Frequency response, Bode

Control Theory, Lecture 5

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- Frequency response
- Partial-fraction expansion with sine input
- Amplitude and phase shift of a steady-state solution
- Bode plot

Sine wave input

Consider a sine wave with a phase shift. It can be presented in these two forms:

$$u(t) = A\sin(\omega t) + B\cos(\omega t) = \tag{1}$$
$$= M\sin(\omega t + \varphi) \tag{2}$$

where $M = \sqrt{A^2 + B^2}$ is the amplitude of the signal and $\varphi = -\tan^{-1}\left(\frac{B}{A}\right)$ is the phase shift.

Consider Laplace transforms:

$$\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2 + \omega^2} \tag{3}$$
$$\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2 + \omega^2} \tag{4}$$
$$\mathcal{L}(A\sin(\omega t) + B\cos(\omega t)) = \frac{A\omega + Bs}{s^2 + \omega^2} \tag{5}$$

Given an ODE with a sine input:

$$a_n y^{(n)} + ... + a_1 \dot{y} + a_0 y = b_m u^{(m)} + ... + b_1 \dot{u} + b_0 u \tag{6}$$

$$u(t) = A \sin(\omega t) \tag{7}$$

we can find its Laplace transform:

$$(a_n s^n + ... + a_1 s + a_0)Y(s) = (b_m s^m + ... + b_1 s + b_0)U(s) \tag{8}$$

$$U(s) = \frac{A\omega}{s^2 + \omega^2} \tag{9}$$

We can find its Laplace representation:

$$Y(s) = \frac{b_m s^m + ... + b_1 s + b_0}{a_n s^n + ... + a_1 s + a_0} \cdot \frac{A\omega}{s^2 + \omega^2} \tag{10}$$

Frequency response

Frequency response is a steady-state output of the system, given a sine input.

Consider a system $Y(s) = G(s)U(s)$.
A sine input $u(t) = A\sin(\omega t)$ in the time domain translates to $U(s) = A\frac{\omega}{\omega^2+s^2}$ in the Laplace domain. So, given a sine input, the system becomes:

$$Y(s) = G(s)\frac{A\omega}{\omega^2 + s^2} \tag{11}$$

Assuming negative real non-repeating poles we can expand the function $Y(s) = G(s)\frac{A\omega}{\omega^2+s^2}$:

$$G(s)\frac{A\omega}{\omega^2+s^2} = \frac{r_1}{s+p_1} + \frac{r_2}{s+p_2} + ... + \frac{r_n}{s+p_n} + \frac{\alpha(s)}{s+j\omega} + \frac{\beta(s)}{s-j\omega}$$

Laplace function of the form $\frac{r_i}{s+p_i}$ corresponds to the following time function:

$$y(t) = r_i e^{-p_i t} \tag{12}$$

So, for a stable transfer function $G(s)$ as time goes to infinity, $r_i e^{-p_i t}$ goes to zero. The only components of the function $Y(s)$ that do not disappear are the last two: $\frac{\alpha(s)}{s+j\omega} + \frac{\beta(s)}{s-j\omega}$.

Fraction expansion, 2

To find $\alpha(s)$ we miltiply the equation by $s + j\omega$:

$$G(s)\frac{A\omega(s+j\omega)}{(s+j\omega)(s-j\omega)} =$$
$$= \left(\frac{r_1}{s+p_1} + ... + \frac{r_n}{s+p_n}\right)(s+j\omega) + \alpha(s) + \frac{\beta(s)(s+j\omega)}{s-j\omega}$$

Considering $s = -j\omega$ we get:

$$\alpha = G(-j\omega)\frac{A}{-2j} \tag{13}$$

To find $\beta(s)$ we multiply the decomposition equation by $s - j\omega$ and then consider $s = j\omega$:

$$\beta = G(j\omega)\frac{A}{2j} \tag{14}$$

Laplace image of the steady-state solution is:

$$Y_{ss}(s) = \frac{\alpha(s)}{s + j\omega} + \frac{\beta(s)}{s - j\omega} \tag{15}$$

$$Y_{ss}(s) = \frac{A}{2j} \left(G(j\omega) \frac{1}{s - j\omega} - G(-j\omega) \frac{1}{s + j\omega} \right) \tag{16}$$

Inverse Laplace transform gives us:

$$y_{ss}(t) = \frac{A}{2j} \left(G(j\omega) e^{j\omega t} - G(-j\omega) e^{-j\omega t} \right) \tag{17}$$

A complex function $W(x)$ can be represented in a polar form:

$$W(x) = r(x)e^{\theta(x)i} \tag{18}$$

where

$$r(x) = |W(x)| = \text{amp}(W(x)) \tag{19}$$

$$\theta(x) = \text{phase}(W(x)) = \text{atan2}(\text{Im}(W(x)),\ \text{Re}(W(x))) \tag{20}$$

We observe the following conjugate identities:

$$|W(jx)| = |W(-jx)| \tag{21}$$

$$\text{phase}(W(-jx)) = -\text{phase}(W(jx)) \tag{22}$$

Finally, we can represent a sine function as a difference of exponentials:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \tag{23}$$

Amplitude and phase, 1

We can define polar coordinates representation:

$$G(j\omega) = r(\omega)e^{\theta(\omega)j} \tag{24}$$

$$G(-j\omega) = r(\omega)e^{-\theta(\omega)j} \tag{25}$$

where $r(\omega) = |G(j\omega)|$ and
$\theta(\omega) = \text{atan2}(\text{Im}(G(j\omega)),\ \text{Re}(G(j\omega)))$.

We can re-write the steady-state solution
$y_{ss}(t) = \frac{A}{2j}\left(G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}\right)$ as:

$$y_{ss}(t) = \frac{A}{2j}\left(r(\omega)e^{j\theta(\omega)}e^{j\omega t} - r(\omega)e^{-j\theta(\omega)}e^{-j\omega t}\right) \tag{26}$$

$$y_{ss}(t) = \frac{Ar(\omega)}{2j}\left(e^{j\theta(\omega)+j\omega t} - e^{-(j\theta(\omega)+j\omega t)}\right) \tag{27}$$

$$y_{ss}(t) = Ar(\omega)\sin(\omega t + \theta(\omega)) \tag{28}$$

Thus we found the steady-state output for the system:

$$y_{ss}(t) = Ar(\omega)\sin(\omega t + \theta(\omega)) \tag{29}$$

Let us find the ratio between amplitude of the input and the output systems:

$$\text{amplification}(\omega) = \frac{Ar(\omega)}{A} = r(\omega) = |G(j\omega)| \tag{30}$$

Let us find the phase shift between the input and the output systems:

$$\text{phase}(\omega) = \theta(\omega) = \text{atan2}(\text{Im}(G(j\omega)),\ \text{Re}(G(j\omega))) \tag{31}$$

The first key idea of a Bode plot is substitution of purely complex variable $j\omega$ in place of Laplace variable s, which can have non-zero real part.

Given a transfer function $W(s)$, $s = \sigma + j\omega$ we can analyse its behaviour when $\sigma = 0$. We can plot:

- its amplitude $a(\omega) = |W(j\omega)|$,
- its phase $\varphi(\omega) = \mathrm{atan2}(\mathrm{im}(W(j\omega)),\ \mathrm{real}(W(j\omega)))$.

Bode plot is actually two plots:

1. $20 \cdot \log(a(\omega))$,
2. $\frac{180}{\pi}\varphi(\omega)$.

The 20 and log() has to do with the vertical axis being in decibels.

Consider $W(s) = \frac{1}{1+s}$. Then $W(j\omega) = \frac{1}{1+j\omega}$. We can transform it as:

$$W(j\omega) = \frac{1 - j\omega}{(1 + j\omega)(1 - j\omega)} = \frac{1 - j\omega}{1 + \omega^2} \tag{32}$$

Thus we have $\text{real}(W(j\omega)) = \frac{1}{1+\omega^2}$ and $\text{im}(W(j\omega)) = -\frac{\omega}{1+\omega^2}$.

Bode plot is then given as:

$$a(\omega) = \sqrt{\frac{1 + \omega^2}{(1 + \omega^2)^2}} = \frac{1}{\sqrt{(1 + \omega^2)}} \tag{33}$$

$$\varphi(\omega) = \text{atan2}\left(-\frac{\omega}{1 + \omega^2},\ \frac{1}{1 + \omega^2}\right) \tag{34}$$

Before we discuss the use of Bode plot, let us remember that closed-loop transfer function has form (when simple feedback is used):

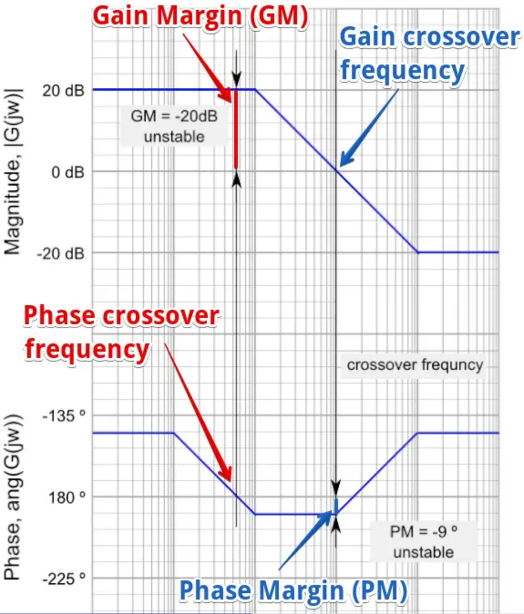
$$W(s) = \frac{G(s)}{1 + G(s)} \tag{35}$$

Substituting $s \longrightarrow j\omega$ we get:

$$W(\omega) = \frac{G(j\omega)}{1 + G(j\omega)} \tag{36}$$

From this we can see that $W(\omega)$ becomes ill-defined if $G(j\omega) = -1$. Meaning, we want to avoid two things happening simultaneously: the amplitude of $G(j\omega)$ being equal to 1, and its phase (argument) being equal to $180°$ (remember, phase of $0°$ is pure positive real number, phase of $90°$ is pure positive imaginary number, $180°$ is pure negative real number, etc.).

Stability margins - example



Consider a system with resonance:

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u} \\ \mathbf{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{cases} \tag{37}$$

The state matrix of this system has eigenvalues $\lambda_i = \pm j$ with resonance frequency $\omega = 1$.

The eigenvalues of the closed-loop system $\mathbf{A} - \mathbf{BC}$ are $\lambda_i = \pm\sqrt{2}j$ with resonance frequency $\omega = \sqrt{2}$.

Bode plot with resonance, 2

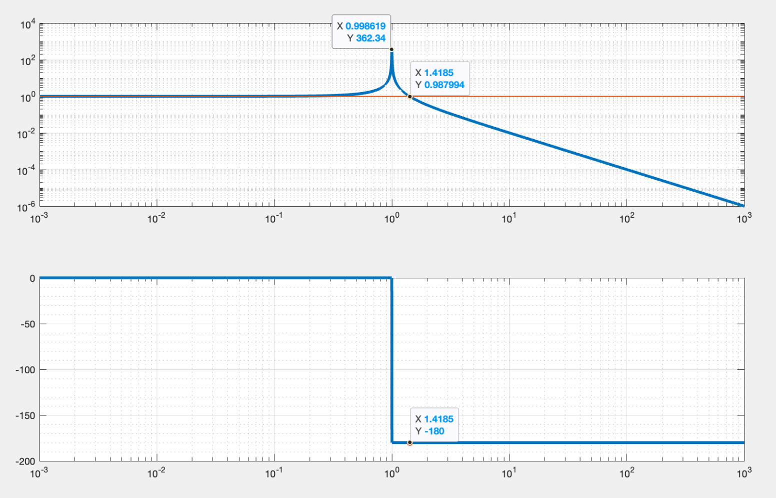


Figure 2: Bode plot of a system with resonance

Literature

- Nise, N.S. Control systems engineering. John Wiley & Sons. (Chapter 10 Frequency Response Techniques)
- Matthew M. Peet; Systems Analysis and Control - Lecture 18: The Frequency Response
- Control System Lectures - Bode Plots, Introduction
- Oxford University Press. s-Domain analysis: poles, zeros, and Bode plots

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025

Laplace and Fourier transforms

- *Fourier series* can be seen as representing a periodic function as a sum of harmonics (sines and cosines). These sines and cosines can be thought of as forming a basis in a linear space. The coefficients of the series can be thought of as a discrete spectrum of the function.
- *Fourier transform* gives a continuous spectrum of the function. The "basis" is still made of harmonic functions.
- *Laplace transform* also gives a continuous spectrum of the function, but in a different basis: the basis is given by complex exponentials. I like to think of this basis as solutions of second order ODEs.

Let's compare. Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi j t \omega} dt, \quad \omega \in \mathbb{R} \tag{38}$$

Laplace transform:

$$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt, \quad s \in \mathbb{C} \tag{39}$$

We can see that Fourier looks like Laplace with purely imaginary number in the exponent.

From analysing solutions of linear ODEs we know that, given harmonic input (sine, cosine, their combination) "after the transient process is over, the solution approaches a harmonic with the same frequency", but possibly different amplitude and phase.

Intuitively we can think of the imaginary part of s as having to do with this frequency response.

The kernel function of the Laplace transform is e^{-st} with $s = \sigma + j\omega$ being a complex variable. If $\sigma = 0$, the kernel becomes $e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$. You can see the similarity with Fourier transform kernel.