

# Controllability, Observability

## Control Theory, Lecture 11

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- Cayley–Hamilton
- Controllability of Discrete LTI
  - ▶ Controllability matrix
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- Observability of Discrete LTI
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- Controllability of Continuous-Time LTI

# DEFINITIONS

## Definition (Controllability)

A system is controllable on time interval  $t_0 \leq t \leq t_f$ , if it is possible to find control input  $u(t)$  that would drive the system to a desired state  $\mathbf{x}(t_f)$  from any initial state  $\mathbf{x}(t_0)$ .

## Definition (Observability)

A system is observable on time interval  $t_0 \leq t \leq t_f$ , if using output  $\mathbf{y}(t)$  on that time interval it is possible to estimate exactly the state of the system  $\mathbf{x}(t_f)$ , given any initial estimation error.

## Definition (Observability, alternative)

A system is observable on time interval  $t_0 \leq t \leq t_f$ , if any initial state  $\mathbf{x}(t_0)$  is uniquely determined by output  $\mathbf{y}(t)$  on that interval.

# CONTROLLABILITY OF DISCRETE LTI

Consider discrete LTI:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \quad (1)$$

Assume the initial state is  $\mathbf{x}_1$ . Then we can deduce that:

$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1$$

$$\mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 + \mathbf{B}\mathbf{u}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2$$

$$\mathbf{x}_4 = \mathbf{A}\mathbf{x}_3 + \mathbf{B}\mathbf{u}_3 = \mathbf{A}(\mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2) + \mathbf{B}\mathbf{u}_3$$

...

$$\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + \mathbf{A}^{n-1} \mathbf{B} \mathbf{u}_1 + \dots + \mathbf{A}^{n-k} \mathbf{B} \mathbf{u}_k + \dots + \mathbf{A} \mathbf{B} \mathbf{u}_{n-1} + \mathbf{B} \mathbf{u}_n$$

# CONTROLLABILITY MATRIX

$\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + \mathbf{A}^{n-1} \mathbf{B} \mathbf{u}_1 + \dots + \mathbf{A}^{n-k} \mathbf{B} \mathbf{u}_k + \dots + \mathbf{A} \mathbf{B} \mathbf{u}_{n-1} + \mathbf{B} \mathbf{u}_n$   
can be re-written as:

$$\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1 = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}] \begin{bmatrix} \mathbf{u}_n \\ \mathbf{u}_{n-1} \\ \mathbf{u}_{n-2} \\ \dots \\ \mathbf{u}_1 \end{bmatrix} \quad (2)$$

Notice that in order for the system to go from  $\mathbf{x}_1$  to  $\mathbf{x}_{n+1}$ , vector  $\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1$  needs to be in the column space of  $\mathcal{C} = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$ .

Since  $\mathbf{x}_{n+1}$  can be anything, and  $\mathbf{x}_1$  might be equal to zero (among other possibilities), we should require that all vectors in  $\mathbb{R}^n$  are in the column space of  $\mathcal{C}$ , meaning  $\mathcal{C}$  needs to be full row rank.

## Controllability

The system  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$ ,  $\mathbf{x} \in \mathbb{R}^n$  is *controllable* if its controllability matrix  $\mathcal{C} = [\mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$  is full row-rank ( $\text{rank}(\mathcal{C}) = n$ ).

Equation  $\det(\mathbf{M} - \lambda \mathbf{I}) = 0$  is called *characteristic equation* of matrix  $\mathbf{M}$ , its roots being eigenvalues of the matrix.

**Theorem (Cayley–Hamilton)**

*A matrix  $\mathbf{M} \in \mathbb{R}^{n,n}$  satisfies its own characteristic equation.*

A characteristic equation can be written as  $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$ , meaning that we can write:

$$\mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + \dots + a_1\mathbf{M} + a_0\mathbf{I} = 0 \quad (3)$$

Meaning that  $\mathbf{M}^n$  is a linear combination of  $\mathbf{M}^{n-1}$ ,  $\mathbf{M}^{n-2}$ , ...,  $\mathbf{I}$ . See Appendix.

What happens if we add more columns to the controllability matrix, for example  $\mathbf{A}^n\mathbf{B}$ ? Consider the matrix:

$$\mathcal{C}_+ = [\mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B} \quad \mathbf{A}^n\mathbf{B}] \quad (4)$$

But from Cayley–Hamilton we know that:

$$\mathbf{A}^n = -a_{n-1}\mathbf{A}^{n-1} - \dots - a_0\mathbf{I} \quad (5)$$

$$\mathbf{A}^n\mathbf{B} = -a_{n-1}\mathbf{A}^{n-1}\mathbf{B} - \dots - a_0\mathbf{B} \quad (6)$$

Meaning that columns of  $\mathbf{A}^n\mathbf{B}$  are expressed as linear combination of columns of  $\mathcal{C}$ , hence the matrix  $\mathcal{C}_+$  has the same rank as  $\mathcal{C}$ .



# OBSERVABILITY OF DISCRETE LTI

Consider discrete LTI:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i \end{cases} \quad (7)$$

And an observer:

$$\hat{\mathbf{x}}_{i+1} = \mathbf{A}\hat{\mathbf{x}}_i + \mathbf{B}\mathbf{u}_i + \mathbf{L}(\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_i) \quad (8)$$

Remember that we can define observation error  $\mathbf{e}_i = \hat{\mathbf{x}}_i - \mathbf{x}_i$  and write its dynamics:

$$\mathbf{e}_{i+1} = \mathbf{A}\mathbf{e}_i - \mathbf{L}\mathbf{C}\mathbf{e}_i \quad (9)$$

Dual system (which is stable if and only if the original is stable), has form:

$$\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i \quad (10)$$

# OBSERVABILITY OF DISCRETE LTI

## Dual system

Dynamical system  $\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i$ , we can be represented as:

$$\begin{cases} \varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i + \mathbf{C}^\top \mathbf{v}_i \\ \mathbf{v}_i = -\mathbf{L}^\top \varepsilon_i \end{cases} \quad (11)$$

Controllability matrix of this system is:

$$\mathcal{O}^\top = [\mathbf{C}^\top \quad (\mathbf{A}^\top)\mathbf{C}^\top \quad \dots \quad (\mathbf{A}^\top)^{n-1}\mathbf{C}^\top] \quad (12)$$

It is easier to represent this matrix in its transposed form:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix} \quad (13)$$

## Observability

The system  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$  and  $\mathbf{y}_i = \mathbf{C}\mathbf{x}_i$ ,  $\mathbf{x} \in \mathbb{R}^n$  is

*observable*, if the observability matrix  $\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$  is full

column-rank ( $\text{rank}(\mathcal{O}) = n$ ).

# CONTROLLABILITY, CONTINUOUS-TIME (1)

Matrix exponential  $e^{\mathbf{A}t}$  is defined as a series:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + \dots \quad (14)$$

Using Cayley–Hamilton we can observe that any powers of  $\mathbf{A}$  higher than  $n$  can be represented as a linear combination of lower powers. This gives us the following expression:

$$e^{\mathbf{A}t} = \phi_0(t)\mathbf{I} + \phi_1(t)\mathbf{A} + \phi_2(t)\mathbf{A}^2 + \dots + \phi_{n-1}(t)\mathbf{A}^{n-1} \quad (15)$$

This allows us to re-write the forced state response:

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau \\ \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t (\phi_0(t-\tau)\mathbf{I} + \phi_1(t-\tau)\mathbf{A} + \dots \\ &\quad + \phi_{n-1}(t-\tau)\mathbf{A}^{n-1})\mathbf{b}u(\tau) d\tau \end{aligned}$$

## CONTROLLABILITY, CONTINUOUS-TIME (2)

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \phi_0(t-\tau)\mathbf{b}u(\tau)d\tau \\ &+ \int_0^t \phi_1(t-\tau)\mathbf{A}\mathbf{b}u(\tau)d\tau + \dots \int_0^t \phi_{n-1}(t-\tau)\mathbf{A}^{n-1}\mathbf{b}u(\tau)d\tau\end{aligned}$$

$$\mathbf{x}(t) - e^{\mathbf{A}t}\mathbf{x}(0) = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{b}] \begin{bmatrix} \int_0^t \phi_0(t-\tau)u(\tau)d\tau \\ \int_0^t \phi_1(t-\tau)u(\tau)d\tau \\ \dots \\ \int_0^t \phi_{n-1}(t-\tau)u(\tau)d\tau \end{bmatrix}$$

If the controllability matrix is rank-deficient, there will exist a state  $\mathbf{x}_f$  and which cannot be reached from some initial conditions  $\mathbf{x}_0$ .

There is an alternative way to test the controllability of a pair  $(\mathbf{A}, \mathbf{B})$ :

## PBH controllability criterion

If for any  $\lambda \in \mathbb{C}$ , the matrix  $[(\mathbf{A} - \lambda\mathbf{I}), \mathbf{B}]$  has full row-rank, then the pair  $(\mathbf{A}, \mathbf{B})$  is controllable.

- If  $\lambda$  is not an eigenvalue of  $\mathbf{A}$ , then  $\det(\mathbf{A} - \lambda\mathbf{I}) \neq 0$  and the matrix has full row rank.
- If  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$  it is sufficient to test the rank of  $[(\mathbf{A} - \lambda\mathbf{I}), \mathbf{B}]$ .

- Controllability and Observability (Rutgers University)  
<https://www.ece.rutgers.edu/~gajic/psfiles/chap5.pdf>

Lecture slides are available via Github:

[github.com/SergeiSa/Control-Theory-2025](https://github.com/SergeiSa/Control-Theory-2025)





# Appendix A: Analytical solution (recap)

Exponential  $e^a$  is defined as a series:

$$e^a = 1 + a + \frac{1}{2}a^2 + \frac{1}{6}a^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}a^n \quad (16)$$

Matrix exponential  $e^{\mathbf{A}}$  is defined as a series:

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}\mathbf{A} + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n \quad (17)$$

# ANALYTICAL SOLUTION TO ODE

An ODE of the form  $\dot{x} = ax$  has analytical solution  $x(t) = e^{at}x(0)$ .

An ODE of the form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  has analytical solution  $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$ .

Let us check that this is a solution:

$$\mathbf{x}(t) = \left( \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + \dots \right) \mathbf{x}(0) \quad (18)$$

$$\dot{\mathbf{x}}(t) = \left( \mathbf{A} + \mathbf{A}\mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}\mathbf{A}t^2 + \dots \right) \mathbf{x}(0) \quad (19)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A} \left( \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \dots \right) \mathbf{x}(0) \quad (20)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) \quad (21)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad \square \quad (22)$$

# FORCED STATE RESPONSE (LTI) (1)

An ODE of the form  $\dot{x} = ax + bu(t)$  also has analytical solution.  
To find it, we first find the following derivative:

$$\frac{d}{dt} (e^{-at}x(t)) = e^{-at}\dot{x}(t) - ae^{-at}x(t) \quad (23)$$

Multiplying  $\dot{x} = ax + bu(t)$  by  $e^{-at}$  we see:

$$e^{-at}\dot{x} = e^{-at}ax + e^{-at}bu(t) \quad (24)$$

$$e^{-at}\dot{x} - e^{-at}ax = e^{-at}bu(t) \quad (25)$$

$$\frac{d}{dt} (e^{-at}x(t)) = e^{-at}bu(t) \quad (26)$$

$$\int_0^t \frac{d}{d\tau} (e^{-a\tau}x(\tau)) d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \quad (27)$$

# FORCED STATE RESPONSE (LTI) (2)

Continuing the derivation:

$$\int_0^t \frac{d}{d\tau} (e^{-a\tau} x(\tau)) d\tau = \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (28)$$

$$e^{-at} x(t) - x(0) = \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (29)$$

$$x(t) = e^{at} x(0) + e^{at} \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (30)$$

$$x(t) = e^{at} x(0) + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \quad (31)$$

## FORCED STATE RESPONSE (LTI) (3)

State-space equation  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t)$  also has an analytical solution:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \quad (32)$$

The same can be re-written as:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \quad (33)$$

# CAYLEY–HAMILTON ILLUSTRATION

Consider matrix  $\mathbf{M}$  and its characteristic equation

$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$  and a decomposition

$\mathbf{M} = \mathbf{V}^{-1}\Lambda\mathbf{V}$ . Let's prove that the following expression is zero:

$$\mathbf{E} = \mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + \dots + a_1\mathbf{M} + a_0\mathbf{I} \quad (34)$$

$$\mathbf{E} = \mathbf{V}^{-1}\Lambda^n\mathbf{V} + a_{n-1}\mathbf{V}^{-1}\Lambda^{n-1}\mathbf{V} + \dots + a_1\mathbf{V}^{-1}\Lambda\mathbf{V} + a_0\mathbf{I} \quad (35)$$

$$\mathbf{VEV}^{-1} = \Lambda^n + a_{n-1}\Lambda^{n-1} + \dots + a_1\Lambda + a_0\mathbf{VV}^{-1} \quad (36)$$

$$\mathbf{VEV}^{-1} = \begin{bmatrix} (\lambda_1^n + a_{n-1}\lambda_1^{n-1} + \dots + a_0) & & \\ & \dots & \\ & & (\lambda_n^n + a_{n-1}\lambda_n^{n-1} + \dots + a_0) \end{bmatrix} \quad (37)$$

$$\mathbf{VEV}^{-1} = \begin{bmatrix} 0 & & \\ & \dots & \\ & & 0 \end{bmatrix} \quad (38)$$

$$\mathbf{VEV}^{-1} = 0 \quad (39)$$

$$0 = \mathbf{E} = \mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + \dots + a_1\mathbf{M} + a_0\mathbf{I} \quad \square \quad (40)$$