ODE and State Space

Control Theory, Lecture 1

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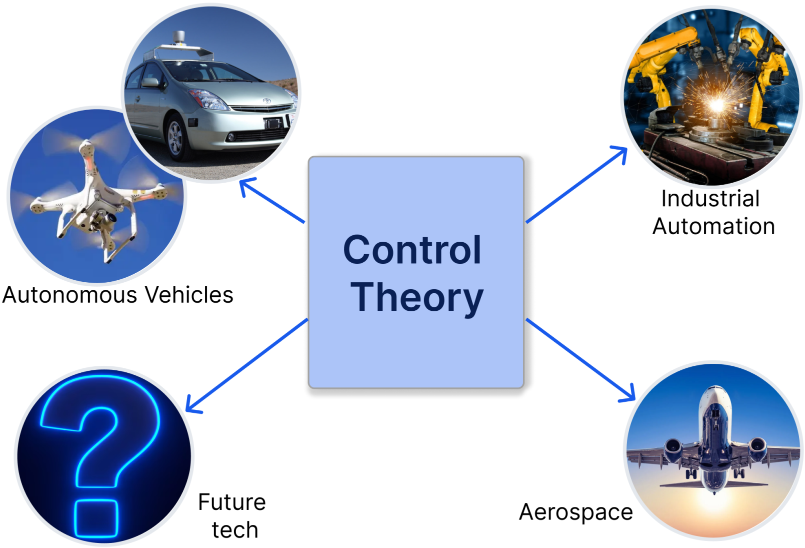
Role of Control Theory



Role of this course

We will learn:

- how to think about objects that we can control;
- what goals we can expect to achieve with control;
- how to control;
- what problems we face when working with control, and how to deal with them.

This is an introductory course, we aim at fulfilling prerequisites for all advanced courses that require basic control background (Control of Autonomous Vehicles, Nonlinear Control, etc)

Many physical objects that we control can be described using **ODE**s.

We begin this course by learning how to write ODEs in a form immideatly useful for control purposes, called **State Space** representation.

Let us remember the normal form of first-order *ordinary differential equations (ODEs):*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where $\mathbf{x} = \mathbf{x}(t)$ is the solution of the equation and t is a free variable (usually - time).

Definition

We can call an ODE a *dynamical system*, and refer to $\mathbf{x}$ as the *state* of the dynamical system.

Example

$$\dot{x} = -2x \tag{2}$$

The *state* of a dynamical system is a minimal set of variables that describes the system, in the sense that knowing the current state and all future inputs one can describe the future behavior of the system.

Example

For a spring-damper system, the state variables could be position and velocity of the mass.

Example

For a double pendulum, the state variables could be joint angles and joint velocities.

The normal form of an *n-th order* ordinary differential equation is:

$$y^{(n)} = f(y^{(n-1)}, y^{(n-2)}, \dots, \dot{y}, y, t) \tag{3}$$

where $y = y(t)$ is the solution of the equation. Same as before, it is a *dynamical system*, but this time we need more variables to describe the state of this system, for example we can use the set $\{y,\ \dot{y}, \dots, y^{(n-1)}\}$.

Example (Pendulum)

$$\ddot{y} = -0.1\dot{y} - 7\sin(y) \tag{4}$$

Example (DC motor under constant voltage)

$$\begin{cases} \dot{y}_1 = -100\dot{y}_2 - 2y_1 + 10 \\ \ddot{y}_2 = -0.1\dot{y}_2 + 100y_1 \end{cases} \tag{5}$$

A system of linear ODEs of the first order can be written as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$

Example

$$\begin{cases} \dot{x}_1 = -20x_1 + 7x_2 \\ \dot{x}_2 = 10x_1 - 3x_2 \end{cases} \tag{7}$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -20 & 7 \\ 10 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{8}$$

A single linear ODE of the n-th order is often written in the form:

$$a_n y^{(n)} + ... + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = 0 \tag{9}$$

Example

$$12\,\dddot{y} + 5\dot{y} + 2y = 0 \tag{10}$$

Example

$$5\ddot{y} - \dot{y} + 10y = 0 \tag{11}$$

ODEs with an input, 1

Sometimes it is convenient to write an ODE in the form with an *input*, for example:

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \tag{12}$$

In this equation $u(t)$ is a function of time. This form offers us many uses:

- We can use $u(t)$ to model the *control input*, (e.g. voltage, motor torque) that we directly control.
- We can use $u(t)$ to model external forces acting on the system.
- We can substitute particular function instead of $u(t)$, e.g. a sine wave or a step function, to study how the system behaves under such input.

First-order ODEs with an input

Some examples of linear ODEs with one input:

Example

$$\begin{cases} \dot{y}_1 = -20y_1 + 7y_2 + u \\ \dot{y}_2 = 10y_1 - y_2 \end{cases} \tag{13}$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -20 & 7 \\ 10 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{14}$$

General form of an n-th order linear ODE with an input can be presented as follows:

$$a_n y^{(n)} + ... + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \tag{15}$$

The state-space representation of a linear system with an input is:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{16}$$

Note that in (15) u is a scalar, while in (16) $\mathbf{u}$ can either be a scalar or a vector.

Equations with an output

Equations can also have an output. What "output" means depends on the particular use-case - it is not a mathematical issue, it is a question of interpretation. For example, an output can mean:

- What we measure (the position and orientation of a quadrotor, angular velocity of a motor's rotor, etc.).
- What we care about and/or what we want to control (the height of a quadrotor, the velocity of a car, etc.)
- etc.

We often denote output as y, and it depends on the state of the system: $y = g(\mathbf{x})$

State-space representation of a linear system with an input and an output is:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{17}$$

If $\mathbf{u} \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}$ (i.e. if they are scalars) and you want to represent the system with an output as a single ODE, it is typical to treat the output as the ODE variable:

$$a_n y^{(n)} + ... + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \tag{18}$$

In this course we will focus entirely on linear dynamical systems, expressed as ODEs:

$$a_n y^{(n)} + ... + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \tag{19}$$

or in state-space form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{20}$$

If $\mathbf{u}$ and $\mathbf{y}$ are scalars, the system is called *single-input single-output (SISO)*, if they are vectors - *multi-input multi-output (MIMO)*.

We can always express a SISO system in either form - ODE or state-space.

Consider eq. $\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = u$.

Make a substitution: $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$. We get:

$$\dot{x}_1 = \dot{y} = x_2 \tag{21}$$

$$\dot{x}_2 = \ddot{y} = x_3 \tag{22}$$

$$\dot{x}_3 = \dddot{y} = u - a_2\ddot{y} - a_1\dot{y} - a_0 y = u - a_2 x_3 - a_1 x_2 - a_0 x_1 \tag{23}$$

Which can be directly put in the state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \tag{24}$$

Literature

- Nise, N.S. Control systems engineering. John Wiley & Sons. (Chapter 3 Modeling in Time Domain)
- 2.14 Analysis and Design of Feedback Control Systems:
  - State-Space Representation of LTI Systems
  - Time-Domain Solution of LTI State Equations
- Linear Physical Systems Analysis:
  - State Space Representations of Linear Physical Systems lpsa.swarthmore.edu/Representations/SysRepSS.html
  - Transformation: Differential Equation to State Space lpsa.swarthmore.edu/.../DE2SS.html

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025

State Space to ODE

Check out the code implementation.