Stability

Control Theory, Lecture 2

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Critical point (node)

Consider the following ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

Let $\mathbf{x}_0$ be such a state that:

$$\mathbf{f}(\mathbf{x}_0, t) = 0 \tag{2}$$

Then such state $\mathbf{x}_0$ is called a *node* or a *critical point*.

Node $\mathbf{x}_0$ is called *stable* iff for any constant δ there exists constant ε such that:

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \ \longrightarrow \ ||\mathbf{x}(t) - \mathbf{x}_0|| < \varepsilon \tag{3}$$

Think of it as "for any initial point that lies at most δ away from $\mathbf{x}_0$, the rest of the trajectory $\mathbf{x}(t)$ will be at most ε away from $\mathbf{x}_0$".

Equivalently we can say "the solutions starting from δ-sized ball do not diverge".

Node $\mathbf{x}_0$ is called *asymptotically stable* iff for any constant δ it is true that:

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \;\longrightarrow\; \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_0 \tag{4}$$

Think of it as "for any initial point that lies at most δ away from $\mathbf{x}_0$, the trajectory $\mathbf{x}(t)$ will asymptotically approach the point $\mathbf{x}_0$".

Equivalently we can say "the solutions starting from δ-sized ball converge to the node".

Stability vs Asymptotic stability

Example

Consider dynamical system $\dot{x} = 0$, and solution $x = 7$. This solution is stable, but not asymptotically stable (a solution corresponding to $x(0) = 7 + \delta$ does not diverge, but does not converge to $x = 7$ either).

Example

Consider dynamical system $\dot{x} = -x$, and solution $x = 0$. This solution is stable and asymptotically stable (all solutions converge to $x = 0$).

Example

Consider dynamical system $\dot{x} = x$, and solution $x = 0$. This solution is unstable (all other solutions diverge from $x = 0$).

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{5}$$

This is called a *linear time-invariant system (LTI)*, indicating that $\mathbf{A}$ and $\mathbf{B}$ are constant.

Removing the input we find an even simpler equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$

This LTI is an *autonomous system*, since its evolution depends only on the state of the system.

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{x} \tag{7}$$

where $\mathbf{D} = \mathrm{diag}(d_1,\ ...,\ d_n)$ is a diagonal matrix. This is the same as a system of independent equations:

$$\begin{cases} \dot{x}_1 = d_1 x_1 \\ ... \\ \dot{x}_n = d_n x_n \end{cases} \tag{8}$$

Each of these equations has an exact solution $x_i = C_i e^{d_i t}$. It diverges from 0 if $d_i > 0$, it does not diverge if $d_i \leq 0$ and it converges to 0 if $d_i < 0$.

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{9}$$

where $\mathbf{A}$ can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

$$\dot{\mathbf{x}} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x} \tag{10}$$

Multiplying it by $\mathbf{V}^{-1}$ we get: $\mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x}$. Defining $\mathbf{z} = \mathbf{V}^{-1}\mathbf{x}$ we transform the equation: $\dot{\mathbf{z}} = \mathbf{D}\mathbf{z}$.

Since elements of $\mathbf{D}$ are real, we can clearly see, that iff they are *all negative* will the system be asymptotically stable. If they are non-positive, the system is stable. And those elements are eigenvalues of $\mathbf{A}$.

Eigenvalues of upper triangular matrices

Eigenvalues of upper triangular matrices are the diagonal elements of these matrices.

Examples of upper triangular matrices are:

$$\begin{bmatrix} 1 & 5 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \quad \begin{bmatrix} -2 & 0 & 8 \\ 0 & -2 & 8 \\ 0 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} \tag{11}$$

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{M}\mathbf{x} \tag{12}$$

where **M** is an upper triangular matrices with negative eigenvalues $m_{1,1}$, ... $m_{n,n}$.

The last equation is $\dot{x}_n = m_{n,n} x_n$, and since $m_{n,n} < 0$ we can observe that $\lim\limits_{t \to \infty} x_n(t) = 0$.

The equation # n-1 is $\dot{x}_{n-1} = m_{n-1,n-1} x_{n-1} + m_{n-1,n} x_n$, and since $m_{n-1,n-1} < 0$ and $\lim\limits_{t \to \infty} x_n(t) = 0$ we can observe that $\lim\limits_{t \to \infty} x_{n-1}(t) = 0$.
This can be repeated for all equations, proving asymptotic stability for the system.

Complex eigenvalues, 2-dimensional case (1)

Let us consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{13}$$

The eigenvalues of the system are $\alpha \pm i\beta$. We denote $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}$.

We start by claiming that the system will be stable iff the $\dot{\mathbf{x}}^\top \mathbf{x} < 0$. Indeed, vector $\dot{\mathbf{x}}$ can always be decomposed into two components, $\dot{\mathbf{x}}_{||}$ parallel to $\mathbf{x}$, and $\dot{\mathbf{x}}_\perp$ perpendicular to $\mathbf{x}$. By definition $\dot{\mathbf{x}}_\perp^\top \mathbf{x} = 0$, and is responsible for the change in orientation of $\mathbf{x}$. The value of $\dot{\mathbf{x}}_{||}$ is responsible for the change in the length of $\mathbf{x}$; the length would shrink iff $\dot{\mathbf{x}}_{||}$ is of opposite direction to $\mathbf{x}$, giving negative value of the dot product $\dot{\mathbf{x}}^\top \mathbf{x}$.

Complex eigenvalues, 2-dimensional case (2)

Let us compute $\dot{\mathbf{x}}^\top \mathbf{x}$:

$$\dot{\mathbf{x}}^\top \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{14}$$

$$\dot{\mathbf{x}}^\top \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha x_1 - \beta x_2 \\ \beta x_1 + \alpha x_2 \end{bmatrix} \tag{15}$$

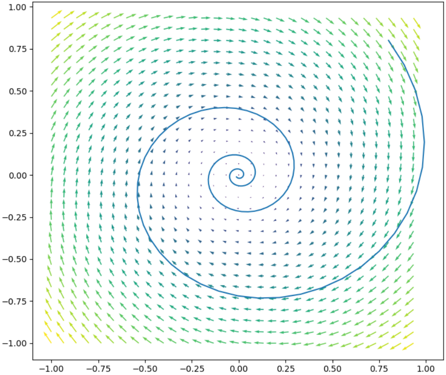
$$\dot{\mathbf{x}}^\top \mathbf{x} = \alpha(x_1^2 + x_2^2) \tag{16}$$

The product $\dot{\mathbf{x}}^\top \mathbf{x} < 0$ is negative iff $\alpha < 0$.

Definition

As long as the *real parts of the eigenvalues* of the system are *strictly negative*, the system is *asymptotically stable.* If the real parts of the eigenvalues of the system are zero, the system is *marginally stable.*

Vector field of $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is shown below:



Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{U}\mathbf{C}\mathbf{U}^{-1}$, where $\mathbf{C}$ is a complex-valued diagonal matrix and $\mathbf{U}$ is a complex-valued inevitable matrix.

$$\dot{\mathbf{x}} = \mathbf{U}\mathbf{C}\mathbf{U}^{-1}\mathbf{x} \tag{17}$$

We multiply both sides by $\mathbf{U}^{-1}$, then define $\mathbf{z} = \mathbf{U}^{-1}\mathbf{x}$ to arrive at:

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} \tag{18}$$

which falls into a set of independent equations, with complex coefficients c_j:

$$\dot{z}_j = c_j z_j \tag{19}$$

Expanding $c_j = \alpha + i\beta$, and $z_j = u + iv$ (we dismiss subscripts for clarity), we find that $\dot{z}_j = c_j z_j$ can be expanded as:

$$\dot{u} + i\dot{v} = \dot{z}_j = c_j z_j = (\alpha + i\beta)(u + iv) \tag{20}$$

$$\dot{u} + i\dot{v} = \alpha u + i\beta u + i\alpha v - \beta v \tag{21}$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{22}$$

As we can see, $\dot{z}_j = c_j z_j$ is asymptotically stable iff $\mathrm{Re}(c_j) < 0$, and marginally stable if $\alpha = \mathrm{Re}(c_j) = 0$. Same is true for $\dot{\mathbf{z}} = \mathbf{C}\mathbf{z}$ and hence, for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, as $\mathbf{U}$ is invertible.

Consider an autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{23}$$

Definition

Eq. (23) is stable iff real parts of eigenvalues of $\mathbf{A}$ are non-positive.

Definition

Eq. (23) is asymptotically stable iff real parts of eigenvalues of $\mathbf{A}$ are negative.

Stability of autonomous LTI

Illustration

Here is an illustration of *phase portraits* of two-dimensional LTIs with different types of stability:

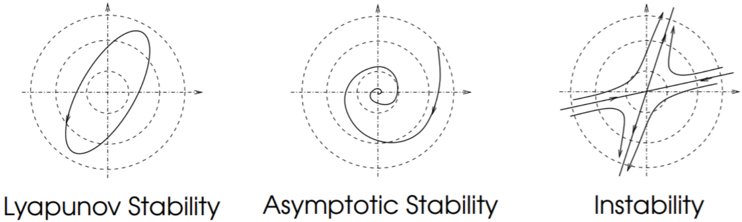
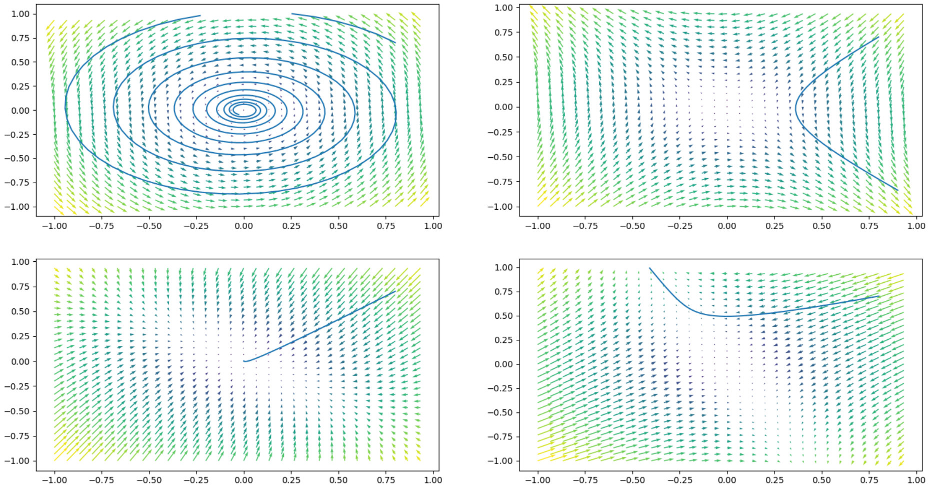


Figure 1: phase portraits for different types of stability

Credit: staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf


1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.00 −0.75 −0.50 −0.25 0.00 0.25 0.50 0.75 1.00

Read/Watch more

- Control Systems Design, by Julio H. Braslavsky staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf
- Stability and Eigenvalues, Steve Brunton youtu.be/h7nJ6ZL4Lf0
- MAE509 (LMIs in Control): Lecture 4, part A - Stability and Eigenvalues youtu.be/8zYOJbpiT38

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025