Linearization

Control Theory, Lecture 12

by Sergei Savin

Spring 2025

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Taylor expansion around node, 1

Consider a non-linear dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{1}$$

If $\mathbf{x}_0$ and $\mathbf{u}_0$ represent a *node*, i.e. $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) = 0$, $\mathbf{x}_0 = \text{const}$, $\mathbf{u}_0 = \text{const}$, we can consider a Taylor expansion around that node:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0) + \text{H.O.T.} \tag{2}$$

Where $\mathbf{x}_0$ and $\mathbf{u}_0$ are expansion point. We define new variables $\mathbf{e}$ and $\mathbf{v}$ as distance from the expansion point:

$$\mathbf{e} = \mathbf{x} - \mathbf{x}_0, \quad \dot{\mathbf{e}} = \dot{\mathbf{x}}, \qquad\qquad \mathbf{v} = \mathbf{u} - \mathbf{u}_0. \tag{3}$$

With that, we can re-write the Taylor expansion:

$$\dot{\mathbf{e}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{e} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\mathbf{v} + \text{H.O.T.} \tag{4}$$

Taylor expansion around node, 2

$$\dot{\mathbf{e}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{e} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\mathbf{v} + \text{H.O.T.} \tag{5}$$

We can introduce notation:

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}, \quad \mathbf{B} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}. \tag{6}$$

If we drop the higher order terms from the Taylor expansion, we obtain *linearization* of the system dynamics:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \tag{7}$$

In this context, $\mathbf{x}_0$ and $\mathbf{u}_0$ is the *linearization point*.

Taylor expansion along a trajectory

Consider a non-linear dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{8}$$

and a trajectory $\dot{\mathbf{x}}_0 = \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)$. We can consider a Taylor expansion along this trajectory:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0) + \text{H.O.T.} \tag{9}$$

Since $\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_0$, we re-write:

$$\dot{\mathbf{e}} \sim \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} + \text{H.O.T.} \tag{10}$$

As before, we drop higher order terms and obtain linearization:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \tag{11}$$

Affine expansion

If we want to maintain our original variables, we can still use Taylor expansion:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) + \mathbf{A}(\mathbf{x} - \mathbf{x}_0) + \mathbf{B}(\mathbf{u} - \mathbf{u}_0) \tag{12}$$

Denoting $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{A}\mathbf{x}_0 - \mathbf{B}\mathbf{u}_0 = \mathbf{c}$ and dropping H.O.T. we approximate the system as affine:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \tag{13}$$

Manipulator equation linearization, 1

Consider Manipulator equation:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau \tag{14}$$

We will attempt to linearize it.

We begin by proposing the following new variables:

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} - \mathbf{q}_0 \\ \dot{\mathbf{q}} - \dot{\mathbf{q}}_0 \end{bmatrix}, \quad \mathbf{u} = \tau - \tau_0 \tag{15}$$

where τ_0 is chosen such that $\mathbf{C}(\dot{\mathbf{q}}_0, \mathbf{q}_0)\dot{\mathbf{q}}_0 + \mathbf{g}(\mathbf{q}_0) = \tau_0$.

Defining selector matrices $\mathbf{S}_q = [\mathbf{I} \;\; 0]$ and $\mathbf{S}_v = [0 \;\; \mathbf{I}]$ we can express conversion from the state to generalized position and velocity:

$$\mathbf{q}(\mathbf{x}) = \mathbf{S}_q\mathbf{x} + \mathbf{q}_0, \quad \dot{\mathbf{q}}(\mathbf{x}) = \mathbf{S}_v\mathbf{x} + \dot{\mathbf{q}}_0 \tag{16}$$

Manipulator equation linearization, 2

We introduce function $\phi(\dot{\mathbf{q}}, \mathbf{q}, \tau) = \ddot{\mathbf{q}}$, expressed as:

$$\phi(\dot{\mathbf{q}}, \mathbf{q}, \tau) = \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \tag{17}$$

Next, we write our dynamics as a first order ODE:

$$\frac{d}{dt}\left(\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix}\right) = \begin{bmatrix}\dot{\mathbf{q}}\\ \phi(\dot{\mathbf{q}}, \mathbf{q}, \tau)\end{bmatrix} \tag{18}$$

$$\dot{\mathbf{x}} = \varphi(\mathbf{x}, \mathbf{u}) = \begin{bmatrix}\dot{\mathbf{q}}(\mathbf{x})\\ \phi(\mathbf{x}, \mathbf{u})\end{bmatrix} \tag{19}$$

With that, we can find matrices $\mathbf{A}$ and $\mathbf{B}$.

Manipulator equation linearization, 3

In this case, state matrices $\mathbf{A}$ and $\mathbf{B}$ become:

$$\mathbf{A} = \begin{bmatrix} \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{q}} & \frac{\partial \dot{\mathbf{q}}}{\partial \dot{\mathbf{q}}} \\ \frac{\partial \phi}{\partial \mathbf{q}} & \frac{\partial \phi}{\partial \dot{\mathbf{q}}} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ \frac{\partial \phi}{\partial \mathbf{q}} & \frac{\partial \phi}{\partial \dot{\mathbf{q}}} \end{bmatrix} \tag{20}$$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial \dot{\mathbf{q}}}{\partial \tau} \\ \frac{\partial \phi}{\partial \tau} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{H}^{-1} \end{bmatrix} \tag{21}$$

Thus, our task is to find the following jacobians: $\frac{\partial \phi}{\partial \mathbf{q}}$ and $\frac{\partial \phi}{\partial \dot{\mathbf{q}}}$.

Manipulator equation linearization, 4

Let us find $\frac{\partial \phi}{\partial \mathbf{q}}$:

$$\frac{\partial \phi}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}} \left(\mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \right) = \tag{22}$$

$$= \frac{\partial \mathbf{H}^{-1}}{\partial \mathbf{q}}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \mathbf{H}^{-1} \frac{\partial}{\partial \mathbf{q}} (\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \tag{23}$$

If we evaluate $\frac{\partial \phi}{\partial \mathbf{q}}$ at the point $\mathbf{q} = \mathbf{q}_0$, $\dot{\mathbf{q}} = \dot{\mathbf{q}}_0$, $\tau = \tau_0$, we can use the fact that $\mathbf{C}(\dot{\mathbf{q}}_0, \mathbf{q}_0)\dot{\mathbf{q}}_0 + \mathbf{g}(\mathbf{q}_0) = \tau_0$ to avoid computing derivative $\frac{\partial \mathbf{H}^{-1}}{\partial \mathbf{q}}$:

$$\frac{\partial \phi}{\partial \mathbf{q}} = \mathbf{H}^{-1} \left(-\frac{\partial \mathbf{C}\dot{\mathbf{q}}}{\partial \mathbf{q}} - \frac{\partial \mathbf{g}}{\partial \mathbf{q}} \right) \tag{24}$$

Manipulator equation linearization, 5

Let us find $\frac{\partial \phi}{\partial \dot{\mathbf{q}}}$:

$$\frac{\partial \phi}{\partial \dot{\mathbf{q}}} = \frac{\partial}{\partial \dot{\mathbf{q}}} \left(\mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \right) = \tag{25}$$

$$= -\mathbf{H}^{-1} \frac{\partial \mathbf{C}\dot{\mathbf{q}}}{\partial \dot{\mathbf{q}}} \tag{26}$$

With that, we expressed all jacobians. The rest is the same as in the general case we studied in the first slides.

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025