Input Response

Control Theory, Lecture 6

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Spring 2025

Steady-State gain: ODE

Given an ODE with a constant input $c = \text{const}$:

$$a_n y^{(n)} + ... + a_1 \dot{y} + a_0 y = b_m u^{(m)} + ... + b_1 \dot{u} + b_0 u \tag{1}$$

$$u(t) = c \tag{2}$$

This is equivalent to:

$$a_n y^{(n)} + ... + a_1 \dot{y} + a_0 y = b_0 c \tag{3}$$

A steady-state solution $y_{ss} = \text{const}$:

$$a_0 y_{ss} = b_0 c \tag{4}$$

$$y_{ss} = \frac{b_0}{a_0} c \tag{5}$$

The quantity $K = \frac{b_0}{a_0}$ is the steady-state gain of the system.

Assume the system $\mathcal{G}$ represented as a transfer function:

$$Y(s) = \frac{b_m s^m + ... + b_0}{a_n s^n + ... + a_0} U(s) \tag{6}$$

Then, as any element multiplied by the differential operator s with power higher than 0 is a derivative of u or y and both are 0 at the steady-state solution, the steady-state gain can be found by setting those to zero:

$$K = \frac{b_0}{a_0} \tag{7}$$

Given an LTI with a constant input $\mathbf{u}_{ss} = \text{const}$:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{8}$$

A steady-state solution $\mathbf{x}_{ss} = \text{const}$:

$$\begin{cases} 0 = \mathbf{A}\mathbf{x}_{ss} + \mathbf{B}\mathbf{u}_{ss} \\ \mathbf{y}_{ss} = \mathbf{C}\mathbf{x}_{ss} \end{cases} \tag{9}$$

$$\mathbf{y}_{ss} = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\mathbf{u}_{ss} \tag{10}$$

The quantity $K = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is the steady-state gain of the system.

For the following LTI:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix} \mathbf{u} \\ \mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases} \tag{11}$$

Then we get:

$$\mathbf{y}_{ss} = -\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -a_1/a_0 & -a_2/a_0 & -1/a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix} \mathbf{u}_{ss} \tag{12}$$

The quantity $K = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B} = \frac{b_0}{a_0}$ is the steady-state gain of the system.

Consider LTI with input $\mathbf{u} = \alpha \cos(\omega t) + \beta \sin(\omega t)$:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{13}$$

with steady-state solution:

$$x_i = g_i \cos(\omega t) + h_i \sin(\omega t) \tag{14}$$

$$\dot{x}_i = \omega h_i \cos(\omega t) - \omega g_i \sin(\omega t) \tag{15}$$

$$y_i = q \cos(\omega t) + r \sin(\omega t) \tag{16}$$

In the vector form:

$$\begin{bmatrix} x_1 \\ .. \\ x_n \end{bmatrix} = \begin{bmatrix} g_1 \\ .. \\ g_n \end{bmatrix} \cos(\omega t) + \begin{bmatrix} h_1 \\ .. \\ h_n \end{bmatrix} \sin(\omega t) \tag{17}$$

$$\mathbf{x} = \mathbf{g} \cos(\omega t) + \mathbf{h} \sin(\omega t) \tag{18}$$

and: $\dot{\mathbf{x}} = \omega \mathbf{h} \cos(\omega t) - \omega \mathbf{g} \sin(\omega t)$

Thus:

$$\mathbf{u} = \alpha \cos(\omega t) + \beta \sin(\omega t) \tag{19}$$

$$\mathbf{x} = \mathbf{g} \cos(\omega t) + \mathbf{h} \sin(\omega t) \tag{20}$$

$$\dot{\mathbf{x}} = \omega \mathbf{h} \cos(\omega t) - \omega \mathbf{g} \sin(\omega t) \tag{21}$$

With that we can re-write $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ as:

$$\omega \mathbf{h} \cos(\omega t) - \omega \mathbf{g} \sin(\omega t) =$$
$$= \mathbf{A}\mathbf{g} \cos(\omega t) + \mathbf{A}\mathbf{h} \sin(\omega t) + \mathbf{B}\alpha \cos(\omega t) + \mathbf{B}\beta \sin(\omega t)$$

Grouping terms in front of cosines we get:

$$\omega \mathbf{h} = \mathbf{A}\mathbf{g} + \mathbf{B}\alpha \tag{22}$$

And grouping terms in front of sines :

$$-\omega \mathbf{g} = \mathbf{A}\mathbf{h} + \mathbf{B}\beta \tag{23}$$

We study the equation $\mathbf{y} = \mathbf{C}\mathbf{x}$ in the same way:

$$q\cos(\omega t) + r\sin(\omega t) = \mathbf{C}\mathbf{g}\cos(\omega t) + \mathbf{C}\mathbf{h}\sin(\omega t) \tag{24}$$

Grouping terms in front of cosines we get:

$$q = \mathbf{C}\mathbf{g} \tag{25}$$

And grouping terms in front of sines :

$$r = \mathbf{C}\mathbf{h} \tag{26}$$

We have the following equations:

$$\omega \mathbf{h} = \mathbf{A}\mathbf{g} + \mathbf{B}\alpha \tag{27}$$

$$-\omega \mathbf{g} = \mathbf{A}\mathbf{h} + \mathbf{B}\beta \tag{28}$$

$$q = \mathbf{C}\mathbf{g} \tag{29}$$

$$r = \mathbf{C}\mathbf{h} \tag{30}$$

This can be re-written in a matrix form:

$$\begin{cases} \begin{bmatrix} -\mathbf{A} & \omega \mathbf{I} \\ -\omega \mathbf{I} & -\mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ \\ \begin{bmatrix} q \\ r \end{bmatrix} = \begin{bmatrix} \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix} \end{cases} \tag{31}$$

This system can be written as:

$$\begin{cases} \begin{bmatrix} -\mathbf{A} & \omega\mathbf{I} \\ -\omega\mathbf{I} & -\mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ \\ \begin{bmatrix} q \\ r \end{bmatrix} = \begin{bmatrix} \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix} \end{cases} \tag{32}$$

Note that we can compute the steady-state solution for all states of this system:

$$\begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} -\mathbf{A} & \omega\mathbf{I} \\ -\omega\mathbf{I} & -\mathbf{A} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{33}$$

We can also solve for the steady-state output:

$$\begin{bmatrix} q \\ r \end{bmatrix} = \begin{bmatrix} \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{bmatrix} \begin{bmatrix} -\mathbf{A} & \omega\mathbf{I} \\ -\omega\mathbf{I} & -\mathbf{A} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{34}$$

The map between the input and the output as:

$$\mathbf{M}(\omega) = \begin{bmatrix} \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{bmatrix} \begin{bmatrix} -\mathbf{A} & \omega \mathbf{I} \\ -\omega \mathbf{I} & -\mathbf{A} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{bmatrix} \quad (35)$$

We can define input coordinates $\zeta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and output coordinates $\mathbf{p} = \begin{bmatrix} q \\ r \end{bmatrix} = \mathbf{M}\zeta$. The amptitude amplification can be defined as the ratio:

$$\text{amp}(\omega) = \frac{\sqrt{q^2 + r^2}}{\sqrt{\alpha^2 + \beta^2}} = \frac{||\mathbf{p}||}{||\zeta||} = \frac{||\mathbf{M}\zeta||}{||\zeta||} \quad (36)$$

Notice that by definition $\max_{\zeta} \frac{||\mathbf{M}\zeta||_2}{||\zeta||_2} = ||\mathbf{M}||_2 = \sigma_{\max}(\mathbf{M})$, where $\sigma_{\max}(\mathbf{M})$ is the largest singular value of the matrix.

The phase shift can be defined as:

$$\text{phase}(\omega) \sim \angle(\mathbf{p}) - \angle(\zeta) = \angle(\mathbf{M}\zeta) - \angle(\zeta) \tag{37}$$

But matrix $\mathbf{M}$ is a scaled orthonormal matrix:

$$\mathbf{M} = \text{amp}(\omega) \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \tag{38}$$

Thus:

$$\text{phase}(\omega) \sim \varphi = \text{atan2}(M_{21}, M_{11}) \tag{39}$$

MIMO Frequency Response, 1

Transfer function-based Bode plot relies on a SISO representation of a system. However, choosing input and output one can use it for a MIMO system as well.

State-space representation naturally points out the connection between inputs and outputs and the resulting responce. Consider a fixed input matrix $\mathbf{B} \in \mathbb{R}^{n \times 1}$; we can ask a question, what choice of the output $\mathbf{C} \in \mathbb{R}^{1 \times n}$ (assuming $||\mathbf{C}|| = 1$) produces the largest amplitude of the output signal.

From the previous slides we saw that:

$$\mathrm{amp}(\omega)\begin{bmatrix}\cos(\varphi) & -\sin(\varphi)\\ \sin(\varphi) & \cos(\varphi)\end{bmatrix} = \begin{bmatrix}\mathbf{C} & 0\\ 0 & \mathbf{C}\end{bmatrix}\begin{bmatrix}-\mathbf{A} & \omega\mathbf{I}\\ -\omega\mathbf{I} & -\mathbf{A}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{B} & 0\\ 0 & \mathbf{B}\end{bmatrix}$$

Defining:

$$\begin{bmatrix}\mathbf{P}_{11} & \mathbf{P}_{12}\\ \mathbf{P}_{21} & \mathbf{P}_{22}\end{bmatrix} = \begin{bmatrix}-\mathbf{A} & \omega\mathbf{I}\\ -\omega\mathbf{I} & -\mathbf{A}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{B} & 0\\ 0 & \mathbf{B}\end{bmatrix}, \quad \mathrm{amp}(\omega) = m \tag{40}$$

We find:

$$\begin{bmatrix}m\cos(\varphi) & -m\sin(\varphi)\\ m\sin(\varphi) & m\cos(\varphi)\end{bmatrix} = \begin{bmatrix}\mathbf{C}\mathbf{P}_{11} & \mathbf{C}\mathbf{P}_{12}\\ \mathbf{C}\mathbf{P}_{21} & \mathbf{C}\mathbf{P}_{22}\end{bmatrix} \tag{41}$$

$$m^2\cos^2(\varphi) + m^2\sin^2(\varphi) = \mathbf{C}\mathbf{P}_{11}\mathbf{P}_{11}^\top\mathbf{C}^\top + \mathbf{C}\mathbf{P}_{21}\mathbf{P}_{21}^\top\mathbf{C}^\top \tag{42}$$

$$m^2 = \mathbf{C}(\mathbf{P}_{11}\mathbf{P}_{11}^\top + \mathbf{P}_{21}\mathbf{P}_{21}^\top)\mathbf{C}^\top \tag{43}$$

Defining $\mathbf{N} = \mathbf{P}_{11}\mathbf{P}_{11}^\top + \mathbf{P}_{21}\mathbf{P}_{21}^\top$ with decomposition $\mathbf{N} = \mathbf{D}\mathbf{D}^\top$ we can find the maximum value of m by maximizing:

$$m = \max_{\mathbf{C}} \frac{\sqrt{\mathbf{C}\mathbf{D}\mathbf{D}^\top\mathbf{C}^\top}}{||\mathbf{C}||} = \max_{\mathbf{C}} \frac{||\mathbf{D}^\top\mathbf{C}^\top||}{||\mathbf{C}||} = \sigma_{\max}(\mathbf{D}) = \sqrt{\sigma_{\max}(\mathbf{N})}$$

This allows us:

- To find the highest amplification ratio in the system's state-space.
- To find the output matrix $\mathbf{C}_{\max}$ which corresponds to this "worse-case scenario"; we find it as the vector in the SVD decomposition of $\mathbf{N}$) matrix corresponding to the largest singular value.

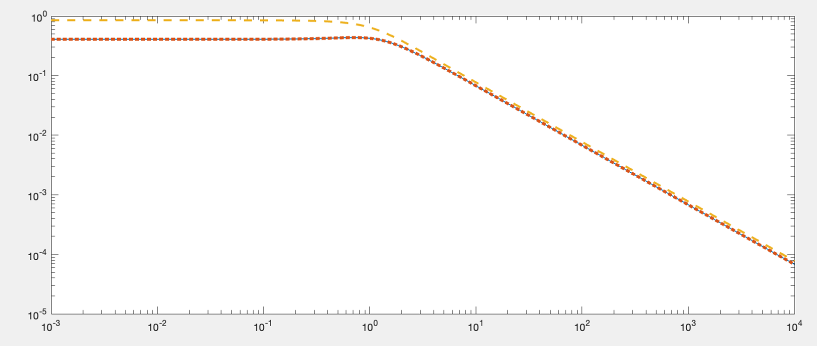
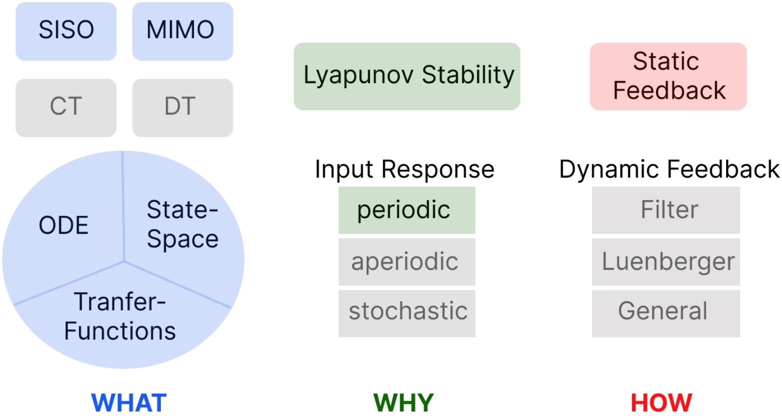


Figure 1: Dashed - worst case amplitude response, other two - a particular bode plot

Where are we


SISO
MIMO
CT
DT
ODE
State-Space
Tranfer-Functions
WHAT
Lyapunov Stability
Input Response
periodic
aperiodic
stochastic
WHY
Static Feedback
Dynamic Feedback
Filter
Luenberger
General
HOW

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025