

# Lyapunov Theory, Lyapunov equations

## Control Theory, Lecture 13

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# LYAPUNOV METHOD: STABILITY CRITERIA

## Asymptotic stability criteria

Autonomous dynamic system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  is asymptotically stable, if there exists a scalar function  $V = V(\mathbf{x}) > 0$ , whose time derivative is negative  $\dot{V}(\mathbf{x}) < 0$ , except  $V(\mathbf{0}) = 0$ ,  $\dot{V}(\mathbf{0}) = 0$ .

## Marginal stability criteria

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  is stable in the sense of Lyapunov,  $\exists V(\mathbf{x}) > 0$ ,  $\dot{V}(\mathbf{x}) \leq 0$ .

## Definition

Function  $V(\mathbf{x}) > 0$  in this case is called *Lyapunov function*.

# LYAPUNOV METHOD: EXAMPLE 1

Take dynamical system  $\dot{x} = -x$ .

We propose a *Lyapunov function candidate*  $V(x) = x^2 \geq 0$ .

Let's find its derivative:

$$\dot{V}(x) = \frac{d}{dt}(x^2) = 2x\dot{x} = 2x(-x) = -x^2 \leq 0 \quad (1)$$

This satisfies the Lyapunov criteria, so the system is stable. It is in fact asymptotically stable, because  $\dot{V}(x) \neq 0$  if  $x \neq 0$ .

## LYAPUNOV METHOD: EXAMPLE 2

Consider pendulum  $\ddot{q} = f(q, \dot{q}) = -\dot{q} - \sin(q)$ .

We propose a *Lyapunov function candidate*

$V(q, \dot{q}) = E(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + 1 - \cos(q) \geq 0$ , where  $E(q, \dot{q})$  is total energy of the system. Let's find its derivative:

$$\dot{V}(q, \dot{q}) = \frac{d}{dt} \left( \frac{1}{2}\dot{q}^2 + 1 - \cos(q) \right) = \dot{q}\ddot{q} + \sin(q)\dot{q} = \quad (2)$$

$$= \dot{q}(-\dot{q} - \sin(q)) + \sin(q)\dot{q} = -\dot{q}^2 \leq 0 \quad (3)$$

This satisfies the Lyapunov criteria, so the system is stable. It is not proven to be asymptotically stable, because  $\dot{V}(q, \dot{q}) = 0$  for any  $q$ , as long as  $\dot{q} = 0$ .

# LASALLE'S INVARIANCE PRINCIPLE, 1

## LaSalle's invariance principle

Autonomous dynamic system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  is asymptotically stable, if there exists a scalar function  $V = V(\mathbf{x}) > 0$ , whose time derivative is negative  $\dot{V}(\mathbf{x}) \leq 0$ , except  $V(\mathbf{0}) = 0$ , where the set  $\{\mathbf{x} : \dot{V}(\mathbf{x}) = 0\}$  does not contain non-trivial trajectories.

A trivial trajectory is  $\mathbf{x}(t) = \mathbf{0}$ . Unlike Lyapunov condition, LaSalle's principle allows us to prove asymptotic stability even for systems with  $\dot{V}(\mathbf{x}) = 0$ .

# LASALLE'S INVARIANCE PRINCIPLE, 2

Local version of LaSalle's invariance principle has the following form:

## Local LaSalle's invariance principle

Autonomous dynamic system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  is asymptotically stable in the neighborhood  $\mathcal{D}$  of the origin, if there exists a scalar function  $V = V(\mathbf{x}) > 0$ , whose time derivative is negative  $\dot{V}(\mathbf{x}) \leq 0$ , except  $V(\mathbf{0}) = 0$ , where the set  $\mathcal{M} = \{\mathbf{x} : \dot{V}(\mathbf{x}) = 0\} \cap \mathcal{D}$  does not contain non-trivial trajectories.

## LASALLE PRINCIPLE: EXAMPLE 2

In our previous example  $\dot{V}(q, \dot{q}) = 0$  for any  $q$ , as long as  $\dot{q} = 0$ . But the set  $\{(q, \dot{q}) : \dot{q} = 0\}$  contains no trajectories of the system  $\ddot{q} = -\dot{q} - \sin(q)$  other than  $q(t) = 0$  in the region  $-\frac{\pi}{2} < q < \frac{\pi}{2}$ . So, LaSalle principle proves local asymptotic stability.



## LASALLE PRINCIPLE: EXAMPLE 3

Consider oscillator  $\ddot{q} = f(q, \dot{q}) = -\dot{q}$ .

We propose a *Lyapunov function candidate*

$V(q, \dot{q}) = T(q, \dot{q}) = \frac{1}{2}\dot{q}^2 \geq 0$ , where  $T(q, \dot{q})$  is kinetic energy of the system. Let's find its derivative:

$$\dot{V}(q, \dot{q}) = \frac{\partial V}{\partial q} \dot{q} + \frac{\partial V}{\partial \dot{q}} f(q, \dot{q}) = \dot{q}(-\dot{q}) = -\dot{q}^2 \leq 0 \quad (4)$$

This satisfies the Lyapunov criteria, so the system is stable.

Note that  $\dot{V}(q, \dot{q}) = 0$  for any  $q$  as long as  $\dot{q} = 0$ . But the set  $\{(q, \dot{q}) : \dot{q} = 0\}$  contains infinitely many trajectories of the system  $\ddot{q} = -\dot{q}$  other than  $q(t) = 0$ , for example  $q(t) = 1$  or  $q(t) = -2$ . So, LaSalle principle does not prove asymptotic stability in this case.

# LINEAR CASE

## Part 1

As you saw, Lyapunov method allows you to deal with nonlinear systems, as well as linear ones. But for linear ones there are additional properties we can use.

### Observation 1

For a linear system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  we can always pick Lyapunov function candidate in the form  $V = \mathbf{x}^\top \mathbf{S} \mathbf{x} > 0$ , where  $\mathbf{S}$  is a positive definite matrix.

Next slides will shows where this leads us.

# LINEAR CASE

## Part 2

Given  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  and  $V = \mathbf{x}^\top \mathbf{S}\mathbf{x} \geq 0$ , let's find its derivative:

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}\dot{\mathbf{x}} \quad (5)$$

$$\dot{V}(\mathbf{x}) = (\mathbf{A}\mathbf{x})^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}\mathbf{A}\mathbf{x} = \mathbf{x}^\top (\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A})\mathbf{x} \quad (6)$$

Notice that  $\dot{V}(x)$  should be negative for all  $\mathbf{x}$  for the system to be stable, meaning that  $\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A}$  should be negative definite. A more strict form of this requirement is *Lyapunov equation*:

$$\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} = -\mathbf{Q} \quad (7)$$

where  $\mathbf{Q}$  is a positive-definite matrix.

# DISCRETE CASE

## Part 1

### Asymptotic stability criteria, discrete case

Given  $\mathbf{x}_{i+1} = \mathbf{f}(\mathbf{x}_i)$ , if  $V(\mathbf{x}_i) > 0$ , and  $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) < 0$ , the system is stable.

Same as before, for linear systems we will be choosing *positive-definite quadratic forms* as Lyapunov function candidates.

# DISCRETE CASE

## Part 2

Consider dynamics  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$  and  $V = \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i \geq 0$ , let's find  $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i)$ :

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) = (\mathbf{A}\mathbf{x}_i)^\top \mathbf{S}\mathbf{A}\mathbf{x}_i - \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i \quad (8)$$

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) = \mathbf{x}_i^\top (\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S})\mathbf{x}_i \quad (9)$$

Notice that  $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i)$  should be negative for all  $\mathbf{x}_i$  for the system to be stable, meaning that  $\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S}$  should be negative definite, giving us *Discrete Lyapunov equation*:

$$\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S} = -\mathbf{Q} \quad (10)$$

where  $\mathbf{Q}$  is a positive-definite matrix.

In practice, you can easily use Lyapunov equations for stability verification. Python and MATLAB have built-in functionality to solve it:

- `scipy: linalg.solve_continuous_lyapunov(A, Q)`
- `MATLAB: lyap(A,Q)`

- 3.9 Liapunov's direct method
- Università degli studi di Padova Dipartimento di Ingegneria dell'Informazione, Nicoletta Bof, Ruggero Carli, Luca Schenato, Technical Report, Lyapunov Theory for Discrete Time Systems

Lecture slides are available via Github:

[github.com/SergeiSa/Control-Theory-2025](https://github.com/SergeiSa/Control-Theory-2025)

