Laplace Transform and Transfer Functions

Control Theory, Lecture 4

by Sergei Savin

Spring 2025

Content

By definition, Laplace transform of a function $f(t)$ is given as:

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{1}$$

where $F(s)$ is called an *image* of the function.

The study of Laplace transform is a separate mathematical field with applications in solving ODEs, which we won't cover. However, we will consider transform of one case of interest - transform of a derivative.

Laplace Transform of a derivative

Consider a derivative $\frac{dx}{dt}$ and its transform:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = \int_0^\infty \frac{dx}{dt} e^{-st} dt \tag{2}$$

we will make use of the integration by parts formula:

Integration by parts

$$\int v \frac{du}{dt} dt = vu - \int \frac{dv}{dt} u dt \tag{3}$$

In our case, $\frac{du}{dt} = \frac{dx}{dt}$, $u = x$, $v = e^{-st}$, $\frac{dv}{dt} = -se^{-st}$:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = \left[xe^{-st}\right]_0^\infty - \int_0^\infty -se^{-st} x dt \tag{4}$$

$$\mathcal{L}\left(\frac{dx}{dt}\right) = -x(0) + s\mathcal{L}(x) \tag{5}$$

Thus, assuming that $x(0) = 0$ and denoting $\mathcal{L}(x) = X(s)$, we can obtain a *derivative operator*:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = s\mathcal{L}(x) = sX(s) \quad (6)$$

This form of a derivative operator is very simple to use in practice.

Consider the following ODE, where u is an input (function of time that influences the solution of the ODE):

$$\ddot{y} + a\dot{y} + by = u \tag{7}$$

We can rewrite it using the derivative operator:

$$s^2 Y(s) + asY(s) + bY(s) = U(s) \tag{8}$$

and then collect $Y(s)$ on the left-hand-side:

$$Y(s) = \frac{1}{s^2 + as + b} U(s) \tag{9}$$

where expression $\frac{1}{s^2+as+b}$ is called a *transfer function*. A transfer function maps an input signal to an output signal.

Example

Given ODE: $2\dddot{y} + 5\dot{y} - 40y = 10u$

Its transfer function representation: $Y(s) = \frac{10}{2s^3+5s-40}U(s)$

Example

Given ODE: $2\dot{y} - 4y = u$

Its transfer function representation: $Y(s) = \frac{1}{2s-4}U(s)$

Example

Given ODE: $3\dddot{y} + 4y = u$

Its transfer function representation: $Y(s) = \frac{1}{2s^3+4}U(s)$

Consider the following ODE:

$$2\ddot{y} + 3\dot{y} + 2y = 10\dot{u} - u \tag{10}$$

In the Laplace domain it takes form:

$$2s^2 Y(s) + 3sY(s) + 2Y(s) = 10sU(s) - U(s) \tag{11}$$

$$(2s^2 + 3s + 2)Y(s) = (10s - 1)U(s) \tag{12}$$

The transfer function representation:

$$Y(s) = \frac{10s - 1}{2s^2 + 3s + 2} U(s) \tag{13}$$

Transfer Functions, 2

Consider the control law:

$$u = -k_p y - k_d \dot{y} \tag{14}$$

Transfer function representation of this control law is:

$$U(s) = -(k_d s + k_p) Y(s) \tag{15}$$

State-Space to Transfer Function conversion

Transfer functions are being used to study the relation between the input and the output of the dynamical system.

Consider standard form state-space dynamical system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \tag{16}$$

We can rewrite it using the derivative operator:

$$\begin{cases} s\mathbf{I}\mathbf{x} - \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \tag{17}$$

and then collect $\mathbf{x}$ on the left-hand-side: $\mathbf{x} = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}$ and finally, express $\mathbf{y}$ out:

$$\mathbf{y} = \left(\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)\mathbf{u} \tag{18}$$

Consider a linear ODE, and its equivalent representations as a state space equation and as a transfer function:

$$a_n y^n + ... + a_1 y = b_m u^m + ... + b_1 u \tag{19}$$

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \tag{20}$$

$$Y(s) = G(s)U(s) \tag{21}$$

We can call it a *system* $\mathcal{G}$ to avoid referencing particular representation.

Closed-loop, 1

Open-loop system representation is $Y(s) = G(s)U(s)$. Let us propose a control law (in time domain):

$$u(t) = k_p(v(t) - y(t)) + k_d(\dot{v}(t) - \dot{y}(t)) \tag{22}$$

where $v(t)$ is a control reference. Laplace transform of this control law takes form:

$$U(s) = (k_p + k_d s)(V(s) - Y(s)) \tag{23}$$

Defining $H(s) = k_p + k_d s$ the closed loop system takes form:

$$Y(s) = G(s)H(s)(V(s) - Y(s)) \tag{24}$$

$$Y(s) = -G(s)H(s)Y(s) + G(s)H(s)V(s) \tag{25}$$

$$(1 + G(s)H(s))Y(s) = G(s)H(s)V(s) \tag{26}$$

$$Y(s) = \frac{G(s)H(s)}{1 + G(s)H(s)}V(s) \tag{27}$$

Closed-loop, 2

Alternatively, we can define a new reference signal $r(t)$:

$$r(t) = k_p v(t) + k_d \dot{v}(t) \tag{28}$$

The previous control law takes form:

$$u(t) = -k_p y(t) - k_d \dot{y}(t) + r(t) \tag{29}$$

Laplace transform of this control law:

$$U(s) = -H(s)Y(s) + R(s) \tag{30}$$

The closed loop system:

$$Y(s) = -G(s)H(s)Y(s) + G(s)R(s) \tag{31}$$

$$Y(s) + G(s)H(s)Y(s) = G(s)R(s) \tag{32}$$

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)} R(s) \tag{33}$$

Below are a few functions and their Laplace transform:

$$f(t) = e^{at} \qquad F(s) = \frac{1}{s-a}$$

$$f(t) = \cos(\omega t) \qquad F(s) = \frac{s}{s^2 + \omega^2}$$

$$f(t) = \sin(\omega t) \qquad F(s) = \frac{\omega}{s^2 + \omega^2}$$

$$f(t) = e^{at}(A\cos(\omega t) + B\sin(\omega t)) \qquad F(s) = \frac{A(s-a) + B\omega}{(s-a)^2 + \omega^2}$$

Laplace transform is a linear operation. If $F(s) = \mathcal{L}(f(t))$ and $G(s) = \mathcal{L}(g(t))$, then:

$$\mathcal{L}\left(af(t) + bg(t)\right) = aF(s) + bG(s) \tag{34}$$

Partial-fraction expansion, 1

Consider a transfer function $G(s)$ which is a rational fraction:

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + ... + b_1 s + b_0}{a_n s^n + ... + a_1 s + a_0} \tag{35}$$

If the denominator polynomial $D(s)$ has purely real non-repeating roots p_i (called *poles*) we can re-write it as:

$$G(s) = \frac{N(s)}{(s - p_1)(s - p_2) \; ... \; (s - p_n)} \tag{36}$$

If the degree of $D(s)$ is higher than the degree of $N(s)$, then we can perform a partial-fraction expansion:

$$G(s) = \frac{N(s)}{(s - p_1)(s - p_2) \; ... \; (s - p_n)} = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + ... + \frac{r_n}{s - p_n}$$

Let us consider $G(s)$ after the expansion:

$$G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + ... + \frac{r_n}{s - p_n}$$

We know that $F(s) = \frac{r_i}{s - p_i}$ is a Laplace image of the time domain function $f(t) = r_i e^{p_i t}$. Hence, the *inverse Laplace transform* of $G(s)$ is given as:

$$y(t) = r_1 e^{p_1 t} + ... + r_n e^{p_n t},$$

As we can see, the Laplace transform and its inverse, paired with partial-fraction expansion allow us to solve ordinary differential equations.

If the denominator polynomial $D(s)$ has both real and imaginary roots we can re-write it as:

$$G(s) = \frac{N(s)}{(s-p_1)\ ...\ (s-p_k)(s^2+\alpha_1 s+\beta_1)\ ...\ (s^2+\alpha_l s+\beta_l)} \tag{37}$$

If the degree of $D(s)$ is lower or equal to that of $N(s)$ we can divide the numerator by the denominator:

$$G(s) = R(s) + \frac{M(s)}{(s-p_1)\ ...\ (s^2+\alpha_l s+\beta_l)}$$

where degree of $M(s)$ is less than that of $D(s)$. We can then perform a partial-fraction expansion:

$$G(s) = R(s) + \frac{r_1}{s-p_1} + ... + \frac{r_k}{s-p_k} + \frac{q_1 s + h_1}{s^2+\alpha_1 s+\beta_1} + ... + \frac{q_l s + h_l}{s^2+\alpha_l s+\beta_l}$$

Note that the Laplace domain function $F(s) = \frac{qs+h}{s^2+\alpha s+\beta}$ is an image of the time function:

$$e^{at}(A\cos(\omega t) + B\sin(\omega t)) = \mathcal{L}^{-1}\left(\frac{qs+h}{s^2+\alpha s+\beta}\right)$$

Thus

Partial-fraction expansion can be perfomed using MATLAB or Python:

- In MATLAB `[r,p,k] = residue(b,a)`
- In Python `r,p,k = scipy.signal.residue(b,a)`

In both cases, `a` are the coefficients of the denominator and `b` are the coefficients of the numerator.

Literature

- Nise, N.S. Control systems engineering. John Wiley & Sons. (Chapter 2 Modeling in the Frequency Domain)
- Chapter 6 Transfer Functions
- Control Systems Lectures - Transfer Functions, by Brian Douglas
- The Laplace Transform - A Graphical Approach, by Brian Douglas

Lecture slides are available via Github:

github.com/SergeiSa/Control-Theory-2025

Steady-State gain

If a system $\mathcal{G}$ is stable and given constant input u_0 its output is approaching some constant value y_0, we can call this pair a *steady-state solution*. The ratio between y_0 and u_0 is a *steady-state gain* - how much does the system increase the input signal.
Assume the system $\mathcal{G}$ represented as a transfer function:

$$Y(s) = \frac{b_m s^m + ... + b_1}{a_n s^n + ... + a_1} U(s) \tag{38}$$

Then, as any element multiplied by the differential operator s with power higher than 0 is a derivative of u or y and both are 0 at the steady-state solution, the steady-state gain can be found by setting those to zero:

$$K = \frac{b_1}{a_1} \tag{39}$$