Controllability, Observability

Control Theory, Lecture 11

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Definition (Controllability)

A system is controllable on time interval $t_0 \le t \le t_f$, if it is possible to find control input $u(t)$ that would drive the system to a desired state $\mathbf{x}(t_f)$ from any initial state $\mathbf{x}(t_0)$.

Definition (Observability)

A system is observable on time interval $t_0 \le t \le t_f$, if using output $\mathbf{y}(t)$ on that time interval it is possible to estimate exactly the state of the system $\mathbf{x}(t_f)$, given any initial estimation error.

Definition (Observability, alternative)

A system is observable on time interval $t_0 \le t \le t_f$, if any initial state $\mathbf{x}(t_0)$ is uniquely determined by output $\mathbf{y}(t)$ on that interval.

Consider discrete LTI:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \tag{1}$$

Assume the initial state is $\mathbf{x}_1$. Then we can deduce that:

$$\begin{aligned}
&\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1 \\
&\mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 + \mathbf{B}\mathbf{u}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2 \\
&\mathbf{x}_4 = \mathbf{A}\mathbf{x}_3 + \mathbf{B}\mathbf{u}_3 = \mathbf{A}(\mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2) + \mathbf{B}\mathbf{u}_3 \\
&\dots \\
&\mathbf{x}_{n+1} = \mathbf{A}^n\mathbf{x}_1 + \mathbf{A}^{n-1}\mathbf{B}\mathbf{u}_1 + \dots + \mathbf{A}^{n-k}\mathbf{B}\mathbf{u}_k + \dots + \mathbf{A}\mathbf{B}\mathbf{u}_{n-1} + \mathbf{B}\mathbf{u}_n
\end{aligned}$$

$\mathbf{x}_{n+1} = \mathbf{A}^n\mathbf{x}_1 + \mathbf{A}^{n-1}\mathbf{B}\mathbf{u}_1 + ... + \mathbf{A}^{n-k}\mathbf{B}\mathbf{u}_k + ... + \mathbf{A}\mathbf{B}\mathbf{u}_{n-1} + \mathbf{B}\mathbf{u}_n$ can be re-written as:

$$\mathbf{x}_{n+1} - \mathbf{A}^n\mathbf{x}_1 = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & ... & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} \begin{bmatrix}\mathbf{u}_n \\ \mathbf{u}_{n-1} \\ \mathbf{u}_{n-2} \\ ... \\ \mathbf{u}_1\end{bmatrix} \tag{2}$$

Notice that in order for the system to go from $\mathbf{x}_1$ to $\mathbf{x}_{n+1}$, vector $\mathbf{x}_{n+1} - \mathbf{A}^n\mathbf{x}_1$ needs be in the column space of $\mathcal{C} = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & ... & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$.
Since $\mathbf{x}_{n+1}$ can be anything, and $\mathbf{x}_1$ might be equal to zero (among other possibilities), we should require that all vectors in $\mathbb{R}^n$ are in the column space of $\mathcal{C}$, meaning $\mathcal{C}$ needs to be full row rank.

Controllability

The system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$, $\mathbf{x} \in \mathbb{R}^n$ is *controllable* if its controllability matrix $\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & ... & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$ is full row-rank ($\text{rank}(\mathcal{C}) = n$).

Equation $\det(\mathbf{M} - \lambda \mathbf{I}) = 0$ is called *characteristic equation* of matrix $\mathbf{M}$, its roots being eigenvalues of the matrix.

Theorem (Cayley–Hamilton)

A matrix $\mathbf{M} \in \mathbb{R}^{n,n}$ satisfies its own characteristic equation.

A characteristic equation can be written as $\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_0 = 0$, meaning that we can write:

$$\mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + ... + a_1\mathbf{M} + a_0\mathbf{I} = 0 \tag{3}$$

Meaning that $\mathbf{M}^n$ is a linear combination of $\mathbf{M}^{n-1}$, $\mathbf{M}^{n-2}$, ..., $\mathbf{I}$. See Appendix.

What happens if we add more columns to the controllability matrix, for example $\mathbf{A}^n\mathbf{B}$? Consider the matrix:

$$\mathcal{C}_+ = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & ... & \mathbf{A}^{n-1}\mathbf{B} & \mathbf{A}^n\mathbf{B}\end{bmatrix} \tag{4}$$

But from Cayley–Hamilton we know that:

$$\mathbf{A}^n = -a_{n-1}\mathbf{A}^{n-1} - ... - a_0\mathbf{I} \tag{5}$$

$$\mathbf{A}^n\mathbf{B} = -a_{n-1}\mathbf{A}^{n-1}\mathbf{B} - ... - a_0\mathbf{B} \tag{6}$$

Meaning that columns of $\mathbf{A}^n\mathbf{B}$ are expressed as linear combination of columns of $\mathcal{C}$, hence the matrix $\mathcal{C}_+$ has the same rank as $\mathcal{C}$.

Observability of Discrete LTI

Consider discrete LTI:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i \end{cases} \tag{7}$$

And an observer:

$$\hat{\mathbf{x}}_{i+1} = \mathbf{A}\hat{\mathbf{x}}_i + \mathbf{B}\mathbf{u}_i + \mathbf{L}(\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_i) \tag{8}$$

Remember that we can define observation error $\mathbf{e}_i = \hat{\mathbf{x}}_i - \mathbf{x}_i$ and write its dynamics:

$$\mathbf{e}_{i+1} = \mathbf{A}\mathbf{e}_i - \mathbf{L}\mathbf{C}\mathbf{e}_i \tag{9}$$

Dual system (which is stable if and only if the original is stable), has form:

$$\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i \tag{10}$$

Dynamical system $\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i$, we can be represented as:

$$\begin{cases} \varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i + \mathbf{C}^\top \mathbf{v}_i \\ \mathbf{v}_i = -\mathbf{L}^\top \varepsilon_i \end{cases} \tag{11}$$

Controllability matrix of this system is:

$$\mathcal{O}^\top = \begin{bmatrix} \mathbf{C}^\top & (\mathbf{A}^\top)\mathbf{C}^\top & ... & (\mathbf{A}^\top)^{n-1}\mathbf{C}^\top \end{bmatrix} \tag{12}$$

It is easier to represent this matrix in its transposed form:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ ... \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{13}$$

Observability

The system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$ and $\mathbf{y}_i = \mathbf{C}\mathbf{x}_i$, $\mathbf{x} \in \mathbb{R}^n$ is *observable*, if the observability matrix $\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ ... \\ \mathbf{CA}^{n-1} \end{bmatrix}$ is full column-rank ($\text{rank}(\mathcal{O}) = n$).

Matrix exponential $e^{\mathbf{A}t}$ is defined as a series:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + ... \tag{14}$$

Using Cayley–Hamilton we can observe that any powers of $\mathbf{A}$ higher than n can be represented as a linear combination of lower powers. This gives us the following expression:

$$e^{\mathbf{A}t} = \phi_0(t)\mathbf{I} + \phi_1(t)\mathbf{A} + \phi_2(t)\mathbf{A}^2 + ... + \phi_{n-1}(t)\mathbf{A}^{n-1} \tag{15}$$

This allows us to re-write the forced state response:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t (\phi_0(t-\tau)\mathbf{I} + \phi_1(t-\tau)\mathbf{A} + ... \\ + \phi_{n-1}(t-\tau)\mathbf{A}^{n-1})\mathbf{b}u(\tau)\, d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \phi_0(t-\tau)\mathbf{b}u(\tau)d\tau + \int_0^t \phi_1(t-\tau)\mathbf{A}\mathbf{b}u(\tau)d\tau + ... \int_0^t \phi_{n-1}(t-\tau)\mathbf{A}^{n-1}\mathbf{b}u(\tau)d\tau$$

$$\mathbf{x}(t) - e^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & ... & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} \begin{bmatrix} \int_0^t \phi_0(t-\tau)u(\tau)d\tau \\ \int_0^t \phi_1(t-\tau)u(\tau)d\tau \\ ... \\ \int_0^t \phi_{n-1}(t-\tau)u(\tau)d\tau \end{bmatrix}$$

If the controllability matrix is rank-deficient, there will exist a state $\mathbf{x}_f$ and which cannot be reached from some initial conditions $\mathbf{x}_0$.

PBH controllability criterion

There is an alternative way to test the controllability of a pair $(\mathbf{A},\ \mathbf{B})$:

PBH controllability criterion

If for any $\lambda \in \mathbb{C}$, the matrix $[(\mathbf{A} - \lambda \mathbf{I}), \mathbf{B}]$ has full row-rank, then the pair $(\mathbf{A},\ \mathbf{B})$ is controllable.

- If λ is not an eigenvalue of $\mathbf{A}$, then $\det(\mathbf{A} - \lambda \mathbf{I}) \neq 0$ and the matrix has full row rank.
- If $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ it is sufficient to test the rank of $[(\mathbf{A} - \lambda \mathbf{I}), \mathbf{B}]$.

Read more

- Controllability and Observability (Rutgers University)
  https://www.ece.rutgers.edu/ gajic/psfiles/chap5.pdf

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025

Appendix A: Analytical solution (recap)

Exponential e^a is defined as a series:

$$e^a = 1 + a + \frac{1}{2}a^2 + \frac{1}{6}a^3 + ... = \sum_{n=0}^{\infty} \frac{1}{n!} a^n \tag{16}$$

Matrix exponential $e^{\mathbf{A}}$ is defined as a series:

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}\mathbf{A} + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A} + ... = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n \tag{17}$$

Analytical solution to ODE

An ODE of the form $\dot{x} = ax$ has analytical solution $x(t) = e^{at}x(0)$.

An ODE of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ has analytical solution $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$.

Let us check that this is a solution:

$$\mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + ...\right)\mathbf{x}(0) \tag{18}$$

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} + \mathbf{A}\mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}\mathbf{A}t^2 + ...\right)\mathbf{x}(0) \tag{19}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + ...\right)\mathbf{x}(0) \tag{20}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) \tag{21}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \qquad \square \tag{22}$$

An ODE of the form $\dot{x} = ax + bu(t)$ also has analytical solution. To find it, we first find the following derivative:

$$\frac{d}{dt}\left(e^{-at}x(t)\right) = e^{-at}\dot{x}(t) - ae^{-at}x(t) \tag{23}$$

Multiplying $\dot{x} = ax + bu(t)$ by e^{-at} we see:

$$e^{-at}\dot{x} = e^{-at}ax + e^{-at}bu(t) \tag{24}$$

$$e^{-at}\dot{x} - e^{-at}ax = e^{-at}bu(t) \tag{25}$$

$$\frac{d}{dt}\left(e^{-at}x(t)\right) = e^{-at}bu(t) \tag{26}$$

$$\int_0^t \frac{d}{d\tau}\left(e^{-a\tau}x(\tau)\right)d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \tag{27}$$

Forced State Response (LTI) (2)

Continuing the derivation:

$$\int_0^t \frac{d}{d\tau}\left(e^{-a\tau}x(\tau)\right)d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \quad (28)$$

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau \quad (29)$$

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau}bu(\tau)d\tau \quad (30)$$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \quad (31)$$

Forced State Response (LTI) (3)

State-space equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t)$ also has an analytical solution:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{32}$$

The same can be re-written as:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{33}$$

Consider matrix $\mathbf{M}$ and its characteristic equation $\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_0 = 0$ and a decomposition $\mathbf{M} = \mathbf{V}^{-1}\Lambda\mathbf{V}$. Let's prove that the following expression is zero:

$$\mathbf{E} = \mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + ... + a_1\mathbf{M} + a_0\mathbf{I} \tag{34}$$

$$\mathbf{E} = \mathbf{V}^{-1}\Lambda^n\mathbf{V} + a_{n-1}\mathbf{V}^{-1}\Lambda^{n-1}\mathbf{V} + ... + a_1\mathbf{V}^{-1}\Lambda\mathbf{V} + a_0\mathbf{I} \tag{35}$$

$$\mathbf{V}\mathbf{E}\mathbf{V}^{-1} = \Lambda^n + a_{n-1}\Lambda^{n-1} + ... + a_1\Lambda + a_0\mathbf{V}\mathbf{V}^{-1} \tag{36}$$

$$\mathbf{V}\mathbf{E}\mathbf{V}^{-1} = \begin{bmatrix} (\lambda_1^n + a_{n-1}\lambda_1^{n-1} + ... + a_0) & & \\ & ... & \\ & & (\lambda_n^n + a_{n-1}\lambda_n^{n-1} + ... + a_0) \end{bmatrix} \tag{37}$$

$$\mathbf{V}\mathbf{E}\mathbf{V}^{-1} = \begin{bmatrix} 0 & & \\ & ... & \\ & & 0 \end{bmatrix} \tag{38}$$

$$\mathbf{V}\mathbf{E}\mathbf{V}^{-1} = 0 \tag{39}$$

$$0 = \mathbf{E} = \mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + ... + a_1\mathbf{M} + a_0\mathbf{I} \quad \square \tag{40}$$