Lyapunov Theory, Lyapunov equations

Control Theory, Lecture 13

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Asymptotic stability criteria

Autonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is assymptotically stable, if there exists a scalar function $V = V(\mathbf{x}) > 0$, whose time derivative is negative $\dot{V}(\mathbf{x}) < 0$, except $V(\mathbf{0}) = 0$, $\dot{V}(\mathbf{0}) = 0$.

Marginal stability criteria

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is stable in the sense of Lyapunov, $\exists V(\mathbf{x}) > 0$, $\dot{V}(\mathbf{x}) \leq 0$.

Definition

Function $V(\mathbf{x}) > 0$ in this case is called *Lyapunov function.*

Take dynamical system $\dot{x} = -x$.

We propose a *Lyapunov function candidate* $V(x) = x^2 \geq 0$.
Let's find its derivative:

$$\dot{V}(x) = \frac{d}{dt}(x^2) = 2x\dot{x} = 2x(-x) = -x^2 \leq 0 \tag{1}$$

This satisfies the Lyapunov criteria, so the system is stable. It is in fact asymptotically stable, because $\dot{V}(x) \neq 0$ if $x \neq 0$.

Consider pendulum $\ddot{q} = f(q, \dot{q}) = -\dot{q} - \sin(q)$.

We propose a *Lyapunov function candidate* $V(q, \dot{q}) = E(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + 1 - \cos(q) \geq 0$, where $E(q, \dot{q})$ is total energy of the system. Let's find its derivative:

$$\dot{V}(q, \dot{q}) = \frac{d}{dt}(\frac{1}{2}\dot{q}^2 + 1 - \cos(q)) = \dot{q}\ddot{q} + \sin(q)\dot{q} = \tag{2}$$

$$= \dot{q}(-\dot{q} - \sin(q)) + \sin(q)\dot{q} = -\dot{q}^2 \leq 0 \tag{3}$$

This satisfies the Lyapunov criteria, so the system is stable. It is not proven to be asymptotically stable, because $\dot{V}(q, \dot{q}) = 0$ for any q, as long as $\dot{q} = 0$.

LaSalle's invariance principle

Autonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is asymptotically stable, if there exists a scalar function $V = V(\mathbf{x}) > 0$, whose time derivative is negative $\dot{V}(\mathbf{x}) \leq 0$, except $V(\mathbf{0}) = 0$, where the set $\{\mathbf{x} : \dot{V}(\mathbf{x}) = 0\}$ does not contain non-trivial trajectories.

A trivial trajectory is $\mathbf{x}(t) = 0$. Unlike Lyapunov condition, LaSalle's principle allows us to prove asymptotic stability even for systems with $\dot{V}(\mathbf{x}) = 0$.

Local version of LaSalle's invariance principle has the following form:

Local LaSalle's invariance principle

Autonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is asymptotically stable in the neighborhood $\mathcal{D}$ of the origin, if there exists a scalar function $V = V(\mathbf{x}) > 0$, whose time derivative is negative $\dot{V}(\mathbf{x}) \leq 0$, except $V(\mathbf{0}) = 0$, where the set $\mathcal{M} = \{\mathbf{x} : \ \dot{V}(\mathbf{x}) = 0\} \cap \mathcal{D}$ does not contain non-trivial trajectories.

In our previous example $\dot{V}(q, \dot{q}) = 0$ for any q, as long as $\dot{q} = 0$. But the set $\{(q, \dot{q}) : \ \dot{q} = 0\}$ contains no trajectories of the system $\ddot{q} = -\dot{q} - \sin(q)$ other than $q(t) = 0$ in the region $-\frac{\pi}{2} < q < \frac{\pi}{2}$. So, LaSalle principle proves local asymptotic stability.

Consider oscillator $\ddot{q} = f(q, \dot{q}) = -\dot{q}$.

We propose a *Lyapunov function candidate* $V(q, \dot{q}) = T(q, \dot{q}) = \frac{1}{2}\dot{q}^2 \geq 0$, where $T(q, \dot{q})$ is kinetic energy of the system. Let's find its derivative:

$$\dot{V}(q, \dot{q}) = \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}} f(q, \dot{q}) = \dot{q}(-\dot{q}) = -\dot{q}^2 \leq 0 \tag{4}$$

This satisfies the Lyapunov criteria, so the system is stable. Note that $\dot{V}(q, \dot{q}) = 0$ for any q as long as $\dot{q} = 0$. But the set $\{(q, \dot{q}) : \ \dot{q} = 0\}$ contains infinitely many trajectories of the system $\ddot{q} = -\dot{q}$ other than $q(t) = 0$, for example $q(t) = 1$ or $q(t) = -2$. So, LaSalle principle does not prove asymptotic stability in this case.

Linear case

Part 1

As you saw, Lyapunov method allows you to deal with nonlinear systems, as well as linear ones. But for linear ones there are additional properties we can use.

Observation 1

For a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ we can always pick Lyapunov function candidate in the form $V = \mathbf{x}^\top \mathbf{S} \mathbf{x} > 0$, where $\mathbf{S}$ is a positive definite matrix.

Next slides will shows where this leads us.

Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and $V = \mathbf{x}^\top \mathbf{S}\mathbf{x} \geq 0$, let's find its derivative:

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}\dot{\mathbf{x}} \tag{5}$$

$$\dot{V}(\mathbf{x}) = (\mathbf{A}\mathbf{x})^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}\mathbf{A}\mathbf{x} = \mathbf{x}^\top (\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A})\mathbf{x} \tag{6}$$

Notice that $\dot{V}(x)$ should be negative for all $\mathbf{x}$ for the system to be stable, meaning that $\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A}$ should be negative definite. A more strict form of this requirement is *Lyapunov equation*:

$$\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} = -\mathbf{Q} \tag{7}$$

where $\mathbf{Q}$ is a positive-definite matrix.

Asymptotic stability criteria, discrete case

Given $\mathbf{x}_{i+1} = \mathbf{f}(\mathbf{x}_i)$, if $V(\mathbf{x}_i) > 0$, and $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) < 0$, the system is stable.

Same as before, for linear systems we will be choosing *positive-definite quadratic forms* as Lyapunov function candidates.

Discrete case

Part 2

Consider dynamics $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$ and $V = \mathbf{x}_i^\top \mathbf{S} \mathbf{x}_i \geq 0$, let's find $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i)$:

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) = (\mathbf{A}\mathbf{x}_i)^\top \mathbf{S} \mathbf{A} \mathbf{x}_i - \mathbf{x}_i^\top \mathbf{S} \mathbf{x}_i \tag{8}$$

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) = \mathbf{x}_i^\top (\mathbf{A}^\top \mathbf{S} \mathbf{A} - \mathbf{S}) \mathbf{x}_i \tag{9}$$

Notice that $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i)$ should be negative for all $\mathbf{x}_i$ for the system to be stable, meaning that $\mathbf{A}^\top \mathbf{S} \mathbf{A} - \mathbf{S}$ should be negative definite, giving us *Discrete Lyapunov equation*:

$$\mathbf{A}^\top \mathbf{S} \mathbf{A} - \mathbf{S} = -\mathbf{Q} \tag{10}$$

where $\mathbf{Q}$ is a positive-definite matrix.

In practice, you can easily use Lyapunov equations for stability verification. Python and MATLAB have built-in functionality to solve it:

- scipy: `linalg.solve_continuous_lyapunov(A, Q)`
- MATLAB: `lyap(A,Q)`

- 3.9 Liapunov's direct method
- Universita degli studi di Padova Dipartimento di Ingegneria dell'Informazione, Nicoletta Bof, Ruggero Carli, Luca Schenato, Technical Report, Lyapunov Theory for Discrete Time Systems

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025