

Laplace Transform and Transfer Functions

Control Theory, Lecture 4

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Spring 2024

- ODE solutions
- Laplace Transform
- Laplace Transform of a derivative
- Derivative operator
- Transfer Functions
- State-Space to Transfer Function conversion
- Steady State Gain
- Read more

By definition, Laplace transform of a function $f(t)$ is given as:

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt \quad (1)$$

where $F(s)$ is called an *image* of the function.

The study of Laplace transform is a separate mathematical field with applications in solving ODEs, which we won't cover.

However, we will consider transform of one case of interest - transform of a derivative.

LAPLACE TRANSFORM OF A DERIVATIVE

Consider a derivative $\frac{dx}{dt}$ and its transform:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = \int_0^{\infty} \frac{dx}{dt} e^{-st} dt \quad (2)$$

we will make use of the integration by parts formula:

Integration by parts

$$\int v \frac{du}{dt} dt = vu - \int \frac{dv}{dt} u dt \quad (3)$$

In our case, $\frac{du}{dt} = \frac{dx}{dt}$, $u = x$, $v = e^{-st}$, $\frac{dv}{dt} = -se^{-st}$:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = [xe^{-st}]_0^{\infty} - \int_0^{\infty} -se^{-st} x dt \quad (4)$$

$$\mathcal{L}\left(\frac{dx}{dt}\right) = -x(0) + s\mathcal{L}(x) \quad (5)$$

Thus, assuming that $x(0) = 0$ and denoting $\mathcal{L}(x) = X(s)$, we can obtain a *derivative operator*:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = s\mathcal{L}(x) = sX(s) \quad (6)$$

This form of a derivative operator is very simple to use in practice.

Consider the following ODE, where u is an input (function of time that influences the solution of the ODE):

$$\ddot{y} + a\dot{y} + by = u \quad (7)$$

We can rewrite it using the derivative operator:

$$s^2Y(s) + asY(s) + bY(s) = U(s) \quad (8)$$

and then collect $Y(s)$ on the left-hand-side:

$$Y(s) = \frac{1}{s^2 + as + b}U(s) \quad (9)$$

This form is called a *transfer function*.

TRANSFER FUNCTION

Examples

Example

Given ODE: $2\ddot{y} + 5\dot{y} - 40y = 10u$

The transfer function for it looks: $Y(s) = \frac{10}{2s^3 + 5s - 40}U(s)$

Example

Given ODE: $2\dot{y} - 4y = u$

The transfer function for it looks: $Y(s) = \frac{1}{2s - 4}U(s)$

Example

Given ODE: $3\ddot{y} + 4y = u$

The transfer function for it looks: $Y(s) = \frac{1}{2s^3 + 4}U(s)$

Consider the following (strange) ODE:

$$2\ddot{y} + 3\dot{y} + 2y = 10\dot{u} - u \quad (10)$$

Using the differential equation:

$$2s^2Y(s) + 3sY(s) + 2Y(s) = 10sU(s) - U(s) \quad (11)$$

...which is the same as:

$$(2s^2 + 3s + 2)Y(s) = (10s - 1)U(s) \quad (12)$$

The transfer function for it looks:

$$Y(s) = \frac{10s - 1}{2s^2 + 3s + 2}U(s) \quad (13)$$

Consider the control law:

$$u = -k_p y - k_d \dot{y} \quad (14)$$

Transfer function representation of this control law is:

$$U(s) = -(k_d s + k_p) Y(s) \quad (15)$$

STATE-SPACE TO TRANSFER FUNCTION CONVERSION

Transfer functions are being used to study the relation between the input and the output of the dynamical system.

Consider standard form state-space dynamical system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \quad (16)$$

We can rewrite it using the derivative operator:

$$\begin{cases} s\mathbf{I}\mathbf{x} - \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \quad (17)$$

and then collect \mathbf{x} on the left-hand-side: $\mathbf{x} = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}$
and finally, express \mathbf{y} out:

$$\mathbf{y} = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}) \mathbf{u} \quad (18)$$

Consider a linear ODE, and its equivalent representations as a state space equation and as a transfer function:

$$a_n y^n + \dots + a_1 y = b_m u^m + \dots + b_1 u \quad (19)$$

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \quad (20)$$

$$Y(s) = G(s)U(s) \quad (21)$$

We can call it a *system* \mathcal{G} to avoid referencing particular representation.

Open-loop system representation is $Y(s) = G(s)U(s)$. Let us propose control law (in time domain):

$$u(t) = k_p(v(t) - y(t)) + k_d(\dot{v}(t) - \dot{y}(t)) \quad (22)$$

where $v(t)$ is a control reference. Laplace transform of this control law takes form:

$$U(s) = (k_p + k_d s)(V(s) - Y(s)) \quad (23)$$

Defining $H(s) = k_p + k_d s$ we find closed loop system takes form:

$$Y(s) = G(s)H(s)(V(s) - Y(s)) \quad (24)$$

$$Y(s) = -G(s)H(s)Y(s) + G(s)H(s)V(s) \quad (25)$$

$$(1 + G(s)H(s))Y(s) = G(s)H(s)V(s) \quad (26)$$

$$Y(s) = \frac{G(s)H(s)}{1 + G(s)H(s)}V(s) \quad (27)$$

Alternatively, we can define a new reference signal $r(t)$:

$$r(t) = k_p v(t) + k_d \dot{v}(t) \quad (28)$$

Control law then takes form:

$$u(t) = -k_p y(t) - k_d \dot{y}(t) + r(t) \quad (29)$$

Laplace transform of the control law takes form:

$$U(s) = -H(s)Y(s) + R(s) \quad (30)$$

The closed loop system takes form:

$$Y(s) = -G(s)H(s)Y(s) + G(s)R(s) \quad (31)$$

$$Y(s) + G(s)H(s)Y(s) = G(s)R(s) \quad (32)$$

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)} R(s) \quad (33)$$

If a system \mathcal{G} is stable and given constant input u_0 its output is approaching some constant value y_0 , we can call this pair a *steady-state solution*. The ratio between y_0 and u_0 is a *steady-state gain* - how much does the system increase the input signal.

Assume the system \mathcal{G} represented as a transfer function:

$$Y(s) = \frac{b_m s^m + \dots + b_1}{a_n s^n + \dots + a_1} U(s) \quad (34)$$

Then, as any element multiplied by the differential operator s with power higher than 0 is a derivative of u or y and both are 0 at the steady-state solution, the steady-state gain can be found by setting those to zero:

$$K = \frac{b_1}{a_1} \quad (35)$$

- Chapter 6 Transfer Functions
- Control Systems Lectures - Transfer Functions, by Brian Douglas
- The Laplace Transform - A Graphical Approach, by Brian Douglas

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2024

