenvalues

Thus, with  $z_{j,i+1} = \lambda_j z_{j,i}$  we can find now the absolute value of the scalars  $z_j$  will dwindle with time iff  $|\lambda_j| < 1$ :

$$\left| \frac{z_{j,i+1}}{z_{j,i}} \right| = |\lambda_j| \quad (6)$$

# STABILITY OF THE DISCRETE DYNAMICS

## 2x2 system

Let us consider stability of the discrete dynamical system with a 2-by-2 matrix  $\mathbf{A}$ :

$$\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \quad (7)$$

Let us find norms of  $\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix}$  and  $\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}$ :

$$\left\| \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \right\|^2 = x_{1,i}^2 + x_{2,i}^2 \quad (8)$$

$$\left\| \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} \right\|^2 = (\alpha^2 + \beta^2)(x_{1,i}^2 + x_{2,i}^2) \quad (9)$$

# STABILITY OF THE DISCRETE DYNAMICS

## 2x2 system

We can find the ratio of the norms of  $\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix}$  and  $\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}$ :

$$\left\| \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} \right\|^2 / \left\| \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \right\|^2 = \alpha^2 + \beta^2 \quad (10)$$

Remembering that eigenvalues of the system are  $\lambda = \alpha \pm j\beta$ , we can rewrite the expression above as:

$$\left\| \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} \right\|^2 / \left\| \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \right\|^2 = |\lambda|^2 \quad (11)$$

We can see that the norm of the variable  $\mathbf{x}$  will dwindle with time iff  $|\lambda| < 1$ .

# STABILITY OF THE DISCRETE DYNAMICS

General stability criterion is given below:

## Stability criterion

In general, discrete system  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$  is stable as long as the eigenvalues of  $\mathbf{A}$  are smaller than 1 by absolute value:

$$|\lambda_i(\mathbf{A})| \leq 1, \forall i.$$



# CT-LTI: Analytical solution

Exponential  $e^a$  is defined as a series:

$$e^a = 1 + a + \frac{1}{2}a^2 + \frac{1}{6}a^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} a^n \quad (12)$$

Matrix exponential  $e^{\mathbf{A}}$  is defined as a series:

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}\mathbf{A} + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n \quad (13)$$

# ANALYTICAL SOLUTION TO ODE

An ODE of the form  $\dot{x} = ax$  has analytical solution  $x(t) = e^{at}x(0)$ .

An ODE of the form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  has analytical solution  $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$ .

Let us check that this is a solution:

$$\mathbf{x}(t) = \left( \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + \dots \right) \mathbf{x}(0) \quad (14)$$

$$\dot{\mathbf{x}}(t) = \left( \mathbf{A} + \mathbf{A}\mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}\mathbf{A}t^2 + \dots \right) \mathbf{x}(0) \quad (15)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A} \left( \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \dots \right) \mathbf{x}(0) \quad (16)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) \quad (17)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad \square \quad (18)$$

# FORCED STATE RESPONSE (LTI) (1)

An ODE of the form  $\dot{x} = ax + bu(t)$  also has analytical solution.  
To find it, we first find the following derivative:

$$\frac{d}{dt} (e^{-at}x(t)) = e^{-at}\dot{x}(t) - ae^{-at}x(t) \quad (19)$$

Multiplying  $\dot{x} = ax + bu(t)$  by  $e^{-at}$  we see:

$$e^{-at}\dot{x} = e^{-at}ax + e^{-at}bu(t) \quad (20)$$

$$e^{-at}\dot{x} - e^{-at}ax = e^{-at}bu(t) \quad (21)$$

$$\frac{d}{dt} (e^{-at}x(t)) = e^{-at}bu(t) \quad (22)$$

$$\int_0^t \frac{d}{d\tau} (e^{-a\tau}x(\tau)) d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \quad (23)$$

# FORCED STATE RESPONSE (LTI) (2)

Continuing the derivation:

$$\int_0^t \frac{d}{d\tau} (e^{-a\tau} x(\tau)) d\tau = \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (24)$$

$$e^{-at} x(t) - x(0) = \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (25)$$

$$x(t) = e^{at} x(0) + e^{at} \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (26)$$

$$x(t) = e^{at} x(0) + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \quad (27)$$

# FORCED STATE RESPONSE (LTI) (3)

State-space equation  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t)$  also has an analytical solution:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \quad (28)$$

The same can be re-written as:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \quad (29)$$

# Discretization

# FROM ANALYTICAL SOLUTION TO DISCRETE DYNAMICS

Given a solution to a state-space system we can consider how the system evolves from the point  $t = 0$  to the point  $t = \Delta t$ , assuming that  $\mathbf{u}(t) = \mathbf{u}_0 = \text{const}$ ,  $t \in [0, \Delta t]$ , and denoting  $\mathbf{x}(0) = \mathbf{x}_0$  and  $\mathbf{x}(\Delta t) = \mathbf{x}_1$ :

$$\mathbf{x}_1 = e^{\mathbf{A}\Delta t}\mathbf{x}_0 + e^{\mathbf{A}\Delta t} \int_0^{\Delta t} e^{-\mathbf{A}\tau} \mathbf{B}\mathbf{u}_0 d\tau \quad (30)$$

Now we denote:

$$\bar{\mathbf{A}} = e^{\mathbf{A}\Delta t}, \quad \bar{\mathbf{B}} = e^{\mathbf{A}\Delta t} \int_0^{\Delta t} e^{-\mathbf{A}\tau} \mathbf{B} d\tau \quad (31)$$

We get:

$$\mathbf{x}_1 = \bar{\mathbf{A}}\mathbf{x}_0 + \bar{\mathbf{B}}\mathbf{u}_0 \quad (32)$$



# DISCRETIZATION VIA FINITE DIFFERENCES

Consider linear time-invariant autonomous system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (33)$$

The time derivative  $\dot{\mathbf{x}}$  can be replaced with a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + \Delta t) - \mathbf{x}(t)) \quad (34)$$

Note that we could have also used other definitions of a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + 0.5\Delta t) - \mathbf{x}(t - 0.5\Delta t)) \quad (35)$$

or

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t) - \mathbf{x}(t - \Delta t)) \quad (36)$$

We can introduce notation:

$$\begin{cases} \mathbf{x}_0 = \mathbf{x}(0) \\ \mathbf{x}_1 = \mathbf{x}(\Delta t) \\ \mathbf{x}_2 = \mathbf{x}(2\Delta t) \\ \dots \\ \mathbf{x}_n = \mathbf{x}(n\Delta t) \end{cases} \quad (37)$$

We say that  $\mathbf{x}_i$  is the value of  $\mathbf{x}$  at the time step  $i$ . Then the finite difference can be written, for example, as follows:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t} (\mathbf{x}_{i+1} - \mathbf{x}_i) \quad (38)$$

We can rewrite our original autonomous LTI as follows:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_i \quad (39)$$

Isolating  $\mathbf{x}_{i+1}$  on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{A}\Delta t + \mathbf{I})\mathbf{x}_i \quad (40)$$

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Or alternatively:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_{i+1} \quad (41)$$

Isolating  $\mathbf{x}_{i+1}$  on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{I} - \mathbf{A}\Delta t)^{-1}\mathbf{x}_i \quad (42)$$

Defining *discrete state space matrix*  $\bar{\mathbf{A}}$  and *discrete control matrix*  $\bar{\mathbf{B}}$  as follows:

$$\bar{\mathbf{A}} = \mathbf{A}\Delta t + \mathbf{I} \quad (43)$$

$$\bar{\mathbf{B}} = \mathbf{B}\Delta t \quad (44)$$

We get discrete dynamics:

$$\mathbf{x}_{i+1} = \bar{\mathbf{A}}\mathbf{x}_i + \bar{\mathbf{B}}\mathbf{u}_i \quad (45)$$

This way of defining discrete dynamics is called *zero order hold (ZOH)*.

## ZERO-ORDER HOLD (4)

Graphically, we can understand what zero order hold is, by comparing it to the first order hold:

Graphically, zero order hold is this:



First order hold is this:

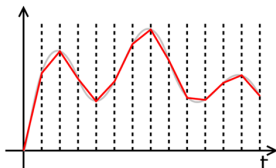


Figure 1: Different types of discretization

- [Automatic Control 1 Discrete-time linear systems](#), Prof. Alberto Bemporad, University of Trento
- [MIT 2.14, State Space Response](#)

Lecture slides are available via Github:

[github.com/SergeiSa/Control-Theory-2024](https://github.com/SergeiSa/Control-Theory-2024)

