# Frequency response, Bode

## Control Theory, Lecture 5

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# Frequency response

**Frequency response**

Frequency response is a steady-state output of the system, given sinusoidal input.

Consider a system $Y(s) = G(s)U(s)$. Sinusoidal input $u(t) = \sin(\omega t)$ in time domain translates to $U(s) = \frac{\omega}{\omega^2+s^2}$ in Laplace domain. So, given a sinusoidal input, the system becomes:

$$Y(s) = G(s)\frac{\omega}{\omega^2 + s^2} \tag{1}$$

# Fraction expansion

If a transfer function $G(s)$ is a rational fraction, it can be represented as:

$$G(s) = \frac{n(s)}{(s+p_1)(s+p_2)\ \dots\ (s+p_n)} \tag{2}$$

where $p_i$ are the roots on the denominator - called *poles* of the transfer function.

In many cases (for example when $p_i$ are real and non-repeating), the fraction can be expanded:

$$G(s) = \frac{n(s)}{(s+p_1)(s+p_2)\ \dots\ (s+p_n)} = \frac{r_1}{s+p_1} + \frac{r_2}{s+p_2} + \dots + \frac{r_n}{s+p_n}$$

# Fraction expansion

We can expand the function $Y(s) = G(s)\frac{\omega}{\omega^2+s^2}$ in a similar way:

$$Y(s) = \frac{r_1}{s+p_1} + \frac{r_2}{s+p_2} + ... + \frac{r_n}{s+p_n} + \frac{\alpha}{s+j\omega} + \frac{\beta}{s-j\omega}$$

Laplace function of the form $\frac{r_i}{s+p_i}$ corresponds to the following time function:

$$y(t) = r_i e^{-p_i t} \tag{3}$$

So, for a stable transfer function as time goes to infinity, $r_i e^{-p_i t}$ goes to zero. The only components of the function $Y(s)$ that do not disappear are the last two: $\frac{\alpha}{s+j\omega} + \frac{\beta}{s-j\omega}$.

# Fraction expansion

One can show that constants in the expansion $\frac{\alpha}{s+j\omega} + \frac{\beta}{s-j\omega}$ can be found in the form:

$$\alpha = -G(j\omega)g \tag{4}$$

$$\beta = G(-j\omega)g \tag{5}$$

In fact, the analysis of the frequency response will involve analyzing the transfer function $G(j\omega)$.

# Laplace and Fourier transforms

- *Fourier series* can be seen as representing a periodic function as a sum of harmonics (sines and cosines). These sines and cosines can be thought of as forming a basis in a linear space. The coefficients of the series can be thought of as a discrete spectrum of the function.
- *Fourier transform* gives a continuous spectrum of the function. The "basis" is still made of harmonic functions.
- *Laplace transform* also gives a continuous spectrum of the function, but in a different basis: the basis is given by complex exponentials. I like to think of this basis as solutions of second order ODEs.

# Laplace and Fourier transforms

Let's compare. Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi j t \omega} dt, \quad \omega \in \mathbb{R} \tag{6}$$

Laplace transform:

$$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt, \quad s \in \mathbb{C} \tag{7}$$

We can see that Fourier looks like Laplace with purely imaginary number in the exponent.

# Laplace and steady state solution

From analysing solutions of linear ODEs we know that, given harmonic input (sine, cosine, their combination) "after the transient process is over, the solution approaches a harmonic with the same frequency", but possibly different amplitude and phase.

Intuitively we can think of the imaginary part of $s$ as having to do with this frequency response.

The kernel function of the Laplace transform is $e^{-st}$ with $s = \sigma + j\omega$ being a complex variable. If $\sigma = 0$, the kernel becomes $e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$. You can see the similarity with Fourier transform kernel.

# Bode plot

The first key idea of a Bode plot is substitution of purely complex variable $j\omega$ in place of Laplace variable $s$, which can have non-zero real part.

Given a transfer function $W(s)$, $s = \sigma + j\omega$ we can analyse its behaviour when $\sigma = 0$. We can plot its amplitude
$a(\omega) = |W(j\omega)|$ and its phase
$\varphi(\omega) = \mathrm{atan2}(\mathrm{im}(W(j\omega)),\ \mathrm{real}(W(j\omega)))$.

Bode plot is actually two plots, 1) $20 \cdot \log(a(\omega))$ and 2) $\frac{180}{\pi}\varphi(\omega)$. The 20 and log has to do with the vertical axis being in decibels.

# Bode plot - example

Consider $W(s) = \frac{1}{1+s}$. Then $W(j\omega) = \frac{1}{1+j\omega}$. We can transform it as:

$$W(j\omega) = \frac{1 - j\omega}{(1 + j\omega)(1 - j\omega)} = \frac{1 - j\omega}{1 + \omega^2} \tag{8}$$

Thus we have $\text{real}(W(j\omega)) = \frac{1}{1+\omega^2}$ and $\text{im}(W(j\omega)) = -\frac{\omega}{1+\omega^2}$.

Bode plot is then given as:

$$a(\omega) = \sqrt{\frac{1 + \omega^2}{(1 + \omega^2)^2}} = \frac{1}{\sqrt{(1 + \omega^2)}} \tag{9}$$

$$\varphi(\omega) = \text{atan2}\left(-\frac{\omega}{1 + \omega^2},\ \frac{1}{1 + \omega^2}\right) \tag{10}$$

# Bode plot - stability margins

Before we discuss the use of Bode plot, let us remember that closed-loop transfer function has form (when simple feedback is used):

$$W(s) = \frac{G(s)}{1 + G(s)} \tag{11}$$

Substituting $s \longrightarrow j\omega$ we get:

$$W(\omega) = \frac{G(j\omega)}{1 + G(j\omega)} \tag{12}$$

From this we can see that $W(\omega)$ becomes ill-defined if $G(j\omega) = -1$. Meaning, we want to avoid two things happening simultaneously: the amplitude of $G(j\omega)$ being equal to 1, and its phase (argument) being equal to 180° (remember, phase of 0° is pure positive real number, phase of 90° is pure positive imaginary number, 180° is pure negative real number, etc.).

# Stability margins - graphical example

Let's check an illustration:

# Code example

Check the colab notebook based on the example above for an illustration of how the Bode plot can be made by hand or via scipy signal library.

# Read more

- Control System Lectures - Bode Plots, Introduction
- Oxford University Press. s-Domain analysis: poles, zeros, and Bode plots

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025