Stabilizing Control

Control Theory, Lecture 3

by Sergei Savin

Spring 2025

Changing stability

Here are two LTIs:

$$\dot{x} = 2x \tag{1}$$

$$\dot{x} = 2x + u \tag{2}$$

First one is autonomous and unstable. Second one is not autonomous, and we won't know whether or not the solution converges to zero, until we know what u is.

If we pick $u = 0$, the result is an unstable equation. But we can also pick u such that the resulting dynamics is stable, such as $u = -3x$:

$$\dot{x} = 2x + u = 2x - 3x = -x \tag{3}$$

So, we can use *control input* u to change stability of the system!

Definition

The problem of finding control law $\mathbf{u}$ that make a certain solution $\mathbf{x}^*$ of dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ stable is called *stabilizing control problem*

This is true for both linear and non-linear systems. But for linear systems we can get a lot more details about this problem, if we restrict our choice of control law.

Consider an LTI system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{4}$$

and let us chose *control as a linear function of the state* x:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \tag{5}$$

We call matrix **K** *control gain*. Thus, we know how the system is going to look when the control is applied:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} \tag{6}$$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \tag{7}$$

Note that (7) is an autonomous system. We call this a *closed loop* system.

We can analyse stability of $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$:

Stability condition for LTI closed-loop system

The real parts of the eigenvalues of the matrix $(\mathbf{A} - \mathbf{B}\mathbf{K})$ should be negative for asymptotic stability, or non-positive for stability in the sense of Lyapunov.

Hurwitz matrix

If square matrix $\mathbf{M}$ has eigenvalues with strictly negative real parts, it is called Hurwitz. We will denote it as $\mathbf{M} \in \mathcal{H}$.

So, all you need to do is to find such $\mathbf{K}$ that $(\mathbf{A} - \mathbf{B}\mathbf{K})$ is Hurwitz, and you made a an asymptotically stable closed-loop system!

Let us consider the following system:

$$\dot{x} = ax + bu \tag{8}$$

we can choose the following linear control law: $u = -kx$. The close loop system for this example is:

$$\dot{x} = (a - bk)x \tag{9}$$

The solution to the closed-loop system is:

$$x(t) = x_0 e^{(a-bk)t} \tag{10}$$

As long as $a - bk < 0$, the solution is converging to zero. Since we can pick k, we can choose it so that $a - bk = -q$, where q is a positive number. Then, we pick $k = \frac{q+a}{b}$, giving us stable system with eigenvalue $-q$.

Let us consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u \tag{11}$$

With control law:

$$u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{12}$$

Close-loop system is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} - bk_1 & a_{12} - bk_2 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{13}$$

The eigenvalues of the closed-loop system are $a_{11} - bk_1$ and a_{22}. The second eigenvalue cannot be influenced by the choice of control gains. If $a_{22} < 0$, we need to pick k_1, such as $a_{11} - bk_1 = -q$, where q is a positive number: $k_1 = \frac{q + a_{11}}{b}$.

Let us consider a spring-mass-dumper:

$$\ddot{y} + \mu\dot{y} + cy = u \tag{14}$$

We can propose the feedback control in the form:

$$u = -k_d\dot{y} - k_p y \tag{15}$$

this is called a *proportional-differential controler*, often shortened as *PD controller*; k_d is a differential coefficient and k_p is a proportional coefficient. The closed-loop system is:

$$\ddot{y} + (\mu + k_d)\dot{y} + (c + k_p)y = 0 \tag{16}$$

In state-space form it is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\mu + k_d) & -(c + k_p) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{17}$$

The stability of the system depends on the eigenvalues of the state matrix of the closed-loop system:

$$\begin{bmatrix} 0 & 1 \\ -(\mu + k_d) & -(c + k_p) \end{bmatrix} \tag{18}$$

It is easy compute eigenvalues of a 2 by 2 matrix, using its determinant det and trace tr:

$$\lambda = \frac{1}{2}\left(\mathrm{tr} \pm \sqrt{\mathrm{tr}^2 - 4\mathrm{det}}\right) \tag{19}$$

In this case $\mathrm{det} = (\mu + k_d)$ and $\mathrm{tr} = -(c + k_p)$.

$$\lambda = \frac{1}{2}\left(-(c + k_p) \pm \sqrt{(c + k_p)^2 - 4(\mu + k_d)}\right) \tag{20}$$

We define $d = \sqrt{(c + k_p)^2 - 4(\mu + k_d)}$:

$$\lambda = \frac{1}{2} \left(-(c + k_p) \pm d \right) \tag{21}$$

- If $\mu + k_d < 0$ then $(c + k_p)^2 - 4(\mu + k_d) > 0$ and $d > (c + k_p)$, so one of the eigenvalues is positive.
- If $\mu + k_d = 0$ then $d = (c + k_p)$, so one of the eigenvalues is zero.
- If $\mu + k_d > 0$ then it is either complex or $0 < d < (c + k_p)$ if it is real;
  - if $c + k_p < 0$ then one of the eigenvalues is positive.
  - if $c + k_p = 0$ then $\lambda = \pm i\sqrt{(\mu + k_d)}$
  - if $c + k_p > 0$ then both eigenvalues are have negative real parts.

$$\ddot{y} + (\mu + k_d)\dot{y} + (c + k_p)y = 0 \tag{22}$$

- Note that the asymptotic stability is achieved when $\mu + k_d > 0$ and $c + k_p > 0$.
- If $\mu + k_d \geq 0$ and $c + k_p \geq 0$ then the system is marginally stable.
- if either $\mu + k_d < 0$ or $c + k_p < 0$ the system is unstable.

Given $c = 8$ and $\mu = 40$, assume that we want the closed-loop system to have eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -20$.

$$16 = \lambda_1 - \lambda_2 = \frac{1}{2}(-(c + k_p) + d) - \frac{1}{2}(-(c + k_p) - d) = d \quad (23)$$

It follows that:

$$-4 = \lambda_1 = \frac{1}{2}(-(c + k_p) + 16) \quad (24)$$

$$-(c + k_p) + 16 = -8 \quad (25)$$

$$k_p = 16 \quad (26)$$

Also we can write:

$$d = \sqrt{(c + k_p)^2 - 4(\mu + k_d)} \quad (27)$$

$$16^2 = 24^2 - 4(40 + k_d) \quad (28)$$

$$k_d = 320/4 - 40 = 40 \quad (29)$$

The method of finding control gains in such a way that the closed-loop system has desired eigenvalues is called *pole placement*.

As the earlier example illustrated, it is not easy to do manually. However, there is software that finds such control gains automatically.

In MATLAB there is a function `K = place(A,B,p)`, where `p` is teh desired aigenvalues of `(A-B*K)`.

Let the function $\mathbf{x}^* = \mathbf{x}^*(t)$ and control $\mathbf{u}^* = \mathbf{u}^*(t)$ be a solution to the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, meaning:

$$\dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* \tag{30}$$

We call $\mathbf{x}^*(t)$ a *reference* or *reference input* and $\mathbf{u}^*(t)$ a *feed-forward control.*

We can try to find control law that would stabilize this reference trajectory. We begin by finding the difference between $\dot{\mathbf{x}}^*$ and $\dot{\mathbf{x}}$:

$$\dot{\mathbf{x}}^* - \dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}^* - \mathbf{x}) + \mathbf{B}(\mathbf{u}^* - \mathbf{u}) \tag{31}$$

We define new variables: $\mathbf{e} = \mathbf{x}^* - \mathbf{x}$ and $\mathbf{v} = \mathbf{u}^* - \mathbf{u}$:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \tag{32}$$

Trajectory tracking (2)

We call $\mathbf{e}$ *control error* and the equation $\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v}$ is *error dynamics*.

With that we are back to the familiar problem - find control law $\mathbf{v} = -\mathbf{K}\mathbf{e}$ that makes closed-loop system stable:

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{e} \tag{33}$$

In the original variables it is:

$$\mathbf{u} = \mathbf{K}(\mathbf{x}^* - \mathbf{x}) + \mathbf{u}^* \tag{34}$$

Consider the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and the reference input $\mathbf{x}^* = \text{const}$ and feed-forward control $\mathbf{u}^* = \text{const}$. This implies:

$$\mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* = 0 \tag{35}$$

We can try to find control law that would stabilize this reference trajectory. The error dynamics and the stabilizing control law are the same as in the previous case. But this time, we can find $\mathbf{u}^*$ if it is not provided:

$$\mathbf{u}^* = -\mathbf{B}^+\mathbf{A}\mathbf{x}^* \tag{36}$$

New input

Consider the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and control law $\mathbf{u} = \mathbf{K}(\mathbf{x}^*(t) - \mathbf{x}) + \mathbf{u}^*(t)$. We can find the expression for the resulting system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{K}(\mathbf{x}^*(t) - \mathbf{x}) + \mathbf{B}\mathbf{u}^*(t) \tag{37}$$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{x}^*(t) + \mathbf{B}\mathbf{u}^*(t) \tag{38}$$

Assuming that $\mathbf{u}^*(t) = 0$ gives us a simplified system:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{x}^*(t) \tag{39}$$

Here we can see that $\mathbf{x}^*(t)$ acts as a new input, and it makes sense to discuss how the system reacts to various inputs.

Read more

- Richard M. Murray Control and Dynamical Systems California Institute of Technology Optimization-Based Control
- Dynamic Simulation in Python

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025