

Stability

Control Theory, Lecture 2

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Consider the following ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \quad (1)$$

Let \mathbf{x}_0 be such a state that:

$$\mathbf{f}(\mathbf{x}_0, t) = 0 \quad (2)$$

Then such state \mathbf{x}_0 is called a *node* or a *critical point*.

Node \mathbf{x}_0 is called *stable* iff for any constant δ there exists constant ε such that:

$$\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta \longrightarrow \|\mathbf{x}(t) - \mathbf{x}_0\| < \varepsilon \quad (3)$$

Think of it as "for any initial point that lies at most δ away from \mathbf{x}_0 , the rest of the trajectory $\mathbf{x}(t)$ will be at most ε away from \mathbf{x}_0 ".

Equivalently we can say "the solutions starting from δ -sized ball do not diverge".

Node \mathbf{x}_0 is called *asymptotically stable* iff for any constant δ it is true that:

$$\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta \longrightarrow \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}_0 \quad (4)$$

Think of it as "for any initial point that lies at most δ away from \mathbf{x}_0 , the trajectory $\mathbf{x}(t)$ will asymptotically approach the point \mathbf{x}_0 ".

Equivalently we can say "the solutions starting from δ -sized ball converge to the node".

STABILITY VS ASYMPTOTIC STABILITY

Example

Consider dynamical system $\dot{x} = 0$, and solution $x = 7$. This solution is stable, but not asymptotically stable (a solution corresponding to $x(0) = 7 + \delta$ does not diverge, but does not converge to $x = 7$ either).

Example

Consider dynamical system $\dot{x} = -x$, and solution $x = 0$. This solution is stable and asymptotically stable (all solutions converge to $x = 0$).

Example

Consider dynamical system $\dot{x} = x$, and solution $x = 0$. This solution is unstable (all other solutions diverge from $x = 0$).

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (5)$$

This is called a *linear time-invariant system (LTI)*, indicating that \mathbf{A} and \mathbf{B} are constant.

Removing the input we find an even simpler equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (6)$$

This LTI is an *autonomous system*, since its evolution depends only on the state of the system.

STABILITY OF AN AUTONOMOUS LTI

Real eigenvalues

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{x} \quad (7)$$

where $\mathbf{D} = \text{diag}(d_1, \dots, d_n)$ is a diagonal matrix. This is the same as a system of independent equations:

$$\begin{cases} \dot{x}_1 = d_1 x_1 \\ \dots \\ \dot{x}_n = d_n x_n \end{cases} \quad (8)$$

Each of these equations has an exact solution $x_i = C_i e^{d_i t}$. It diverges from 0 if $d_i > 0$, it does not diverge if $d_i \leq 0$ and it converges to 0 if $d_i < 0$.

STABILITY OF AUTONOMOUS LTI

Real eigenvalues

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (9)$$

where \mathbf{A} can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, where \mathbf{D} is a diagonal matrix.

$$\dot{\mathbf{x}} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x} \quad (10)$$

Multiplying it by \mathbf{V}^{-1} we get: $\mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x}$.
Defining $\mathbf{z} = \mathbf{V}^{-1}\mathbf{x}$ we transform the equation: $\dot{\mathbf{z}} = \mathbf{D}\mathbf{z}$.

Since elements of \mathbf{D} are real, we can clearly see, that iff they are *all negative* will the system be asymptotically stable. If they are non-positive, the system is stable. And those elements are eigenvalues of \mathbf{A} .

Eigenvalues of upper triangular matrices

Eigenvalues of upper triangular matrices are the diagonal elements of these matrices.

Examples of upper triangular matrices are:

$$\begin{bmatrix} 1 & 5 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \quad \begin{bmatrix} -2 & 0 & 8 \\ 0 & -2 & 8 \\ 0 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} \quad (11)$$

UPPER TRIANGULAR MATRICES

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{M}\mathbf{x} \quad (12)$$

where \mathbf{M} is an upper triangular matrices with negative eigenvalues $m_{1,1}, \dots, m_{n,n}$.

The last equation is $\dot{x}_n = m_{n,n}x_n$, and since $m_{n,n} < 0$ we can observe that $\lim_{t \rightarrow \infty} x_n(t) = 0$.

The equation $\# n-1$ is $\dot{x}_{n-1} = m_{n-1,n-1}x_{n-1} + m_{n-1,n}x_n$, and since $m_{n-1,n-1} < 0$ and $\lim_{t \rightarrow \infty} x_n(t) = 0$ we can observe that $\lim_{t \rightarrow \infty} x_{n-1}(t) = 0$.

This can be repeated for all equations, proving asymptotic stability for the system.

STABILITY OF AUTONOMOUS LTI

Complex eigenvalues, 2-dimensional case (1)

Let us consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (13)$$

The eigenvalues of the system are $\alpha \pm i\beta$. We denote $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}$.

We start by claiming that the system will be stable iff the $\dot{\mathbf{x}}^\top \mathbf{x} < 0$. Indeed, vector $\dot{\mathbf{x}}$ can always be decomposed into two components, $\dot{\mathbf{x}}_{\parallel}$ parallel to \mathbf{x} , and $\dot{\mathbf{x}}_{\perp}$ perpendicular to \mathbf{x} . By definition $\dot{\mathbf{x}}_{\perp}^\top \mathbf{x} = 0$, and is responsible for the change in orientation of \mathbf{x} . The value of $\dot{\mathbf{x}}_{\parallel}$ is responsible for the change in the length of \mathbf{x} ; the length would shrink iff $\dot{\mathbf{x}}_{\parallel}$ is of opposite direction to \mathbf{x} , giving negative value of the dot product $\dot{\mathbf{x}}^\top \mathbf{x}$.

STABILITY OF AUTONOMOUS LTI

Complex eigenvalues, 2-dimensional case (2)

Let us compute $\dot{\mathbf{x}}^\top \mathbf{x}$:

$$\dot{\mathbf{x}}^\top \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (14)$$

$$\dot{\mathbf{x}}^\top \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha x_1 - \beta x_2 \\ \beta x_1 + \alpha x_2 \end{bmatrix} \quad (15)$$

$$\dot{\mathbf{x}}^\top \mathbf{x} = \alpha(x_1^2 + x_2^2) \quad (16)$$

The product $\dot{\mathbf{x}}^\top \mathbf{x} < 0$ is negative iff $\alpha < 0$.

Definition

As long as the *real parts of the eigenvalues* of the system are *strictly negative*, the system is *asymptotically stable*. If the real parts of the eigenvalues of the system are zero, the system is *marginally stable*.

STABILITY OF AUTONOMOUS LTI

Complex eigenvalues, 2-dimensional case (3)

Vector field of $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is shown below:



STABILITY OF AUTONOMOUS LTI

General case (1)

Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where \mathbf{A} can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{U}\mathbf{C}\mathbf{U}^{-1}$, where \mathbf{C} is a complex-valued diagonal matrix and \mathbf{U} is a complex-valued inevitable matrix.

$$\dot{\mathbf{x}} = \mathbf{U}\mathbf{C}\mathbf{U}^{-1}\mathbf{x} \quad (17)$$

We multiply both sides by \mathbf{U}^{-1} , then define $\mathbf{z} = \mathbf{U}^{-1}\mathbf{x}$ to arrive at:

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} \quad (18)$$

which falls into a set of independent equations, with complex coefficients c_j :

$$\dot{z}_j = c_j z_j \quad (19)$$

STABILITY OF AUTONOMOUS LTI

General case (2)

Expanding $c_j = \alpha + i\beta$, and $z_j = u + iv$ (we dismiss subscripts for clarity), we find that $\dot{z}_j = c_j z_j$ can be expanded as:

$$\dot{u} + i\dot{v} = \dot{z}_j = c_j z_j = (\alpha + i\beta)(u + iv) \quad (20)$$

$$\dot{u} + i\dot{v} = \alpha u + i\beta u + i\alpha v - \beta v \quad (21)$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad (22)$$

As we can see, $\dot{z}_j = c_j z_j$ is asymptotically stable iff $\operatorname{Re}(c_j) < 0$, and marginally stable if $\alpha = \operatorname{Re}(c_j) = 0$. Same is true for $\dot{\mathbf{z}} = \mathbf{C}\mathbf{z}$ and hence, for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, as \mathbf{U} is invertible.

STABILITY OF AUTONOMOUS LTI

Condition

Consider an autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (23)$$

Definition

Eq. (23) is stable iff real parts of eigenvalues of \mathbf{A} are non-positive.

Definition

Eq. (23) is asymptotically stable iff real parts of eigenvalues of \mathbf{A} are negative.

STABILITY OF AUTONOMOUS LTI

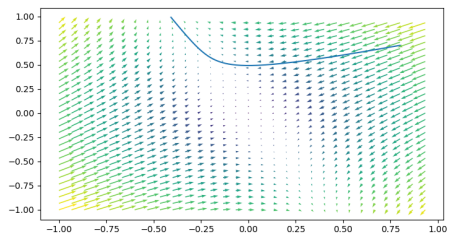
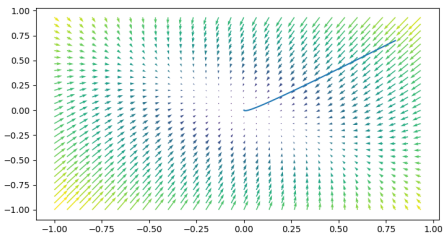
Illustration

Here is an illustration of *phase portraits* of two-dimensional LTIs with different types of stability:



Figure 1: phase portraits for different types of stability

Credit: staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf



- Control Systems Design, by Julio H. Braslavsky
staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf
- Stability and Eigenvalues, Steve Brunton
youtu.be/h7nJ6ZL4Lf0
- MAE509 (LMIs in Control): Lecture 4, part A - Stability and Eigenvalues youtu.be/8zYOJbpiT38

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025

