

Controllability, Observability

Control Theory, Lecture 11

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- Cayley–Hamilton
- Controllability of Discrete LTI
 - ▶ Controllability matrix
 - ▶ Controllability criterion
- Observability of Discrete LTI
 - ▶ Dual system
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- Controllability of Continuous-Time LTI

Equation $\det(\mathbf{M} - \lambda \mathbf{I}) = 0$ is called *characteristic equation* of matrix \mathbf{M} , its roots being eigenvalues of the matrix.

Theorem (Cayley–Hamilton)

A matrix $\mathbf{M} \in \mathbb{R}^{n,n}$ satisfies its own characteristic equation.

A characteristic equation can be written as $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$, meaning that we can write:

$$\mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + \dots + a_0\mathbf{I} = 0 \quad (1)$$

Meaning that \mathbf{M}^n is a linear combination of \mathbf{M}^{n-1} , \mathbf{M}^{n-2} , ..., \mathbf{I} .

DEFINITIONS

Definition (Controllability)

A system is controllable on time interval $t_0 \leq t \leq t_f$, if it is possible to find control input $u(t)$ that would drive the system to a desired state $\mathbf{x}(t_f)$ from any initial state $\mathbf{x}(t_0)$.

Definition (Observability)

A system is observable on time interval $t_0 \leq t \leq t_f$, if using output $\mathbf{y}(t)$ on that time interval it is possible to estimate exactly the state of the system $\mathbf{x}(t_f)$, given any initial estimation error.

Definition (Observability, alternative)

A system is observable on time interval $t_0 \leq t \leq t_f$, if any initial state $\mathbf{x}(t_0)$ is uniquely determined by output $\mathbf{y}(t)$ on that interval.

CONTROLLABILITY OF DISCRETE LTI

Consider discrete LTI:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \quad (2)$$

Assume the initial state is \mathbf{x}_1 . Then we can deduce that:

$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1$$

$$\mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 + \mathbf{B}\mathbf{u}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2$$

$$\mathbf{x}_4 = \mathbf{A}\mathbf{x}_3 + \mathbf{B}\mathbf{u}_3 = \mathbf{A}(\mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2) + \mathbf{B}\mathbf{u}_3$$

...

$$\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + \mathbf{A}^{n-1} \mathbf{B} \mathbf{u}_1 + \dots + \mathbf{A}^{n-k} \mathbf{B} \mathbf{u}_k + \dots + \mathbf{B} \mathbf{u}_n$$

CONTROLLABILITY MATRIX

Equation $\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + \mathbf{A}^{n-1} \mathbf{B} \mathbf{u}_1 + \dots + \mathbf{A}^{n-k} \mathbf{B} \mathbf{u}_k + \dots \mathbf{B} \mathbf{u}_n$ can be re-written as:

$$\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1 = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}] \begin{bmatrix} \mathbf{u}_n \\ \mathbf{u}_{n-1} \\ \mathbf{u}_{n-2} \\ \dots \\ \mathbf{u}_1 \end{bmatrix} \quad (3)$$

Notice that in order for the system to go from \mathbf{x}_1 to \mathbf{x}_{n+1} , vector $\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1$ needs to be in the column space of $\mathcal{C} = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$.

Since \mathbf{x}_{n+1} can be anything, and \mathbf{x}_1 might be equal to zero (among other possibilities), we should require that all vectors in \mathbb{R}^n are in the column space of \mathcal{C} , meaning \mathcal{C} needs to be full row rank.

Controllability

For a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$, where $\mathbf{x} \in \mathbb{R}^n$, if the matrix $\mathcal{C} = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$ is full row rank (i.e. $\text{rank}(\mathcal{C}) = n$), any state can be reached, which means that *the system is controllable*.

What happens if we add more columns to the controllability matrix, for example $\mathbf{A}^n\mathbf{B}$? Consider the matrix:

$$\mathcal{C}_+ = [\mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B} \quad \mathbf{A}^n\mathbf{B}] \quad (4)$$

But from Cayley–Hamilton we know that:

$$\mathbf{A}^n = -a_{n-1}\mathbf{A}^{n-1} - \dots - a_0\mathbf{I} \quad (5)$$

$$\mathbf{A}^n\mathbf{B} = -a_{n-1}\mathbf{A}^{n-1}\mathbf{B} - \dots - a_0\mathbf{B} \quad (6)$$

Meaning that columns of $\mathbf{A}^n\mathbf{B}$ are expressed as linear combination of columns of \mathcal{C} , hence the matrix \mathcal{C}_+ has the same rank as \mathcal{C} .

OBSERVABILITY OF DISCRETE LTI

Consider discrete LTI:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i \end{cases} \quad (7)$$

And an observer:

$$\hat{\mathbf{x}}_{i+1} = \mathbf{A}\hat{\mathbf{x}}_i + \mathbf{B}\mathbf{u}_i + \mathbf{L}(\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_i) \quad (8)$$

Remember that we can define observation error $\mathbf{e}_i = \hat{\mathbf{x}}_i - \mathbf{x}_i$ and write its dynamics:

$$\mathbf{e}_{i+1} = \mathbf{A}\mathbf{e}_i - \mathbf{L}\mathbf{C}\mathbf{e}_i \quad (9)$$

Dual system (which is stable if and only if the original is stable), has form:

$$\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i \quad (10)$$

OBSERVABILITY OF DISCRETE LTI

Dual system

Dynamical system $\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i$, we can be represented as:

$$\begin{cases} \varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i + \mathbf{C}^\top \mathbf{v}_i \\ \mathbf{v}_i = -\mathbf{L}^\top \varepsilon_i \end{cases} \quad (11)$$

Controllability matrix of this system is:

$$\mathcal{O}^\top = [\mathbf{C}^\top \quad (\mathbf{A}^\top)\mathbf{C}^\top \quad \dots \quad (\mathbf{A}^\top)^{n-1}\mathbf{C}^\top] \quad (12)$$

It is easier to represent this matrix in its transposed form:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix} \quad (13)$$

Observability

For a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$ and $\mathbf{y}_i = \mathbf{C}\mathbf{x}_i$, where $\mathbf{x} \in \mathbb{R}^n$, if

the matrix $\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix}$ is full column rank (i.e.

$\text{rank}(\mathcal{O}) = n$), observation error can go to zero from any initial position, which means that *the system is observable*.

CONTROLLABILITY, CONTINUOUS-TIME (1)

Let us consider matrix exponential $e^{\mathbf{A}t}$ is defined as a series:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + \dots \quad (14)$$

Using Cayley–Hamilton we can observe that any powers of \mathbf{A} higher than n can be represented as a linear combination of lower powers. This gives us the following expression:

$$e^{\mathbf{A}t} = \phi_0(t)\mathbf{I} + \phi_1(t)\mathbf{A} + \phi_2(t)\mathbf{A}^2 + \dots + \phi_{n-1}(t)\mathbf{A}^{n-1} \quad (15)$$

This allows us to re-write the forced state response:

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau \\ \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t (\phi_0(t-\tau)\mathbf{I} + \phi_1(t-\tau)\mathbf{A} + \dots \\ &\quad + \phi_{n-1}(t-\tau)\mathbf{A}^{n-1})\mathbf{b}u(\tau) d\tau \end{aligned}$$

CONTROLLABILITY, CONTINUOUS-TIME (2)

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \phi_0(t-\tau)\mathbf{b}u(\tau)d\tau \\ &\quad + \int_0^t \phi_1(t-\tau)\mathbf{A}\mathbf{b}u(\tau)d\tau + \dots \int_0^t \phi_{n-1}(t-\tau)\mathbf{A}^{n-1}\mathbf{b}u(\tau)d\tau\end{aligned}$$

$$\mathbf{x}(t) - e^{\mathbf{A}t}\mathbf{x}(0) = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{b}] \begin{bmatrix} \int_0^t \phi_0(t-\tau)u(\tau)d\tau \\ \int_0^t \phi_1(t-\tau)u(\tau)d\tau \\ \dots \\ \int_0^t \phi_{n-1}(t-\tau)u(\tau)d\tau \end{bmatrix}$$

This shows that if controllability matrix is rank-deficient, it would not be possible to achieve some state from some initial condition.

PBH CONTROLLABILITY CRITERION

There is an alternative way to test if pair (\mathbf{A}, \mathbf{B}) is controllable:

PBH controllability criterion

If for any $\lambda \in \mathbb{C}$, the the matrix $[(\mathbf{A} - \lambda\mathbf{I}), \mathbf{B}]$ has full row rank, then the pair (\mathbf{A}, \mathbf{B}) is controllable.

- If λ is not an eigenvalue of \mathbf{A} , then $\det(\mathbf{A} - \lambda\mathbf{I}) \neq 0$ and the matrix has full row rank.
- If $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and $\mathcal{V} = \text{null}(\mathbf{A} - \lambda\mathbf{I})$, $\dim(\mathcal{V}) = 1$ and $\mathbf{v} \in \mathcal{V}$, meaning λ, \mathbf{v} are eigenvalue and eigenvector of \mathbf{A} , then in order for the criterion to hold the columns of \mathbf{B} should not all be orthogonal to \mathbf{v} : $\mathbf{v}^\top \mathbf{B} \neq 0$.
- If eigenspace \mathcal{V} is k -dimensional, the projection of \mathbf{B} onto that eigenspace should also be k -dimensional.

- Controllability and Observability (Rutgers University)
<https://www.ece.rutgers.edu/~gajic/psfiles/chap5.pdf>

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025



Appendix A: Analytical solution (recap)

Exponential e^a is defined as a series:

$$e^a = 1 + a + \frac{1}{2}a^2 + \frac{1}{6}a^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}a^n \quad (16)$$

Matrix exponential $e^{\mathbf{A}}$ is defined as a series:

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}\mathbf{A} + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n \quad (17)$$

ANALYTICAL SOLUTION TO ODE

An ODE of the form $\dot{x} = ax$ has analytical solution $x(t) = e^{at}x(0)$.

An ODE of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ has analytical solution $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$.

Let us check that this is a solution:

$$\mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + \dots \right) \mathbf{x}(0) \quad (18)$$

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} + \mathbf{A}\mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}\mathbf{A}t^2 + \dots \right) \mathbf{x}(0) \quad (19)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A} \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \dots \right) \mathbf{x}(0) \quad (20)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) \quad (21)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad \square \quad (22)$$

FORCED STATE RESPONSE (LTI) (1)

An ODE of the form $\dot{x} = ax + bu(t)$ also has analytical solution. To find it, we first find the following derivative:

$$\frac{d}{dt} (e^{-at}x(t)) = e^{-at}\dot{x}(t) - ae^{-at}x(t) \quad (23)$$

Multiplying $\dot{x} = ax + bu(t)$ by e^{-at} we see:

$$e^{-at}\dot{x} = e^{-at}ax + e^{-at}bu(t) \quad (24)$$

$$e^{-at}\dot{x} - e^{-at}ax = e^{-at}bu(t) \quad (25)$$

$$\frac{d}{dt} (e^{-at}x(t)) = e^{-at}bu(t) \quad (26)$$

$$\int_0^t \frac{d}{d\tau} (e^{-a\tau}x(\tau)) d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \quad (27)$$

FORCED STATE RESPONSE (LTI) (2)

Continuing the derivation:

$$\int_0^t \frac{d}{d\tau} (e^{-a\tau} x(\tau)) d\tau = \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (28)$$

$$e^{-at} x(t) - x(0) = \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (29)$$

$$x(t) = e^{at} x(0) + e^{at} \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (30)$$

$$x(t) = e^{at} x(0) + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \quad (31)$$

FORCED STATE RESPONSE (LTI) (3)

State-space equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t)$ also has an analytical solution:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \quad (32)$$

The same can be re-written as:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \quad (33)$$