Luenberger Observer

Control Theory, Lecture 9

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Before we considered systems and control laws of the following type:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{u} = \mathbf{K}\mathbf{x} \end{cases} \tag{1}$$

But when we implement that control law, how do we know the current value of $\mathbf{x}$?

In practice, we can *estimate* it using *measurement*.

Why information is imperfect?

There are a number of reasons why we can not directly measure the state of the system. Here are some:

- **Lack of sensors.**
- Digital measurements are done in discrete time intervals.
- Disruption of measurements.
- Un-modelled kinematics or dynamics (links bending, gear box backlash, friction, etc.) making the very definition of the state disconnected from reality we want to represent.
- Imprecise, nonlinear and biased sensors.
- Other physical effects.

Let us introduce new notation. We have an LTI system of the following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \hat{\mathbf{x}}(t) = \text{estimate}\left(\mathbf{y}(t)\right) \\ \mathbf{u} = -\mathbf{K}\hat{\mathbf{x}} \end{cases} \tag{2}$$

Then:

- $\mathbf{x}$ and $\mathbf{y}$ are the state and output (actual, true)
- $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}}$ are the estimated (observed) state and output.

Notice that we never know true state $\mathbf{x}$, and therefore for the control purposes we have to use the estimated state $\hat{\mathbf{x}}$.

Estimation error

We define state estimation error:

$$\varepsilon = \hat{\mathbf{x}} - \mathbf{x} \tag{3}$$

But this is impossible to compute, since we do not know $\mathbf{x}$. However, we can compare measured output $\mathbf{y}$ with estimated output $\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}}$:

$$\tilde{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}} - \mathbf{y} \tag{4}$$

This can always be computed.

Let us consider autonomous dynamical system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{5}$$

with measurements $\mathbf{y}$. We want to get as good an estimate of the state $\hat{\mathbf{x}}$ as we can.

Proposal: the dynamics should also hold for the observed state:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} \tag{6}$$

If we knew the initial conditions exactly, we could find the exact state of the system without using measurements $\mathbf{y}$. We can call it an open loop observation. Unfortunately, we do not know the initial conditions precisely.

We propose *observer* that takes into account measurements in a linear way; similar to linear control $-\mathbf{K}\mathbf{x}$, here we propose a linear correction law $-\mathbf{L}\tilde{\mathbf{y}}$. Remembering that $\tilde{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}} - \mathbf{y}$ we get:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{7}$$

This is called *Luenberger observer*. The next task is to find suitable observer gain $\mathbf{L}$.

We subtract $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ from (7), to get *observer error dynamics*:

$$\dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{A}\mathbf{x} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{8}$$

$$\dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}} = \mathbf{A}(\hat{\mathbf{x}} - \mathbf{x}) - \mathbf{L}(\mathbf{C}\hat{\mathbf{x}} - \mathbf{C}\mathbf{x}) \tag{9}$$

$$\dot{\varepsilon} = (\mathbf{A} - \mathbf{L}\mathbf{C})\varepsilon \tag{10}$$

Observer gains, 1

The observer $\dot{\varepsilon} = (\mathbf{A} - \mathbf{LC})\varepsilon$ is *stable* (i.e., the state estimation error tends to zero), as long as the following matrix has eigenvalues with negative real parts:

$$\mathbf{A} - \mathbf{LC} \in \mathbb{H} \tag{11}$$

We need to find $\mathbf{L}$. Let us observe a slight difference between observer design and controller design:

- Controller design: find such $\mathbf{K}$ that $\mathbf{A} - \mathbf{BK} \in \mathbb{H}$.
- Observer design: find such $\mathbf{L}$ that: $\mathbf{A} - \mathbf{LC} \in \mathbb{H}$

The gain is on the left side for the observer, preventing us from using any of our tools (pole placement, LQR) to tune it.

Observer gains, 2

If $\mathbf{A} - \mathbf{L}\mathbf{C} \in \mathbb{H}$, then $(\mathbf{A} - \mathbf{L}\mathbf{C})^\top \in \mathbb{H}$ (eigenvalues of a matrix and its transpose are the same, see Appendix).

Therefore, we can solve the following *dual problem*:

- find such $\mathbf{L}$ that $\mathbf{A}^\top - \mathbf{C}^\top \mathbf{L}^\top \in \mathbb{H}$.

The dual problem is *equivalent* to the control design problem. We can solve it by producing and solving algebraic Riccati equation, as in the LQR formulation. In pseudo-code it can be represented the following way:

$\mathbf{L}^\top$ `= lqr(`$\mathbf{A}^\top$, $\mathbf{C}^\top$, $\mathbf{Q}$, $\mathbf{R}$`)`.
where $\mathbf{Q}$ and $\mathbf{R}$ are weight matrices, determining the "sensitivity" or "aggressiveness" of the observer.

Thus we get dynamics+observer combination:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \mathbf{u} = -\mathbf{K}\hat{\mathbf{x}} \end{cases} \tag{12}$$

where $\mathbf{A} - \mathbf{B}\mathbf{K} \in \mathbb{H}$ and $\mathbf{A}^\top - \mathbf{C}^\top\mathbf{L}^\top \in \mathbb{H}$. 9 Let us re-write the dynamics:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}\mathbf{C}\mathbf{x} - \mathbf{L}\mathbf{C}\hat{\mathbf{x}} \end{cases} \tag{13}$$

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$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}\mathbf{C}\mathbf{x} - \mathbf{L}\mathbf{C}\hat{\mathbf{x}} \end{cases} \tag{14}$$

in a matrix form:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{C} & (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} \tag{15}$$

We can't directly reason about eigenvalues of this matrix. But it can be simplified with a change of variables.

Let us use the following substitution: $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, which implies $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{e}$:
Our system had form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}\mathbf{C}\mathbf{x} - \mathbf{L}\mathbf{C}\hat{\mathbf{x}} \end{cases} \tag{16}$$

Since $\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}$, we get:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} - (\mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}\mathbf{C}\mathbf{x} - \mathbf{L}\mathbf{C}\hat{\mathbf{x}})$$

$$\dot{\mathbf{e}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) - \mathbf{L}\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}})$$

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}$$

Equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}$ takes form:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e}$$

Collecting $\dot{\mathbf{x}}$ and $\dot{\mathbf{e}}$ we get:

$$\begin{cases} \dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BKe} \\ \dot{\mathbf{e}} = (\mathbf{A} - \mathbf{LC})\mathbf{e} \end{cases} \tag{17}$$

In matrix form it becomes:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} (\mathbf{A} - \mathbf{BK}) & \mathbf{BK} \\ 0 & (\mathbf{A} - \mathbf{LC}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{18}$$

Eigenvalues of a upper block-triangular matrices equal to the union of the eigenvalues of the blocks on the main diagonal (see Appendix B). Hence here, the eigenvalues of the system are equal to the union of eigenvalues of $(\mathbf{A} - \mathbf{BK})$ and $(\mathbf{A} - \mathbf{LC})$.

Since the eigenvalues of the system are equal to the union of eigenvalues of $(\mathbf{A} - \mathbf{BK})$ and $(\mathbf{A} - \mathbf{LC})$, we can make the following observation:

Separation principle

As long as the observer and the controller are stable independently, the overall system is stable too.

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025

Given matrix $\mathbf{M}$ and its eigenvalue λ and eigenvector $\mathbf{v}$. we can prove that λ is an eigenvector of $\mathbf{M}^\top$:

$$\mathbf{M}\mathbf{v} = \lambda\mathbf{v} \tag{19}$$

$$\det\left(\mathbf{M} - \mathbf{I}\lambda\right) = 0 \tag{20}$$

$$\det\left(\mathbf{M}^\top - \mathbf{I}\lambda\right) = 0 \tag{21}$$

$$\mathbf{M}^\top\mathbf{u} = \lambda\mathbf{u} \tag{22}$$

We used the fact that determinant of a matrix is equal to the determinant of its transpose: $\det\left(\mathbf{A}\right) = \det\left(\mathbf{A}^\top\right)$.

Given matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \tag{23}$$

Let λ, $\mathbf{v}$ be an eigenvalue and eigenvector of $\mathbf{A}$ and μ, $\mathbf{u}$ be an eigenvalue and eigenvector of $\mathbf{C}$. We can prove that λ, $\mathbf{v}_M = \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix}$ are eigenvalue and eigenvector of $\mathbf{M}$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{v} \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix}. \tag{24}$$

Eig. values of block-diagonal matrices

If μ is not an eigenvalue of $\mathbf{A}$, we can prove that μ, $\mathbf{u}_M = \begin{bmatrix} (\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{u} \end{bmatrix}$ are eigenvalue and eigenvector of $\mathbf{M}$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} (\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} + \mathbf{B}\mathbf{u} \\ \mathbf{C}\mathbf{u} \end{bmatrix} =$$

$$= \begin{bmatrix} (\mathbf{I} + \mathbf{A}(\mathbf{I}\mu - \mathbf{A})^{-1})\mathbf{B}\mathbf{u} \\ \mu\mathbf{u} \end{bmatrix} = \begin{bmatrix} (\mathbf{I}\mu - \mathbf{A} + \mathbf{A})(\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mu\mathbf{u} \end{bmatrix} =$$

$$= \begin{bmatrix} \mu(\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mu\mathbf{u} \end{bmatrix} = \mu \begin{bmatrix} (\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{u} \end{bmatrix}.$$

Counting the number of eigenvalues we observe that eigenvalues of $\mathbf{M}$ include only eigenvalues of $\mathbf{A}$ and $\mathbf{B}$.