Dynamic Controllers, Filters

Control Theory, Lecture 10

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Content

- Systems - static, dynamic
- Dynaic Controller
- PID
- Filters

A system is characterized by how it relates its input to its output.

- Controller $\mathbf{u} = \mathbf{K}\mathbf{x}$ is a system that relates $\mathbf{x}$ - its input - to $\mathbf{u}$, its output.
- Plant (in state-space representation) $\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$ relates its input $\mathbf{u}$ to its output $\mathbf{y}$.
- Plant (in Laplace representation) $Y(s) = W(s)U(s)$ with a transfer function $W(s)$ is a system relating its input signal $U(s)$ to its output signal $Y(s)$.

For some systems, the relation between input and output is proportional. Such system we can call *static*.

Examples of static systems:

- (State-Space) controller $\mathbf{u} = \mathbf{K}\mathbf{x}$.
- (Laplace) gain function $Y(s) = 10U(s)$.

Alternatively, there are linear systems for which relation between input and output depends on derivatives of the input and the output. Such system we can call *dynamic*.

Examples of dynamic systems:

- (State-Space) plant $\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$.
- (Laplace) plant $Y(s) = \frac{1}{s^2+2s+7}U(s)$.
- (ODE) plant $\ddot{y} + 5\dot{y} + y = u$.

Static vs dynamic

We can think of static systems as a form of linear algebraic equations, while dynamic systems are linear differential equations.

The distinguishing quality of dynamic systems is that they have a *state*. We can think of the state as an internal variable which changes with time, affecting the relation between the input and the output of the system.

Controller + Luenberger observer is also a dynamic system with input $\mathbf{y}$, output $\mathbf{u}$ and state $\hat{\mathbf{x}}$:

$$\begin{cases} \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \\ \mathbf{u} = -\mathbf{K}\hat{\mathbf{x}} \end{cases} \tag{1}$$

The form of this system resebles the plant with output equation.

We can think of this system as a *dynamic controller* (rather then separating it into a dynamic observer and a static controller).

A generic form of dynamic controller is:

$$\begin{cases} \dot{\hat{\mathbf{x}}} = \mathbf{A}_k \hat{\mathbf{x}} + \mathbf{B}_k \mathbf{y} \\ \mathbf{u} = \mathbf{C}_k \hat{\mathbf{x}} + \mathbf{D}_k \mathbf{y} \end{cases} \tag{2}$$

where matrices $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{C}_k$ and $\mathbf{D}_k$ can be tuned to achieve better performance.

A Luenberger observer + static controller is a particular instance of dynamic controller.

One of the better known examples of dynamic controlllers is a *proportional-integral-derivative controller (PID)*. Scaler case:

$$u(t) = k_p e(t) + k_d \frac{d}{dt} e(t) + k_i \int_0^t e(\tau) d\tau \tag{3}$$

Vector case:

$$\mathbf{u}(t) = \mathbf{K}_p \mathbf{e}(t) + \mathbf{K}_d \frac{d}{dt} \mathbf{e}(t) + \mathbf{K}_i \int_0^t \mathbf{e}(\tau) d\tau \tag{4}$$

This controller works for second-order systems (in $\mathbf{e}(t)$) where $\mathbf{e}$ and $\dot{\mathbf{e}}$ form the state of the system (or control error).

This controller is equivalent to the linear controllers we studied in the course (e.g. LQR) but with integral part allowing for *robustness* - the ability to compensate for static additive disturbance in the model.

We can re-write PID in Laplace space:

$$U(s) = \left(k_p + k_d s + k_i \frac{1}{s}\right) E(s) \tag{5}$$

$$U(s) = \frac{k_d s^2 + k_p s + k_i}{s} E(s) \tag{6}$$

It is not instantly obvious that the controller has state.

To re-write it in the state-space form, we introduce input to the controller (output to the plant) $\mathbf{y} = \begin{bmatrix} \mathbf{e} \\ \dot{\mathbf{e}} \end{bmatrix}$. We define a state $\mathbf{z} = \int_0^t \mathbf{e}(\tau) d\tau$, giving us the following state-space form of the controller:

$$\begin{cases} \dot{\mathbf{z}} = \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix} \mathbf{y} \\ \mathbf{u} = \mathbf{K}_i \mathbf{z} + \begin{bmatrix} \mathbf{K}_p & \mathbf{K}_d \end{bmatrix} \mathbf{y} \end{cases} \tag{7}$$

Consider a system with a static disturbance $\mathbf{d}$:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{d} \tag{8}$$

Applying a stabilizing linear control law $\mathbf{u} = -\mathbf{K}\mathbf{x}$ we find that the closed-loop system $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{d}$ has a steady-state solution $\mathbf{x}_s$:

$$(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}_s + \mathbf{d} = 0 \tag{9}$$

$$\mathbf{x}_s = -(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{d} \tag{10}$$

We can achieve the same result using Laplace representation:

$$\begin{cases} s\mathbf{X} = \mathbf{AX} + \mathbf{BU} + \mathbf{D} \\ \mathbf{U} = -\mathbf{KX} \end{cases} \tag{11}$$

$$s\mathbf{X} - (\mathbf{A} - \mathbf{BK})\mathbf{X} = \mathbf{D} \tag{12}$$

$$\mathbf{X} = (s\mathbf{I} - (\mathbf{A} - \mathbf{BK}))^{-1}\mathbf{D} \tag{13}$$

With a steady-state gain $-(\mathbf{A} - \mathbf{BK}))^{-1}$, same as above.

If we know static disturbance $\mathbf{d}$, we could chose a control law $\mathbf{u} = -\mathbf{K}\mathbf{x} - \mathbf{B}^{+}\mathbf{d}$ which will cancel the disturbance (if it lies in the range of the control matrix $\mathbf{B}$).

But if we don't know $\mathbf{d}$, we can use an *integral* component in a control law to compensate the disturbance:

$$\mathbf{u} = -\mathbf{K}_p\mathbf{x} - \mathbf{K}_i\int_0^t \mathbf{x}d\tau \tag{14}$$

In the Laplace domain it can be represented as:

$$\begin{cases} s\mathbf{X} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U} + \mathbf{D} \\ \mathbf{U} = -\mathbf{K}_p\mathbf{X} - \frac{1}{s}\mathbf{K}_i\mathbf{X} \end{cases} \tag{15}$$

$$\begin{cases} s\mathbf{X} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U} + \mathbf{D} \\ \mathbf{U} = -\mathbf{K}_p\mathbf{X} - \frac{1}{s}\mathbf{K}_i\mathbf{X} \end{cases} \tag{16}$$

We can write it as a transfer function:

$$s\mathbf{X} = \mathbf{A}\mathbf{X} - \mathbf{B}\mathbf{K}_p\mathbf{X} - \frac{1}{s}\mathbf{B}\mathbf{K}_i\mathbf{X} + \mathbf{D} \tag{17}$$

$$s^2\mathbf{X} = s\mathbf{A}\mathbf{X} - s\mathbf{B}\mathbf{K}_p\mathbf{X} - \mathbf{B}\mathbf{K}_i\mathbf{X} + s\mathbf{D} \tag{18}$$

$$\mathbf{X} = s\left(s^2\mathbf{I} - s(\mathbf{A} - \mathbf{B}\mathbf{K}_p) + \mathbf{B}\mathbf{K}_i\right)^{-1}\mathbf{D} \tag{19}$$

Its static-state gain is equal to 0.

Consider a system:

$$\begin{cases} \dot{x} = ax + bu \\ y = cx + d\sin(wt) \end{cases} \quad (20)$$

where $w >> 0$ is a frequency of the external disturbance and d is it amplitute.

We can first pass the measurement y through a system that attenuates high-frequency component of a signal; such system is called a *low-pass filter* and then pass it through the controller:

$$\begin{cases} \dot{z} = -z + y \\ u = kz \end{cases} \quad (21)$$

The filter $\dot{z} = -z + y$ has a Laplace form:

$$Z(s) = \frac{1}{s+1} Y(s) \tag{22}$$

Frequency responce of this transfer function is:

$$W(\omega) = \frac{1}{\sqrt{\omega^2 + 1}} Y(s) \tag{23}$$

As frequency increases, the signal is attenuated more. For frequencies close to zero, the signal passes nearly unchanged.

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025